UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XLIV
2006

A NOTE ON PLURIPOLAR EXTENSIONS OF UNIVALENT FUNCTIONS

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Abstract. In this note we present a detailed proof of a recent result due to Edlund and Jöricke (see Corollary 2 in [**1**]) saying that there exists a univalent function f in the unit disc $\boldsymbol{D} := \{|z| < 1\}$ smooth up to the boundary such that f does not have analytic continuation across any point of the unit circle while the pluripolar hull of its graph over $\boldsymbol{D}$ contains the graph of the function $f_e(z) := 1/\overline{f(1/\bar{z})}$ univalent in $\boldsymbol{D}_e := \{|z| > 1\}$.

1. Introduction. Given a pluripolar subset of $\boldsymbol{C}^N$, its (global) *pluripolar hull* E^* is defined by the formula

$$E^* := \bigcap_{U \in \mathcal{F}_E} \{U(z) = -\infty\}, \tag{1}$$

where $\mathcal{F}_E := \{U \in PSH(\boldsymbol{C}^N); U(z) = -\infty \quad on \quad E\}$. A pluripolar set E is called *complete pluripolar* if there exists $U \in PSH(\boldsymbol{C}^N)$ such that $E = \{U(z) = -\infty\}$.

We say that a function $f_2 \in \mathcal{O}(D_2)$ holomorphic in a domain $D_2 \subset \boldsymbol{C}^N$ is a *pluripolar continuation* of a function $f_1 \in \mathcal{O}(D_1)$ holomorphic on a domain $D_1 \subset \boldsymbol{C}^N$, if $\Gamma^*_{f_1}(D_1) \supset \Gamma_{f_2}(D_2)$, i.e. if for for every function $U \in PSH(\boldsymbol{C}^{N+1})$ such that $U(z, f_1(z)) = -\infty$ on D_1 we have $U(z, f_2(z)) = -\infty$ on D_2.

If $f \in \mathcal{O}(D)$ is a holomorphic function in a domain D in $\boldsymbol{C}^N$ then its graph $\Gamma_f(D)$ is a pluripolar subset of $\boldsymbol{C}^{N+1}$. Given $f \in \mathcal{O}(D)$, let $\tilde{f}$ be the complete multivalued analytic function defined on a domain $\tilde{D} \supset D$ such that f is its

1991 *Mathematics Subject Classification.* 30B40, 32D15, 32F05.

Key words and phrases. Plurisubharmonic function, pluripolar set, pluripolar hull, pseudo-continuation.

Supported by KBN grant no. P03A 047 22.

holomorphic branch on D. One can easily check that the pluripolar hull of $\Gamma_f(D)$ contains

$$\Gamma_{\tilde f}(\tilde D) := \{(z,w) \in \boldsymbol{C}^N \times \boldsymbol{C}; z \in D,\, w \in \tilde f(z)\},$$

the graph of $\tilde f$ over $\tilde D$, i.e. $\Gamma^*_f(D) \supset \Gamma_{\tilde f}(\tilde D)$.

The aim of this note is to prove the following slight improvement of Corollary 2 in [**1**].

Theorem 1.1. *Let E be a non-empty nowhere dense compact subset of the unit circle. There exists a conformal $\mathcal{C}^\infty$-diffeomorphism*

$$f : \bar{\boldsymbol{D}} \mapsto \bar G, \quad f(0) = 0,$$

of the closure of the unit disk $\boldsymbol{D}$ onto the closure of a domain $G \subset \boldsymbol{D}$, strictly starlike with respect to 0, such that the following conditions are satisfied:

(a) f does not have analytic continuation across any point of the unit circle;

(b) the set $E_1 := \bar G \cap \partial \boldsymbol{D}$ has positive Lebesgue measure, $E \subset E_1$ and the function $f_e(z) := 1/\overline{f(1/\bar z)}$, $z \in \boldsymbol{D}_e := \{\frac{1}{\bar z}\,;\, |z| < 1\}$, is a pseudo-continuation of f across the set $f^{-1}(E_1)$; [1]

(c) $\Gamma^*_f(\boldsymbol{D}) = \Gamma^*_{f_e}(\boldsymbol{D}_e \setminus \{\infty\}) \supset \Gamma_f(f^{-1}(E))$, i.e. the functions f and f_e are pluripolar continuations of each other across the graph of f over the set $f^{-1}(E)$. In other words: if $P \in PSH(\boldsymbol{C}^2)$ and $P(z, f(z)) = -\infty$ on $\boldsymbol{D}$ (resp., $P(z, f_e(z)) = -\infty$ on $\boldsymbol{D}_e \setminus \{\infty\}$) then $P(z, f_e(z)) = -\infty$ on $(\boldsymbol{D}_e \setminus \{\infty\}) \cup f^{-1}(E)$ (resp., $P(z, f(z)) = -\infty$ on $\boldsymbol{D} \cup f^{-1}(E)$).

2. Proof of Theorem 1.1.

First we shall prove the following

Lemma 2.1. *Given a non-empty compact nowhere dense subset E of the unit circle, one can find a domain $G \subset \boldsymbol{D}$, strictly starlike with respect to 0, such that the following conditions are satisfied:*

(a) ∂G is a $\mathcal{C}^\infty$-smooth Jordan curve which is real analytic at no of its points;

(b) $E \subset E_1 := \bar G \cap \partial \boldsymbol{D}$, $\lambda(E_1) > 0$ (λ – the Lebesgue measure on $\partial \boldsymbol{D}$);

(c) There exists a positive constant m_1 such that

$$V_U(z) \equiv V_{\tilde U}(z) \geq m_1, \quad z \in E,$$

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[1] It is clear that f_e maps conformally the closure of $\boldsymbol{D}_e$ onto the closure of $G_e := \{\frac{1}{\bar w}\,;\, w \in G\}$, and $f(z) = f_e(z)$ for all $z \in f^{-1}(E_1)$, which implies that f and f_e are pseudo-continuations of each other across $f^{-1}(E_1)$. More information on pseudo-continuation may be found in [**3**] .

where $U := \boldsymbol{C} \setminus (\bar{G} \cup \overline{G_e})$, $G_e := \{1/\bar{z}\,;\, z \in G\}$, V_U *is the global extremal function of* U *(for the definition see* [**2**] *or* [**4**]*), and* $\tilde{U} := \bigcup_{j=1}^{\infty} \overline{U_j}$*, where the union is taken over all connected components of the open set* U.[2]

Proof of Lemma 2.1. First we shall prove

Claim 1. *Let* E *be a non-empty nowhere dense closed subset of the unit circle. There exists a sequence of open arcs* $\{I_j\}$ *of the unit circle with the following properties:*

(1) $\bar{I}_j \cap \bar{I}_k = \emptyset\, (j \neq k)$*;*

(2) the set $S := \bigcup_1^{\infty} I_j$ *is dense on the unit circle;*

(3) the set $\tilde{S} := \bigcup_1^{\infty} \bar{I}_j$ *does not intersect* E*, and there exists* $m_1 > 0$ *such that* $V_S(z) = V_{\tilde{S}}(z) \geq m_1$, $z \in E$*. In particular, the set* $\tilde{S}$ *is thin at each point of* E*;*

(4) $\lambda(E_1) > 0$*, where* $E_1 := \partial \boldsymbol{D} \setminus S$*.*

Proof of Claim 1. Let $W = \{w_n\}$ be a countable dense subset of $\partial \boldsymbol{D} \setminus E$. We shall choose arcs of the sequence $\{I_j\}$ inductively.

Let I_1 be an open arc with center w_1 such that no of its endpoints belongs to W, and $\bar{I}_1 \cap E = \emptyset$. The number $2m_1 := \min\{V_{I_1}(z);\, z \in E \cap \{0\}\}$ is positive.[3]

Fix $k \geq 1$. Suppose arcs $I_1, \ldots, I_k$ with centers $w_{n_1}, \ldots, w_{n_k}$ $(n_1 = 1 < n_2 < \cdots < n_k)$ are already chosen in such a way that the following conditions are satisfied: $\bar{I}_j \cap \bar{I}_l = \emptyset$ $(j \neq l,\ j, l \leq k)$, no endpoint of I_j lies in W, $w_{n_{j+1}}$ is the element of $W \setminus (I_1 \cup \cdots \cup I_j)$ with the smallest index, and

$$V_{I_1 \cup \cdots \cup I_j}(z) \geq m_1\Big(2 - \frac{1}{2} - \cdots - \frac{1}{2^j}\Big), \quad z \in E \cap \{0\}, \quad j = 1, \ldots, k.$$

Let $w_{n_{k+1}}$ be the element of $W \setminus (I_1 \cup \cdots \cup I_k)$ with the smallest index. Let I_{k+1} be an open arc with center $w_{n_{k+1}}$ whose endpoints do not belong to W and which is so short that

$$V_{I_1 \cup \cdots \cup I_{k+1}}(z) \geq m_1\Big(2 - \frac{1}{2} - \cdots - \frac{1}{2^{k+1}}\Big), \quad z \in E \cap \{0\}.$$

It is clear that the sequence $\{I_k\}$ satisfies (1) and (2).

To show (3) it is sufficient to observe that

$$V_S(z) = V_{\tilde{S}}(z) = \lim_{n \to \infty} V_{I_1 \cup \cdots \cup I_n}(z), \quad z \in \boldsymbol{C},$$

is a subharmonic function with logarithmic pole at ∞, harmonic on $\boldsymbol{C} \setminus \bar{S}$, continuous on $\boldsymbol{D} \cup \tilde{S}$, $V_S(z) = 0$ on $\tilde{S}$, and $V_S(z) \geq m_1$ for all $z \in E$.

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[2]It is clear that for every $j \geq 1$ the component U_j is a simple connected Jordan domain symmetric with respect to the unit circle. One may assume that $I_j \subset U_j$.

[3]Recall that V_{I_1} is identical with the Green function of $\hat{\boldsymbol{C}} \setminus \bar{I}_1$ with pole at ∞.

To show (4) observe that

$$V_S(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{it}-z|^2} V_S(e^{it})dt = \frac{1}{2\pi}\int_{E_1} \frac{1-|z|^2}{|e^{it}-z|^2} V_S(e^{it})dt, \quad z \in \boldsymbol{D},$$

which implies $\lambda(E_1) > 0$.

The proof of our claim is completed. □

Now we pass to the proof of Lemma 2.1.

Let $\{I_j\}$ be a sequence of arcs satisfying the conditions of Claim 1. Let $p \in \mathcal{C}^\infty(\boldsymbol{R})$ be a positive real-valued function of class $\mathcal{C}^\infty$ on the real line such that $0 < p(t) \le 1$ on $\boldsymbol{R}$, and p is nowhere $\boldsymbol{R}$-analytic, e.g. we can take $p(t) = \frac{1}{1+|h(e^{it})|^2}$, $t \in \boldsymbol{R}$, where

$$h(z) = \sum_1^\infty 2^{-2^{\sqrt{n}}} z^{2^n}, \quad |z| \le 1.$$

Without loss of generality we may assume $1 \in E$. Let e^{α_j}, e^{β_j} be endpoints of I_j, where $0 < \alpha_j < \beta_j < 2\pi$. Put

$$r_j(t) := p(t)\exp\left[-\frac{1}{1-(\frac{2(t-\alpha_j)}{\beta_j-\alpha_j}-1)^2}\right], \quad \alpha_j \le t \le \beta_j,$$

$$r_j(t) := 0, \quad t \in [0\,,2\pi] \setminus (\alpha_j, \beta_j).$$

One can check that $r_j \in \mathcal{C}^\infty([0,\,2\pi])$ and $r_j^{(k)}(t) = 0$ for all $k \ge 1$ and for all $t \in [0\,,2\pi] \setminus (\alpha_j, \beta_j)$, i.e. the function r_j is flat at every point of the last set. Moreover, r_j is positive at every point of the open interval $(\alpha_j,\,,\,\beta_j)$ and not $\boldsymbol{R}$-analytic at any point of the closed interval $[\alpha_j, \beta_j]$. It is clear that r_j can be extended to $\boldsymbol{R}$ as a $\mathcal{C}^\infty$ periodic function with period 2π.

Put

$$r(t) := \sum_1^\infty \epsilon_j r_j(t), \quad t \in \boldsymbol{R}, \tag{2}$$

where $\epsilon_j > 0$ is chosen so small that

$$\epsilon_j |r_j^{(k)}(t)| < \frac{1}{2^j}, \quad k = 0, \cdots, j, \quad j \ge 1, \quad t \in \boldsymbol{R}. \tag{3}$$

It is clear that $0 \le r(t) < 1$, $t \in \boldsymbol{R}$, $r \in \mathcal{C}^\infty(\boldsymbol{R})$, r is periodic with period 2π, and nowhere $\boldsymbol{R}$-analytic. Observe that if s is a boundary point of I_k then $r(s) = \epsilon r_k(s) = 0$. Each point t of E is a limit point of such points s. Hence $\{r(t) = 0\} = \partial\boldsymbol{D} \setminus S =: E_1$.

The domain G containing 0 in its interior and bounded by the curve γ with the parametric representation

$$z = \gamma(t) \equiv (1 - r(t))\, e^{it}, \quad 0 \le t \le 2\pi,$$

is strictly starlike with respect to 0. Moreover, $E \subset E_1 := \bar{G} \cap \partial\boldsymbol{D} \equiv \partial\boldsymbol{D} \setminus S$, and ∂G is a $\mathcal{C}^\infty$-smooth Jordan curve nowhere $\boldsymbol{R}$-analytic.

We shall show that, given $0 < m < m_1$, the coefficients ϵ_j in the formula (2) can be chosen so small that

$$V_U(z) \equiv V_{\boldsymbol{C}\setminus(\bar{G}\cup\bar{G}_e)}(z) \ge m, \quad z \in E.$$

The function V_S, given by Claim 1, is non-negative in $\boldsymbol{C}$, continuous at each point of $\tilde{S}$, and $V_S(z) = 0$ on $\tilde{S}$. It follows that, given $0 < \delta < m_1$, the set $U_\delta := \{z;\, V_S(z) < \delta\}$ is an open neighborhood of $\tilde{S}$. In particular, $\bar{I}_j \subset U_\delta$ for every $j \ge 1$.

Hence one can choose coefficients ϵ_j so small that both (3) and the following condition (4) are satisfied

$$\{(1-\epsilon_j r_j(t))e^{it}\,,\ \frac{e^{it}}{1-\epsilon_j r_j(t)}\} \subset U_\delta, \quad \alpha_j \le t \le \beta_j,\ j \ge 1. \tag{4}$$

It is clear that $\tilde{U} = \cup_1^\infty \bar{U}_j \subset U_\delta$, where U_j is the connected component of U such that $I_j \subset U_j$. Hence $V_U(z) \ge V_{U_\delta} \equiv V_S - \delta \ge m := m_1 - \delta > 0, \quad z \in E$. This ends the proof of Lemma 2.1. □

We shall need the following

LEMMA 2.2. *Given $0 < \rho < 1 < R$ and a closed subset E of the unit circle, assume that U is an open subset of $\{\rho < |z| < R\}$ such that $V_U(z) \ge m = const > 0$ on E. Then for every $0 < \theta < 1$ there exists $0 < r_0 < \rho$ such that*

$$V_{D(0,r_0)\cup U}(z) \ \ge\ \theta m, \quad z \in E.$$

PROOF. Put $M := \sup\{V_U(z);\, |z| \le R\}$. Given $0 < \epsilon < 1$,

$$\varphi_\epsilon(z) \ :=\ (1-\epsilon)\log\frac{|z|}{R} + \epsilon V_U(z)$$

is a subharmonic function of the class $\mathcal{L}$ such that

$$\varphi_\epsilon(z) \le \begin{cases} 0, & z \in U, \\ (1-\epsilon)\log\frac{r}{R} + \epsilon M, & |z| \le r, \end{cases}$$

where $0 < r < \rho$. Hence, if $(1-\epsilon)\log\frac{r}{R} + \epsilon M \le 0$ (i.e. if $0 < \epsilon \le \frac{\log\frac{R}{r}}{M + \log\frac{R}{r}}$) then $\varphi_\epsilon(z) \le V_{D(0,r)\cup U}(z)$ on $\boldsymbol{C}$. Fix $0 < \theta < 1$. Then $\varphi_\epsilon(z) \ge \theta m$ on E, if

$\epsilon \geq \frac{\theta m+\log R}{m+\log R}$. Choose $r_0 = r$ with $0 < r < \rho$ so small that

$$\frac{\theta m + \log R}{m + \log R} < \frac{\log \frac{R}{r}}{M + \log \frac{R}{r}}.$$

Then $V_{D(0,r_0)\cup U}(z) \geq \varphi_\epsilon(z) \geq \theta m$ on E for $\epsilon \in \left(\frac{\theta m+\log R}{m+\log R}, \frac{\log \frac{R}{r}}{M+\log \frac{R}{r}}\right)$ which ends the proof of Lemma 2.2 □

We shall also need the following Theorem due to Vitushkin [**5**].

Let K be a compact subset of $\boldsymbol{C}$. Then $\boldsymbol{C} \setminus K$ is a (at most) countable union of open sets $\{U_j\}$. The set $\partial' K := \cup_j \partial U_j$ is called *exterior boundary of* K. Remaining part of the boundary ∂K is denoted by $\partial_0 K$ and called *interior boundary of* K.

Theorem 2.1. (*Vitushkin* [**5**]). *If the interior boundary of a compact set* K *is located on a countable union of Lyapunov's arcs then* $\mathcal{A}(K) = \mathcal{R}(K)$*, where* $\mathcal{A}(K) := \mathcal{C}(K) \cap \mathcal{O}(intK)$ *and* $\mathcal{R}(K) := \{f \in \mathcal{C}(K); f$ *is a uniform limit of a sequence of rational functions*$\}$.

Now we pass to the proof of Theorem 1.1. Let $g : \bar{G} \mapsto \bar{\boldsymbol{D}}, \quad g(0) = 0$, be the $\mathcal{C}^\infty$-smooth conformal mapping of the closure of the domain G given by Lemma 2.1 onto the closure of the unit disk. The function $g_e(z) = 1/\overline{g(1/\bar{z})}, \quad z \in \boldsymbol{D}_e$, is $\mathcal{C}^\infty$-smooth and maps $\bar{G}_e$ conformally onto $\bar{\boldsymbol{D}}_e$, $g_e(\infty) = \infty$. Moreover, $g(z) = g_e(z)$ on E_1.

The function $\mathcal{F} := g \cup g_e$ is continuous on $\bar{G} \cup \bar{G}_e$ and holomorphic in $G \cup G_e$.

Fix $R > 1$ so large that $\{|z| = R\} \subset G_e$, and put $U := \boldsymbol{C} \setminus (\bar{G} \cup \bar{G}_e)$. By Lemma 2.1, given m_1 with $0 < m_1 < m$, there exists $r_0 > 0$ such that $\frac{1}{r_0} > R$, $\overline{D(0,r_0)} \subset G$ and $V_{U\cup D(0,r_0)}(z) \geq m_1$ on E. It is clear that $V_{U\cup D(0,r_0)}(z) \leq \log^+ \frac{|z|}{r_0}$ on $\boldsymbol{C}$. Since $U = \{\frac{1}{\bar{z}}; z \in U\}$, the function $v(z) := V_{U\cup D(0,r_0)}(\frac{1}{\bar{z}})/\log \frac{R}{r_0}$ is subharmonic on $\boldsymbol{C} \setminus \{0\}$, $v(z) = 0$ on $U \cup D(0, 1/r_0)$, $v(z) \leq 1$ for $|z| \geq 1/R$, $v(z) \geq \frac{m_1}{\log \frac{R}{r_0}} > 0$ on E, and $v(z) > 0$ for all $z \in G_e \cup E$ with $|z| < 1/r_0$. Hence

$$v(z) \leq h(z) \equiv h(z, U \cup D(\infty, \frac{1}{r_0}), D(\infty, \frac{1}{R})), \quad |z| \geq \frac{1}{R},$$

where h denotes the (0-1)-extremal function for the domain $D(\infty, 1/R)$ and its subset $U \cup D(\infty, 1/r_0)$.[4] Here $D(\infty, \rho) := \{z \in \hat{\boldsymbol{C}}; |z| > \rho\}$, $\rho > 0$.

[4]Recall that if E is a subset of a domain D, we put $h(z, E, D) := \sup\{u(z); u \in SH(D), u \leq 0 \text{ on } E, u \leq 1 \text{ on } D\}$, $z \in D$.

Put $K := (\bar{G} \cup \bar{G}_e) \cap \{|z| \leq \frac{1}{r_0}\}$. By the Vitushkin Theorem there exists a sequence of rational functions $\{\mathcal{F}_n\}$ with poles in $U \cup D(\infty, \frac{1}{r_0})$ uniformly convergent to $\mathcal{F}$ on K.

Fix a function $P \in PSH(\boldsymbol{C}^2)$ such that $P(z, g(z)) = -\infty$ on G. Let a be a fixed point of $G_e \cup E$ with $|a| < 1/r_0$. It remains to show that $P(a, g_e(a)) = -\infty$.

Observe that $f_n(z) := \mathcal{F}_n(z) + \mathcal{F}(a) - \mathcal{F}_n(a) \to g(z)$ uniformly on $\{|z| = \frac{1}{R}\}$. The sequence $\{f_n\}$ is uniformly bounded on the set $D(0, 1/r_0) \setminus U$. Therefore the sequence $v_n(z) := P(z, f_n(z))$ is uniformly upper bounded on this set.

Put $\Omega_n := \cup_{j=1}^{k_n} U_j$, where k_n is so large that all poles of the function f_n, lying in U, are located in Ω_n. By the maximum principle

$$\sup\{|f_n(z)|;\, z \in D(0, 1/r_0) \setminus \Omega_n\} \;=\; \sup\{|f_n(z)|; \zeta \in D(0, 1/r_0) \setminus U\}$$

for all $n \geq 1$. The function v_n is subharmonic on an open neighborhood of the set $\overline{D}(0, 1/r_0) \setminus \Omega_n$. Put $C := \sup_{n \geq 1} \sup\{v_n(z);\, z \in D(0, 1/r_0) \setminus U\}$, and $M_n := \max\{v_n(z);\, |z| = \frac{1}{R}\}$. Then C is finite and $M_n \to -\infty$ as $n \to \infty$.

The function $h(z, D(\infty, 1/r_0) \cup \Omega_n, D(\infty, 1/R))$ is harmonic in the domain $\{\frac{1}{R} < |z| < \frac{1}{r_0}\} \setminus \bar{\Omega}_n$ and continuous in its closure, vanishes on $\{|z| = 1/r_0\} \cup \partial\Omega_n$, and is equal to 1 on $\{|z| = 1/R\}$. Hence, by two constant theorem

$$v_n(z) \;\leq\; C + (M_n - C) h(z, D(\infty, \frac{1}{r_0}) \cup \Omega_n, D(\infty, \frac{1}{R}))$$

for all z in $\{\frac{1}{R} \leq |z| \leq \frac{1}{r_0}\} \setminus \Omega_n$.

One can check that $h(z, D(\infty, 1/r_0) \cup \Omega_n, D(\infty, 1/R)) \geq h(z, D(\infty, 1/r_0) \cup U, D(\infty, 1/R)) \geq v(z)$, $n \geq 1\,, |z| \geq 1/R$. Therefore

$$P(a, g_e(a)) = P(a, f_n(a)) \;\leq\; C + (M_n - C) v(a), \quad n \geq n_1(a),$$

where $n_1(a)$ is so large that $M_n - C \; < \; 0$ for $n \; \geq \; n_1(a)$. It follows that $P(a, g_e(a)) = -\infty$.

By the same method one can show that if $P(z, g_e(z)) = -\infty$ on G_e then $P(z, g(z)) = -\infty$ on $G \cup E$. Namely, it is sufficient to observe that the function $v(z) = V_{U \cup D(0, r_0)}(z) / \log \frac{R}{r_0}$ is subharmonic in $\boldsymbol{C}$, harmonic on $\boldsymbol{C} \setminus \overline{D(0, r_0)} \cup \bar{U}$, $v(z) = 0$ on $U \cup D(0, r_0)$, $v(z) \leq 1$ on $\{|z| \leq R\}$, $v(z) \geq m_1 / \log \frac{R}{r_0}$ on E, and $v(z) > 0$ for all $z \in G \cup E$ with $|z| > r_0$. Hence

$$v(z) \;\leq\; h(z,\, U \cap D(0, r_0), D(0, R)), \quad |z| \leq R.$$

Put $K := (\bar{G} \cup \bar{G}_e) \cap \{|z| \leq R\}$. By Vitushkin Theorem there exists a sequence of rational functions $\{\mathcal{F}_n\}$ with poles in $U \cup D(\infty, R)$ uniformly convergent to $\mathcal{F}$ on K. Now, we can repeat the reasoning of the last part of the proof of the former case.

Corollary. *Put* $f := g^{-1}$, $f_e := g_e^{-1}$. *Then*

$$f : \bar{\boldsymbol{D}} \mapsto \bar{G}, \qquad f_e : \bar{\boldsymbol{D}} \mapsto \bar{G}_e$$

are conformal diffeomorphisms satisfying all the assertions of Theorem 1.1.

Acknowledgements. The author thanks Włodzimierz Zwonek for his careful reading of the preliminary version of the manuscript.

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Received November 24, 2006

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