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REAL ALGEBRAIC VERSIONS OF CARTAN'S THEOREMS A AND B

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Contents

Chapter 1

Introduction

The main aim of the thesis is to establish real algebraic versions of Cartan's Theorems A and B after blowing up. These results were provided in our recent papers [22, 23]. Let X be a non-singular real affine variety and $\mathcal{F}$ be a coherent sheaf on X. Our version of Cartan's Theorem A states that there exists a multi-blowup $\alpha : X_\alpha \to X$ such that $\alpha^*\mathcal{F}$ is generated by global sections on X_α. Cartan's Theorem B is formulated for blown-up Čech cohomology introduced by means of a directed set of multi-blowups. It is valid for coherent sheaves of homological dimension ≤ 1 and for quasi-coherent sheaves of global presentation with stalks of projective dimension ≤ 1.

There are several cases where Cartan's Theorems A and B hold: in complex analytic geometry, in algebraic geometry over an algebraically closed field (Serre [27]), in scheme theory (Grothendieck [12], Hartshorne [14]) as well as recent versions in real regulous geometry (Fichou–Huisman–Mangolte–Monnier [7]) and in regulous geometry over Henselian valued fields (Nowak [25, 26]). Note also that the theory of regulous functions is closely related to that of continuous hereditarily rational functions, developed by Kollár–Nowak [21]. They used blowups to improve the functions under study, which is also a basic tool in this thesis.

We organize the thesis as follows. In Chapter 2 we present some basic definitions concerning real affine varieties, quasi-coherent sheaves and simple normal crossing. We then prove that the multi-blowups of a non-singular real affine variety form a directed system. In Chapter 4 we prove Cartan's Theorem A. In the next chapter we provide basic properties of homological dimension of quasi-coherent sheaves. We are then able to define blown-up

Čech cohomology and prove a version of Cartan's Theorem B. Chapter 7 contains application of the blown-up Čech cohomology to a real algebraic version of the first Cousin problem. We show there that the first Cousin problem is universally solvable after blowing up. We then describe the local structure of coherent sheaves of homological dimension 1. The last chapter contains various examples. An explicit usage of Cartan's Theorem A to the sheaf $\mathcal{F}_{1,1}$ is shown, we then provide a counterexample to the quasi-coherent version of our Cartan's Theorem A. We also give a concrete example of an additive Cousin data on $\mathbb{R}^2$ which is not solvable and we deduce that $H^1(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2}) \neq 0$. In Appendix A.1 we prove that each pre-algebraic vector bundle splits into a direct sum of algebraic line bundles by multi-blowups. In Appendix A.2 we show that the construction from A.1 does not lead to the trivial line bundles in general.

Our approach combines the technique of coherent algebraic sheaves and their Čech cohomology, developed by Serre [27] on algebraic varieties over algebraically closed fields, and transformation to a simple normal crossing by blowing up.

Notation and conventions

Throughout the paper, X will be a quasi-projective (hence affine) real algebraic variety with the structure sheaf $\mathcal{O}_X$ of regular functions. If $\sigma : Y \to X$ is a morphism of real affine varieties, by $\sigma^* s$ or f^σ we denote the pull-back of a section s or a function f, respectively. For $U \subset X$, $U^\sigma := \sigma^{-1}(U)$ denotes the preimage under σ of U; similarly, for an open covering $\mathcal{U} = \{U_i\}_{i=1}^n$, we put $\mathcal{U}^\sigma = \{U_i^\sigma\}_{i=1}^n$.

We use the following notation. For a section s of a sheaf $\mathcal{F}$ on X, by $s(x) \in \mathcal{F}_x$, $x \in X$, we mean a corresponding germ. For a regular function f on X, however, $f(x) \in \mathbb{R}$ stands for the value of f at x. The germ of f at x is denoted by f_x. Nevertheless, usually in this paper superscript x in f_x refers only to a certain function germ at x, considered prior to a function itself. But next, by abuse of notation, f_x may often denote a representative of the germ f_x as well. This does not lead to confusion. Let I be a set of indices. We define $\mathcal{F}^{\oplus I} := \bigoplus_{i \in I} \mathcal{F}$ to be the direct sum of I copies of a sheaf $\mathcal{F}$.

Besides Section 2.1, every real affine variety X is assumed to be irreducible and non-singular. In case of a non-singular reducible variety, every reasoning can be carried out on each component separately.

Throughout the paper, greek letters $\alpha, \beta, \gamma, \sigma$ will denote multi-blowups of X, $X_\alpha, X_\beta, X_\gamma, X_\sigma$ their domains and calligraphic letters $\mathcal{F}, \mathcal{G}, \mathcal{H}$ sheaves on X.

Chapter 2

Preliminaries

2.1 Real algebraic varieties

In this section we will present basic informations concerning real algebraic varieties. Reader is referred to [4] for more details. We begin with defining an affine real algebraic variety.

Definition 2.1. *An affine real algebraic variety over $\mathbb{R}$ is a topological space X equipped with a sheaf $\mathcal{O}_X$ of functions with values in $\mathbb{R}$ isomorphic to an algebraic set $V \subset \mathbb{R}^n$ with its Zariski topology, equipped with its sheaf of regular functions $\mathcal{O}_V$. The sheaf $\mathcal{O}_X$ is called the sheaf of regular functions on X.*

Let X be a real affine variety.

Definition 2.2. *We say that $x \in X$ is a non-singular point if the local ring $\mathcal{O}_{x,X}$ is a regular ring. We say that X is a non-singular real affine variety, if every point of X is non-singular.*

Definition 2.3. *We say that X is irreducible, if the ring of global regular functions $\mathcal{O}_X(X)$ is an integral domain.*

Every real affine variety bears both the Zariski and Euclidean topologies. The latter will only be used in Chapter 3. We list a few differences between complex and real algebraic geometry.

- Irreducible real affine varieties can be compact (a circle $x^2 + y^2 = 1$) or disconnected (a hyperbola $xy = 1$) in the Euclidean topology, while the irreducible complex affine varieties are always connected and non-compact (except for a point).

- Real projective spaces $\mathbb{P}^n(\mathbb{R})$, and Grassmanianns $\mathbb{G}_{n,k}(\mathbb{R})$ are real affine varieties (cf. [4, Theorem 3.4.4]).

- Let $U \subset X$ be a Zariski open subset of a real affine variety X. Then, U is not necessarily dense in X in the Euclidean topology. However, it is so whenever X is non-singular.

- If X is an algebraic subset of $\mathbb{R}^n$, then X is a zero set of a single polynomial.

We now briefly present the construction of blowing up. Let X be a real affine variety and Y be a Zariski closed subset of X with its real ideal $I_Y = \{f \in \mathcal{O}_X(X) : f|_Y = 0\}$. The ring $\mathcal{O}_X(X)$ is noetherian, hence we can choose finitely many functions $f_1, f_2, \dots f_k \in \mathcal{O}_X(X)$ such that $(f_1, f_2, \dots, f_k) = I_Y$. Define

$$Z := \{(x, (f_1(x) : f_2(x) : \cdots : f_k(x)) \in X \times \mathbb{P}^{k-1}(\mathbb{R}) \mid x \in X \setminus Y\}.$$

Denote by $E(X,Y)$ the Zariski closure of Z in $X \times \mathbb{P}^{k-1}(\mathbb{R})$, and $\sigma : E(X,Y) \to X$, the projection map. Then the following holds (cf. [4, Prop 3.5.8]):

i) The algebraic variety $E(X,Y)$ does not depend on the choice of generators of I_Y, up to a biregular isomorphism compatible with σ.

ii) We have $Z = \sigma^{-1}(X \setminus Y)$ and $\sigma|_Z : Z \to X \setminus Y$ is a biregular isomorphism.

iii) $E(X,Y)$ is a real affine variety.

Definition 2.4. *We call $E(X,Y)$ a blowup of X with centre Y.*

If X and Y are non-singular, then so is $E(X,Y)$. Note that the map σ is proper in Euclidean topology.

2.2 Coherent and quasi-coherent sheaves

In this section we recall definitions concerning coherent and quasi-coherent sheaves. The reader is referred to [27] for a thorough exposition. Let X be a real affine variety with a structure sheaf $\mathcal{O}_X$ of regular functions and $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules ($\mathcal{O}_X$-sheaf).

Definition 2.5. *We say that $\mathcal{F}$ is a sheaf of finite type if locally $\mathcal{F}$ is generated by finitely many sections i.e. for any $x \in X$ there exist a Zariski open neighbourhood U_x of x and sections $s_1, s_2, \ldots, s_k \in \mathcal{F}(U_x)$ such that $s_1(y), s_2(y), \ldots, s_k(y)$ generate $\mathcal{F}_y$ as $\mathcal{O}_{y,X}$-module, for every $y \in U_x$.*

We recall the following useful

Proposition 2.6. *Let $\mathcal{F}$ be a sheaf of finite type and $s_1, s_2, \ldots, s_p$ be sections of $\mathcal{F}$ on a neighbourhood U of $a \in X$. If $\mathcal{F}_a$ is generated by $s_1, s_2, \ldots, s_p$, so is $\mathcal{F}_y$ for y sufficiently close to a.*

Proof. See [27, I, §2, Proposition 1]

□

Remark 2.7. Under assumptions of Proposition 2.6, if $U = X \setminus \{Q = 0\}$ with some $Q \in \mathcal{O}_X(X)$, then for any n the sections $Q^n s_1, \ldots, Q^n s_k$ generate every stalk $\mathcal{F}_x$, for x sufficiently close to a, because the function Q is invertible at a.

Consider now a regular map between real affine varieties $f : X \to Y$. Then the pull-back $f^*\mathcal{G}$ of a given sheaf of $\mathcal{O}_Y$-modules $\mathcal{G}$ is given by the formula

$$f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}(\mathcal{O}_Y)} \mathcal{O}_X,$$

where $f^{-1}\mathcal{G}$ is the inverse image of the sheaf $\mathcal{G}$. It is well known that the functor $\mathcal{G} \mapsto f^{-1}\mathcal{G}$ is exact and $(f^{-1}\mathcal{G})_x \cong \mathcal{G}_{f(x)}$ for every $x \in X$. Consequently, the functor $\mathcal{G} \mapsto f^*\mathcal{G}$ is right exact and if $\mathcal{G}$ is of finite type, coherent or quasi-coherent, so is $f^*\mathcal{G}$. Moreover, it is easy to check the following

Lemma 2.8. *Under the above assumptions, if $\mathcal{G}$ is a sheaf of $\mathcal{O}_Y$-modules generated by sections $s_1, s_2, \ldots, s_k \in \mathcal{G}(Y)$, then the pull-back $f^*\mathcal{G}$ is generated by the pull-back $f^*s_1, f^*s_2, \ldots, f^*s_k \in (f^*\mathcal{G})(X)$.* □

Let $\alpha : X_\alpha \to X$ be a regular map between real affine varieties and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. For any Zariski open set $U \subset X$ we have the canonical and functorial homomorphism

$$\alpha^* : \mathcal{F}(U) \to \alpha^*\mathcal{F}(U^\alpha)$$

such that

$$(\alpha^* s)(y) = s(\alpha(y)) \otimes 1 \in (\alpha^{-1}\mathcal{F})_{\alpha(y)} \otimes_{\mathcal{O}_{\alpha(y),X}} \mathcal{O}_{y,X_\alpha} = (\alpha^*\mathcal{F})_y,$$

for $s \in \mathcal{F}(U)$ and each $y \in U^\alpha$.

Definition 2.9. *We say that $\mathcal{F}$ is a quasi-coherent sheaf on X if there exists a finite Zariski open covering $\{U_i\}_{i=1}^n$ of X such that for each i there is an exact sequence of sheaves:*

$$\mathcal{O}_X^{\oplus J_i}|_{U_i} \xrightarrow{\phi_i} \mathcal{O}_X^{\oplus I_i}|_{U_i} \xrightarrow{\psi_i} \mathcal{F}|_{U_i} \to 0.$$

A sheaf $\mathcal{F}$ is coherent if each J_i and each I_i can be taken finite.

It is clear that one can take a common J with the biggest cardinality among J_i for $i = 1, 2, \ldots n$. A sequence of sheaves is exact if it is exact on stalks.

It is not difficult to check that $\mathcal{F}$ is coherent iff the two conditions are satisfied

i) $\mathcal{F}$ is a sheaf of finite type,

ii) if $s_1, s_2, \ldots, s_p$ are sections of $\mathcal{F}$ over an open set $U \subset X$, then the sheaf of relations between $s_1, s_2, \ldots, s_p$ is of finite type over the set U.

2.3 Simple normal crossing

We now present necessary notions concerning transformation a function to a simple normal crossing (see e.g. [20]). Consider a regular (noetherian) local ring $(R, \mathfrak{m})$. A *regular system of parameters of* R is any minimal set of generators $x_1, x_2, \ldots, x_d$ of $\mathfrak{m}$; obviously d is the Krull dimension of R.

Let X be non-singular real affine variety of dimension d. By *local coordinates* near a point $a \in X$ we mean a regular system of parameters of $\mathcal{O}_{a,X}$.

Definition 2.10. *We say that $g \in \mathcal{O}_{a,X}$ is a simple normal crossing at a if in a neighbourhood of $a \in X$, one has*

$$g(x) = u(x)x^{\alpha} = u(x)x_1^{\alpha_1}x_2^{\alpha_2}\dots x_d^{\alpha_d},$$

where $u(x)$ is a unit at a, $\alpha \in \mathbb{N}^d$ and $x = (x_1, x_2, \dots, x_d)$ are local coordinates near a. We say that $g \in \mathcal{O}_X(X)$ is a simple normal crossing if g is a simple normal crossing at each point $a \in X$.

We say that functions $g_1, g_2, \dots, g_k \in \mathcal{O}_X(X)$ are simultaneously simple normal crossing if in a neighbourhood of each point a we have

$$g_i(x) = u_i(x)x^{\alpha_i}$$

in the same local coordinates $x = (x_1, x_2, \dots, x_d)$.

By a multi-blowup $\sigma : X_\sigma \to X$ we mean a finite composition of blowups with smooth centres. One of the basic tools applied in this paper is transformation to a simple normal crossing recalled below.

Theorem 2.11. *Let $f : X \to \mathbb{R}$ be a regular function on a non-singular real affine variety X. Then there exists a multi-blowup $\sigma : X_\sigma \to X$ such that $f^\sigma := f \circ \sigma$ is a simple normal crossing.*

Remark 2.12. In Kollár's notation, however, $\sigma^*\mathcal{I} = \sigma^{-1}\mathcal{I} \cdot \mathcal{O}_{\widetilde{X}}$ for any sheaf $\mathcal{I}$ of ideals on X. Note that $\sigma^{-1}\mathcal{I} \cdot \mathcal{O}_{\widetilde{X}}$ and $\sigma^{-1}\mathcal{I} \otimes_{\sigma^{-1}\mathcal{O}_X} \mathcal{O}_{\widetilde{X}}$ are canonically isomorphic if $\mathcal{I}$ is a sheaf of locally principal ideals.

A useful strengthening of Theorem 2.11, stated below, relies on the following elementary result.

Lemma 2.13. *[3, Lemma 4.7] Let $x = (x_1, x_2, \dots, x_p)$ be a regular system of parameters of $\mathcal{O}_{x,X}$. Let $\alpha, \beta, \gamma \in \mathbb{N}^p$ and let $a(x), b(x), c(x)$ be invertible elements in $\mathcal{O}_{x,X}$. If*

$$a(x)x^{\alpha} - b(x)x^{\beta} = c(x)x^{\gamma},$$

then either $\alpha \le \beta$ or $\beta \le \alpha$. Here inequality $\alpha \le \beta$ means that $\alpha_j \le \beta_j$ for all $j = 1, 2, \dots, p$.

We immediately obtain

Corollary 2.14. *Let $f_1, f_2, \ldots, f_k$ be regular functions on X. Then there exists a multi-blowup $\sigma : X_\sigma \to X$ such that $f_1^\sigma, f_2^\sigma, \ldots, f_k^\sigma$ are simultaneously simple normal crossings which locally are linearly ordered by divisibility relation near each point $b \in X_\sigma$.*

Proof. Apply Theorem 2.11 to the function

$$f := f_1 f_2 \ldots f_k \prod_{i<j} (f_i - f_j).$$

Then all of the functions f_j^σ and $f_i^\sigma - f_j^\sigma,\ i, j = 1, 2, \ldots, k, i < j$, are simultaneously simple normal crossings. Now the conclusion follows directly from the above lemma.

□

Chapter 3

Directed set of multi-blowups

We now show that the set of multi-blowups of a non-singular real affine variety X has a structure of a directed set.

Given two multi-blowups $\alpha : X_\alpha \to X$, $\beta : X_\beta \to X$ we say that $X_\alpha \succeq X_\beta$ if there is a (unique) regular map $f_{\alpha\beta}$ making the following diagram commute

$$\begin{tikzcd} X_\alpha \arrow[r, "f_{\alpha\beta}"] \arrow[dr, "\alpha"'] & X_\beta \arrow[d, "\beta"] \\ & X \end{tikzcd}$$

Obviously, $\succeq$ is a reflexive and transitive relation on the set of all multi-blowups of X.

Proposition 3.1. *With the relation given above, the set of multi-blowups of X is a directed set.*

Proof. We need to show that for any two multi-blowups $\sigma_1 : X_1 \to X$ and $\sigma_2 : X_2 \to X$ there is a multi-blowup $\sigma_3 : X_3 \to X$ such that $X_3 \succeq X_1$ and $X_3 \succeq X_2$. Let $\phi : X_1 \dashrightarrow X_2$ be a rational map which makes the diagram

$$\begin{tikzcd} X_1 \arrow[r, dashed, "\phi"] \arrow[dr, "\sigma_1"'] & X_2 \arrow[d, "\sigma_2"] \\ & X \end{tikzcd}$$

commutative. Let $\mathrm{dom}(\phi)$ be the biggest Zariski open subset of X_1 on which ϕ is a regular map. Clearly, X_2 is an affine variety embedded into $\mathbb{R}^N \subset \mathbb{P}^N(\mathbb{R})$ for some N; embed $\mathbb{R}^N$ into $\mathbb{P}^N(\mathbb{R})$ by the map

$$(x_1, x_2, \ldots, x_N) \mapsto (x_1 : x_2 : \cdots : x_N : 1).$$

Then ϕ can be treated as a map into $\mathbb{R}^N$, $\phi : X_1 \dashrightarrow \mathbb{R}^N$, with a presentation

$$\phi(x) = \left(\frac{\varphi_1(x)}{q_1(x)}, \frac{\varphi_2(x)}{q_2(x)}, \ldots, \frac{\varphi_N(x)}{q_N(x)} \right),$$

where $\varphi_1, \ldots, \varphi_N, q_1, \ldots, q_N$ are regular functions on X_1 and

$$\mathrm{dom}(\phi) = \bigcap_{i=1}^{N} \mathrm{dom}(\frac{\varphi_i}{q_i}).$$

Since the functions $q_1, q_2, \ldots, q_N$ are nowhere vanishing on $\mathrm{dom}(\phi)$, we can assume that ϕ have the following presentation

$$\phi(x) = \left(\frac{\phi_1(x)}{q(x)}, \frac{\phi_2(x)}{q(x)}, \ldots, \frac{\phi_N(x)}{q(x)} \right),$$

where $\phi_1, \ldots, \phi_N, q$ are regular on X_1 and $\{q = 0\} \cap \mathrm{dom}(\phi) = \emptyset$. Consider a multi-blowup $\tau : X_3 \to X_1$ from Corollary 2.14 applied to the regular functions $\phi_1, \ldots, \phi_N, q$ on X_1. Then we get the commutative diagram

$$\begin{tikzcd}
 & X_3 \arrow[dl, "\tau"'] \arrow[d, dashed, "\phi\circ\tau"] \arrow[drr, "f"] & & \\
X_1 \arrow[r, dashed, "\phi"] \arrow[dr, "\sigma_1"'] & X_2 \arrow[r, hook] \arrow[d, "\sigma_2"] & \mathbb{R}^N \arrow[r, hook] & \mathbb{P}^N \\
 & X & &
\end{tikzcd}$$

with the function $f : X_3 \to \mathbb{P}^N(\mathbb{R})$ given by the formula

$$f = (\phi_1 \circ \tau : \cdots : \phi_N \circ \tau : q \circ \tau).$$

It follows immediately from the conclusion of Corollary 2.14 that the map f is regular. Note that the maps σ_1, σ_2, τ are proper.

We must show that $f(X_3) \subset X_2$. If $y \in X_3$, $\tau(y) \in \mathrm{dom}(\phi)$, then $f(y) \in X_2$ by the commutativity of the upper left triangle diagram. Assume that $\tau(y) \notin \mathrm{dom}(\phi)$. Let $\{y_n\} \subset X_3$ be a sequence converging to y in the Euclidean topology such that $\tau(y_n) \in \mathrm{dom}(\phi)$. Such a sequence can always be found because the blowups are biregular on a Zariski open subset and a Zariski open subsets of a non-singular irreducible variety are dense in the Euclidean topology. Let

$$K := \{(\sigma_1 \circ \tau)(y_n) : n \in \mathbb{N}\} \cup \{(\sigma_1 \circ \tau)(y)\} \subset X$$

and

$$V := \sigma_2^{-1}(K) \subset X_2.$$

Obviously K is a compact set, so is V as σ_2 is proper. We get

$$\sigma_2(f(y_n)) = \sigma_2(\phi(\tau(y_n))) = \sigma_1(\tau(y_n)) \in K.$$

Hence by the definition of V, $f(y_n) \in V$. Thus the sequence $\{f(y_n)\}$ is convergent to $f(y) \in V \subset X_2$, as desired.

□

Proposition 3.1 will be crucial for the construction of blown-up Čech cohomology given in Chapter 6.

Remark 3.2. The above construction for proper schemes was described in [29]. It was broadly used in the theory of real holomorphy rings (see e.g. [6]). Let F be a field of transcendence degree n over $\mathbb{R}$. We say that the field F is *formally real* if it can be ordered. Then the *absolute holomorphy ring* $H(F)$ of F is defined as the intersection of all valuation rings of F with formally real residue field. Let A be finitely generated $\mathbb{R}$-subalgebra of F with quotient field F. The intersection $H(F|A)$ of all those valuation rings of F with formally real residue field which contain A is called *real holomorphy ring of F over A*. It is known (cf. [28]) that $H(F|A)$ is a a Prüfer domain of Krull dimension $\leq n$ [30, Theorem 5.4], and every finitely generated ideal of $H(F|A)$ can be generated by $n+1$ elements [15, Theorem 3.1].

Let X be a non-singular real affine variety, and $K(X)$ be its field of rational functions. A geometric criterion [6, Theorem 1.2] says that $H(K(X)|\mathcal{O}_X(X))$ is the ring of all locally bounded (in the Euclidean topology) rational functions on X. If in addition X is a complete (compact) variety we get $H(K(X)) = H(K(X)|\mathcal{O}_X(X))$.

It follows from the Proposition 4.11 that the ring $\widetilde{\mathcal{O}_X}(X)$, defined on an arbitrary non-singular real affine variety X, coincide with the ring of all locally bounded rational functions on X. Hence $H(K(X)|\mathcal{O}_X(X)) = \widetilde{\mathcal{O}_X}(X)$.

Chapter 4

Cartan's Theorem A

4.1 Right exactness of a section functor after blowing up

The main aim of this section is to prove

Theorem 4.1. *Let*

$$\mathcal{G} \xrightarrow{\theta} \mathcal{H} \to 0$$

be an exact sequence of quasi-coherent sheaves on X. Then for any Zariski open subset $U \subset X$ and any section $u \in \mathcal{H}(U)$ there exists a multi-blowup $\alpha : X_\alpha \to X$ such that $\alpha^* u \in \operatorname{im} \theta^\alpha$.

A crucial role in the proof of Theorem 4.1 is played by Lemmas 4.7 and 4.8. Before we prove them, we need some preparatory lemmas. We begin with the following well known

Lemma 4.2. *Let $U = X \setminus \{Q = 0\}$ be a Zariski open subset of X. Every regular function f on U can be written in the form $f = \frac{g}{P}$ where g, P are global regular functions on X and $V(P) \subset V(Q)$.*

Proof. See [4, Proposition 3.2.3]. □

Lemma 4.3. *Let P, Q be regular functions on X such that $V(P) \subset V(Q)$. Then there exist a multi-blowup $\sigma : X_\sigma \to X$ and a positive integer N such that $\frac{(Q^N)^\sigma}{P^\sigma}$ can be extended to a global regular function on X_σ or, equivalently,*

$$(Q^N)^\sigma \in P^\sigma \cdot \mathcal{O}_{X_\sigma}(X_\sigma).$$

Proof. By Corollary 2.14, there exist a multi-blowup $\sigma : X_\sigma \to X$ and a finite covering $\{U_i\}_{i=1}^n$ of X_σ such that $Q^\sigma|_{U_i}$ and $P^\sigma|_{U_i}$ are functions of the form $u_i(x)x^{\alpha_i}$ and $v_i(x)x^{\beta_i}$, where u_i, v_i are units on U_i and $x = (x_1, x_2, \dots, x_d)$ are local coordinates on U_i, $i = 1, 2, \dots, n$. Since $U_i \cap V(P^\sigma) \subset U_i \cap V(Q^\sigma)$, for any non-zero entry of the multi-index β_i the corresponding entry of α_i is also non-zero. Consequently, $P^\sigma | (Q^N)^\sigma$ on every U_i for N large enough, and thus

$$(Q^N)^\sigma \in P^\sigma \cdot \mathcal{O}_{X_\sigma}(X_\sigma),$$

as asserted. □

Corollary 4.4. *Let Q be a regular function on X. Then for any finite number of regular functions $P_i \in \mathcal{O}_X(X)$ such that $V(P_i) \subset V(Q)$ for $i = 1, 2, \dots, s$, there exist a multi-blowup $\sigma : X_\sigma \to X$ and a positive integer N such that*

$$(Q^N)^\sigma \in P_i^\sigma \cdot \mathcal{O}_{X_\sigma}(X_\sigma)$$

for each $i = 1, 2, \dots, s$.

Proof. This follows easily via repeated application of Lemma 4.3. □

Remark 4.5. We can generalize the above remark as follows. If

$$V(P_i) \cap U_i \subset V(Q) \cap U_i, \quad i = 1, 2, \dots, n,$$

for some non-empty Zariski open sets $U_1, \dots, U_n \subset X$, then there exist a multi-blowup $\sigma : X_\sigma \to X$ and a positive integer N such that

$$(Q^N|_{U_i})^\sigma \in (P_i|_{U_i})^\sigma \cdot \mathcal{O}_{X_\sigma}(U_i^\sigma).$$

Corollary 4.6. *Let Q be a regular function on X and $U = X \setminus \{Q = 0\}$. Then for any $f \in \mathcal{O}_X(U)$ there exist a multi-blowup $\sigma : X_\sigma \to X$ and a positive integer N such that $(Q^N f)^\sigma$ can be extended to a global regular function on X_σ.*

Proof. It follows immediately from Lemmas 4.3 and 4.2. □

Lemma 4.7. *Let $\mathcal{F}$ be a quasi-coherent sheaf on X. For any $Q \in \mathcal{O}_X(X)$ and a section $s \in \mathcal{F}(X)$ such that $s|_U = 0$ with $U = X \setminus \{Q = 0\}$, there exist a multi-blowup $\sigma : X_\sigma \to X$ and a positive integer N such that $(Q^N)^\sigma \sigma^* s = 0$ in $\sigma^*\mathcal{F}(X_\sigma)$.*

Proof. By quasi-compactness, there is a finite Zariski open covering $\{U_i\}_{i=1}^n$ of X such that for each i we have:

1) a presentation
$$\mathcal{O}_X^{\oplus J}|_{U_i} \xrightarrow{\phi_i} \mathcal{O}_X^{\oplus I_i}|_{U_i} \xrightarrow{\psi_i} \mathcal{F}|_{U_i} \to 0.$$

2) $s|_{U_i} = \psi_i(t_i)$ for some $t_i \in \mathcal{O}_X^{\oplus I_i}(U_i)$.

Put
$$\mathrm{Rel}\,(t_i, \phi_i(e_j) : j \in J; \mathcal{O}_X(U_i)) :=$$
$$= \left\{ (q, (q_j)_{j\in J}) \in \mathcal{O}_X(U_i) \oplus \mathcal{O}_X(U_i)^{\oplus J} \ : qt_i + \sum_{j\in J} q_j\phi_i(e_j) = 0 \right\} =$$
$$= \varinjlim_{\substack{J_f\subset J\\ \#J_f<\infty}} \mathrm{Rel}\,(t_i, \phi_i(e_j) : j \in J_f; \mathcal{O}_X(U_i)).$$

In other words, we express $\mathrm{Rel}\,(t_i, \phi_i(e_j) : j \in J; \mathcal{O}_X(U_i))$ as a direct limit of modules of relations of t_i and finitely many elements indexed by J. Let

$$\mathcal{I}_i := \left\{ q \in \mathcal{O}_X(U_i) : \exists (q_j)_{j\in J_f}, \#J_f < \infty : qt_i + \sum_{j\in J_f} q_j\phi_i(e_j) = 0 \right\} =$$
$$= \pi_1\left(\mathrm{Rel}\,(t_i, \phi_i(e_j) : j \in J; \mathcal{O}_X(U_i))\right) \subset \mathcal{O}_X(U_i)$$

for each $i = 1, 2, \ldots, n$; here π_1 is the natural projection onto the first factor, e_j is an element of $\mathcal{O}_X(U_i)^{\oplus J}$ which has 1 on j-th entry and zero elsewhere. Then, for every $x \in U_i$ we have

$$\mathrm{Rel}\,(t_i(x), \phi_i(e_j)(x), j \in J; \mathcal{O}_{x,X}) =$$
$$= \varinjlim_{\substack{J_f\subset J\\ \#J_f<\infty}} \mathrm{Rel}\,(t_i(x), \phi_i(e_j)(x), j \in J; \mathcal{O}_{x,X}) =$$
$$= \varinjlim_{\substack{J_f\subset J\\ \#J_f<\infty}} \Big(\mathrm{Rel}\,(t_i, \phi_i(e_j) : j \in J_f; \mathcal{O}_X(U_i)) \cdot \mathcal{O}_{x,X}\Big) =$$
$$= \Big(\varinjlim_{\substack{J_f\subset J\\ \#J_f<\infty}} \mathrm{Rel}\,(t_i, \phi_i(e_j) : j \in J_f; \mathcal{O}_X(U_i))\Big) \cdot \mathcal{O}_{x,X} =$$

$$= \Big(\mathrm{Rel}\,(t_i, \phi_i(e_j) : j \in J; \mathcal{O}_X(U_i))\Big) \cdot \mathcal{O}_{x,X}$$

because modules of relations commute with flat base change (see e.g. [5, Chap. I, §2, Remark 2 after Prop. 1]). Therefore,

$$\mathcal{I}_i \cdot \mathcal{O}_{x,X} = \{q_x \in \mathcal{O}_{x,X} : \exists (q_{jx})_{j\in J_f}, \#J_f < \infty, qt_i(x) + \sum_{j\in J_f} q_j\phi_i(e_j)(x) = 0\}$$

and thus, $1 \in \mathcal{I}_i \cdot \mathcal{O}_{x,X}$ for every $x \in U_i \cap U$. Hence we get

$$U_i \cap V(\mathcal{I}_i) \subset U_i \cap V(Q).$$

Clearly, there exists $p_i \in \mathcal{I}_i$ such that $V(p_i) = V(\mathcal{I}_i)$. Each p_i may be written in the form

$$p_i = \frac{P_i}{R_i} \text{ with } P_i, R_i \in \mathcal{O}_X(X) \text{ and } V(R_i) \cap U_i = \emptyset$$

for $i = 1, 2, \ldots, n$. Hence

$$V(p_i) \cap U_i = V(P_i) \cap U_i$$

and by Remark 4.5 there exist a multi-blowup $\sigma : X_\sigma \to X$ and a positive integer N such that

$$(Q^N)^\sigma|_{U_i^\sigma} \in P_i^\sigma|_{U_i^\sigma} \cdot \mathcal{O}_{X_\sigma}(U_i^\sigma), \quad i = 1, \ldots, n.$$

As p_i and P_i differ only by a unit on U_i, we get

$$(Q^N)^\sigma|_{U_i^\sigma} \in p_i^\sigma|_{U_i^\sigma} \cdot \mathcal{O}_{X_\sigma}(U_i^\sigma), \quad i = 1, \ldots, n.$$

Therefore

$$\begin{aligned}(Q^N)^\sigma|_{U_i^\sigma} \cdot \sigma^*(t_i) \in p_i^\sigma|_{U_i^\sigma} \cdot \sigma^*(t_i) \cdot \mathcal{O}_{X_\sigma}(U_i^\sigma) = \\ = \sigma^*(p_i|_{U_i} t_i) \cdot \mathcal{O}_{X_\sigma}(U_i^\sigma) \subset \sigma^*(\phi_i(\mathcal{O}_X^{\oplus J}(U_i))) = \phi_i^\sigma(\mathcal{O}_{X_\sigma}^{\oplus J}(U_i^\sigma)),\end{aligned}$$

whence,

$$(Q^N)^\sigma|_{U_i^\sigma} \cdot \sigma^* s|_{U_i^\sigma} = \psi_i^\sigma((Q^N)^\sigma \sigma^* t_i) = 0|_{U_i^\sigma}.$$

This finishes the proof.

□

Lemma 4.8. *Let $\mathcal{F}$ be a quasi-coherent sheaf on X with local presentations*

$$\mathcal{O}_X^{\oplus J}|_{U_i} \xrightarrow{\phi_i} \mathcal{O}_X^{\oplus I_i}|_{U_i} \xrightarrow{\psi_i} \mathcal{F}|_{U_i} \to 0 \quad i = 1, 2, \ldots, n$$

on a finite Zariski open covering $\{U_i\}_{i=1}^n$ of X. Consider a finite number of sections $s_j \in \mathcal{F}(V_j)$ on Zariski open sets

$$V_j = X \setminus \{Q_j = 0\}, \quad j = 1, 2, \ldots, m$$

where Q_j are regular functions on X. Assume that every V_j is contained in $U_{i(j)}$ for some $i(j) = 1, 2, \ldots, n$ and that for each j there is a section $t_j \in \mathcal{O}_X^{I_{i(j)}}(V_j)$ such that $\psi_{i(j)}(t_j) = s_j$. Then there exist a positive integer N and a multi-blowup $\sigma : X_\sigma \to X$ such that every section $(Q_j^N)^\sigma \sigma^* s_j$, $j = 1, 2, \ldots, m$, extends to a global section on X_σ.

Proof. Since taking pull-back under a multi-blowup does not affect the assumptions, it suffices to consider only one $j = 1, 2, \ldots, m$. So fix an index j, and let $t_{ji} = t_j|_{U_i \cap V_j} = (t_{jil})_{l \in I_i}$. Since t_{ji} is an element of the direct sum, all but finitely many t_{jil} are zero, so every t_{ji} can be identified with finitely many non-vanishing regular functions on $U_i \cap V_j$, indexed by some subset $J_i \subset I_i$ such that $\# J_i < \infty$ i.e. $t_{ji} = (t_{jil})_{l \in J_i}$. We have

$$t_{jil} = \frac{t_{jil1}}{t_{jil2}}, \; t_{jil1}, t_{jil2} \in \mathcal{O}_X(X)$$

and

$$V(t_{jil2}) \cap U_i \subset V(Q_j) \cap U_i.$$

Using Remark 4.5 we can find a positive integer N_1 and a multi-blowup $\alpha : X_\alpha \to X$ such that

$$((Q_j)^{N_1} t_{jil})^\alpha \in \mathcal{O}_{X_\alpha}(U_i^\alpha) \;\text{ for all }\; j, i, l.$$

Now define $\widetilde{s_{ji}} := \psi_i^\alpha(((Q_j)^{N_1} t_{ji})^\alpha)$. Then for any two distinct indices i_0, $i_1 \in \{1, 2, \ldots, n\}$ we have

$$(\widetilde{s_{ji_0}} - \widetilde{s_{ji_1}})|_{U_{i_0}^\alpha \cap U_{i_1}^\alpha} \in (\alpha^* \mathcal{F})(U_{i_0}^\alpha \cap U_{i_1}^\alpha)$$

and

$$(\widetilde{s_{ji_0}} - \widetilde{s_{ji_1}})|_{U_{i_0}^\alpha \cap U_{i_1}^\alpha \cap V_j^\alpha} = 0.$$

By Lemma 4.7, we can find a multi-blowup $\beta : X_\beta \to X_\alpha$ and a positive integer N_2 such that

$$((Q_j^{N_1+N_2})^{\alpha\circ\beta}\beta^*\widetilde{s_{ji_0}} - (Q_j^{N_1+N_2})^{\alpha\circ\beta}\beta^*\widetilde{s_{ji_1}})|_{U_{i_0}^{\alpha\circ\beta}\cap U_{i_1}^{\alpha\circ\beta}} = 0.$$

Considering all distinct pairs of indices i_0, i_1, we can assume that the differences as above vanish for all those pairs. Therefore the sections

$$((Q_j^{N_1+N_2})^{\alpha\circ\beta}\beta^*\widetilde{s_{ji}})|_{U_i^{\alpha\circ\beta}}, \quad i = 1, 2, \ldots, n,$$

glue together to a global section on X_β. Thus

$$\sigma := \alpha \circ \beta : X_\beta = X_\sigma \to X$$

is the multi-blowup we are looking for.

□

Proof of Theorem 4.1 Let $\{V_i\}_{i=1}^n$ be a finite Zariski open covering of U satisfying following two conditions

a)
$$\mathcal{O}_X^{\oplus J}|_{V_i} \xrightarrow{\phi_i} \mathcal{O}_X^{\oplus I_i}|_{V_i} \xrightarrow{\psi_i} \mathcal{G}|_{V_i} \to 0 \quad i = 1, 2, \ldots, n,$$

b) there exist sections $s_i \in \mathcal{G}(V_i)$ and $t_i \in \mathcal{O}_X^{\oplus I_i}(V_i)$ such that $\theta(s_i) = u|_{V_i}$ and $\psi_i(t_i) = s_i$.

Each V_j is of the form $V_j = U \setminus \{Q_j = 0\}$ for some $Q_j \in \mathcal{O}_X(X)$. By Lemma 4.8, there exist a multi-blowup $\alpha : X_\alpha \to X$ and a positive integer N such that every section $(Q_j^N)^\alpha \alpha^* s_j$ extends to U^α:

$$(Q_j^N)^\alpha \alpha^* s_j \in \alpha^*\mathcal{G}(U^\alpha).$$

Obviously,
$$\left(\theta^\alpha((Q_j^N)^\alpha\alpha^* s_j) - (Q_j^N)^\alpha\alpha^* u\right)|_{V_j^\alpha} = 0$$
for each $j = 1, 2, \ldots, m$. By Lemma 4.7 there exists a multi-blowup $\beta : X_\beta \to X_\alpha$ and a positive integer M such that

$$\theta^{\alpha\circ\beta}\left((Q_j^{N+M})^{\alpha\circ\beta}(\alpha\circ\beta)^* s_j\right) = (Q_j^{N+M})^{\alpha\circ\beta}(\alpha\circ\beta)^* u$$

for each $j = 1, 2, \ldots, m$. We may, of course, assume that $N + M$ is even. Then the function

$$R := \frac{1}{\sum_{j=1}^{m} (Q_j^{N+M})^{\alpha\circ\beta}}$$

is regular on $U^{\alpha\circ\beta}$, because V_j was a covering of U. We have

$$R\sum_{j=1}^{m} \theta^{\alpha\circ\beta}((Q_j^{N+M})^{\alpha\circ\beta}(\alpha\circ\beta)^* s_j) = R\sum_{j=1}^{m}(Q_j^{N+M})^{\alpha\circ\beta}(\alpha\circ\beta)^* u = (\alpha\circ\beta)^* u.$$

This concludes the proof. □

Now we state two direct consequences of Theorem 4.1.

Corollary 4.9. *Let*

$$\mathcal{G} \xrightarrow{\theta} \mathcal{H} \to 0$$

be an exact sequence of quasi-coherent sheaves on X. Consider $\{U_i\}_{i=1}^m$ a finite collection of Zariski open subsets and sections $u_i \in \mathcal{H}(U_i)$ for $i = 1, 2, \ldots, m$. Then there exists a multi-blowup $\alpha : X_\alpha \to X$ such that for each i, $\alpha^* u_i \in \operatorname{im} \theta^\alpha$.

Proof. This can be obtained by repeated application of Theorem 4.1. □

Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Put

$$\widetilde{\mathcal{F}}(U) = \varinjlim_{\alpha} \alpha^* \mathcal{F}(U^\alpha),$$

for any Zariski open subset U of X; direct limit is taken over the directed set of multi-blowups of X. Obviously, $\widetilde{\mathcal{F}}(U)$ has a structure of $\mathcal{O}_X(U)$-module.

It is clear that any element of $\widetilde{\mathcal{F}}(U)$ can be represented as a class $[s]$ for some multi-blowup $\alpha : X_\alpha \to X$ and some section $s \in \alpha^* \mathcal{F}(U^\alpha)$.

Corollary 4.10. *Let*

$$\mathcal{G} \xrightarrow{\theta} \mathcal{H} \to 0$$

be an exact sequence of quasi-coherent sheaves on X. Then, for any Zariski open subset $U \subset X$, the induced sequence of $\mathcal{O}_X(U)$-modules

$$\widetilde{\mathcal{G}}(U) \to \widetilde{\mathcal{H}}(U) \to 0$$

is exact.

Proof. The above exact sequence is well defined since α^* is functorial. Let $[s] \in \widetilde{\mathcal{H}}(U)$ with a representative $s \in \alpha^*\mathcal{H}(U^\alpha)$ for some multi-blowup $\alpha : X_\alpha \to X$. To finish the proof, it is enough to apply Theorem 4.1 to s.

□

Finally we give the following characterization of the ring $\widetilde{\mathcal{O}_X}(X)$.

Proposition 4.11. *$\widetilde{\mathcal{O}_X}(X)$ coincides with the ring of all locally bounded rational functions on X.*

Proof. Let $f \in K(X)$ be a rational function on X, we have $f = \frac{p}{q}$ for some $p, q \in \mathcal{O}_X(X)$. Assume that f is locally bounded on X. Let $\sigma : X_\sigma \to X$ be a multi-blowup such that both p^σ and q^σ are simultaneously simple normal crossing. For any $a \in X_\sigma$ we have

$$\frac{p^\sigma(a)}{q^\sigma(a)} = \frac{u_1 x^{\alpha_1}(a)}{u_2 x^{\alpha_2}(a)}.$$

By local boundedness $\alpha_2 \leq \alpha_1$, hence $f^\sigma \in \mathcal{O}_{\widetilde{X}}(\widetilde{X})$ and $f \in \widetilde{\mathcal{O}_X}(X)$.

Assume to the contrary that f is not locally bounded. For any multi-blowup $\sigma : X_\sigma \to X$ the corresponding function f^σ cannot be regular nor locally bounded on X_σ, hence $f \notin \mathcal{O}_{\widetilde{X}}(\widetilde{X})$.

□

4.2 Cartan's Theorem A

We are now ready to prove a version of Cartan's Theorem A relying on the coherent version of Lemma 4.8. It is a main result of our paper [22]. Note that the quasi-coherent versions of Lemmas 4.7 and 4.8 will be used in the proof of Cartan's Theorem B in Chapter 6.

Theorem 4.12. *Let $\mathcal{F}$ be a coherent sheaf on X. Then there exist a multi-blowup $\sigma : X_\sigma \to X$ and finitely many global sections $s_1, s_2, \ldots, s_k$ on X_σ which generate every stalk $(\sigma^*\mathcal{F})_y$, $y \in X_\sigma$.*

Proof. Consider a finite Zariski open covering $\{U_i\}_{i=1}^n$ of X with local presentation of the sheaf $\mathcal{F}$

$$\mathcal{O}_X^p|_{U_i} \xrightarrow{\phi_i} \mathcal{O}_X^{q_i}|_{U_i} \xrightarrow{\psi_i} \mathcal{F}|_{U_i} \to 0.$$

By Proposition 2.6, for any point $a \in X$ there are finitely many sections

$$s_{a1}, s_{a2}, \dots, s_{am_a} \in \mathcal{F}(V_a), \quad m_a \in \mathbb{N},$$

on a Zariski open neighbourhood V_a of a, contained in U_i for some $i = 1, 2, \dots, n$, which generate $\mathcal{F}$ over V_a. After shrinking V_a, we can also assume that $s_{ak} = \psi_i(t_{ak})$ for some $t_{ak} \in \mathcal{O}_X^{q_i}(V_a)$, $k = 1, 2, \dots, m_a$. By quasi-compactness, we can find a finite covering of X

$$V_j := V_{a_j}, \ \ j = 1, 2, \dots, m.$$

Clearly, each V_j is contained in $U_{i(j)}$ for some $i(j) = 1, 2, \dots, n$. Put

$$s_{jk} = s_{a_j k} \quad \text{and} \quad t_{jk} = t_{a_j k}$$

for $j = 1, 2, \dots, m$, $k = 1, 2, \dots, m_j = m_{a_j}$. Then $s_{jk} = \psi_{i(j)}(t_{jk})$ and the sections s_{jk}, $k = 1, 2, \dots, m_j$, generate $\mathcal{F}$ over V_j. The sets $X \setminus V_j$ are Zariski closed and thus are of the form

$$X \setminus V_j = \{Q_j = 0\}, \ \ j = 1, 2, \dots, m,$$

for some regular functions Q_j on X. It follows from Lemma 4.8 that there exist a multi-blowup $\sigma : X_\sigma \to X$ and a positive integer N such that for each $j = 1, 2, \dots, m$ the sections

$$(Q_j^N)^\sigma \sigma^* s_{jk}, \quad k = 1, 2, \dots, m_j,$$

extend to global sections $\widetilde{s_{jk}} \in \sigma^*\mathcal{F}(X_\sigma)$. Since $\{\sigma^{-1}(V_j)\}_{j=1}^m$ is a Zariski open covering of X_σ, it follows easily from Lemma 2.8 and Remark 2.7 that the global sections

$$\widetilde{s_{jk}}, \ \ j = 1, 2, \dots, m, \quad k = 1, 2, \dots, m_j$$

generate the pull-back $(\sigma^*\mathcal{F})_y$ for every $y \in X_\sigma$. This completes the proof.

□

Remark 4.13. In the case if $\mathcal{F}$ is a locally free coherent sheaf on X, the above theorem was proven in [2].

Remark 4.14. The above theorem could also be formulated for a coherent sheaf $\mathcal{F}$ on a singular variety X via resolution of singularities. In this case, however, we would control the sheaf $\mathcal{F}$ only over the set of so-called central points, because in real algebraic geometry any resolution of singularities is a map onto that set.

We immediately obtain

Corollary 4.15. *Let $\mathcal{F}$ be a coherent sheaf on X. Then there exists a multi-blowup $\sigma : X_\sigma \to X$ such that the pull-back $\sigma^*\mathcal{F}$ admits a global presentation:*

$$\mathcal{O}^p_{X_\sigma} \to \mathcal{O}^q_{X_\sigma} \to \sigma^*\mathcal{F} \to 0$$

Remark 4.16. Tognoli [31, 32] introduced a notion of an A-coherent sheaf on a real affine variety in terms of extending a given sheaf to the scheme associated to the variety. It is then shown, that the sheaf $\mathcal{F}$ is A-coherent iff there exists a global exact sequence of sheaves:

$$\mathcal{O}^p_X \to \mathcal{O}^q_X \to \mathcal{F} \to 0.$$

We can thus reformulate Corollary 4.15 in terms of A-coherent sheaves.

Corollary 4.17. *Let $\mathcal{F}$ be a coherent sheaf on X. Then there exists a multi-blowup $\sigma : X_\sigma \to X$ such that $\sigma^*\mathcal{F}$ is an A-coherent sheaf.*

Chapter 5

Homological dimension

In this section, we assume all rings to be commutative with unity. By a local ring we mean a ring with unique maximal ideal. Let $\mathcal{H}$ be a quasi-coherent sheaf on X and $x \in X$.

Definition 5.1. *We say that $\mathcal{H}$ is of homological dimension k at x, $\operatorname{hdim}_x \mathcal{H} = k$, if k is the smallest integer such that there exist a Zariski open neighbourhood U and sets of indices $I_0, I_1, \ldots, I_k$ for which there is an exact sequence of sheaves:*

$$0 \to \mathcal{O}_X^{\oplus I_k}|_U \to \mathcal{O}_X^{\oplus I_{k-1}}|_U \to \cdots \to \mathcal{O}_X^{\oplus I_0}|_U \to \mathcal{H}|_U \to 0.$$

We define the homological dimension of $\mathcal{H}$ as

$$\operatorname{hdim} \mathcal{H} = \sup_{x \in X} \operatorname{hdim}_x \mathcal{H}.$$

Obviously, $\operatorname{hdim} \mathcal{H} = 0$ iff $\mathcal{H}$ is a locally free sheaf and $\operatorname{hdim} \mathcal{H} = 1$ means that $\mathcal{H}$ is locally a quotient of free sheaves.

Remark 5.2. In the case of a coherent sheaf $\mathcal{H}$, all sets I_j can be finite.

Remark 5.3. If $\mathcal{H}$ is of homological dimension k at x and U is a sufficiently small open neighbourhood of x, then $\operatorname{hdim}_y \mathcal{H} \leq k$ for all $y \in U$. Then for any point $y \in U$ the $\mathcal{O}_{y,X}$-module $\mathcal{H}_y$ is of projective dimension $\operatorname{pd} \mathcal{H}_y \leq k$, because the notions of free and projective modules coincide over a local ring (cf. [19]).

Proposition 5.4. *Let $\mathcal{H}$ be a coherent sheaf on X and $x \in X$, then*

$$\operatorname{pd} \mathcal{H}_x = \operatorname{hdim}_x \mathcal{H},$$

here $\operatorname{pd} \mathcal{H}_x$ is a projective dimension of $\mathcal{O}_{x,X}$-module.

Proof. The inequality $\leq$ follows from Remark 5.3. Conversely, let $k = \operatorname{pd} \mathcal{H}_x$. Then there is an exact sequence of $\mathcal{O}_{x,X}$-modules

$$0 \longrightarrow \mathcal{O}_{x,X}^{p_k} \xrightarrow{\phi_{xk}} \dots \xrightarrow{\phi_{x2}} \mathcal{O}_{x,X}^{p_1} \xrightarrow{\phi_{x1}} \mathcal{O}_{x,X}^{p_0} \xrightarrow{\phi_{x0}} \mathcal{H}_x \longrightarrow 0.$$

By [27, I, §2, Proposition 5] the homomorphisms ϕ_{xj} can be lifted to homomorphisms of sheaves on some common Zariski open neighbourhood V of x:

$$\mathcal{O}_X^{p_k}|_V \xrightarrow{\phi_k} \dots \xrightarrow{\phi_2} \mathcal{O}_X^{p_1}|_V \xrightarrow{\phi_1} \mathcal{O}_X^{p_0}|_V \xrightarrow{\phi_0} \mathcal{H}|_V.$$

By [27, I, §2, Theorem 2], $\operatorname{im}\phi_i$ and $\ker\phi_i$ are coherent for all $i = 1, 2, \dots, k$ and so are the sheaves $\ker\phi_i/\operatorname{im}\phi_{i+1}$ for all $i = 0, 1, \dots, k-1$. By assumption, $(\ker\phi_i/\operatorname{im}\phi_{i+1})_x = 0$. Therefore the above equality propagates to a common Zariski open neighbourhood $U \subset V$ of x. Consequently, the sequence is exact

$$0 \to \mathcal{O}_X^{p_k}|_U \xrightarrow{\phi_k} \dots \xrightarrow{\phi_2} \mathcal{O}_X^{p_1}|_U \xrightarrow{\phi_1} \mathcal{O}_X^{p_0}|_U \xrightarrow{\phi_0} \mathcal{H}|_U \to 0.$$

Hence $\operatorname{pd} \mathcal{H}_x \geq \operatorname{hdim}_x \mathcal{H}$, as desired.

$\square$

Remark 5.5. If $\mathcal{H}$ is a quasi-coherent sheaf we can get only $\operatorname{pd} \mathcal{H}_x \leq \operatorname{hdim}_x \mathcal{H}$, as not every morphism on stalks can be lifted to a neighbourhood.

Moreover, we need the following result of homological algebra (cf. [18, Part III, Theorem 2]).

Theorem 5.6. *Let*

$$0 \to F \to G \to H \to 0$$

be a short exact sequence of R-modules over a ring R.

a) If $\operatorname{pd} G > \operatorname{pd} F$*, then* $\operatorname{pd} H = \operatorname{pd} G$*.*

b) If $\operatorname{pd} G < \operatorname{pd} F$*, then* $\operatorname{pd} H = \operatorname{pd} F + 1$*.*

c) If $\operatorname{pd} G = \operatorname{pd} F$*, then* $\operatorname{pd} H \leq \operatorname{pd} G + 1$*.*

We can reformulate the above theorem into a more condensed version due to P.M. Cohn, namely

$$\operatorname{pd} G \leq \max(\operatorname{pd} F, \operatorname{pd} H)$$

with equality unless $\operatorname{pd} H = \operatorname{pd} F + 1$.

As a corollary, we obtain the following proposition, which will be useful further in the paper.

Proposition 5.7. *Let*

$$0 \to F \to G \to H \to 0$$

be a short exact sequence of R-modules over a ring R.

a) If $\operatorname{pd} G \leq 1$ *and* $\operatorname{pd} H \leq 1$ *then* $\operatorname{pd} F \leq 1$.

b) If $\operatorname{pd} F \leq 1$ *and* $\operatorname{pd} H \leq 1$ *then* $\operatorname{pd} G \leq 1$.

c) If $\operatorname{pd} F = 0$ *and* $\operatorname{pd} G \leq 1$ *then* $\operatorname{pd} H \leq 1$.

d) If $\operatorname{pd} G = 0$ *and* $\operatorname{pd} H \leq 1$ *then* $\operatorname{pd} F = 0$.

Proof. This follows directly from Theorem 5.6. □

Given a short exact sequence of coherent sheaves, the corollary below indicates the cases where if two of them are of homological dimension ≤ 1, so is the third one. Actually, this property may not be preserved only in the case where both $\mathcal{F}$ and $\mathcal{G}$ are of homological dimension 1.

Corollary 5.8. *Let*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

be a short exact sequence of coherent sheaves on X.

a) If $\operatorname{hdim} \mathcal{G} \leq 1$ *and* $\operatorname{hdim} \mathcal{H} \leq 1$*, then* $\operatorname{hdim} \mathcal{F} \leq 1$.

b) If $\operatorname{hdim} \mathcal{F} \leq 1$ *and* $\operatorname{hdim} \mathcal{H} \leq 1$*, then* $\operatorname{hdim} \mathcal{G} \leq 1$.

c) If $\operatorname{hdim} \mathcal{F} = 0$ *and* $\operatorname{hdim} \mathcal{G} \leq 1$*, then* $\operatorname{hdim} \mathcal{H} \leq 1$.

d) If $\operatorname{hdim} \mathcal{G} = 0$ *and* $\operatorname{hdim} \mathcal{H} \leq 1$*, then* $\operatorname{hdim} \mathcal{F} = 0$.

Proof. Apply Propositions 5.4 and 5.7. □

Proposition 5.9. *Let R be an integral domain and K its field of fractions. Consider an exact sequence of R-modules*

$$0 \to F \to G$$

such that F is free and the projective dimension of G is ≤ 1. Then for any ring S such that $R \subset S \subset K$ the induced sequence

$$0 \to F \otimes_R S \to G \otimes_R S$$

is exact.

Proof. Clearly, $G = F_2/E$ and $F = F_1/E$ where $E \subset F_1 \subset F_2$ are R-modules and F_2, E are free. We have the following short exact sequence of R-modules

$$0 \to E \to F_1 \to F_1/E \to 0$$

where E and F_1/E are free. Since free module is projective, the above exact sequence splits and F_1 is also a free module. Therefore the canonical homomorphisms

$$E \otimes S \to F_1 \otimes S \to F_2 \otimes S$$

are injective, because $E \hookrightarrow E \otimes S \hookrightarrow E \otimes K$ and the problem of injectivity of free modules reduces to the one for K-vector spaces. Consequently, we can regard $E \otimes S$ and $F_1 \otimes S$ as submodules of $F_2 \otimes S$. Hence the canonical homomorphism

$$(F_1/E) \otimes S \cong (F_1 \otimes S)/(E \otimes S) \to (F_2/E) \otimes S \cong (F_2 \otimes S)/(E \otimes S)$$

is injective as asserted. □

In the proof of Proposition 6.5, we shall use the following

Corollary 5.10. *Let*

$$0 \to \mathcal{F} \to \mathcal{G}$$

be an exact sequence of quasi-coherent sheaves on X such that $\operatorname{pd} \mathcal{F}_x = 0$ *and* $\operatorname{pd} \mathcal{G}_x \leq 1$ *for all $x \in X$.*

Then for any multi-blowup $\sigma : X_\sigma \to X$ the induced sequence

$$0 \to \sigma^* \mathcal{F} \to \sigma^* \mathcal{G}$$

is exact.

Proof. Let $\sigma : X_\sigma \to X$ be a multi-blowup. Take any $x \in X$, $y \in X_\sigma$ such that $\sigma(y) = x$. We have an exact sequence

$$0 \to (\sigma^{-1}\mathcal{F})_y = \mathcal{F}_x \to (\sigma^{-1}\mathcal{G})_y = \mathcal{G}_x.$$

Hence by Proposition 5.9, we get

$$0 \to (\sigma^*\mathcal{F})_y = \mathcal{F}_x \otimes_{\mathcal{O}_{x,X}} \mathcal{O}_{y,X_\sigma} \to (\sigma^*\mathcal{G})_y = \mathcal{G}_x \otimes_{\mathcal{O}_{x,X}} \mathcal{O}_{y,X_\sigma};$$

the above sequence is exact because $\mathcal{O}_{y,X_\sigma}$ is a localization of $\mathcal{O}_{x,X}$.

□

Lemma 5.11. *Let $\alpha : X_\alpha \to X$ be a multi-blowup of X and $\mathcal{H}$ a quasi-coherent sheaf on X. If $\mathcal{H}$ is of homological dimension ≤ 1, so is the pull-back $\alpha^*\mathcal{H}$.*

Proof. Let $\alpha : X_\alpha \to X$ be a multi-blowup of X, take any $x \in X$ and $U \subset X$ as in the definition of homological dimension. We have a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_X^{\oplus I_1}|_U \longrightarrow \mathcal{O}_X^{\oplus I_0}|_U \longrightarrow \mathcal{H}|_U \longrightarrow 0.$$

By the above corollary and right exactness of pull-back we obtain a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_{X_\alpha}^{\oplus I_1}|_{U^\alpha} \longrightarrow \mathcal{O}_{X_\alpha}^{\oplus I_0}|_{U^\alpha} \longrightarrow \alpha^*\mathcal{H}|_{U^\alpha} \longrightarrow 0.$$

Since x was arbitrary we get $\operatorname{hdim} \mathcal{H} \leq 1$, as asserted. $\square$

Corrollary 5.10 along with Lemma 5.11 and Remark 5.5 yield immediately the following

Corollary 5.12. *Let*

$$0 \to \mathcal{F} \to \mathcal{G}$$

be an exact sequence of quasi-coherent sheaves on X such that $\operatorname{hdim} \mathcal{F} = 0$ *and* $\operatorname{hdim} \mathcal{G} \leq 1$. *Then for any multi-blowup $\sigma : X_\sigma \to X$ the induced sequence*

$$0 \to \sigma^*\mathcal{F} \to \sigma^*\mathcal{G}$$

is exact.

Chapter 6

Cartan's Theorem B

6.1 Blown-up Čech cohomology

In this section, we introduce the concept of blown-up Čech cohomology for quasi-coherent sheaves. Our construction combines the classical one due to Serre [27] with direct limit with respect to the directed set of multi-blowups described in Chapter 3.

Let $\mathcal{F}$ be a sheaf on X and $\mathcal{U} = \{U_i\}_{i=1}^n$ be a finite Zariski open covering of X. Put $U_{i_0 \ldots i_q} = U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_q}$ and

$$C^q(\mathcal{U}, \mathcal{F}) := \prod_{1 \le i_0, i_1, \ldots, i_q \le n} \mathcal{F}(U_{i_0 i_1 \ldots i_q}).$$

$C^q(\mathcal{U}, \mathcal{F})$ is called the abelian group of q-cochains. We have a chain complex

$$\ldots \longrightarrow C^{q-1}(\mathcal{U}, \mathcal{F}) \xrightarrow{d^{q-1}} C^q(\mathcal{U}, \mathcal{F}) \xrightarrow{d^q} C^{q+1}(\mathcal{U}, \mathcal{F}) \longrightarrow \ldots$$

where

$$(d^q f)_{i_0 i_1, \ldots i_{q+1}} = \sum_{j=0}^{q+1} (-1)^j f_{i_0 i_1 \ldots \widehat{i_j} \ldots i_{q+1}}|_{U_{i_0 i_1 \ldots i_{q+1}}}$$

for any $f = (f_{i_0 \ldots i_q}) \in C^q(\mathcal{U}, \mathcal{F})$, where $\widehat{i_j}$ means that we omit the index i_j. If $\sigma : Y \to X$ is a morphism of real affine varieties, we get the induced chain complex

$$\ldots \longrightarrow C^{q-1}(\mathcal{U}^\sigma, \sigma^* \mathcal{F}) \xrightarrow{d^{q-1}} C^q(\mathcal{U}^\sigma, \sigma^* \mathcal{F}) \xrightarrow{d^q} C^{q+1}(\mathcal{U}^\sigma, \sigma^* \mathcal{F}) \longrightarrow \ldots.$$

and a canonical chain complex homomorphism

$$\sigma^* : C^\bullet(\mathcal{U}, \mathcal{F}) \to C^\bullet(\mathcal{U}^\sigma, \sigma^*\mathcal{F});$$

the canonical homomorphism $\mathcal{F}(U) \to \sigma^*\mathcal{F}(U^\sigma)$ was described in the Section 2.2. Therefore σ induces a homomorphism of cohomology complexes

$$\sigma^* : H^\bullet(\mathcal{U}, \mathcal{F}) \to H^\bullet(\mathcal{U}^\sigma, \sigma^*\mathcal{F}).$$

Let $\widetilde{C}^\bullet(\mathcal{U}, \mathcal{F})$ be the chain complex defined by the formula

$$\widetilde{C}^\bullet(\mathcal{U}, \mathcal{F}) = \varinjlim_{\alpha} C^\bullet(U^\alpha, \alpha^*\mathcal{F}).$$

where the limit is taken over a directed system of multi-blowups of X.

Definition 6.1. *The q-th blown-up Čech cohomology group $\widetilde{H}^q(\mathcal{U}, \mathcal{F})$ of $\mathcal{F}$ with respect to $\mathcal{U}$ is the q-th cohomology group of the chain complex $\widetilde{C}^\bullet(\mathcal{U}, \mathcal{F})$.*

Remark 6.2. It is well known that direct limit functor commutes with cohomology functor (see e.g. [1, Theorem 4.14]). Hence

$$\widetilde{H}^q(\mathcal{U}, \mathcal{F}) = \varinjlim_{\alpha} H^q(\mathcal{U}^\alpha, \alpha^*\mathcal{F}).$$

We now establish long exact cohomology sequence for some quasi-coherent sheaves, which plays a crucial role in the cohomology theory developed in this thesis. The general case of arbitrary quasi-coherent sheaves is not at our disposal, as pull-back functor $\mathcal{F} \mapsto \alpha^*\mathcal{F}$ (along with tensor product functor) is not left exact. But we have of course the following

Proposition 6.3. *Let $\mathcal{U}$ be a finite Zariski open covering of X and*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

be a short exact sequence of quasi-coherent sheaves on X. Suppose that for any multi-blowup $\sigma : X_\sigma \to X$ the induced sequence

$$0 \to \sigma^*\mathcal{F} \to \sigma^*\mathcal{G}$$

is exact. Then there is a short exact sequence of chain complexes

$$0 \to \widetilde{C}^\bullet(\mathcal{U}, \mathcal{F}) \to \widetilde{C}^\bullet(\mathcal{U}, \mathcal{G}) \to \widetilde{C}^\bullet(\mathcal{U}, \mathcal{H}) \to 0$$

which induces a long exact sequence of blown-up Čech cohomology with respect to $\mathcal{U}$

$$\cdots \to \widetilde{H}^p(\mathcal{U}, \mathcal{F}) \to \widetilde{H}^p(\mathcal{U}, \mathcal{G}) \to \widetilde{H}^p(\mathcal{U}, \mathcal{H}) \to \widetilde{H}^{p+1}(\mathcal{U}, \mathcal{F}) \to$$

$$\widetilde{H}^{p+1}(\mathcal{U}, \mathcal{G}) \to \widetilde{H}^{p+1}(\mathcal{U}, \mathcal{H}) \to \widetilde{H}^{p+2}(\mathcal{U}, \mathcal{F}) \to \dots$$

Proof. Of course the induced short sequence

$$0 \to \sigma^*\mathcal{F} \to \sigma^*\mathcal{G} \to \sigma^*\mathcal{H} \to 0$$

is exact, and thus the short exact sequence of chain complexes

$$0 \to \widetilde{C}^\bullet(\mathcal{U}, \mathcal{F}) \to \widetilde{C}^\bullet(\mathcal{U}, \mathcal{G}) \to \widetilde{C}^\bullet(\mathcal{U}, \mathcal{H}) \to 0$$

is exact by Corollary 4.10. Hence the proposition follows directly.

□

We immediately obtain

Corollary 6.4. *Let* $\mathcal{U}$ *be a finite Zariski open covering of* X *and*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

be a short exact sequence of quasi-coherent sheaves on X*. Assume that one of the following conditions holds*

i) $\operatorname{pd} \mathcal{F}_x = 0$ *and* $\operatorname{pd} \mathcal{G}_x \leq 1$ *for all* $x \in X$*.*

ii) $\operatorname{hdim} \mathcal{F} = 0$ *and* $\operatorname{hdim} \mathcal{G} \leq 1$.

Then there is an induced long exact sequence of blown-up Čech cohomology with respect to $\mathcal{U}$

$$\cdots \to \widetilde{H}^p(\mathcal{U}, \mathcal{F}) \to \widetilde{H}^p(\mathcal{U}, \mathcal{G}) \to \widetilde{H}^p(\mathcal{U}, \mathcal{H}) \to \widetilde{H}^{p+1}(\mathcal{U}, \mathcal{F}) \to$$

$$\widetilde{H}^{p+1}(\mathcal{U}, \mathcal{G}) \to \widetilde{H}^{p+1}(\mathcal{U}, \mathcal{H}) \to \widetilde{H}^{p+2}(\mathcal{U}, \mathcal{F}) \to \dots$$

6.2 Cartan's Theorem B

We now prove a version of Cartan's Theorem B for blown-up Čech cohomologies for some quasi-coherent subsheaves of $\mathcal{O}_X^{\oplus I}$. These results come from our paper [23].

Proposition 6.5. *Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-subsheaf of $\mathcal{O}_X^{\oplus I}$ with all stalks $\mathcal{F}_x$, $x \in X$ being free over $\mathcal{O}_{x,X}$. For any finite Zariski open covering of X we have*

$$\widetilde{H}^q(\mathcal{U}, \mathcal{F}) = 0.$$

Proof. We have a short exact sequence of quasi-coherent sheaves

$$0 \to \mathcal{F} \to \mathcal{O}_X^{\oplus I} \to \mathcal{O}_X^{\oplus I}/\mathcal{F} \to 0.$$

By assumptions and Corollary 5.10, for any multi-blowup $\alpha : X_\alpha \to X$ we have a short exact sequence

$$0 \to \alpha^*\mathcal{F} \to \mathcal{O}_{X_\alpha}^{\oplus I} \to \mathcal{O}_{X_\alpha}^{\oplus I}/\alpha^*\mathcal{F} = \alpha^*(\mathcal{O}_X^{\oplus I}/\mathcal{F}) \to 0,$$

hence $\alpha^*\mathcal{F} \subset \mathcal{O}_{X_\alpha}^{\oplus I}$.

Any $[f] \in \widetilde{C}^q(\mathcal{U}, \mathcal{F})$, has a representative $f \in C^q(\mathcal{U}^\alpha, \alpha^*)$ for some multi-blowup $\alpha : X_\alpha \to X$. Our objective is to find a multi-blowup $\beta : X_\beta \to X_\alpha$ and a $(q-1)$-cocycle k such that $dk = \beta^* f$. To simplify the proof we assume that $f \in C^q(\mathcal{U}, \mathcal{F})$. Each of the sets U_i is of the form $U_i = X \setminus \{Q_i = 0\}$ for some $Q_i \in \mathcal{O}_X(X)$. Recall that $U_{i_0 i_1 \dots i_q} = U_{i_1} \cap U_{i_2} \cap \dots \cap U_{i_q}$ and $f = (f_{i_0 i_1 \dots i_q})$ over all $(q+1)$-tuples of indices for which $i_0, i_1, \dots, i_q \in \{1, 2, \dots, n\}$.

By the assumptions every $f_{i_0 i_1 \dots i_q}$ can be identified with a finite tuple of non-vanishing regular functions on $U_{i_0 i_1 \dots i_q}$ i.e

$$f_{i_0 i_1 \dots i_q} \in \mathcal{F}(U_{i_0 i_1 \dots i_q}) \subset \mathcal{O}_X^{\oplus I}(U_{i_0 i_1 \dots i_q})$$

where

$$U_{i_0 i_1 \dots i_q} = X \setminus \{Q_{i_0 i_1 \dots i_q} = 0\}$$

and

$$Q_{i_0 i_1 \dots i_q} = Q_{i_0} Q_{i_1} \dots Q_{i_q}.$$

By Lemma 4.8 there exists a multi-blowup $\alpha : X_\alpha \to X$, a positive integer N_1 and a global section $g_{i_0 i_1 \dots i_q} \in \mathcal{O}_{X_\alpha}^{\oplus I}(X_\alpha)$ such that for every $(q+1)$-tuple

$$g_{i_0 i_1 \dots i_q}|_{U^\alpha_{i_0 i_1 \dots i_q}} = (Q^{N_1}_{i_0 i_1 \dots i_q})^\alpha \alpha^* f_{i_0 i_1 \dots i_q}.$$

Consider the image of $g_{i_0 i_1 \ldots i_q}$ in the quasi-coherent sheaf $\mathcal{O}_{X_\alpha}^{\oplus I}/\alpha^*\mathcal{F}$. It is a global section which vanishes on $U^{\alpha}_{i_0 i_1 \ldots i_q}$ and thus, by Lemma 4.7, there exist a multi-blowup $\beta : X_\beta \to X_\alpha$ and a positive integer M such that

$$(Q^M_{i_0 i_1 \ldots i_q})^{\alpha\circ\beta}\beta^* g_{i_0 i_1 \ldots i_q}$$

is a zero section in $(\beta^*(\mathcal{O}^I_{X_\alpha}/\alpha^*\mathcal{F}))(X_\beta)$. Setting $N = N_1 + M$, we get

$$h_{i_0 i_1 \ldots i_q} := (Q^M_{i_0 i_1 \ldots i_q})^{\alpha\circ\beta}\beta^* g_{i_0 i_1 \ldots i_q} =$$

$$(Q^N_{i_0 i_1 \ldots i_q})^{\alpha\circ\beta}(\alpha\circ\beta)^* f_{i_0 i_1 \ldots i_q} \in (\alpha\circ\beta)^*\mathcal{F}(X_\beta).$$

Since these sections are global, one can always increase N and assume that the number N is even. Define the global regular function

$$R := \frac{1}{\sum_{i=1}^n (Q_i^N)^{\alpha\circ\beta}}.$$

Now we are able to define $k \in C^{q-1}(\mathcal{U}^{\alpha\circ\beta}, (\alpha\circ\beta)^*\mathcal{F})$ by the formula

$$k_{i_0 i_1 \ldots i_{q-1}} = R\sum_{i=1}^n \frac{h_{i i_0 i_1 \ldots i_{q-1}}}{(Q^N_{i_0 i_1 \ldots i_{q-1}})^{\alpha\circ\beta}}\Big|_{U^{\alpha\circ\beta}_{i_0 i_1 \ldots i_{q-1}}}.$$

By the very definition of the operator d we have

$$(dk)_{i_0 i_1 \ldots i_q} = \sum_{j=0}^q (-1)^j R \sum_{i=1}^n \frac{h_{i i_0 i_1 \ldots \widehat{i_j} \ldots i_{q-1}}}{(Q^N_{i_0 i_1 \ldots \widehat{i_j} \ldots i_{q-1}})^{\beta}}\Big|_{U^{\beta}_{i_0 i_1 \ldots i_q}}$$

the right hand side is a finite tuple of regular functions on $U^{\alpha\circ\beta}_{i_0 i_1 \ldots i_q}$. To finish the proof we have to show that $dk = (\alpha\circ\beta)^* f$. It is enough to show it on $\bigcap \mathcal{U}^{\alpha\circ\beta} = U$, since we are dealing only with rational functions and U is Zariski open and dense in $U^{\alpha\circ\beta}_{i_0 i_1 \ldots i_q}$. Recall that, since $(\alpha\circ\beta)^* f$ is a cocycle, we have

$$0 = (d(\alpha\circ\beta)^* f)_{i i_0 i_1 \ldots i_q} = (\alpha\circ\beta)^* f_{i_0 i_1 \ldots i_q} + \sum_{j=0}^q (-1)^{j+1}(\alpha\circ\beta)^* f_{i i_0 i_1 \ldots \widehat{i_j} \ldots i_q}.$$

The following equality holds on U:

$$k_{i_0 i_1 \ldots i_{q-1}} = R\sum_{i=1}^n (Q_i^N)^{\alpha\circ\beta}(\alpha\circ\beta)^* f_{i i_0 i_1 \ldots i_{q-1}},$$

whence

$$(dk)_{i_0 i_1 \dots i_q} = \sum_{j=0}^{q} (-1)^j R \sum_{i=1}^{n} (Q_i^N)^{\alpha\circ\beta} (\alpha\circ\beta)^* f_{i i_0 i_1 \dots \widehat{i_j} \dots i_q} \tag{6.1}$$

Combining the equality (6.1) with the fact that $(\alpha\circ\beta)^* f$ is a cocycle, we obtain

$$(dk)_{i_0 i_1 \dots i_q} = R \sum_{i=1}^{n} (Q_i^N)^{\beta} \sum_{j=0}^{q} (-1)^j (\alpha\circ\beta)^* f_{i i_0 i_1 \dots \widehat{i_j} \dots i_q} =$$

$$= R \sum_{i=1}^{n} (Q_i^N)^{\alpha\circ\beta} (\alpha\circ\beta)^* f_{i_0 i_1 \dots i_q} = (\alpha\circ\beta)^* f_{i_0 i_1 \dots i_q};$$

the last equality follows from the definition of the function R. This completes the proof.

□

Now we can readily prove the following real algebraic version of Cartan's Theorem B.

Theorem 6.6. *Let $\mathcal{U} = \{U_i\}_{i=1}^n$ be a finite Zariski open covering of X and let $\mathcal{H}$ be a quasi-coherent sheaf which admits a global presentation such that* $\operatorname{pd} \mathcal{H}_x \le 1$ *for all $x \in X$. Then $\widetilde{H}^q(\mathcal{U}, \mathcal{H}) = 0$ for $q > 0$.*

Proof. We have a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{O}_X^{\oplus I} \to \mathcal{H} \to 0.$$

Obviously, $\operatorname{hdim}(\mathcal{O}_X^{\oplus I}) = 0$ and $\operatorname{pd} \mathcal{H}_x \le 1$ by assumption. Hence $\operatorname{pd} \mathcal{F}_x = 0$ for all $x \in X$ by Proposition 5.7. By Corollary 6.4, the above exact sequence induces a long exact sequence of blown-up Čech cohomology

$$\dots \to \widetilde{H}^q(\mathcal{U}, \mathcal{F}) \to \widetilde{H}^q(\mathcal{U}, \mathcal{O}_X^{\oplus I}) \to \widetilde{H}^q(\mathcal{U}, \mathcal{H}) \to \widetilde{H}^{q+1}(\mathcal{U}, \mathcal{F})$$

$$\to \widetilde{H}^{q+1}(\mathcal{U}, \mathcal{O}_X^{\oplus I}) \to \widetilde{H}^{q+1}(\mathcal{U}, \mathcal{H}) \to \dots$$

By Proposition 6.5

$$\widetilde{H}^q(\mathcal{U}, \mathcal{F}) = \widetilde{H}^q(\mathcal{U}, \mathcal{O}_X^{\oplus I}) = \widetilde{H}^{q+1}(\mathcal{U}, \mathcal{F}) = \widetilde{H}^{q+1}(\mathcal{U}, \mathcal{O}_X^{\oplus I}) = 0$$

for any $q > 0$. Hence $\widetilde{H}^q(\mathcal{U}, \mathcal{H}) = 0$ as asserted.

□

Corollary 6.7. *Let $\mathcal{U} = \{U_i\}_{i=1}^n$ be a finite Zariski open covering of X and let $\mathcal{H}$ be a coherent sheaf of homological dimension ≤ 1. Then $\widetilde{H}^q(\mathcal{U}, \mathcal{H}) = 0$ for $q > 0$.*

Proof. By Lemma 4.15 there exists a multi-blowup $\alpha : X_\alpha \to X$ such that $\alpha^*\mathcal{H}$ admits a global presentation. We can thus apply Theorem 6.6.

□

The family of finite Zariski open coverings of X can be directed by refinement relation $\succeq$ defined as follows. A Zariski open covering $\mathcal{U} = \{U_i\}_{i=1}^n$ is finer that a covering $\mathcal{V} = \{V_j\}_{j=1}^m$, $\mathcal{U} \succeq \mathcal{V}$, if there is

$$\tau : \{1, 2, \dots, n\} \to \{1, 2, \dots, m\}$$

such that $U_i \subset V_{\tau(i)}$ for all $i = 1, 2, \dots, n$. Consider the homomorphism

$$\tau : C^\bullet(\mathcal{V}, \mathcal{F}) \to C^\bullet(\mathcal{U}, \mathcal{F}),$$
$$(\tau_q(f))_{i_0 i_1 \dots i_q} := f_{\tau(i_0)\tau(i_1)\dots\tau(i_q)}|_{U_{i_0 i_1 \dots i_q}}$$

which does depend on τ. Yet the induced homomorphism on the Čech cohomology

$$\tau_q : H^q(\mathcal{V}, \mathcal{F}) \to H^q(\mathcal{U}, \mathcal{F})$$

does not depend on the choice of τ (cf. [27, I, §3, Prop. 3]). Likewise in the classical case of Čech cohomology, we get the induced homomorphism on the cohomology

$$\tau_q : \widetilde{H}^q(\mathcal{V}, \mathcal{F}) \to \widetilde{H}^q(\mathcal{U}, \mathcal{F})$$

independent of the choice of τ. Therefore we can give the following

Definition 6.8. *The q-th blown-up Čech cohomology group $\widetilde{H}^q(X, \mathcal{F})$ of $\mathcal{F}$ is the direct limit*

$$\widetilde{H}^q(X, \mathcal{F}) = \lim_{\overrightarrow{\mathcal{U}}} \widetilde{H}^q(\mathcal{U}, \mathcal{F}).$$

Summing up, Theorem 6.6, Proposition 6.5 and Corollary 6.7 yield immediately the following general version of Cartan's Theorem B.

Theorem 6.9. *Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules and let $\mathcal{U}$ be a finite Zariski open covering of X. Assume that one of the following conditions hold* *a)* $\mathcal{F}$ *is a quasi-coherent subsheaf of* $\mathcal{O}_X^{\oplus I}$ *such that* $\operatorname{pd}\mathcal{F}_x = 0$ *for all* $x \in X$.

b) $\mathcal{F}$ *is a quasi-coherent sheaf of global presentation such that* $\operatorname{pd}\mathcal{F}_x \leq 1$ *for all* $x \in X$.

c) $\mathcal{F}$ *is a coherent sheaf and* $\operatorname{hdim}\mathcal{F} \leq 1$.

Then $\widetilde{H}^q(\mathcal{U}, \mathcal{F}) = 0$ *and, a fortiori,* $\widetilde{H}^q(X, \mathcal{F}) = 0$ *for* $q \geq 1$.

Some examples concerning the above Cartan's Theorem B will be given in Chapter 9.

Chapter 7

Additive Cousin problem

In this section we deal with the first Cousin problem. The classical complex analytic version of the first Cousin problem is treated e.g. in [10]. Before discussing details, we give an outline of the problem. Let $\mathcal{U} = \{U_i\}_{i=1}^n$ be a finite Zariski open covering of X. Assume that for each i we have a rational function f_i on U_i such that $f_i - f_j$ is regular on $U_i \cap U_j$ for each two distinct indices $i, j = 1, 2, \dots n$. Then we call $\{(U_i, f_i)\}_{i=1}^n$ data of the first Cousin problem or an additive Cousin distribution on X.

Let U be a Zariski open subset of X.

Definition 7.1. *We say that two rational functions f, g on X have the same principal part on U if $f - g \in \mathcal{O}_X(U)$.*

Definition 7.2. *We say that the first Cousin data $\{(U_i, f_i)\}_{i=1}^n$ is solvable if $\{(U_i, f_i)\}_{i=1}^n$ have the principal part of a rational function on X, i.e. there exists a rational function f on X such that $f - f_i$ are regular on U_i, $i = 1, 2, \dots, n$. If every first Cousin data on X is solvable, we say that the first Cousin problem is universally solvable on X (see e.g. [10, Introduction, §2]).*

The first Cousin problem consists in characterizing those data which are solvable. We are going to describe the above problem in terms of sheaves. Let $\mathcal{K}_X$ be the constant sheaf of rational functions on X. Consider the short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{K}_X \xrightarrow{\varphi} \mathcal{H}_X := \mathcal{K}_X/\mathcal{O}_X \to 0$$

of quasi-coherent sheaves on X. The data $\{(U_i, f_i)\}_{i=1}^n$ of the first Cousin problem can be related to a unique global section $s \in \mathcal{H}_X(X)$; every such section is called a principal part distribution on X. Then we also say that $\{(U_i, f_i)\}_{i=1}^n$ is an s-representing distribution. In particular, for every rational function $f \in \mathcal{K}_X(X)$ we have its principal part distribution $\varphi(f)$ on X.

Let $\{(U_i, f_i)\}_{i=1}^n$ be an s-representing distribution. Then any rational function $f \in \mathcal{K}_X(X)$ satisfying $\varphi(f) = s$ is one such that $f - f_i \in \mathcal{O}_X(U_i)$ for each $i = 1, 2, \dots, n$.

In general, the short exact sequence of quasi-coherent sheaves

$$0 \to \mathcal{O}_X \to \mathcal{K}_X \to \mathcal{H}_X \to 0$$

induces only the following exact sequence of chain complexes

$$0 \to C^\bullet(\mathcal{U}, \mathcal{O}_X) \to C^\bullet(\mathcal{U}, \mathcal{K}_X) \to C^\bullet(\mathcal{U}, \mathcal{H}_X).$$

Under the circumstances, we have the following exact sequence for the classical Čech cohomologies (cf. [27, Chap I, §3, section 24])

$$0 \to H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{K}_X) \to H^0(X, \mathcal{H}_X) \xrightarrow{\zeta}$$
$$H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{K}_X) \to H^1(X, \mathcal{H}_X).$$

The above sequence of chain complexes induces a long exact sequence for Čech cohomologies whenever the topology of X is paracompact (cf. [27, Chap I, §3, section 25]). This can be applied in the classical case of complex analytic geometry (see e.g. [10]).

Clearly, every s-representing distribution $\{(U_i, f_i)\}_{i=1}^n$ determines a 1-cocycle $g = (g_{ij})$, $g_{ij} := f_i - f_j$, which induces a cohomology class $\zeta(s)$ in $H^1(\mathcal{U}, \mathcal{O}_X)$. The solvability of a given Cousin data can be rephrased in terms of vanishing $\zeta(s)$ in $H^1(\mathcal{U}, \mathcal{O}_X)$.

Lemma 7.3. *An s-representing distribution $\{(U_i, f_i)\}_{i=1}^n$ is solvable iff $\zeta(s) = 0 \in H^1(\mathcal{U}, \mathcal{O}_X)$.*

Proof. The necessary condition. Let f be a rational function such that $f - f_i \in \mathcal{O}_X(U_i)$. Then (g_{ij}) is a coboundary of a 0-cocycle h given by $h_i = f_i - f$, hence $\zeta(s) = 0$.

Conversely, if $\zeta(s)$=0 then $g = dh$ for some 0-cocycle $h = (h_i)_i$, $h_i \in \mathcal{O}_X(U_i)$. We then have $(h_i - h_j)|_{U_i \cap U_j} = (f_i - f_j)|_{U_i \cap U_j}$, and obviously $(f_i - h_i)|_{U_i \cap U_j} = (f_j - h_j)|_{U_i \cap U_j}$ for any pair of two indices i, j. It follows that the system $(f_i - h_i)$ for $i = 1, 2, \dots, n$ can be glued to a global rational function f such that $f - f_i \in \mathcal{O}_X(U_i)$, as asserted.

□

No natural map from $H^q(X, \mathcal{O}_X)$ to $H^q(\mathcal{U}, \mathcal{O}_X)$ exist for arbitrary covering $\mathcal{U}$. Therefore it is necessary to refine a given covering to solve the first Cousin problem in complex analytic geometry (cf [10], Chapter V). In view of Theorem 6.9, it turns out to be superfluous in real algebraic geometry after blowing up. This is stated in the following

Theorem 7.4. *Let $\{(U_i, f_i)\}_{i=1}^n$ be an s-representing distribution. Then there exists a multi-blowup $\alpha : X_\alpha \to X$ such that pull-back $\{(U_i^\alpha, f_i^\alpha)\}_{i=1}^n$ is solvable i.e. there exists a rational function f on X_α such that $f - f_i^\alpha \in \mathcal{O}_{X_\alpha}(U_i^\alpha)$.*

We need an elementary lemma

Lemma 7.5. *For any multi-blowup $\alpha : X_\alpha \to X$, we have a short exact sequence*

$$0 \to \mathcal{O}_{X_\alpha} \to \mathcal{K}_{X_\alpha} \to \mathcal{H}_{X_\alpha} \to 0$$

of quasi-coherent sheaves on X_α.

Proof. For any $y \in X_\alpha, x = \alpha(y) \in X$ the inclusion $\mathcal{O}_{x,X} \hookrightarrow K_x = K$ induces the inclusion

$$\mathcal{O}_{x,X} \otimes_{\mathcal{O}_{x,X}} \mathcal{O}_{y,X_\alpha} = \mathcal{O}_{y,X_\alpha} \hookrightarrow K_x \otimes_{\mathcal{O}_{x,X}} \mathcal{O}_{y,X_\alpha} = K = K_y.$$

Hence the conclusion follows.

□

Proof of Theorem 7.4 The above lemma yields the following short exact sequence of chain complexes

$$0 \to \widetilde{C}^\bullet(\mathcal{U}, \mathcal{O}_X) \to \widetilde{C}^\bullet(\mathcal{U}, \mathcal{K}_X) \to \widetilde{C}^\bullet(\mathcal{U}, \mathcal{H}_X) \to 0$$

which induces a long exact sequence in blown-up Čech cohomology

$$0 \to \widetilde{H}^0(\mathcal{U}, \mathcal{O}_X) \to \widetilde{H}^0(\mathcal{U}, \mathcal{K}_X) \to \widetilde{H}^0(\mathcal{U}, \mathcal{H}_X) \to \widetilde{H}^1(\mathcal{U}, \mathcal{O}_X)$$
$$\to \widetilde{H}^1(\mathcal{U}, \mathcal{K}_X) \to \widetilde{H}^1(\mathcal{U}, \mathcal{H}_X) \to \dots$$

By Proposition 6.5, $\widetilde{H}^1(\mathcal{U}, \mathcal{O}_X) = 0$. Since

$$\widetilde{H}^1(\mathcal{U}, \mathcal{O}_X) = \varinjlim_{\alpha} H^1(\mathcal{U}^\alpha, \mathcal{O}_{X_\alpha}),$$

for any class $\omega \in H^1(\mathcal{U}, \mathcal{O}_X)$ there exists a multi-blowup $\alpha : X_\alpha \to X$ such that $\alpha^*\omega = 0$ in $H^1(\mathcal{U}^\alpha \, \mathcal{O}_{X_\alpha})$. To finish the proof it is enough to take $\omega = \zeta(s)$. □

Remark 7.6. It follows from the above proof that there is an isomorphism of $\mathcal{O}_X(X)$-modules

$$\widetilde{H}^q(\mathcal{U}, \mathcal{K}_X) \cong \widetilde{H}^q(\mathcal{U}, \mathcal{H}_X)$$

for $q \geq 1$.

Some examples concerning the additive Cousin problem will be given in Chapter 9.

Chapter 8

Local Structure of sheaves of homological dimension 1

In this chapter, we provide a description of the local structure of coherent sheaves of homological dimension 1 after blowing up. Fix a coherent sheaf $\mathcal{H}$ of homological dimension 1 on X. Since we are interested in what happens after blowing-up, we may assume, by Theorem 4.12 that $\mathcal{H}$ admits a global presentation

$$0 \to \mathcal{F} \to \mathcal{O}_X^q \to \mathcal{H} \to 0.$$

By Corollary 5.8 $\mathcal{F}$ is a locally free sheaf of rank, say p. Our goal is to prove the following theorem:

Theorem 8.1. *For $\mathcal{H}$ as above, there exist a multi-blowup $\alpha : X_\alpha \to X$, a finite Zariski open covering $\{V_k\}_{k=1}^m$ of X_α and regular functions $h_{ik} \in \mathcal{O}_{X_\alpha}(V_k)$ for $i = 1, 2, \ldots, p$ and $k = 1, 2, \ldots, m$ such that*

$$\alpha^*\mathcal{H}|_{V_k} \cong \mathcal{O}_{X_\alpha}^{q-p}|_{V_k} \oplus \bigoplus_{i=1}^{p} (\mathcal{O}_{X_\alpha}|_{V_k})/(h_{ik})$$

and $h_{1k}|h_{2k}|\ldots|h_{pk}$.

Before we move to the proof of the above theorem, we need a local analogue of a Smith normal form of a matrix over a PID.

Definition 8.2. *Let $A = (a_{ij})$, where $1 \leq i \leq p$ and $1 \leq j \leq q$, $p \leq q$ be a matrix with entries in a commutative ring R. We say that A is in Smith* *normal form if $a_{ij} = 0$ for $i \neq j$, $a_{11}|a_{22}|\dots|a_{rr}$, $\quad a_{r+1\,r+1} = \dots = a_{pp} = 0$ i.e.*

$$A = \begin{bmatrix} a_{11} & 0 & 0 & & \dots & & 0 \\ 0 & a_{22} & 0 & & \dots & & 0 \\ 0 & 0 & \ddots & & & & \\ \vdots & \vdots & & a_{rr} & & & \vdots \\ & & & & 0 & & \\ & & & & & \ddots & \\ 0 & 0 & & \dots & & & 0 \end{bmatrix}$$

Let $U \subset X$ be a Zariski open subset of X, $A = (a_{ij})$ be a $p \times q$ matrix with entries in $\mathcal{O}_X(U)$ and $r = \min(p, q)$.

Definition 8.3. *We say that A is locally equivalent to a matrix in Smith normal form if there exist a finite Zariski open covering $\{U_k\}_{k=1}^n$ of U and invertible matrices P_k and Q_k of appropriate sizes with entries in $\mathcal{O}_X(U_k)$, such that the matrix*

$$P_k A|_{U_k} Q_k$$

is in Smith normal form.

We remind the reader that by elementary operations on rows of a matrix A we mean substituting s-th row w_s by $w_s + cw_t$, for $1 \leq t \leq p$ and $c \in \mathcal{O}_X(U)$, or interchanging two rows with each other. Ditto for columns. Such operations are realized by multiplying A by invertible matrices from left or right.

Lemma 8.4. *For any $p \times q$ matrix A with entries in $\mathcal{O}_X(U)$ there exists a multi-blowup $\alpha : X_\alpha \to X$ such that A is locally equivalent to a matrix in Smith normal form considered as a matrix with entries in $\mathcal{O}_{X_\alpha}(U^\alpha)$.*

Proof. We shall combine the algorithm for Smith normal form with transformation to a simple normal crossing, proceeding with induction on the $\min(p, q)$.

Let $\sigma : X_\sigma \to X$ be a multi-blowup such that all of the functions a_{ij}^σ are simultaneously simple normal crossings which are linearly ordered by divisibility relation on each U_k, where $\{U_k\}_{k=1}^n$ is a finite Zariski open covering of U^σ.

If $\min(p, q) = 1$ then the matrix A has only one row or column and possesses a smallest element among entries. Hence the assertion is true.

Assume that $\min(p,q) > 1$. Without loss of generality we may assume that a_{11}^{σ} is the smallest function with respect to the divisibility relation among all a_{ij}^{σ}. Now, since for any two functions their greatest common divisor is well defined, we may proceed as in the case of matrices over integers. By performing elementary operations on rows and columns, we can find invertible matrices P_{1k} and Q_{1k} such that

$$P_{1k}A|_{U_k}Q_{1k} = \begin{bmatrix} a_{11}^{\sigma} & 0 \\ 0 & \widetilde{A_k} \end{bmatrix}$$

and $\widetilde{A_k}$ is a $p-1 \times q-1$ matrix with entries in $\mathcal{O}_{X_\sigma}(U_k)$. By construction, every entry of $\widetilde{A_k}$ is divisible by a_{11}^{σ}, however, entries of $\widetilde{A_k}$ may not be locally linearly ordered by divisibility relation. Assertion then follows from the induction for $\widetilde{A_k}$, for $k = 1, 2, \dots, n$.

□

Proof of Theorem 8.1. Let $\{U_k\}_{k=1}^{n}$ be a finite Zariski open covering of X with presentation of $\mathcal{H}$ as in the definition of homological dimension (cf. Definition 5.1), i.e.

$$0 \to \mathcal{O}_X^p|_{U_k} \xrightarrow{\phi_k} \mathcal{O}_X^q|_{U_k} \to \mathcal{H}|_{U_k} \to 0.$$

Each ϕ_k is given by a $p \times q$ matrix with entries in $\mathcal{O}_X(U_k)$. Hence, by the above lemma, there exists a multi-blowup $\alpha : X_\alpha \to X$ such that each ϕ_k^{α} is locally a Smith normal form on U_k^{α}. This finishes the proof.

□

A global strengthening of Theorem 8.1 can be formulated as below.

Conjecture 8.5. *Let $\mathcal{H}$ be a coherent sheaf of homological dimension 1. Then there exist a multi-blowup $\alpha : X_\alpha \to X$, locally free sheaves $\mathcal{L}_1, \mathcal{L}_2, \dots \mathcal{L}_q$ of rank 1 on X_α and global sections $s_i \in \mathcal{L}_i(X_\alpha)$, $i = 1, 2, \dots, q$, such that*

$$\alpha^*\mathcal{H} \cong \bigoplus_{i=1}^{q} \mathcal{L}_i/(s_i \cdot \mathcal{O}_{X_\alpha}).$$

It seems to be a quite nontrivial task to verify this conjecture. Note that coherent sheaves of homological dimension 0 split into a direct sum of locally free sheaves of rank 1. We discuss the issues of splitting and triviality of locally free coherent sheaves in Appendices A.1 and A.2.

Chapter 9

Examples

In this chapter we present various examples, let $X = \mathbb{R}^2$. In Section 9.1.1 we discuss a locally free coherent sheaf $\mathcal{F}_{1,1}$ on $\mathbb{R}^2$ which is not generated by global sections. We then show how to find a multi-blowup $\sigma : X_\sigma \to \mathbb{R}^2$ such that $\sigma^*\mathcal{F}_{1,1}$ is generated by global sections. Section 9.1.2 contains a counterexample to a quasi-coherent version of Cartan's Theorem A. We construct a quasi-coherent locally free sheaf on $\mathbb{R}^2$ such that for any multi-blowup $\sigma : X_\sigma \to \mathbb{R}^2$ the pull-back $\sigma^*\mathcal{F}$ is not generated by global sections.

Section 9.2 treats an example of an additive Cousin problem on a plane. We present a Cousin data which is not solvable and show how to blowup a plane such that the induced Cousin data is solvable and we deduce that $H^1(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2}) \neq 0$.

9.1 Cartan's Theorem A

We now give an example of a family of locally free coherent sheaves on $\mathbb{R}^2$ (based on [4, Example 12.1.5]) which are not generated by global sections. In the subsequent sections, we show how to blowup the plane in order to obtain generation by global sections and we provide a counterexample to a quasi-coherent version of Cartan's Theorem A.

For $k, l \in \mathbb{N}$, the irreducible polynomial

$$P_{k,l}(x, y) = x^{2k}(x-1)^{2l} + y^2$$

has at most two zeros, $c_1 = (0, 0)$ and $c_2 = (1, 0)$. Put $U_i := \mathbb{R}^2 \backslash \{c_i\},\ \ i = 1, 2$.

Then the transition function

$$g_{2,1} : U_1 \cap U_2 \to \mathrm{GL}(1, \mathbb{R}) = \mathbb{R}^*$$

$$(x, y) \mapsto P_{k,l}(x, y)$$

determines a vector bundle $\xi_{k,l}$ of rank 1 on $\mathbb{R}^2$. The sheaf $\mathcal{F}_{k,l}$ of its sections is locally free of rank 1. A global section s of $\xi_{k,l}$ can be described as a pair of regular functions

$$s_i : U_i \to \mathbb{R}, \quad i = 1, 2$$

such that $g_{2,1}s_1 = s_2$. It is clear that the bundles $\xi_{0,l}$ and $\xi_{k,0}$ are trivial line bundles.

Suppose now that $k, l > 0$ and set $s_i = \frac{f_i}{h_i}$ where f_i and h_i are relatively prime polynomials. Then h_1, h_2 have no zeros on U_1, U_2, respectively and $P_{k,l}f_1h_2 = f_2h_1$. Since $P_{k,l}$ cannot divide either h_1 or h_2, we get $f_2 = \lambda P_{k,l}f_1$ and $h_1 = \lambda^{-1}h_2$ with some $\lambda \in \mathbb{R}, \lambda \neq 0$. Therefore any global section s has to vanish at $c_1 = (0, 0)$, and thus $\mathcal{F}_{k,l}$ cannot be generated by global sections.

Similarly, if we put $g_{1,2} = P_{k,l}(x, y)$ in the above construction, then we obtain a locally free sheaf of rank one such that every global section vanishes at c_2.

9.1.1 The sheaf $\mathcal{F}_{1,1}$

Consider the sheaf $\mathcal{F}_{1,1}$. We will now show that in order to obtain generation by global sections, it is enough to blow up the origin. However, this is not the only way, blowing up the point $(1, 0)$ is sufficient as well.

Let $\sigma : X_\sigma \to \mathbb{R}^2$ be the blowup of $\mathbb{R}^2$ at the origin,

$$X_\sigma = \{(x, y, u : v) \in \mathbb{R}^2 \times \mathbb{P}^1(\mathbb{R}) : xv = uy\}.$$

$$\sigma(x, y, u : v) = (x, y)$$

and let $\sigma^*\mathcal{F}_{1,1}$ be the pullback of $\mathcal{F}_{1,1}$. We can cover X_σ with two open charts

$$\Omega_1 = \{(x, y, u : v) \in X_\sigma : u \neq 0\} \text{ and } \Omega_2 = \{(x, y, u : v) \in X_\sigma : v \neq 0\},$$

with local coordinates

$$(x, \frac{v}{u}) \text{ and } (\frac{u}{v}, y)$$

on Ω_1 and Ω_2 respectively. In these local coordinates σ is expressed by the formulas

$$\sigma(r,s) = (r, rs) \quad \text{and} \quad \sigma(r,s) = (rs, s),$$

respectively. Obviously $c_2 \in \sigma(\Omega_1)$ and $c_2 \notin \sigma(\Omega_2)$.

A crucial observation is that the function

$$F := \frac{P^\sigma(x,y)}{(x^2+y^2)^\sigma}$$

is regular on X_σ. Indeed,

$$F(r,s) = \frac{r^2((r-1)^2+s^2)}{r^2(1+s^2)} = \frac{(r-1)^2+s^2}{1+s^2}$$

on the first chart, and

$$F(r,s) = \frac{s^2(r^2(rs-1)^2+1)}{s^2(r^2+1)} = \frac{r^2(rs-1)^2+1}{r^2+1}$$

on the other chart. Moreover, the set

$$\{c \in X_\sigma : F(c) = 0\} = \sigma^{-1}(c_2) \in \Omega_1$$

is a singleton. Consequently, the pair $\widetilde{s} = (\widetilde{s_1}, \widetilde{s_2})$ with

$$\widetilde{s_1} = \frac{1}{(x^2+y^2)^\sigma} \text{ on } U_1^\sigma, \text{ and } \widetilde{s_2} = F \text{ on } U_2^\sigma$$

is a nowhere vanishing global section of a line bundle $\sigma^*\xi_{1,1}$ and of the locally free sheaf $\sigma^*\mathcal{F}_{1,1}$ of rank 1. Therefore, $\widetilde{s}$ generates the sheaf $\sigma^*\mathcal{F}_{1,1}$ as desired.

In fact, this implies even more, namely that $\sigma^*\mathcal{F}_{1,1}$ is isomorphic to the structure sheaf $\mathcal{O}_{X_\sigma}$ or, equivalently, $\sigma^*\xi_{1,1}$ is a trivial algebraic line bundle.

Let $\tau : X_\tau \to \mathbb{R}^2$ be a blowup of the plane at the point $(1,0)$. In a similar way, the global section $\widetilde{t} = (\widetilde{t_1}, \widetilde{t_2})$ given by

$$\widetilde{t_1} = \frac{((x-1)^2+y^2)^\tau}{P_{1,1}^\tau(x,y)} \text{ and } \widetilde{t_2} = ((x-1)^2+y^2)^\tau$$

is a nowhere vanishing global section of $\tau^*\mathcal{F}_{1,1}$.

Remark 9.1. In real algebraic geometry, the projective spaces and Grassmannians are affine varieties. Let $\mathcal{F}$ be a locally free coherent sheaf on X, ξ the corresponding vector bundle and E its total space. Huisman [16] showed that $\mathcal{F}$ is generated by global sections iff E is a real affine variety. The total space of the line bundle $\xi_{1,1}$ is thus a non-affine real algebraic variety. Another example is given by Galbiati [9].

9.1.2 A counterexample to the quasi-coherent version of Cartan's Theorem A

We now provide an example of a quasi-coherent locally free sheaf $\mathcal{F}$ on $\mathbb{R}^2$ of infinite rank such that for any multi-blowup $\alpha : X_\alpha \to \mathbb{R}^2$ the pull-back sheaf $\alpha^*\mathcal{F}$ is not generated by global sections. This shows that the version of Cartan's Theorem A cannot be generalized to quasi-coherent sheaves.

Lemma 9.2. *Let $k, l \in \mathbb{N}, k, l > 0$, $d_1, d_2, \ldots, d_r$ be a finite number of points of $\mathbb{R}^2$ distinct from c_1 and c_2, $\sigma : X_\sigma \to \mathbb{R}^2$ be a multi-blowup at $d_1, d_2, \ldots, d_r$ and $D = \{d_1, d_2, \ldots, d_r\}$. Then*

$$\sigma^*\xi_{k,l}|_{X_\sigma \setminus \sigma^{-1}(D)} \cong \xi_{k,l}|_{\mathbb{R}^2 \setminus D}$$

and the pull-back $\sigma^*\xi_{k,l}$ is not generated by global sections.

Proof. Since a blowup is biregular off the exceptional divisor, it is enough to show that $\xi_{k,l}|_{\mathbb{R}^2\setminus D}$ is not generated by global sections. Let $V_i := U_i \setminus D$, $i = 1, 2$. Then the reasoning from the beginning of this chapter can be repeated verbatim with the U_i replaced by the V_i, $i = 1, 2$. □

Let $\sigma : X_\sigma \to \mathbb{R}^2$ be the blowup at the origin.

Lemma 9.3. *The pull-back $\sigma^*\xi_{k,l}$ is a trivial line bundle on Ω_2 and $\sigma^*\xi_{k,l}|_{\Omega_1}$ is isomorphic to $\xi_{k-1,l}$ on $\Omega_1 \cong \mathbb{R}^2_{r,s}$.*

Proof. The first assertion is obvious since $c_2 \notin \sigma(\Omega_2)$. To verify the second one, compute the transition function of $\sigma^*\xi_{k,l}|_{\Omega_1}$:

$$\widetilde{g_{2,1}}(r, s) = P^\sigma_{k,l}(r, s) = r^2(r^{2k-2}(r - 1)^{2l} + s^2) = r^2 P_{k-1,l}(r, s)$$

on Ω_1. But the line bundle η on Ω_1 with the transition function r^2 is trivial on Ω_1. Therefore

$$\sigma^*\xi_{k,l}|_{\Omega_1} \cong \xi_{k-1,l} \otimes \eta \cong \xi_{k-1,l}$$

as asserted.

□

Similarly, let $\tau : X_\tau \to \mathbb{R}^2$ be a blowup at the point c_2. We have

$$X_\tau = \{(x, y, u : v) \in \mathbb{R}^2 \times \mathbb{P}^1(\mathbb{R}) : (x - 1)v = (u - 1)y\}.$$

We can cover X_τ by two Zariski open sets

$$\Omega_1 = \{(x, y, u : v) \in X_\tau : u \neq 1\} \text{ and } \Omega_2 = \{(x, y, u : v) \in X_\tau : v \neq 0\}$$

with local coordinates

$$(x, \frac{v}{u-1}) \text{ and } (\frac{u-1}{v}, y)$$

respectively. In these local coordinates τ is expressed by the formulas

$$\tau(r, s) = (r, (r-1)s) \text{ and } \tau(r, s) = (rs+1, s)$$

respectively. As before, we obtain

Lemma 9.4. *The pull-back $\tau^*\xi_{k,l}$ is a trivial line bundle on Ω_2 and $\tau^*\xi_{k,l}|_{\Omega_1}$ is isomorphic to $\xi_{k,l-1}$ on $\Omega_1 \cong \mathbb{R}^2_{r,s}$.*

□

In view of Lemmas 9.2, 9.3, 9.4, it is clear that blowing up at a point $d \neq c_1, c_2$ is immaterial and what improves the bundle $\xi_{k,l}$ is only successive blowing up at the points $(0, 0)$ or $(1, 0)$ on the chart Ω_1. Each such blowup transforms the initial line bundle (isomorphic to $\xi_{p,q}$) to the line bundle $\xi_{p-1,q}$ or $\xi_{p,q-1}$, respectively, on the chart Ω_1. We must continue until we attain the line bundle $\xi_{0,q}$ or $\xi_{p,0}$, with some $0 \leq p \leq k, 0 \leq q \leq l$, which is generated by global sections. In this manner, we have proven the following

Proposition 9.5. *Let $\sigma : X_\sigma \to \mathbb{R}^2$ be a composition of r blowups and $k, l \in \mathbb{N}$. If the pull-back $\sigma^*\xi_{k,l}$ or, equivalently, $\sigma^*\mathcal{F}_{k,l}$ is generated by global sections, then $r \geq \min(k, l)$.*

□

Remark 9.6. The composition α or β of k or l blowups σ or τ at the points $(0, 0)$ or $(1, 0)$, respectively, on the successive charts Ω_1 transforms $\xi_{k,l}$ to a trivial line bundle. Indeed, it is not difficult to check that the nowhere vanishing section on X_α is given by $\widetilde{s} = (\widetilde{s_1}, \widetilde{s_2})$

$$\widetilde{s_1} = \frac{1}{(x^{2k} + y^2)^\alpha} \text{ and } \widetilde{s_2} = \frac{P^\alpha_{k,l}}{(x^{2k} + y^2)^\alpha}.$$

And that the global nowhere vanishing section on X_β is given by $\widetilde{t} = (\widetilde{t_1}, \widetilde{t_2})$

$$\widetilde{t_1} = \frac{((x-1)^{2l} + y^2)^\beta}{P^\beta_{k,l}(x, y)} \text{ and } \widetilde{t_2} = ((x-1)^{2k} + y^2)^\beta.$$

We immediately obtain

Corollary 9.7. *Let*

$$\mathcal{F} := \bigoplus_{k=1}^{\infty} \mathcal{F}_{k,k}$$

be a quasi-coherent locally free sheaf of infinite rank on $\mathbb{R}^2$. Then for any multi-blowup $\alpha : X_\alpha \to \mathbb{R}^2$ the pull-back $\alpha^*\mathcal{F}$ is not generated by global sections.

9.2 Additive Cousin problem on a plane

In this section we are going to present an explicit example of an additive Cousin data on $\mathbb{R}^2$ which is not solvable. We then provide a multi-blowup of $\mathbb{R}^2$ such that its pull-back is solvable. In particular, we show that $H^1(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2}) \neq 0$.

Let $i = 1, 2, 3$. Consider points $c_i = (i-1, 0) \in \mathbb{R}^2$, irreducible polynomials $\omega_i \in \mathbb{R}[x, y]$ and open sets $U_i \ni c_i$

$$\begin{aligned} \omega_1 &= x^2(x-2)^2 + y^2, \quad & U_1 &= \mathbb{R}^2 \setminus \{c_2, c_3\}, \\ \omega_2 &= x^2(x-1)^2 + y^2, \quad & U_2 &= \mathbb{R}^2 \setminus \{c_1, c_3\}, \\ \omega_3 &= (x-1)^2(x-2)^2 + y^2, \quad & U_3 &= \mathbb{R}^2 \setminus \{c_1, c_2\}. \end{aligned}$$

Each polynomial ω_i has exactly two zeros; one of them belongs to U_i while the other does not. Note that the family $\{U_i\}_{i=1}^3$ is a Zariski open covering of $\mathbb{R}^2$.

Define the rational functions

$$f_i := \frac{1}{\omega_i}$$

on U_i for $i = 1, 2, 3$. It is clear that $f_i - f_j$ is a regular function on $U_i \cap U_j$, hence, the data $\{(U_i, f_i)\}_{i=1}^3$, is an additive Cousin data.

Lemma 9.8. *With the notation as above, there does not exist a rational function f on $\mathbb{R}^2$ such that*

$$f - f_i \in \mathcal{O}_{\mathbb{R}^2}(U_i) \quad \textit{for} \quad i = 1, 2, 3.$$

Proof. Assume by contrary that there exists a rational function f on $\mathbb{R}^2$ such that $f - f_i$ is a regular function on U_i for $i = 1, 2, 3$. f can be presented as a quotient $f = \frac{p}{q}$ of two coprime polynomials $p, q \in \mathbb{R}[x, y]$.

By the definition of f we have

$$f - f_i = \frac{p}{q} - \frac{1}{\omega_i} = \frac{p\omega_i - q}{q\omega_i}.$$

Since the elements ω_i are irreducible and have a zero on U_i, it follows that each ω_i divides q. Hence we may write $q = \omega_1\omega_2\omega_3\widetilde{q}$ for some polynomial $\widetilde{q} \in \mathbb{R}[x, y]$.

Consider now $i = 1$. We have the following equality

$$\frac{p}{\omega_1\omega_2\omega_3\widetilde{q}} - \frac{1}{\omega_1} = \frac{p - \omega_2\omega_3\widetilde{q}}{\omega_1\omega_2\omega_3\widetilde{q}}$$

on U_1. The polynomial ω_2 is irreducible and has a zero on U_1, hence in order for the above function to be regular, ω_2 has to divide p. This is a contradiction with p and q being coprime. $\square$

Hence and by Lemma 7.3, we obtain

Corollary 9.9. *The cohomology class Φ in $H^1(\mathcal{U}, \mathcal{O}_{\mathbb{R}^2})$ represented by the 1-cocycle $f = (f_{ij})$ with*

$$f_{ij} = (f_i - f_j)|_{U_{ij}} \quad \textit{for} \quad i, j = 1, 2, 3$$

does not vanish. Hence $H^1(\mathcal{U}, \mathcal{O}_{\mathbb{R}^2}) \neq 0$.

We can show even more

Lemma 9.10. *We have the following*

$$H^1(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2}) \neq 0.$$

Proof. Let $\{(U_i, f_i)\}_{i=1}^3$ be a Cousin data as above and $\mathcal{V}$ be any finite Zariski open covering of $\mathbb{R}^2$ finer than $\mathcal{U}$. We have $c_1 \in V_1$ for some $V_1 \in \mathcal{V}$. It is clear that $V_1 \not\subset U_2$ and $V_1 \not\subset U_3$, hence $V_1 \subset U_1$. Choose $V_2, V_3 \in \mathcal{V}$ with $c_2 \in V_2$, $c_3 \in V_3$ in a similar way. We have a map induced by the refinement

$$H^1(\mathcal{U}, \mathcal{O}_{\mathbb{R}^2}) \to H^1(\mathcal{V}, \mathcal{O}_{\mathbb{R}^2}).$$

Then the cohomology class Ψ corresponding to Φ, via the above map, is represented by a 1-cocycle including, among others,

$$f_{ij} = (f_i - f_j)|_{V_{ij}} \quad \text{for} \quad i, j = 1, 2, 3.$$

The same reasoning as in Lemma 9.8 applied to V_1, V_2, V_3 instead of U_1, U_2, U_3 implies that the Cousin data $\{(V_i, f_i)\}_{i=1}^n$ is not solvable. By Corollary 9.9 we get $H^1(\mathcal{V}, \mathcal{O}_{\mathbb{R}^2}) \neq 0$. Since $\mathcal{V}$ was arbitrary fine, we get the conclusion. □

We may now prove the following

Lemma 9.11. *Let $\alpha : X_\alpha \to \mathbb{R}^2$ be a blow up of the plane at the points c_1, c_2, c_3. Then $\alpha^*\Phi = 0 \in H^1(\mathcal{U}^\alpha, \mathcal{O}_{X_\alpha})$.*

Proof. We have a 1-cocycle $f = (f_{ij})$ with

$$f_{ij} = (f_i - f_j)|_{U_{ij}}$$

for $i, j = 1, 2, 3$. Clearly $f_{ii} = 0$ and $f_{ij} = -f_{ji}$. Our aim is to show that $\alpha^* f$ is a coboundary. Define $Q_1 = \omega_3$, $Q_2 = \omega_1$, $Q_3 = \omega_2$. We have $U_i = \mathbb{R}^2 \setminus \{Q_i = 0\}$ for $i = 1, 2, 3$. It is then not difficult to check that the functions

$$\alpha^* f_{12}(Q_{12}^2)^\alpha = \frac{\omega_2^\alpha - \omega_1^\alpha}{\omega_1^\alpha \omega_2^\alpha}(\omega_3^\alpha \omega_1^\alpha)^2$$

$$\alpha^* f_{23}(Q_{23}^2)^\alpha = \frac{\omega_3^\alpha - \omega_2^\alpha}{\omega_2^\alpha \omega_3^\alpha}(\omega_1^\alpha \omega_2^\alpha)^2$$

$$\alpha^* f_{13}(Q_{13}^2)^\alpha = \frac{\omega_3^\alpha - \omega_1^\alpha}{\omega_1^\alpha \omega_3^\alpha}(\omega_3^\alpha \omega_2^\alpha)^2$$

extend to regular functions on X_α :

$$h_{ij}|_{U_{ij}^\alpha} = \alpha^* f_{ij}(Q_{ij}^\alpha)^2.$$

Then

$$R = \frac{1}{(Q_1^\alpha)^2 + (Q_2^\alpha)^2 + (Q_3^\alpha)^2}$$

is a global nowhere vanishing function on X_α. Likewise in the proof of Proposition 6.5, we can define a 0-cocycle $k = (k_i)$

$$k_1 = R\frac{h_{11} + h_{21} + h_{31}}{(Q_1^\alpha)^2}|_{U_1^\alpha} = R(-\alpha^* f_{12}(Q_2^\alpha)^2 - \alpha^* f_{13}(Q_3^\alpha)^2)|_{U_1^\alpha}$$

$$k_2 = R\frac{h_{12}+h_{22}+h_{32}}{(Q_2^\alpha)^2}|_{U_2^\alpha} = R(\alpha^* f_{12}(Q_1^\alpha)^2 - \alpha^* f_{23}(Q_3^\alpha)^2)|_{U_2^\alpha}$$

$$k_3 = R\frac{h_{13}+h_{23}+h_{33}}{(Q_3^\alpha)^2}|_{U_3^\alpha} = R(\alpha^* f_{13}(Q_1^\alpha)^2 + \alpha^* f_{23}(Q_2^\alpha)^2)|_{U_3^\alpha}.$$

Since $\alpha^* f$ is a cocycle, simple computation gives

$$(dk)_{ij} = \alpha^* f_{ij}.$$

Hence, $\alpha^*\Phi = 0 \in H^1(\mathcal{U}^\alpha, \mathcal{O}_{X_\alpha})$, as asserted. □

Remark 9.12. There is another way of seeing that the functions $f_{ij}^\alpha (Q_{ij}^2)^\alpha$ extend to global functions. Observe that the functions $f_{ij}Q_{ij}^2$ are continuous on $\mathbb{R}^2$. Furthermore, the denominator of $f_{ij}Q_{ij}^2$ is locally positive definite (cf. [8, Definition 3.8]) at c_1, c_2, c_3. Then by [8, Theorem 3.12] there is a blowup $\alpha : X_\alpha \to \mathbb{R}^2$ at a finite number of points such that the functions $\alpha^* f_{ij}(Q_{ij}^2)^\alpha$ extends to global regular functions on X_α. It is clear that blowup at a point different from c_1, c_2, c_3 is immaterial. However, if we blowup in less than three points, at least one of the functions $\alpha^* f_{ij}(Q_{ij}^2)^\alpha$ remains continuous instead of being regular. Hence, $\alpha : X_\alpha \to \mathbb{R}^2$ can be chosen to be the blowup at c_1, c_2, c_3.

Appendix

We begin with the definitions of pre-algebraic and algebraic vector bundles. In Section A.1 we are going to prove that any pre-algebraic vector bundle splits into a direct sum of algebraic line bundles after a suitable multi-blowup. In Section A.2, we show that line bundles obtained in this way are not trivial in general. All vector bundles are assumed to have a finite rank.

Definition A.1. *A pre-algebraic vector bundle over X is a triple $\xi = (E, p, X)$, where:*

i) E is a real algebraic variety (not necessarily affine), and $p : E \to X$, a regular mapping,

ii) for each $x \in X$, the fibre $p^{-1}(x)$ is a finite dimensional $\mathbb{R}$-vector space,

iii) there exist a finite Zariski open covering $\{U_i\}_{i=1}^n$ and, for each $i = 1, 2, \ldots, n$, an integer k and a biregular isomorphism $\varphi_i : U_i \times \mathbb{R}^k \to p^{-1}(U_i)$, such that $p \circ \varphi_i$ is the canonical projection of $U_i \times \mathbb{R}^k$ onto U_i and, for every $x \in U_i$, the restriction $\{x\} \times \mathbb{R}^k \to p^{-1}(x)$ of φ_i is an $\mathbb{R}$-linear isomorphism.

The classical correspondence between vector bundles and locally free sheaves holds for pre-algebraic vector bundles and locally free sheaves of $\mathcal{O}_X$-modules (cf. [4, Proposition 12.1.3]).

Definition A.2. *A pre-algebraic vector bundle ξ over X is said to be algebraic if there exists an injective algebraic morphism from ξ to a trivial bundle ϵ_X^n (i.e. ξ is algebraically isomorphic to a pre-algebraic vector subbundle of a trivial bundle).*

It is clear that there exists an algebraic bundle η such that $\epsilon_X^n \cong \xi \oplus \eta$. One can take η to be the orthogonal complement $\xi^\perp$ of ξ in ϵ_X^n because algebraic bundles can be equipped with the inner product induced by the one from $\mathbb{R}^n$. Therefore, if ζ is an algebraic subbundle of an algebraic bundle ξ on X, then there exists and algebraic subbundle $\eta \subset \xi$ such that $\xi \cong \zeta \oplus \eta$, actually we can take $\eta = \zeta^\perp$.

Let ξ be a pre-algebraic vector bundle with the corresponding locally free sheaf $\mathcal{F}$. [4, Theorem 12.1.7] gives several equivalent conditions for a pre-algebraic bundle ξ to be an algebraic bundle. They imply, in particular, that ξ is algebraic iff the corresponding locally free sheaf $\mathcal{F}$ is generated by global sections. Hence by Theorem 4.12 (or [2, Theorem 1.1]) we immediately obtain

Proposition A.3. *Let ξ be a pre-algebraic vector bundle on X. Then there exists a multi-blowup $\sigma : X_\sigma \to X$ such that $\sigma^*\xi$ is an algebraic vector bundle on X_σ.*

A.1 Splitting Theorem for vector bundles

Let ξ be a pre-algebraic vector bundle on X and $s_i : U_i \to E$ be two sections on a Zariski open subsets of X.

Definition A.4. *We say that the sections s_1 and s_2 are equivalent if there exists a nowhere vanishing regular function $g_{12} \in \mathcal{O}_X(U_1 \cap U_2)$ such that*

$$s_1|_{U_1\cap U_2} = g_{12} s_2|_{U_1\cap U_2}.$$

We first prove an easy lemma

Lemma A.5. *Let ξ be a pre-algebraic bundle on X. Assume there is a finite Zariski open covering $\{U_i\}_{i=1}^n$ of X and nowhere vanishing sections $s_i : U_i \to E$ (i.e. $s_i(x) \neq 0$ for all $x \in U_i$) such that s_i is equivalent with s_j for any $i, j = 1, 2, \ldots, n$. Then there exist a pre-algebraic line bundle $\eta = (L, p_L, X)$, an embedding $\iota : \eta \to \xi$ and a section $s : X \to L$ such that the section $\iota \circ s$ is equivalent with s_i for $i = 1, 2, \ldots, n$.*

Proof. The line bundle η is given by the transition functions g_{ij} where

$$s_i|_{U_i\cap U_j} = g_{ij} s_j|_{U_i\cap U_j}.$$

It is clear that η embeds into ξ. The desired section s of η is the gluing of s_i by means of g_{ij}. $\square$

Remark A.6. If, moreover, ξ is an algebraic vector bundle of rank k, then there exists an algebraic vector bundle η' of rank $k-1$ such that $\xi \cong \eta \oplus \eta'$.

We may now prove the following splitting theorem for pre-algebraic vector bundles

Theorem A.7. *Let $\xi = (E, p, X)$ be a pre-algebraic bundle on X of rank p. Then there exist a multi-blowup $\sigma : X_\sigma \to X$ such that*

$$\sigma^*\xi \cong \bigoplus_{j=1}^{p} \eta_j$$

where each η_j is an algebraic line bundle on X_σ.

Proof. By Proposition A.3 we may assume that ξ is an algebraic vector bundle. It is enough to find a multi-blowup $\sigma : X_\sigma \to X$ such that $\sigma^*\xi \cong \eta \oplus \eta'$ where η is an algebraic line bundle, and η' is an algebraic vector bundle of rank $p-1$. The conclusion follows by repeating this argument

Denote by $\mathcal{F}$ the locally free coherent $\mathcal{O}_X$-sheaf of sections of ξ. Let s be a global section of ξ with nowhere vanishing germs (one can always take a rational section and multiply it by a suitable regular function). For any $x \in X$, choose a basis $f_{1x}, f_{2x}, \ldots, f_{px}$ of $\mathcal{F}_x$, take $h_{1x}, h_{2x}, \ldots, h_{px} \in \mathcal{O}_{x,X}$ such that $\sum_{i=1}^{p} h_{ix} f_{ix} = s_x$, and consider the ideal

$$\mathcal{J}_x := (h_{1x}, h_{2x}, \ldots, h_{px}) \subset \mathcal{O}_{x,X};$$

it is easy to check that $\mathcal{J}_x$ is independent of the choice of the basis $f_{1x}, f_{2x}, \ldots, f_{px}$. Equality $\sum_{i=1}^{p} h_{ix} f_{ix} = s_x$ propagates to a Zariski open neighbourhood V_x of x. By quasi-compactness, we get a finite Zariski open covering $U_i := U_{x_i}$ for $i = 1, 2, \ldots, n$. Put $h_{ji} := h_{jx_i}$ for $j = 1, 2, \ldots, p$, $i = 1, 2, \ldots, n$.

Let $\mathcal{J} := \bigcup_{x \in X} \mathcal{J}_x$. It is easy to see that $\mathcal{J}$ is an open subset of $\mathcal{O}_X$, and is thus a sheaf of ideals of finite type, whence coherent. We have

$$\mathcal{J}|_{U_i} = (h_{1i}, h_{2i}, \ldots, h_{pi}) \cdot \mathcal{O}_X|_{U_i}$$

with $h_{ji} \in \mathcal{O}_X(U_i)$. By Lemma 4.2, there is a presentation

$$h_{ji} = \frac{h_{ji1}}{h_{ji2}}$$

where $h_{jil} \in \mathcal{O}_X(X)$, $l = 1, 2$ and $V(h_{ji2}) \cap U_i = \emptyset$.

By quasi-compactness and Corollary 2.14, there exist a multi-blowup $\sigma : X_\sigma \to X$ and a refinement $\{V_k\}_{i=1}^m$ of $\{U_i^\sigma\}_{i=1}^n$ such that the functions h_{1ji}^σ are simple normal crossings linearly ordered by divisibility relation on each V_k. If $V_k \subset U_{i(k)}^\sigma$ for $k = 1, 2, \ldots, m$, let h_k denote the smallest function among $h_{1\,i(k)}^\sigma, \ldots, h_{p\,i(k)}^\sigma$ with respect to the divisibility relation, for $k = 1, 2, \ldots, m$.

Then the sections

$$\widetilde{s_k} := \frac{\sigma^* s}{h_k} : V_k \to \sigma^* E$$

are equivalent, nowhere vanishing sections. Hence and by Lemma A.5, an algebraic line bundle η can be embedded into $\sigma^*\xi$, $\eta \subset \sigma^*\xi$. Then there exists an algebraic vector subbundle η' of $\sigma^*\xi$ of rank $p-1$ such that

$$\sigma^*\xi \cong \eta \oplus \eta'.$$

This finishes the proof.

□

Remark A.8. In the paper [24], a vector bundle ξ was split by blowing up along the zeros of a section transverse to the zero section of ξ.

Corollary A.9. *Let $\mathcal{F}$ be a locally free coherent sheaf of rank p. Then there exists a multi-blowup $\sigma : X_\sigma \to X$ such that*

$$\sigma^*\mathcal{F} \cong \bigoplus_{i=1}^{p} \mathcal{F}_i,$$

where $\mathcal{F}_i$ is a locally free coherent sheaf of rank 1 on X_σ.

A.2 Lack of triviality after blowing up

In this section we will show that the procedure from the previous section does not lead in general to trivial line bundles, even in the category of topological vector bundles. A line bundle generated by a global nowhere vanishing section is trivial. However, such section may not exist. To demonstrate this, we make use of some tools of algebraic topology (for instance, Chern classes, Euler class, etc.); for the rudiments of algebraic topology the reader is referred to e.g. [13].

For a given smooth manifold M of dimension m, the blowup of M at a point is diffeomorphic to the connected sum $M\#\mathbb{RP}^m$. (In real algebraic geometry this diffeomorphism can be taken to be Nash, see e.g. [4, Remark 3.5.13], and for the complex case [17, Proposition 2.5.8].) Hence, in the case of surfaces, one can analyze connected sums instead of blowups.

Let $M_0 := S^2$ and $M_k := M_{k-1} \# \mathbb{RP}^2$, i.e. M_k is the connected sum of k copies of $\mathbb{RP}^2$. M_k can be seen as the blow up of S^2 at k points (possibly nested). The homology groups of M_k, $k > 0$ are given by the formula

$$H_i(M_k; \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}^{k-1} \oplus \mathbb{Z}/2 & i = 1 \\ 0 & i > 1 \end{cases}$$

This formula can be found in [11, Example 19.31], and a more general version in [13, Section 3.3]. The cohomology groups of M_k can be computed by means of the universal coefficient theorem for cohomology [13, Theorem 3.2]:

Let X be a smooth manifold with homology groups $H_i(X; \mathbb{Z})$. Then the cohomology groups of X over $\mathbb{Z}$ are determined by split exact sequences

$$0 \to \mathrm{Ext}^1(H_{n-1}(X; \mathbb{Z}); \mathbb{Z}) \to H^n(X; \mathbb{Z}) \to \mathrm{Hom}(H_n(X; \mathbb{Z}), \mathbb{Z}) \to 0.$$

Since $\mathrm{Ext}^1(\mathbb{Z}^{k-1} \oplus \mathbb{Z}/2, \mathbb{Z}) \cong \mathbb{Z}/2$ we get

$$H^2(M_k; \mathbb{Z}) \cong \mathbb{Z}/2 \quad \text{for} \quad k > 0.$$

We still need the following

Lemma A.10. *The canonical map $H^2(M_{k-1}; \mathbb{Z}) \to H^2(M_k; \mathbb{Z})$ is surjective for any $k > 0$.*

Proof. Recall that $M_k = M_{k-1} \# \mathbb{RP}^2$. Denote by $\mathbb{D}$ a small open 2-dimensional ball. Consider the following pair

$$(M_{k-1} \# \mathbb{RP}^2, \mathbb{RP}^2 \setminus \mathbb{D}; \mathbb{Z}).$$

We have an isomorphism of cohomology groups

$$H^i(M_{k-1} \# \mathbb{RP}^2, \mathbb{RP}^2 \setminus \mathbb{D}; \mathbb{Z}) \cong H^i(M_{k-1}; \mathbb{Z}),$$

which can be easily deduced from [11, Theorem 19.14]. Consider the following part of the long exact cohomology sequence

$$\cdots \to H^2(M_{k-1} \# \mathbb{RP}^2, \mathbb{RP}^2 \setminus \mathbb{D}; \mathbb{Z}) \to$$

$$H^2(M_{k-1}\#\mathbb{RP}^2;\mathbb{Z}) \to H^2(\mathbb{RP}^2 \setminus \mathbb{D};\mathbb{Z}) \to \dots$$

Since $\mathbb{RP}^2\setminus\mathbb{D} = M_1\setminus\mathbb{D}$ is homotopic to S^1, it is clear that $H^2(\mathbb{RP}^2\setminus\mathbb{D};\mathbb{Z}) = 0$. Therefore we can rewrite the long exact sequence as

$$\dots \to H^2(M_{k-1};\mathbb{Z}) \to H^2(M_k;\mathbb{Z}) \to 0 \to \dots$$

which proves surjectivity.

□

Let $\xi = (E, p, X)$ be a real oriented vector bundle of rank r on a smooth manifold X. We then have its Euler class $e(\xi) \in H^r(X;\mathbb{Z})$. One of the axioms of Euler class states that $e(\xi) = 0$ iff ξ has a nowhere vanishing global section. If ξ is a complex vector bundle then the $e(\xi)$ is given by the k-th Chern class, where k is the complex rank of ξ.

Now we can readily turn to our main goal.

Proposition A.11. *There exists an algebraic oriented real vector bundle ξ of rank 2 on S^2 with the following property: if $\alpha^*\xi \cong \xi_1 \oplus \xi_2$ is a decomposition of $\alpha^*\xi$ into topological line bundles for some multi-blowup $\alpha : X_\alpha \to S^2$, then both of these line bundles are non-trivial.*

Proof. Let ξ be the tautological line bundle of $\mathbb{CP}^1$; ξ can be considered as an algebraic oriented real vector bundle of rank 2 on S^2. Then $c_1(\xi) = e(\xi)$ is a generator of $H^2(S^2;\mathbb{Z}) \cong \mathbb{Z}$.

Let $\alpha : X_\alpha \to S^2$ be a multi-blowup such that $\alpha^*\xi \cong \xi_1 \oplus \xi_2$ (whose existence is established in Section A.1). By Lemma A.10 the map

$$\alpha^* : H^2(S^2;\mathbb{Z}) = \mathbb{Z} \to H^2(X_\alpha;\mathbb{Z}) = \mathbb{Z}/2$$

is surjective as the composition of finitely many surjective maps. Hence

$$e(\alpha^*\xi) = \alpha^*(e(\xi)) \neq 0.$$

By the Whitney sum formula, $e(\xi_1), e(\xi_2) \neq 0$, and thus none of the line bundles ξ_1 or ξ_2 is trivial.

□

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