



Semialgebraic version of Whitney’s extension theorem

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Abstract. In this note we prove a semialgebraic counterpart of Whitney’s extension theorem.

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Semialgebraic geometry plays an important role in areas of pure mathematics such as singularity theory and mathematical logic. It appears also very valuable in areas of applied mathematics such as robotics and CAD. Together with its generalization (subanalytic geometry, o-minimal geometry) it serves as a tool for problems in analysis [4, 7].

It is therefore a very natural question if the famous Whitney extension theorem admits a semialgebraic counterpart. Let us first recall this fundamental result of analysis.

For a subset A of \mathbb{R}^n , denote by $\mathcal{C}(A)$ the algebra of real continuous functions on A .

Definition 1. Let p be a positive integer. A \mathcal{C}^p -Whitney field on the set A is a polynomial

$$F(u, X) = \sum_{|\kappa| \leq p} \frac{1}{\kappa!} F^\kappa(u) X^\kappa \in \mathcal{C}(A)[X] = \mathcal{C}(A)[X_1, \dots, X_n],$$

which fulfills the following condition: for each $c \in A$ and every $\alpha \in \mathbb{N}^n$ such that $|\alpha| \leq p$,

$$D_X^\alpha F(a, 0) - D_X^\alpha F(b, a - b) = o(|a - b|^{p-|\alpha|}),$$

when $A \times A \ni (a, b) \rightarrow (c, c)$.

Whenever A is open in \mathbb{R}^n , any \mathcal{C}^p -Whitney field can be identified with a \mathcal{C}^p -function on A .

Whitney’s extension theorem (cf. [11]) asserts that given a closed subset A of \mathbb{R}^n and any \mathcal{C}^p -Whitney field F on A , there exists a \mathcal{C}^p -function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $D^\kappa f = F^\kappa$ on A , if $\kappa \in \mathbb{N}^n$, $|\kappa| \leq p$, and the restriction $f|_{\mathbb{R}^n \setminus A} : \mathbb{R}^n \setminus A \rightarrow \mathbb{R}$ is analytic.

We refer the reader to [1] or [10] for basic definitions and properties of semialgebraic sets.

In this note we establish the semialgebraic counterpart of this theorem. Firstly, let us remark that using the so-called Λ_p -regular stratifications, the second author and K. Kurdyka proved an o-minimal analogue of Whitney’s extension theorem, containing in particular the following semialgebraic version (cf. [5] and [6, Theorem 1]).

Theorem 2. *Let E be a semialgebraic closed subset of an open semialgebraic subset Ω of \mathbb{R}^n , and let p and q be positive integers such that $p \leq q$. Let*

$$F(x, X) = \sum_{|\kappa| \leq p} \frac{1}{\kappa!} F^\kappa(x) X^\kappa$$

be a semialgebraic \mathcal{C}^p -Whitney field on E (i.e. all F^κ are semialgebraic functions). Then there exists a semialgebraic \mathcal{C}^p -function $f : \Omega \rightarrow \mathbb{R}$, \mathcal{C}^q on $\Omega \setminus E$, such that $D^\kappa f = F^\kappa$ on E whenever $\kappa \in \mathbb{N}^n$ with $|\kappa| \leq p$.

We want to use Shiota’s approximation theorem to get the function f Nash on $\Omega \setminus E$. Recall that a semialgebraic subset M of \mathbb{R}^n is called a *Nash submanifold* if it is an analytic submanifold of \mathbb{R}^n and a map $f : M \rightarrow N$ of Nash submanifolds is called a *Nash map* if it is an analytic map with semialgebraic graph. Actually, in the semialgebraic case the analyticity is equivalent to \mathcal{C}^∞ (cf. [8, Proposition 3.11]). It is well known that every Nash submanifold admits a finite system of Nash charts (cf. [10]).

Let p be a positive integer. We briefly recall the definition of the *semialgebraic \mathcal{C}^p -topology*. Let $M \subset \mathbb{R}^n$ and $N \subset \mathbb{R}^m$ be Nash submanifolds of dimension k and l , respectively. Denote by $\mathcal{N}^p(M, N)$ the set of \mathcal{C}^p semialgebraic maps from M to N , i.e. \mathcal{C}^p maps with semialgebraic graphs.

Fix any $f \in \mathcal{N}^p(M, N)$. Then there exist finite systems $\Phi = \{\varphi_i, U_i\}_{i \in S}$ (resp. $\Psi = \{\psi_i, V_i\}_{i \in S}$) of Nash charts on M (resp. on N) and a family of closed sets $\{Z_i\}_{i \in S}$ of M such that $\bigcup_{i \in S} Z_i = M$ and for each $i \in S$, $Z_i \subset U_i$, and $f(Z_i) \subset V_i$. Now basic neighborhoods of f in the semialgebraic \mathcal{C}^p -topology in $\mathcal{N}^p(M, N)$ are of the form

$$U_\varepsilon(f) = \{g \in \mathcal{N}^p(M, N) : \text{for each } i \in S, \lambda \in \mathbb{N}^k, |\lambda| \leq p, \xi \in \varphi_i(Z_i), g(Z_i) \subset V_i, |D^\lambda(\psi_i f \varphi_i^{-1})(\xi) - D^\lambda(\psi_i g \varphi_i^{-1})(\xi)| \leq \varepsilon(\varphi_i^{-1}(\xi))\},$$

where $\varepsilon : M \rightarrow (0, +\infty)$ is any semialgebraic positive continuous function on M .

Clearly, if $M = \Omega$ is an open semialgebraic subset of \mathbb{R}^n and $N = \mathbb{R}$, then

$$U_\varepsilon(f) = \{g \in \mathcal{N}^p(\Omega, \mathbb{R}) : |D^\lambda f(x) - D^\lambda g(x)| \leq \varepsilon(x) \text{ whenever } \lambda \in \mathbb{N}^n, |\lambda| \leq p, x \in \Omega\}.$$

Theorem 3. (Shiota's approximation theorem, [9, 10]). *Let $M \subset \mathbb{R}^n$ and $N \subset \mathbb{R}^m$ be Nash submanifolds and $f : M \rightarrow N$ a semialgebraic \mathcal{C}^p map. Then f can be approximated by Nash maps in the semialgebraic \mathcal{C}^p -topology.*

Now the semialgebraic Whitney extension theorem can be formulated as follows.

Theorem 4. *Let E be a semialgebraic closed subset of an open semialgebraic subset Ω of \mathbb{R}^n , $p \in \mathbb{N}$, and let*

$$F(x, X) = \sum_{|\kappa|=0}^p \frac{1}{\kappa!} F^\kappa(x) X^\kappa$$

be a semialgebraic \mathcal{C}^p Whitney field on E . Then there exists a \mathcal{C}^p semialgebraic function $f : \Omega \rightarrow \mathbb{R}$ such that $D^\kappa f = F^\kappa$ on E for $|\kappa| \leq p$ and f is Nash on the set $\Omega \setminus E$.

Proof. Thanks to Theorem 2, there exists a semialgebraic \mathcal{C}^p function $\tilde{f} : \Omega \rightarrow \mathbb{R}$ such that on the set E the equality $D^\kappa \tilde{f} = F^\kappa$ holds for any $|\kappa| \leq p$. Applying Shiota's approximation theorem to the function \tilde{f} , and $\varepsilon(x) := \text{dist}(x, E)$ we get a function $f : \Omega \rightarrow \mathbb{R}$ such that f is Nash on the set $\Omega \setminus E$ and

$$|D^\kappa f(x) - D^\kappa \tilde{f}(x)| < \varepsilon(x)$$

for every $x \in \Omega \setminus E$ and any $|\kappa| \leq p$. Hence $D^\kappa \tilde{f} = D^\kappa f = F^\kappa$ for all $|\kappa| \leq p$ on the set E as required. \square

Remark 5. A counterpart of this theorem is also valid for non polynomially bounded o-minimal structures and more generally on every o-minimal structure on which Shiota's approximation theorem holds true ([3]). It is however unclear whether Theorem 4 is true for all polynomially bounded o-minimal structures as it is still unknown whether Shiota's theorem holds in this case. In particular, we do not know whether it is possible to generalize the Whitney extension theorem to the globally subanalytic category which is an interesting open problem.

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