# NEW REDUCTION IN THE JACOBIAN CONJECTURE 

By Ludwik M. Drużkowski<br>Dedicated to Professor Tadeusz Winiarski on the occasion of his 60th birthday


#### Abstract

It is sufficient to consider in the Jacobian Conjecture (for every $n>1)$ only polynomial mappings of cubic linear form $F(x)=x+(A x)^{* 3}$, i. e. $F(x)=\left(x_{1}+\left(a_{1}^{1} x_{1}+\ldots+a_{n}^{1} x_{n}\right)^{3}, \ldots, x_{n}+\left(a_{1}^{n} x_{1}+\ldots+a_{n}^{n} x_{n}\right)^{3}\right)$ where the matrix $F^{\prime}(x)-I=3 \Delta\left((A x)^{* 2}\right) A$ is nilpotent for every $x=\left(x_{1}, \ldots, x_{n}\right)$. In the paper we give a new contributions to the Jacobian Conjecture, namely we show that it is sufficient in this problem to consider (for every $n>1$ ) only cubic linear mappings $F(x)=x+(A x)^{* 3}$ such that $A^{2}=0$.


1. Introduction and notation. Let $\mathbb{K}$ denote either the field of complex numbers $\mathbb{K}$ or the field of reals $\mathbb{R}$. Basis in the domain and codomain vector spaces $\mathbb{K}^{n}$ are assumed to be fixed and identical, so a linear mapping $A$ from $\mathbb{K}^{n}$ into $\mathbb{K}^{n}$ is identified with its matrix and denoted by the same letter $A$ (I denotes the identity matrix). Let $M_{n}$ denote the set of $n \times n$ square matrices with entries in $\mathbb{K}$. A vector $x \in \mathbb{K}^{n}$ is treated as one column matrix and $x^{T}$ denotes its transpose, i. e. $x^{T}=\left(x_{1}, \ldots, x_{n}\right) \in \mathbb{K}^{n}$. Let $a_{j}, b_{j}, c_{j}: \mathbb{K}^{n} \rightarrow \mathbb{K}$ be linear forms and let the symbol $a_{j} x$ (resp. $b_{j} x, c_{j} x$ ) denote the value of the linear form $a_{j}$ (resp. $b_{j}, c_{j}$ ) at a point $x \in \mathbb{K}^{n}$, i. e. $a_{j} x=a_{j}^{1} x_{1}+\ldots a_{j}^{n} x_{n}$, $j=1, \ldots, n$. Denote for short the square matrix $A:=\left[a_{i}^{j}: i, j=1, \ldots, n\right]$ and the vector $(A x)^{T}:=\left(a_{1} x, \ldots, a_{n} x\right)$, i.e. $A x$ is one column matrix. If $v=\left(v_{1}, \ldots, v_{n}\right)^{T}$ is a column vector, then we denote the $k$ power of $v$ by $v^{* k}:=\left(\left(v_{1}\right)^{k}, \ldots,\left(v_{n}\right)^{k}\right)^{T}$ and by $\Delta\left(v^{* k}\right)$ we denote the diagonal $n \times n$ matrix

$$
\Delta\left(v^{* k}\right):=\left[\begin{array}{cccccc}
\left(v_{1}\right)^{k} & 0 & 0 & \ldots & 0 & 0 \\
0 & \left(v_{2}\right)^{k} & 0 & \ldots & 0 & 0 \\
\ldots \ldots \ldots . & & & & & \\
0 & 0 & \ldots & 0 & \left(v_{n-1}\right)^{k} & 0 \\
0 & 0 & \ldots & 0 & 0 & \left(v_{n}\right)^{k}
\end{array}\right]
$$

If $F=\left(F_{1}, \ldots, F_{n}\right): \mathbb{K}^{n} \rightarrow \mathbb{K}^{n}$ is a polynomial mapping, then we denote $\operatorname{Jac} F(x):=\operatorname{det}\left[\frac{\partial F_{i}}{\partial x_{j}}(x): i, j=1, \ldots, n\right]$. Let a polynomial mapping $F=$ $\left(F_{1}, \ldots, F_{n}\right)$ have a cubic linear form $F(x)=x+(A x)^{* 3}$ that is $F_{j}(x)=$ $x_{j}+\left(a_{j} x\right)^{3}, x=\left(x_{1}, \ldots, x_{n}\right) \in \mathbb{K}^{n}, j=1, \ldots, n$.

We recall that the n-dimensional Jacobian Conjecture $(J C)_{n}(n>1)$ asserts

If $F$ is any polynomial mapping of $\mathbb{K}^{n}$ and $\operatorname{Jac} F(x)=$ const $\neq 0$, then $F$ is injective.
By the Jacobian Conjecture (for short (JC)) we mean that $(J C)_{n}$ holds for each $n>1$.

If $F$ is injective polynomial transformation of $\mathbb{C}^{n}$, then $F$ is a polynomial automorphism, cf. [1, 8]. Therefore the Jacobian Conjecture is sometimes formulated with the requirement that $F$ has to be a polynomial automorphism. We have the following reduction theorem.

Theorem 1. [2] In order to verify the Jacobian Conjecture (for every $n>1$ ) it is sufficient to check the Jacobian Conjecture (for every $n>1$ ) only for polynomial mappings $F=\left(F_{1}, \ldots, F_{n}\right)$ of a cubic linear form

$$
F(x)=x+(A x)^{* 3}, \quad \text { i.e. } F_{j}(x)=x_{j}+\left(a_{j} x\right)^{3}, \quad j=1, \ldots, n .
$$

It is known ( $\mathbf{1}, \mathbf{2}$ ) that $\operatorname{Jac} F=1$ if and only if the matrix $A_{x}:=$ $\left[\left(a_{j} x\right)^{2} a_{j}^{i}: i, j=1, \ldots, n\right]=\Delta\left((A x)^{* 2}\right) A$ is nilpotent for every $x \in \mathbb{K}^{n}$. Some interesting applications of Th. 1 to the Jacobian Conjecture can be found in [4, 5, 7]. Note that

$$
\begin{gathered}
F(x)=x+A_{x}(x)=x+\Delta\left((A x)^{* 2}\right)(A x) \\
F^{\prime}(x)=I+3 A_{x}=I+3 \Delta\left((A x)^{* 2}\right) A,
\end{gathered}
$$

and call $A$ the matrix of the cubic linear mapping $F$. Hence, for every $x \in \mathbb{K}^{n}$ there exists an index of nilpotency of the matrix $A_{x}$, i.e. a number $p(x) \in \mathbb{N}$ such that $A_{x}^{p(x)}=0$ and $A_{x}{ }^{p(x)-1} \neq 0$. We define the index of nilpotency of the mapping $F$ to be the number ind $F:=\sup \left\{p(x) \in \mathbb{N}: x \in \mathbb{K}^{n}\right\}$. Obviously ind $F \leq n$.

## 2. We will prove the following.

Theorem 2. (new reduction theorem) In order to verify the Jacobian Conjecture (for every $n>1$ ) it is sufficient to check the Jacobian Conjecture (for every $n>1$ ) only for polynomial mappings $F=\left(F_{1}, \ldots, F_{n}\right)$ of the cubic linear form

$$
F_{j}(x)=x_{j}+\left(a_{j} x\right)^{3}, \quad j=1, \ldots, n,
$$

having an additional nilpotent property of the matrix $A:=\left[a_{i}^{j}: i, j=1, \ldots, n\right]$, namely $A^{2}=0$.

Proof. Due to Th. 1 we can take $F: \mathbb{K}^{n} \rightarrow \mathbb{K}^{n}$ of the form $F(x)=$ $x+(A x)^{* 3}, x \in \mathbb{K}^{n}$. Evidently $F$ is a polynomial automorphism if and only if $x+\delta(A x)^{* 3}$ is a polynomial automorphism for every (some) $\delta \in \mathbb{K} \backslash\{0\}$. Put $\widehat{F}(x, y):=\left(x+\delta(A x)^{* 3}, y\right), \delta \neq 0,(x, y) \in \mathbb{K}^{n} \times \mathbb{K}^{n}$. Obviously $F$ is a polynomial automorphism of $\mathbb{K}^{n}$ if and only if $\widehat{F}: \mathbb{K}^{2 n} \rightarrow \mathbb{K}^{2 n}$ is an automorphism of $\mathbb{K}^{2 n}$. We define polynomial automorphisms of $\mathbb{K}^{2 n}$ by the formulas:

$$
Q(x, y):=\left(\alpha x-\beta y, y+(\alpha A x-\beta A y)^{* 3}\right) \quad \text { where } \alpha \beta \neq 0
$$

and

$$
P(x, y):=\left(\frac{1}{\alpha} x+\frac{\beta}{\alpha} y, y\right) \quad \text { where } \alpha \beta \neq 0
$$

Put $G:=P \circ \widehat{F} \circ Q: \mathbb{K}^{2 n} \rightarrow \mathbb{K}^{2 n}$. It not difficult to verify that

$$
G(x, y)=\left(x+\frac{(\delta+\beta) \alpha^{2}}{\beta^{3}}\left(\beta A x-\frac{\beta^{2}}{\alpha} y\right)^{* 3}, y+(\alpha A x-\beta y)^{* 3}\right)
$$

The mapping $F$ is a polynomial automorphism if and only if $G$ is a polynomial automorphism. Now we choose $\alpha \neq 0, \beta \neq 0$ such that $\frac{(\delta+\beta) \alpha^{2}}{\beta^{3}}=1$ (it is always possible if $\frac{\alpha^{2}}{\beta^{2}} \neq 1$ ). Hence we get

$$
G(x, y)=\left(x+\left(\beta A x-\frac{\beta^{2}}{\alpha} y\right)^{* 3}, y+(\alpha A x-\beta y)^{* 3}\right)
$$

Denote by $N$ a block matrix (with entries in $M_{n}$ ) of the form

$$
N:=\left(\begin{array}{cc}
\beta A & -\frac{\beta^{2}}{\alpha} A \\
\alpha A & -\beta A
\end{array}\right)
$$

Observe that we can write $G(w)=w+(N w)^{* 3}, w \in \mathbb{K}^{2 n}$. It is easy to check that $N^{2}=0$. Therefore the theorem is proved.

Remark 1. Since $A^{2}=0, \operatorname{rank} A \leq \frac{n}{2}$.
In the example given in [3, Ex. 7.8], and also investigated in [6, Ex. 6.1], the matrix $A$ of an automorphism $F(x)=x+(A x)^{* 3}: \mathbb{K}^{15} \rightarrow \mathbb{K}^{15}$ has the form

$$
\left(\begin{array}{ccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & -1 & 1 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & -1 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\
0 & 0 & 1 & -2 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & -2 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & -2 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 1 \\
1 & 0 & -1 & 0 & -1 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\
1 & 0 & 0 & -2 & -1 & 1 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & -2 & -1 & 1 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\
0 & 1 & -1 & 2 & 0 & 0 & 0 & -1 & 1 & 1 & 1 & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 2 & 1 & -1 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & -2 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & -2 & -1 & 1 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0
\end{array}\right)
$$

It is easy to check that ind $A=2$, rank $A=5$ and ind $F=5$.
Remark 2. It was proved earlier ([2]) that in Th. 1 we can additionally assume that $(*)$ the matrix $A=A_{c}$ for some point $c \in \mathbb{K}^{n}$ and ind $A=$ ind $F$. If we investigated the Jacobian Conjecture for cubic linear assuming ind $A=2$, then the property $(*)$ usually does not hold (cf. the mentioned above example where ind $A=2<5=$ ind $F$ ).

## References

1. Bass H., Connell E.H., Wright D., The Jacobian Conjecture: reduction of degree and formal expansion of the inverse, Bull. Amer. Math. Soc. 7 (1982), 287-330.
2. Drużkowski L.M., An effective approach to Keller's Jacobian Conjecture, Math. Ann. 264 (1983), 303-313.
3. $\qquad$ , The Jacobian Conjecture, preprint 492, Institute of Mathematics, Polish Academy of Sciences, Warsaw, 1991.
4. _, The Jacobian Conjecture in case of rank or corank less than three, J. Pure Appl. Algebra 85 (1993), 233-244.
5. Gorni G., Tutaj-Gasińska H., Zampieri G., Drużkowski matrix search and D-nilpotent automorphisms, Indag. Math. 10(2) (1999), 235-245.
6. Gorni G., Zampieri G., On cubic-linear polynomial mappings, Indag. Math. 8(4) (1997), 471-492.
7. Hubbers E.-M. G. M., Nilpotent Jacobians, Universal Press, Veenendal 1998, ISBN 90-9012143-9.
8. Rusek K., Winiarski T., Polynomial automorphisms of $\mathbb{C}^{n}$, Univ. Iagell. Acta Math. 24 (1984), 143-149.

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