On $m$-Subharmonic Ordering of Measures

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Abstract. We study an order-relation induced by $m$-subharmonic functions. We shall consider maximality with respect to this order and a related notion of minimality for certain $m$-subharmonic functions. This concept is then applied to the problem of convergence of measures in the weak*-topology, in particular Hessian measures.

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1. Introduction

In this paper we study an order-relation between measures on an $m$-hyperconvex domain $\Omega$ in $\mathbb{C}^n$. Let $\mu$ and $\nu$ be measures on $\Omega$. We say that $\mu$ is $m$-subharmonically greater than $\nu$ if $\int_{\Omega}(-\varphi)d\mu \geq \int_{\Omega}(-\varphi)d\nu$, $\forall \varphi \in \mathcal{E}_{0,m}(\Omega) \cap C(\overline{\Omega})$ and write $\mu \succcurlyeq \nu$, where $\mathcal{E}_{0,m}(\Omega)$ is the Cegrell class of negative $m$-subharmonic functions defined in Sect. 2. It is easy to see that the condition $\mu \geq \nu$ implies $\mu \succcurlyeq \nu$. But the inverse is not true (see Example 1). We also show that if $u, v$ are functions in the Cegrell class $\mathcal{F}_m(\Omega)$ such that $u \leq v$, then their complex Hessian measures are in the relation $H_m(u) \succcurlyeq H_m(v)$ (see Proposition 2). But the inverse is not true (see Example 2).

In Sect. 4, we study maximality with respect to the $\succcurlyeq$-ordering, and a related notion of minimality for $m$-subharmonic functions in the class $\mathcal{F}_m(\Omega)$. A finite measure $\mu$ on $\Omega$ is said to be maximal if for any measure $\nu$ on $\Omega$ such that $\nu(\Omega) = \mu(\Omega)$, the relation $\nu \succcurlyeq \mu$ implies that $\nu = \mu$. The Dirac measure is a maximal measure. Theorem 9 shows that each finite measure on $\Omega$ with compact support is majorized in the $\succcurlyeq$-ordering by a maximal measure with the same total mass. A function $u \in \mathcal{F}_m(\Omega)$ is said to be minimal if for any...
function \( v \in \mathcal{F}_m(\Omega) \) with the same total Hessian mass, the relation \( v \leq u \) implies that \( v = u \). We show that if a function \( u \in \mathcal{F}_m(\Omega) \) and \( H_m(u) \) is maximal measure, then \( u \) is minimal function (see Proposition 5). But the converse is still unknown. Theorem 10 shows that if \( u \in \mathcal{F}_m(\Omega) \) is such that \( H_m(u) \) is carried by an \( m \)-polar set, then \( u \) is a minimal function. However, there are functions in \( \mathcal{F}_m(\Omega) \) whose Hessian measure are maximal and are not carried by an \( m \)-polar set. We also prove that each function in \( \mathcal{F}_m(\Omega) \) is minorized by a minimal function with the same total Hessian mass.

In Sect. 5, we apply the \( m \)-subharmonic ordering to the problem of convergence in the weak*-topology. First, we prove that if \( \{\mu_j\} \) is an \( m \)-subharmonically increasing sequence of measures on \( \Omega \) with uniformly bounded total mass then \( \mu_j \) converges to a measure \( \mu \) in the weak*-topology. And finally, we use the notion of maximal measure to prove a sufficient condition of convergence in the weak*-topology for the class \( \mathcal{F}_m(\Omega) \) (see Theorem 14).

2. Preliminaries

Let \( \Omega \) be an open set in \( \mathbb{C}^n \) and let \( m \) be a natural number \( 1 \leq m \leq n \).

As usual let \( d = \partial + \bar{\partial}, \) \( d^c = i(\bar{\partial} - \partial), \) and let \( \beta = \frac{dd^c|z|^2}{2} \) be the canonical Kähler form in \( \mathbb{C}^n \). Denote by \( SH_m(\Omega) \) the set of all \( m \)-subharmonic functions in \( \Omega \), and \( SH_m^-(\Omega) \) for the set of all nonpositive \( m \)-subharmonic functions in \( \Omega \). For \( u_1, \ldots, u_m \in SH_m(\Omega) \cap L^\infty_{\text{loc}}(\Omega) \), the operator

\[
H_m(u_1, \ldots, u_m) := dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge \beta^{n-m}
\]

is a nonnegative Radon measure. In particular, when \( u = u_1 = \cdots = u_m \), the Hessian measures

\[
H_m(u) := (dd^c u)^m \wedge \beta^{n-m}
\]

are well-defined for \( u \in SH_m(\Omega) \cap L^\infty_{\text{loc}}(\Omega) \) (see [4]).

**Definition 1.** Let \( E \) be a subset of \( \Omega \). The \( m \)-relative extremal function \( h_{m,E,\Omega} \) is defined by

\[
h_{m,E,\Omega}(z) = \sup \{ u(z) : u \in SH_m(\Omega), u \leq 0 \text{ and } u \leq -1 \text{ on } E \}.
\]

By [11, Proposition 1.5], we have that \( h_{m,E,\Omega}^* \) is \( m \)-subharmonic on \( \Omega \).

**Definition 2.** Let \( \Omega \) be an open set. A function \( u \in SH_m(\Omega) \) is called \( m \)-maximal if \( v \in SH_m(\Omega), v \leq u \) outside a compact set subset of \( \Omega \) implies that \( v \leq u \) in \( \Omega \).

**Theorem 1** [4]. Assume that \( u \in SH_m(\Omega) \cap L^\infty_{\text{loc}}(\Omega) \). Then \( H_m(u) = 0 \) in \( \Omega \) if and only if \( u \) is \( m \)-maximal.

Now let us recall the definition of \( m \)-hyperconvex domain.
Definition 3. A bounded domain $\Omega \subset \mathbb{C}^n$ is called an $m$-hyperconvex if there exists an $m$-subharmonic function $\rho: \Omega \to (-\infty, 0)$ such that the closure of the set $\{z \in \Omega: \rho(z) < c\}$ is compact in $\Omega$ for every $c \in (-\infty, 0)$. In other words, the sublevel set $\{z \in \Omega: \rho(z) < c\}$ is relatively compact in $\Omega$. Such a function $\rho$ is called the exhaustion function.

Theorem 2 [9, Proposition 1.4.11]. Let $\Omega$ be an $m$-hyperconvex bounded domain and $K \subset \Omega$ is compact. Then $h_{m,K,\Omega}^\ast$ is $m$-maximal in $\Omega \setminus K$.

Let us recall the definition of $m$-polar sets.

Definition 4. A set $E \subset \mathbb{C}^n$ is called $m$-polar if for any $z \in E$ there exists a neighbourhood $V$ of $z$ and $v \in \text{SH}_m(V)$ such that $E \cap V \subset \{v = -\infty\}$.

The following theorem was proved by Lu.

Theorem 3 [9, Theorem 1.6.5]. If $E$ is $m$-polar, then there exists $u \in \text{SH}_m^{-}(\mathbb{C}^n)$ such that $E \subset \{u = -\infty\}$.

Throughout this paper $\Omega$ will denote a bounded $m$-hyperconvex domain in $\mathbb{C}^n$. Now we recall the definitions of the Cegrell classes.

Definition 5. (1) We let $\mathcal{E}_{0,m}(\Omega)$ denote the class of bounded functions in $\text{SH}_m(\Omega)$ such that

$$\lim_{z \to \partial \Omega} u(z) = 0 \text{ and } \int_{\Omega} H_m(u) < +\infty.$$  

(2) A function $u \in \text{SH}_m(\Omega)$ belongs to $\mathcal{E}_m(\Omega)$ if for each $z_0 \in \Omega$, there exists an open neighborhood $U \subset \Omega$ of $z_0$ and a decreasing sequence $\{u_j\} \subset \mathcal{E}_{0,m}(\Omega)$ such that $u_j \downarrow u$ in $U$ and $\sup_j \int_{\Omega} H_m(u_j) < +\infty$.

(3) Denote $\mathcal{F}_m(\Omega)$ be the class of functions $u \in \text{SH}_m(\Omega)$ such that there exists a sequence $\{u_j\} \subset \mathcal{E}_{0,m}(\Omega)$ decreases to $u$ in $\Omega$ and $\sup_j \int_{\Omega} H_m(u_j) < +\infty$.

We have the following inclusions

$$\mathcal{E}_{0,m} \subset \mathcal{F}_m \subset \mathcal{E}_m \text{ and } \text{SH}_m^{-}(\Omega) \cap L_{\text{loc}}^{\infty}(\Omega) \subset \mathcal{E}_m.$$  

Below we present some of the basic properties of the Cegrell classes.

Theorem 4 [2,9]. For each $u \in \text{SH}_m^{-}(\Omega)$, there exists a sequence $\{u_j\} \in \mathcal{E}_{0,m}(\Omega) \cap C(\bar{\Omega})$ such that $u_j \downarrow u$ in $\Omega$.

Proposition 1. Let $\mathcal{K}$ be one of the classes $\mathcal{E}_{0,m}, \mathcal{F}_m, \mathcal{E}_m$. Then $\mathcal{K}$ is a convex cone. Moreover, if $u \in \mathcal{K}$ and $v \in \text{SH}_m^{-}(\Omega)$ then $\max\{u,v\} \in \mathcal{K}$.

The following lemma explains why the functions in $\mathcal{E}_{0,m}(\Omega)$ are sometimes called test functions.

Theorem 5 [2,9]. For $\varphi \in C_0^{\infty}(\Omega)$, there exist two functions $u,v$ in $\mathcal{E}_{0,m} \cap C(\bar{\Omega})$ such that $\varphi(z) = u(z) - v(z)$, $\forall z \in \Omega$.
Following Cegrell’s idea Lu proved that the Hessian operator is well-defined for the functions in the class \( E_m(\Omega) \).

**Theorem 6** [9, Theorem 1.7.14]. Let \( u^k \in E_m(\Omega), k = 1, \ldots, m \) and \( \{u^k_j\}_j \) be sequences in \( E_{0,m}(\Omega) \) such that \( u^k_j \downarrow u^k \), for each \( 1 \leq k \leq m \). Then the sequence of measures

\[
\mathrm{dd}^c u^1_1 \wedge \cdots \wedge \mathrm{dd}^c u^m_j \wedge \beta^{n-m}
\]

converge to a Radon measure in weak*-topology independent to the choice of sequences \( \{u^k_j\} \). We define \( \mathrm{dd}^c u^1_1 \wedge \cdots \wedge \mathrm{dd}^c u^m_j \wedge \beta^{n-m} \) to be this limit.

Integration by parts formula is true for the function from the Cegrell class \( F_m(\Omega) \).

**Theorem 7** [9, Theorem 1.7.18]. Assume that \( u, v, w, w_1, \ldots, w_{m-1} \in F_m(\Omega) \). Then we have

\[
\int_{\Omega} u \mathrm{dd}^c v \wedge T = \int_{\Omega} v \mathrm{dd}^c u \wedge T,
\]

where \( T = \mathrm{dd}^c w_1 \wedge \cdots \wedge \mathrm{dd}^c w_{m-1} \wedge \beta^{n-m} \) and the equality means that if one of the two terms is finite then they are equal.

The following theorem is sometimes called the Cegrell decomposition theorem.

**Theorem 8.** Let \( \mu \) be a finite, positive measure on \( \Omega \). Then there exist \( \varphi \in E_{0,m}(\Omega) \) and \( 0 \leq f \in L^1(H_m(\varphi)) \) such that

\[
\mu = f H_m(\varphi) + \nu,
\]

where \( \nu \) is carried by a \( m \)-polar set.

**Proof.** By the proof of [10, Theorem 4.14], we can find a function \( u \in E_{1,m}(\Omega) \) and \( 0 \leq f \in L^1(H_m(u)) \) such that \( \mu = f H_m(u) + \nu \), where \( \nu \) is charged by an \( m \)-polar subset of \( \Omega \). The rest of the proof goes verbatim as the proof of [10, Theorem 5.3].

\[ \square \]

### 3. The \( m \)-Subharmonic Ordering

Let \( \mu_j, \mu \) be measures on \( \Omega \). By Theorem 5, we can see that following conditions are equivalent

1. \( \lim_{j \to \infty} \int_{\Omega} \varphi d\mu_j = \int_{\Omega} \varphi d\mu, \quad \forall \varphi \in C_0(\Omega) \);
2. \( \lim_{j \to \infty} \int_{\Omega} \varphi d\mu_j = \int_{\Omega} \varphi d\mu, \quad \forall \varphi \in C_0^\infty(\Omega) \);
3. \( \lim_{j \to \infty} \int_{\Omega} \varphi d\mu_j = \int_{\Omega} \varphi d\mu, \quad \forall \varphi \in E_{0,m}(\Omega) \cap C_0(\Omega) \).

If one of above assertion is satisfied, we say that \( \mu_j \) tends to \( \mu \) on \( \Omega \) in the weak*-topology.
Remark 1. (1) If $\mu_j \to \mu$ in the weak*-topology on $\Omega$, then
\[
\mu(\Omega) \leq \liminf_{j \to \infty} \mu_j(\Omega).
\]
(2) Assume that $\{\mu_j\}_j$ is a sequence measures on $\Omega$ and $\sup_j \mu_j(\Omega) < \infty$, then there exists a subsequence $\{\mu_{j_k}\}_k \subset \{\mu_j\}_j$ such that $\mu_{j_k}$ converges to a measure $\mu$ in the weak*-topology as $k \to \infty$.

Definition 6. Let $\mu$ and $\nu$ be measures on $\Omega$. We write $\mu \succ \nu$ if and only if
\[
\int_{\Omega} -\varphi \, d\mu \geq \int_{\Omega} -\varphi \, d\nu, \quad \forall \varphi \in E_{0,m}(\Omega) \cap C(\overline{\Omega}).
\]
And we say that $\mu$ is $m$-subhamonically greater than $\nu$.

Remark 2. (1) If $\mu \succ \nu$, then $\int_{\Omega} -\varphi \, d\mu \geq \int_{\Omega} -\varphi \, d\nu, \quad \forall \varphi \in SH_m(\Omega)$ by Theorem 4. In particular, $\mu(\Omega) \geq \nu(\Omega)$.
(2) If $\mu \geq \nu$, then $\mu \succ \nu$. But Example 1 shows that the opposite implication is not true.

Example 1. For $a \in \Omega$, let $\delta_a$ be the Dirac measure at $a$. Let $\sigma_r$ be the normalized measure on the sphere $\partial B(a, r)$, where $r$ enough small such that $B(a, r) \subset \Omega$. Then for each $\varphi \in SH_m(\Omega)$, by the subharmonicity we have
\[
\int_{\Omega} \varphi \, d\delta_a = \varphi(a) \leq \int_{\partial B(a, r)} \varphi \, d\sigma_r = \int_{\Omega} \varphi \, d\sigma_r.
\]
Thus $\delta_a \succ \sigma_r$, but it is clear that $\delta_a$ is not greater than $\sigma_r$ even though $\delta_a(\Omega) = \sigma_r(\Omega) = 1$.

Proposition 2. If $u, v \in F_m(\Omega)$ and $u \geq v$, then $H_m(v) \succ H_m(u)$.

Proof. For $\varphi \in E_{0,m}(\Omega)$, by Theorem 7
\[
\int_{\Omega} -\varphi H_m(u) = \int_{\Omega} -udd^c \varphi \wedge dd^c u^{m-1} \wedge \beta^{n-m}
\leq \int_{\Omega} -vdd^c \varphi \wedge (dd^c u)^{m-1} \wedge \beta^{n-m}
= \int_{\Omega} -\varphi dd^c v \wedge (dd^c u)^{m-1} \wedge \beta^{n-m}
\leq \cdots \leq \int_{\Omega} -\varphi (dd^c v)^{m} \wedge \beta^{n-m} = \int_{\Omega} -\varphi H_m(v).
\]
Thus $H_m(v) \succ H_m(u)$. 

The following example shows that the converse implication to the statement given in Proposition 2 is not true.

Example 2. Let $\Omega$ is the unit ball $B$ in $\mathbb{C}^n$, $n \geq 2$ and define the functions
\[
v(z) = \frac{2}{3}(t^3 - 1), \quad w(z) = t^2 - 1.
\]
Then $v, w \in E_{0,2}(B) \cap C^2(B)$ and $w \leq v$ on
\[ H_2(w)(z) = 4^n n!dV, \quad H_2(v)(z) = 2^{2n-1}(2n+2)(n-1)!|z|^2dV, \]

where \( dV \) is the Lebesgue measure on \( \mathbb{C}^n \). By \cite{12} one can compute the solution \( u \) to the equation

\[ H_2(u) = \frac{H_2(v) + H_2(w)}{2}, \quad u \in \mathcal{E}_{0,2}(\mathbb{B}) \cap C^2(\mathbb{B}). \]

The solution is given by

\[ u(z) = \frac{\sqrt{2}}{3} (|z|^2 + 1)^\frac{3}{2} - \frac{4}{3}. \]

We have \( H_2(u) \succ H_2(v) \) by (1). Otherwise, \( u(0) > -1 = v(0) \), so \( v \not\succ u \).

**Remark 3.** The relation \( \succ \) defines a partial order on the set of positive Borel measures on \( \Omega \). But it is not a total order. To see that consider the Dirac measures \( \delta_z \) and \( \delta_w \), where \( z, w \in \Omega \) and \( z \neq w \). Choose \( \varphi, \psi \in SH_m(\Omega) \) such that \( \varphi(z) < \varphi(w) \) and \( \psi(z) > \psi(w) \). Then \( \int_{\Omega} -\varphi d\delta_z > \int_{\Omega} -\varphi d\delta_w \) and \( \int_{\Omega} -\psi d\delta_z < \int_{\Omega} -\psi d\delta_w \), so \( \delta_z \) and \( \delta_w \) are not comparable with respect to \( \succ \).

**Definition 7.** For a set \( E \subset \Omega \), we define the convex hull of \( E \) in \( \Omega \) with respect to the family \( SH_m(\Omega) \cap C(\overline{\Omega}) \), denoted by \( \hat{E} \) as followed

\[ \hat{E} = \{ z \in \Omega : \varphi(z) \leq \sup_{\overline{E}} \varphi, \forall \varphi \in SH_m(\Omega) \cap C(\overline{\Omega}) \}. \]

**Remark 4.** We have that \( \hat{E} \) is closed in \( \Omega \). Moreover, if \( E \) is relatively compact in \( \Omega \), so is \( \hat{E} \).

**Proposition 3.** Let \( \mu, \nu \) be finite regular measures on \( \Omega \) such that \( \mu(\Omega) = \nu(\Omega) \). If \( \nu \succ \mu \) then \( \text{supp} \nu \subset \text{supp} \mu \).

**Proof.** Put \( K = \text{supp} \mu \). If \( \hat{K} = \Omega \) then Proposition 3 is clear. Therefore we assume that \( \Omega \setminus \hat{K} \neq \emptyset \). Suppose that \( \text{supp} \nu \not\subset \hat{K} \). Since \( \hat{K} \) is closed in \( \Omega \), it follows that \( \nu(\Omega \setminus \hat{K}) > 0 \). By the regularity of \( \nu \), we can find a compact set \( L \in \Omega \setminus \hat{K} \) such that \( \nu(L) > 0 \). From the definition of \( \hat{K} \), for each \( z \in L \), there exist a neighborhood \( U(z) \) of \( z \) and a function \( \varphi \in SH_m(\Omega) \cap C(\overline{\Omega}) \) such that \( \varphi(\xi) > \sup_{\hat{K}} \varphi, \forall \xi \in U(z) \). We choose \( z_1, \ldots, z_k \in L \) such that \( L \subset \bigcup_{i=1}^{k} U(z_i) \). Let \( \varphi_1, \ldots, \varphi_k \) be the associated functions and \( M_i = \sup_{\hat{K}} \varphi_i \), \( M = M_1 + \cdots + M_k \).

Define

\[ \psi = \max\{\varphi_1, M_1\} + \cdots + \max\{\varphi_k, M_k\}. \]

Then we have \( \psi \in SH_m(\Omega) \cap C(\overline{\Omega}) \), \( \psi \geq M \) on \( \overline{\Omega} \), \( \psi = M \) on \( \hat{K} \) and \( \psi > M \) on \( L \). Define \( \psi_0 = \psi - \max_{\Omega} \psi \) and let \( M_0 = M - \max_{\overline{\Omega}} \psi \). Then \( \psi_0 \in SH_m(\Omega) \cap C(\overline{\Omega}) \), \( \psi_0 \geq M_0 \) on \( \Omega \), \( \psi_0 = M_0 \) on \( \hat{K} \) and \( \psi_0 > M_0 \) on \( L \). Hence,
\[ \int_{\Omega} -\psi_0 d\nu < -M_0\nu(\Omega) = -M_0\mu(\Omega) = \int_{\Omega} -\psi_0 d\mu. \]

Proposition 3 is proved by a contradiction. \( \square \)

4. Maximal Measures and Minimal Functions

We want to study the maximality with respect to the \( m \)-subharmonic ordering by using some kind of normalization.

Definition 8. A finite measure \( \mu \) on \( \Omega \) is said to be maximal if for any measure \( \nu \) on \( \Omega \) such that \( \nu(\Omega) = \mu(\Omega) \), the relation \( \nu \succcurlyeq \mu \) implies that \( \nu = \mu \).

Example 3. For \( 1 \leq m < n \), we define
\[ \varphi_j(z) = \max \left\{ -\frac{1}{j}|z|^2 - \frac{2^n}{m}, -1 \right\} \in SH_m(\mathbb{B}) \]
and \( \delta_0 \) is the Dirac measure defined on the unit ball \( \mathbb{B} \) in \( \mathbb{C}^n \). Then for each measure \( \nu \), \( \nu(\Omega) = 1 \) and \( \nu \succcurlyeq \delta_0 \) we have
\[ \lim_{j \to \infty} \int_{\mathbb{B}} -\varphi_j d\nu = -\nu(\{0\}) \]
and
\[ -1 \leq \int_{\mathbb{B}} -\varphi_j d\delta_0 \leq \int_{\mathbb{B}} -\varphi_j d\nu \leq 1, \forall j. \]
Thus we get \( \nu(\{0\}) = 1 \), so \( \nu = \delta_0 \) which implies \( \delta_0 \) is maximal.

Remark 5. (1) If we can write a maximal measure as the sum \( \mu = \mu_1 + \mu_2 \) of two finite measures, then these are maximal too. To prove this, assume that \( \mu_1 \) is not maximal. Then there is a finite measure \( \nu \neq \mu_1 \) such that \( \nu(\Omega) = \mu_1(\Omega) \) and \( \nu \succcurlyeq \mu \). We have \( (\nu + \mu_2)(\Omega) = \mu(\Omega) \) and \( \nu + \mu_2 \succcurlyeq \mu \), but \( \nu + \mu_2 \neq \mu \), which is a contradiction.
(2) If \( \mu \) is maximal measure, so is \( c\mu \), for \( c > 0 \).
(3) We will show that the condition \( \mu_1, \mu_2 \) are maximal does not imply the maximality of \( \mu_1 + \mu_2 \) (see Example 5). This implies that the set of maximal measures on \( \Omega \) is not a convex cone.

Definition 9. We say that a set \( K \subseteq \Omega \) is an interpolation set for \( SH_m(\Omega) \) if for each \( f \in C(K), f < 0 \) there exists a function \( \varphi \in SH_m(\Omega) \) such that \( \varphi = f \) on \( K \).

Proposition 4. If \( \mu \) is a finite measure on \( \Omega \) such that \( \text{supp} \mu \) is contained in some interpolation set \( K \) for \( SH_m(\Omega) \), then \( \mu \) is maximal.
Proof. Assume that \( \nu \) is a measure on \( \Omega \) such that \( \nu(\Omega) = \mu(\Omega) \) and \( \nu \succ \mu \). By Proposition 3, we have \( \text{supp} \nu \subset \text{supp} \mu \subset K \). For a given \( f \in C(K), f \leq 0 \), there exists a function \( \varphi \in \text{SH}^- \) such that \( \varphi = f \) on \( K \). We get

\[
\int_{\Omega} -f \, d\nu = \int_{\Omega} -\varphi \, d\nu \leq \int_{\Omega} -\varphi \, d\mu = \int_{\Omega} -f \, d\mu.
\]

This implies that \( \int_{\Omega} f \, d\mu \geq \int_{\Omega} f \, d\nu \) holds for any \( f \in C_0(\Omega), f \leq 0 \). Hence \( \mu \leq \nu \), so \( \mu = \nu \). \( \square \)

Example 4. Let \( a_1, \ldots, a_k \in \Omega \). For \( 1 \leq j \leq k \), we choose \( M_j \) such that

\[
\psi_j(z) = \sum_{l \neq j} \ln |z - a_l| + M_j \in \text{SH}^-.
\]

For each value \( c_j < 0 \), we take \( d_j > 0 \) such that \( d_j \psi_j(a_j) = c_j \). Define \( \varphi = \max(d_1 \psi_1, \ldots, d_k \psi_k) \). Then we have \( \varphi \in \text{SH}^- \) and \( \varphi(a_j) = c_j \). Thus the finite set \( \{a_1, \ldots, a_k\} \) is an interpolation set for \( \text{SH}^- \). And Proposition 4 implies that the measure \( \sum_{j=1}^{k} b_j \delta_{a_j} \) is maximal, where \( \delta_{a_j} \) is the Dirac measure at the point \( a_j \) and \( b_1, \ldots, b_k \) are given nonnegative numbers.

We will show that each finite measure with compacted support is majorized by a maximal measure with the same total mass.

Lemma 1. Assume that \( \mu \) and \( \nu \) are measures on \( \Omega \) such that \( \nu \succ \mu \). If \( \int_{\Omega} \varphi \, d\mu = \int_{\Omega} \varphi \, d\nu > -\infty \) for some negative strictly \( m \)-subharmonic function \( \varphi \). Then \( \mu = \nu \).

Proof. For given \( f \in C_0^\infty(\Omega) \), choose a constant \( c > 0 \) so that \( (\pm f + c \varphi) \in \text{SH}^- \). Then we have

\[
\int_{\Omega} (\pm f + c \varphi) \, d\mu = \int_{\Omega} \pm f \, d\mu + c \int_{\Omega} \varphi \, d\mu \geq \int_{\Omega} (\pm f + c \varphi) \, d\nu
\]

\[
= \int_{\Omega} \pm f \, d\nu + c \int_{\Omega} \varphi \, d\nu,
\]

which implies that \( \int_{\Omega} \pm f \, d\mu \geq \int_{\Omega} \pm f \, d\nu \). So \( \mu = \nu \). \( \square \)

Theorem 9. Let \( \mu \) be a finite measure on \( \Omega \) with compact support. Then there is a maximal measure \( \mu_0 \) such that \( \mu_0 \succ \mu \) and \( \mu_0(\Omega) = \mu(\Omega) \).

Proof. Put \( K = \text{supp} \mu \) and

\[
\mathcal{M}_\mu = \{ \nu: \nu \succ \mu, \nu(\Omega) = \mu(\Omega) \}.
\]

Because \( \mu \in \mathcal{M}_\mu \), so \( \mathcal{M}_\mu \neq \emptyset \). By Proposition 3, \( \text{supp} \nu \subset K \) for each \( \nu \in \mathcal{M}_\mu \). Let \( \rho \) be the exhaustion function of \( \Omega \) that is negative, continuous strictly \( m \)-subharmonic. We define

\[
A = \sup_{\nu \in \mathcal{M}_\mu} \int_{\Omega} (-\rho) \, d\nu.
\]
Since \( \rho \) is bounded on \( K \), it follows that \( A \) is finite. Let \( \{ \nu_j \}_j \) be a sequence in \( \mathcal{M}_\mu \) such that \( \int_\Omega (-\rho) d\nu_j \to A \), as \( j \to \infty \). By Remark 1, we may assume that \( \nu_j \) tend to some measure \( \mu_0 \) in the weak*-topology and \( \mu_0(\Omega) \leq \mu(\Omega) \). For each \( \varphi \in \mathcal{E}_{0,m} \cap C(\Omega) \),

\[
\int_\Omega (-\varphi) d\mu_0 = \lim_{j \to \infty} \int_\Omega (-\varphi) d\nu_j \geq \int_\Omega (-\varphi) d\mu,
\]

which implies that \( \mu_0 \ni \mu \). By Remark 2 and the fact \( \mu_0 \leq \mu(\Omega) \), we get \( \mu_0(\Omega) = \mu(\Omega) \). Thus \( \mu_0 \in \mathcal{M}_\mu \). Take a function \( f \in C_0(\Omega) \), \( f = 1 \) on \( K \). We get

\[
\int_\Omega (-\rho) d\mu_0 = \int_\Omega (-\rho) f d\mu_0 = \lim_{j \to \infty} \int_\Omega (-\rho) f d\nu_j = \lim_{j \to \infty} \int_\Omega (-\rho) d\nu_j = A.
\]

Suppose that \( \nu \) be any measure on \( \Omega \) such that \( \nu \geq \mu_0 \) and \( \nu(\Omega) = \mu(\Omega) \). Then \( \nu \in \mathcal{M}_\mu \) and \( A \geq \int_\Omega (-\rho) d\nu \geq \int_\Omega (-\rho) d\mu_0 = A \). Hence \( \int_\Omega (-\rho) d\nu = \int_\Omega (-\rho) d\mu_0 = A \). Lemma 1 implies that \( \nu = \mu_0 \), so Theorem 9 is finished. \( \square \)

**Definition 10.** A function \( u \in \mathcal{F}_m(\Omega) \) is said to be minimal if for any function \( v \in \mathcal{F}_m(\Omega) \), the conditions \( H_m(u)(\Omega) = H_m(v)(\Omega) \) and \( v \leq u \) imply \( v = u \).

**Proposition 5.** Let \( u \in \mathcal{F}_m(\Omega) \) be such that \( H_m(u) \) is a maximal measure. Then \( u \) is minimal.

To prove this proposition we need the following lemma.

**Lemma 2.** If \( u, v \in \mathcal{F}_m(\Omega) \), \( H_m(u) = H_m(v) \) and \( u \leq v \) then \( u = v \).

**Proof.** We use a method from [7]. Using integration by parts, we have

\[
\int_\Omega -(u - v)(dd^c \rho)^m \wedge \beta^{n-m} = \int_\Omega d(u - v) \wedge d^c \rho \wedge (dd^c \rho)^{m-1} \wedge \beta^{n-m} \\
\leq \left[ \int_\Omega d(u - v) \wedge d^c (u - v) \wedge (dd^c \rho)^{m-1} \wedge \beta^{n-m} \right]^\frac{1}{2} \\
\times \left[ \int_\Omega d\rho \wedge d^c \rho \wedge (dd^c \rho)^{m-1} \wedge \beta^{n-m} \right]^\frac{1}{2},
\]

where \( \rho \in \mathcal{E}_{0,m}(\Omega) \cap C^\infty(\Omega) \) is a strictly \( m \)-subharmonic exhaustion function of \( \Omega \) (see [2]). Hence, to prove \( u = v \) it is enough to show that

\[
\int_\Omega d(u - v) \wedge d^c (u - v) \wedge (dd^c \rho)^{m-1} \wedge \beta^{n-m} = 0. \quad (2)
\]
If $m = 1$ then (2) is clear. For $m \geq 2$ and $j + k = m - 1$, we have

$$0 \leq \int_{\Omega} -(u - v)(dd^c u)^j \wedge (dd^c v)^k \wedge dd^c \rho \wedge \beta^{n-m}$$

$$= \int_{\Omega} -\rho dd^c (u - v) \wedge (dd^c u)^j \wedge (dd^c v)^k \wedge \beta^{n-m}$$

$$\leq \int_{\Omega} -(u - v) \sum_{a+b=m-1} (dd^c u)^a \wedge (dd^c v)^b \wedge dd^c \rho \wedge \beta^{n-m}$$

$$= \int_{\Omega} -\rho dd^c (u - v) \wedge \sum_{a+b=m-1} (dd^c u)^a \wedge (dd^c v)^b \wedge \beta^{n-m}$$

$$= \int_{\Omega} -\rho (H_m(u) - H_m(v)) = 0.$$  

Thus, for every couple $j, k, j + k = m - 2$ we have

$$\int_{\Omega} -udd^c (u - v) \wedge (dd^c u)^j \wedge (dd^c v)^k \wedge dd^c \rho \wedge \beta^{n-m}$$

$$= \int_{\Omega} -rho dd^c (u - v) \wedge (dd^c u)^j+1 \wedge (dd^c v)^k \wedge \beta^{n-m} = 0.$$  

Similarly, $\int_{\Omega} -vdd^c (u - v) \wedge (dd^c u)^j \wedge (dd^c v)^k \wedge dd^c \rho \wedge \beta^{n-m} = 0$. So

$$\int_{\Omega} -(u - v)dd^c (u - v) \wedge (dd^c u)^j \wedge (dd^c v)^k \wedge dd^c \rho \wedge \beta^{n-m}$$

$$= \int_{\Omega} d(u - v) \wedge d^c(u - v) \wedge (dd^c u)^j \wedge (dd^c v)^k \wedge dd^c \rho \wedge \beta^{n-m} = 0, \quad (3)$$

for every couple $j, k, j + k = m - 2$. Assume that

$$\int_{\Omega} d(u - v) \wedge d^c(u - v) \wedge (dd^c u)^j \wedge (dd^c v)^k \wedge (dd^c \rho)^l \wedge \beta^{n-m} = 0 \quad (4)$$

for $j + k = m - l - 1$. By (3), (4) is true for $l = 1$. For $j + k = m - l - 2$ we have

$$\int_{\Omega} d(u - v) \wedge d^c(u - v) \wedge (dd^c u)^j \wedge (dd^c v)^k \wedge (dd^c \rho)^l+1 \wedge \beta^{n-m}$$

$$= \int_{\Omega} -\rho(dd^c (u - v))^2 \wedge (dd^c u)^j \wedge (dd^c v)^k \wedge (dd^c \rho)^l \wedge \beta^{n-m}$$

$$= \int_{\Omega} d\rho \wedge d^c(u - v) \wedge dd^c(u - v) \wedge (dd^c u)^j \wedge (dd^c v)^k \wedge (dd^c \rho)^l \wedge \beta^{n-m}$$

$$ \leq \left| \int_{\Omega} d\rho \wedge d^c(u - v) \wedge (dd^c u)^j+1 \wedge (dd^c v)^k \wedge (dd^c \rho)^l \wedge \beta^{n-m} \right|$$

$$+ \left| \int_{\Omega} d\rho \wedge d^c(u - v) \wedge (dd^c u)^j \wedge (dd^c v)^{k+1} \wedge (dd^c \rho)^l \wedge \beta^{n-m} \right|$$
\[ \leq \left[ \int_\Omega d\rho \wedge d^c \rho \wedge (dd^c u)^{j+1} \wedge (dd^c v)^k \wedge (dd^c \rho)^l \wedge \beta^{n-m} \right]^{\frac{1}{2}} \times \left[ \int_\Omega d(u - v) \wedge d^c (u - v) \wedge (dd^c u)^{j+1} \wedge (dd^c v)^k \wedge (dd^c \rho)^l \wedge \beta^{n-m} \right]^{\frac{1}{2}} + \left[ \int_\Omega d\rho \wedge d^c \rho \wedge (dd^c u)^j \wedge (dd^c v)^{k+1} \wedge (dd^c \rho)^l \wedge \beta^{n-m} \right]^{\frac{1}{2}} \times \left[ \int_\Omega d(u - v) \wedge d^c (u - v) \wedge (dd^c u)^j \wedge (dd^c v)^{k+1} \wedge (dd^c \rho)^l \wedge \beta^{n-m} \right]^{\frac{1}{2}} \]

= 0,

by assumption (4). So (2) is true by taking \( l = m - 1 \) in (4).

\[ \square \]

**Proof of Proposition 5.** Assume that \( v \in \mathcal{F}_m(\Omega), H_m(v)(\Omega) = H_m(u)(\Omega) \) and \( v \leq u \). Since \( v \leq u \), Proposition 2 implies that \( H_m(v) \geq H_m(u) \). From the assumption \( H_m(u) \) is maximal, we get \( H_m(u) = H_m(v) \). Now Proposition 5 follows from Lemma 1.

\[ \square \]

**Lemma 3.** Assume that \( u,v \in \mathcal{E}_m(\Omega) \) and \( u \geq v \). Then \( \chi_{\{u=-\infty\}} H_m(u) \leq \chi_{\{v=-\infty\}} H_m(v) \).

**Proof.** We use a method from [1]. For \( \epsilon > 0 \) small enough, set \( w_j = \max\{(1 - \epsilon)u - j, v\} \). Then we have \( w_j = (1 - \epsilon)u - j \) on the open set \( \{v < -\frac{j}{\epsilon}\} \). Therefore

\[ H_m(w_j) = (1 - \epsilon)^m H_m(u) \text{ on } \{v < -\frac{j}{\epsilon}\}. \]

Hence \( H_m(w_j) \geq (1 - \epsilon)^m \chi_{\{u=-\infty\}} H_m(u) \). Letting \( j \to \infty \), then we get \( H_m(v) \geq (1 - \epsilon)^m \chi_{\{u=-\infty\}} H_m(u) \). The proof is complete by letting \( \epsilon \to 0^+ \).

\[ \square \]

**Lemma 4.** For each \( u \in \mathcal{F}_m(\Omega) \), if \( H_m(u) \) is carried by an \( m \)-polar set, then \( H_m(u) = \chi_{\{u=-\infty\}} H_m(u) \).

**Proof.** We use the same idea as in [5]. We choose a sequence \( \{u_j\} \in \mathcal{E}_{0,m}(\Omega) \cap C(\Omega), u_j \downarrow u \). Then \( \frac{u_j}{1 - u_j} \downarrow \frac{u}{1 - u} \in \mathcal{F}_m(\Omega) \cap L^\infty(\Omega) \). For each \( v \in C^2(\Omega), \)

\[ \left. \frac{\partial}{\partial z_l \partial \bar{z}_k} \left( \frac{v}{1 - v} \right) \right| = \frac{v_l \bar{v}_k}{(1 - v)^2} + \frac{2v_l v_k}{(1 - v)^3}, \quad 1 \leq l, k \leq n. \]

This implies that

\[ \frac{H_m(u_j)}{(1 - u_j)^{2m}} \leq H_m \left( \frac{u_j}{1 - u_j} \right). \]
The function \( \frac{1}{(1-t)^{2m}} \) is convex on \([-\infty, 0]\), hence by [11, Proposition 2.1],

\[
\frac{1}{(1-u)^{2m}} - 1 \leq SH_m^- (\Omega).
\]

For every fixed \( k \),

\[
\left( \frac{1}{(1-u_k)^{2m}} - 1 \right) H_m(u) \geq \lim_{j \to \infty} \left( \frac{1}{(1-u_k)^{2m}} - 1 \right) H_m(u_j)
\]

\[
\geq \lim_{j \to \infty} \left( \frac{1}{(1-u_j)^{2m}} - 1 \right) H_m(u_j) \geq \lim_{j \to \infty} \left( \frac{1}{(1-u)^{2m}} - 1 \right) H_m(u_j)
\]

\[
= \left( \frac{1}{(1-u)^{2m}} - 1 \right) H_m(u).
\]

Letting \( k \to \infty \), we get \( \frac{H_m(u_j)}{(1-u_j)^{2m}} \) tends weakly to \( \frac{H_m(u)}{(1-u)^{2m}} \). Moreover, \( H_m \left( \frac{u_j}{1-u_j} \right) \) tends weakly to \( H_m \left( \frac{u}{1-u} \right) \). Hence,

\[
\frac{H_m(u)}{(1-u)^{2m}} \leq H_m \left( \frac{u}{1-u} \right).
\]

Theorem 8 shows that there exist \( \varphi \in \mathcal{E}_{0,m}(\Omega) \) and \( f \in L^1(H_m(\varphi)) \) such that

\[
H_m(u) = f H_m(\varphi) + \nu,
\]

where \( \nu \) is carried by an \( m \)-polar set. Moreover, (5) implies that \( \frac{H_m(u)}{(1-u)^{2m}} \) has no mass on \( m \)-polar sets. Hence, \( \frac{\nu}{(1-u)^{2m}} = 0 \), so \( \nu \) is carried by the set \( \{ u = -\infty \} \).

**Theorem 10.** Let \( u \in \mathcal{F}_m(\Omega) \) be such that \( H_m(u) \) is carried by an \( m \)-polar set. Then \( u \) is a minimal function.

**Proof.** Assume that \( v \in \mathcal{F}_m(\Omega) \), \( v \leq u \) and \( H_m(v)(\Omega) = H_m(u)(\Omega) \). By Lemmas 3 and 4,

\[
\int_{\Omega} H_m(v) \geq \int_{\{ v = -\infty \}} H_m(v) \geq \int_{\{ u = -\infty \}} H_m(u) = \int_{\Omega} H_m(u).
\]

Hence, \( H_m(v) = \chi_{\{ v = -\infty \}} H_m(v) \). By Lemma 3 again, \( H_m(u) \leq H_m(v) \). Combine this with \( H_m(u)(\Omega) = H_m(v)(\Omega) \), we get \( H_m(u) = H_m(v) \). Lemma 2 implies that \( u = v \). 

**Proposition 6.** Assume that \( \mu \) is a finite measure on \( \Omega \) such that \( \text{supp} \mu \) is contained in a level set \( \{ z \in \Omega: \psi(z) = c \} \), where \( c > -\infty \) and \( \psi < 0 \) is a strictly \( m \)-subharmonic function on \( \Omega \). Then \( \mu \) is maximal.

**Proof.** Suppose that \( \nu \succ \mu \) and \( \nu(\Omega) = \mu(\Omega) \). By Proposition 3, \( \text{supp} \nu \subset \{ z \in \Omega: \psi(z) = c \} \). Thus,

\[
\int_{\Omega} -\psi d\nu = \int_{\Omega} -cd\nu = \int_{\Omega} -cd\mu = \int_{\Omega} -\psi d\mu < \infty.
\]

Therefore, Lemma 1 implies that \( \nu = \mu \), and the proof is complete. 

The following example confirms Remark 5(3).
Example 5 [3, Examples 4.15, 4.16]. We consider the unit disc $\mathbb{D}$ in $\mathbb{C}$. Define the sets $S_1 = \{z = \frac{1}{2} e^{i\theta} : 0 \leq \theta \leq \pi\}$ and $S_2 = \{z = \frac{1}{2} e^{i\theta} : \pi < \theta < 2\pi\}$. Let $\sigma$ be the area measure on the circle $\partial \mathbb{D}(0, \frac{1}{2})$ and define $\mu_j = \sigma|_{S_j}$, for $j = 1, 2$. We have $S_j \subset \{\psi = |z|^2 - 1 = -\frac{3}{4}\}$. Let $h_j = h_{1, S_j, \mathbb{D}}$ be the 1-relative extremal function for $S_j$ over $\mathbb{D}$. Then $h_j \in \mathcal{E}_{0,1}(\mathbb{D}) \cap C(\mathbb{D})$ and $h_j = -1$ on $S_j$. Moreover, $h_j$ is harmonic on the connected set $\mathbb{D}\backslash S_j$, which implies that $h > -1$ on $\mathbb{D}\backslash S_j$. Hence $\hat{S}_j = S_j$ and Proposition 6 deduces that $\mu_1$ and $\mu_2$ are maximal measures. But $\sigma = \mu_1 + \mu_2$ is not maximal (see Example 1).

We will show that each function in $\mathcal{F}_m(\Omega)$ is minorized by a minimal function with the same total Hessian mass.

Proposition 7. Let $\{u_j\}$ be a decreasing sequence in $\mathcal{F}_m(\Omega)$ such that $u_j \downarrow u$ and $H_m(u_j)(\Omega) = H_m(u_{j+1})(\Omega)$ for all $j$. Then $u \in \mathcal{F}_m(\Omega)$ and $H_m(u)(\Omega) = H_m(u_j)(\Omega)$.

Proof. We have $u \in SH_m^{-}(\Omega)$, and by Theorem 4, there exists a sequence $\{w_j\} \subset \mathcal{E}_{0,m}(\Omega) \cap C(\Omega)$ such that $w_j \downarrow u$ as $j \to \infty$. Set $v_j = \max(w_j, u_j)$. Then $v_j \geq u_j, v_j \in \mathcal{E}_{0,m}(\Omega)$ and $v_j \downarrow u$ as $j \to \infty$. Theorem [10, Theorem 3.22] implies that

$$\sup_j \int_{\Omega} H_m(v_j)(\Omega) \leq \sup_j H_m(u_j) = H_m(u_1) < \infty,$$

Thus, $u \in \mathcal{F}_m(\Omega)$. Since the sequence of measures $H_m(v_j)$ converges to the measure $H_m(u)$ in the weak* topology, we get

$$\lim_{j \to \infty} \inf H_m(v_j)(\Omega) \geq H_m(u)(\Omega).$$

Moreover, by [10, Theorem 3.22] again, we obtain $H_m(u)(\Omega) \geq H_m(u_j)$ since $u, u_j \in \mathcal{F}_m(\Omega), u \leq u_j$.

Theorem 11. For each $u \in \mathcal{F}_m(\Omega)$, there exists a minimal function $u_0 \in \mathcal{F}_m(\Omega)$ such that $u_0 \leq u$ and $H_m(u_0)(\Omega) = H_m(u)(\Omega)$.

Proof. Define $S = \{v \in \mathcal{F}_m(\Omega) : v \leq u, H_m(v)(\Omega) = H_m(u)(\Omega)\}$. Let $T$ be the totally ordered subset of $S$ and let $t(z) = \inf_{v \in T} v(z)$. We shall prove that $t \in S$. It is obvious that $t \leq u$. Let $\{K_i\}$ be a compact exhaustion sets of $\Omega$ and let $\{t_j\}$ be a sequence of continuous functions such that $t_j \geq t$ and $t_j \downarrow t$ as $j \to \infty$. For each $z \in K_i$, choose $v_z \in T$ such that $v_z(z) < t_j(z)$ and define the open set $U_z = \{w \in \Omega : v_z(w) < t_j(w)\}$. Take $z_1, \ldots, z_N \in K_i$ such that $\bigcup_{k=1}^{N} U_{z_k} \supset K_i$. Since $T$ is totally ordered, we may choose $v_1^j$ to be the smallest of the functions $v_{z_1}, \ldots, v_{z_N}$, which implies that $v_1^j < t_j$ on $K_i$. Now let $u_1 = v_1^j$ and $u_j$ be the smallest of the functions $\{u_1, \ldots, u_{j-1}, v_1^j\}$ if $j \geq 2$, since $T$ is totally ordered. Then $\{u_j\}$ is a decreasing sequence of functions in $T$ such that $u_j \leq v_1^j < t_j$ on $K_j$. Therefore $u_j \in \mathcal{F}_m(\Omega), H_m(u_j)(\Omega) = H_m(u)(\Omega)$ and $u_j \downarrow t$, as $j \to \infty$. Proposition 7 implies $t \in \mathcal{F}_m(\Omega)$ and $H_m(t)(\Omega) = H_m(u)(\Omega)$. 

Hence \( t \in S \). Since \( T \) is arbitrary, Zorn’s lemma deduces that there is a minimal element \( u_0 \) of \( S \), so the proof is complete. \( \square \)

5. Convergence in the Weak*-Topology

We will use the \( m \)-subharmonic ordering to obtain some results on weak*-convergence of measures. If \( \Omega \) is a bounded domain in \( \mathbb{C}^n \) and \( \{u_j\} \) is a sequence of locally bounded \( m \)-subharmonic functions on \( \Omega \) which is decreasing to a function \( u \in SH_m(\Omega) \cap L^\infty_{\text{loc}}(\Omega) \), then \( H_m(u_j) \) converges to \( H_m(u) \) in the weak* topology (see [4]). The same conclusion holds if \( SH_m(\Omega) \cap L^\infty_{\text{loc}}(\Omega) \) is replaced by the class \( \mathcal{E}_m(\Omega) \), where \( \Omega \) is a bounded \( m \) hyperconvex domain (see [9]).

The following example shows that Hessian operator is discontinuous with respect to the convergence in \( L^1_{\text{loc}} \). This example follows the idea in [8].

**Example 6.** For \( n \geq 2 \), we define

\[
u_j(z_1, \ldots, z_n) = \left| \sum_{k=1}^{n} z_k^{2j} \right|^{\frac{1}{2j}}
\]

We can compute

\[
\frac{\partial^2 u}{\partial z_p \partial z_q} = \frac{1}{4} \left| \sum_{k=1}^{n} z_k^{2j} \right|^{\frac{1}{2j}-2} z_p^{2j-1} z_q^{2j-1}, \forall 1 \leq p, q \leq n.
\]

Thus, \( H_m(u_j) = 0 \), for all \( j \). We have \( 0 \leq u_j \leq n \frac{1}{2j} u \), where \( u(z_1, \ldots, z_n) = \max\{|z_1|, \ldots, |z_n|\} \). Hence, we get \( u_j \to u \) in \( L^1_{\text{loc}}(\mathbb{C}^n) \) as \( j \to \infty \). We can show that \( H_m(u) \neq 0 \). Assume the contrary. Then \( H_m(u) = 0 \) on the polydisc \( \Delta_n(r) = \mathbb{D}(0, r) \times \cdots \times \mathbb{D}(0, r) \), i.e., \( u \) is \( m \)-maximal function on \( \Delta_n(r) \). Note that \( u \geq r_1 \) outside the compact subset \( \overline{\Delta_n(r_1)} \), where \( r_1 < r \) but we do not have \( u \geq r_1 \) on \( \Delta_n(r) \).

The following theorem give us a sufficient condition for weak*-convergence for the class \( \mathcal{F}_m(\Omega) \).

**Theorem 12.** If \( u_j \to u \) in \( L^1_{\text{loc}}(\Omega) \) and there is a strictly \( m \)-subharmonic function \( v \in \mathcal{E}_{0,m}(\Omega) \) such that

\[
\int_{\Omega} v H_m(u_j) \to \int_{\Omega} v H_m(u) \text{ as } j \to \infty,
\]

then \( H_m(u_j) \) tends to \( H_m(u) \) in the weak*-topology.

**Proof.** We use the idea from [6]. For \( w \in \mathcal{E}_{0,m}(\Omega) \), using integration by parts (Theorem 7) we have
\begin{equation}
\int_\Omega wH_m(u_j) \leq \int_\Omega wH_m[(\sup_{s \geq j} u_s)] \downarrow \int_\Omega wH_m(u) \text{ as } j \to \infty.
\end{equation}

Hence,
\begin{equation}
\limsup_{j \to \infty} \int_\Omega wH_m(u_j) \leq \int_\Omega wH_m(u). \tag{6}
\end{equation}

Theorem 4 implies that (6) is true for \( w \in S H_m^-(\Omega) \). Let \( \varphi \in C_0^\infty(\Omega) \) be given. By assumption \( v \) is strictly \( m \)-subharmonic we can choose \( A > 0 \) large enough such that \((\pm \varphi + Av) \in \mathcal{E}_{0,m}(\Omega) \). By (6) we have
\begin{equation}
\limsup_{j \to \infty} \int_\Omega (\pm \varphi + Av)H_m(u_j) \leq \int_\Omega (\pm \varphi + Av)H_m(u).
\end{equation}
Combining this with assumption \( \lim_{j \to \infty} \int_\Omega vH_m(u_j) = \int_\Omega vH_m(u) \) we obtain
\begin{equation}
\limsup_{j \to \infty} \int_\Omega \pm \varphi H_m(u_j) \leq \int_\Omega \pm \varphi H_m(u),
\end{equation}
which implies the desired result. \( \Box \)

**Definition 11.** If \( \{\mu_j\} \) is a sequence of measures such that \( \mu_{j+1} \succeq \mu_j \) for all \( j \), then we say that \( \{\mu_j\} \) is \( m \)-subharmonically increasing.

**Theorem 13.** Let \( \{\mu_j\} \) be an \( m \)-subharmonically increasing sequence of measures on \( \Omega \) such that \( \sup_{j} \mu_j(\Omega) < \infty \). Then \( \mu_j \) converges to a measure \( \mu \) in the weak*-topology. Moreover, \( \int_{\Omega} (-\varphi) d\mu_j \uparrow \int_{\Omega} (-\varphi) d\mu \) for each \( \varphi \in S H_m^- (\Omega) \).

**Proof.** Let \( \varphi \in S H_m^- (\Omega) \cap L^\infty (\Omega) \). Then
\[ 0 \leq \int_{\Omega} (-\varphi) d\mu_1 \leq \int_{\Omega} (-\varphi) d\mu_2 \leq \cdots \leq \sup_{j}(-\varphi) \sup_{j} \mu_j(\Omega) < \infty. \]
so \( \lim_{j \to \infty} \int_{\Omega} (-\varphi) d\mu_j < \infty \). Thus the limit exists for each \( \varphi \in C_0(\Omega) \). It follows that this defines a measure \( \mu \) on \( \Omega \) that \( \mu_j \) converges to \( \mu \) in the weak*-topology. Moreover, we know that \( \lim_{j \to \infty} \int_{\Omega} (-\varphi) d\mu_j = \int_{\Omega} (-\varphi) d\mu \) for each \( \varphi \in \mathcal{E}_{0,m}(\Omega) \cap C(\Omega) \). Now, let \( \varphi \in S H_m^- (\Omega) \). As above \( \{\int_{\Omega} (-\varphi) d\mu_j\} \) is an increasing sequence. We always have
\begin{equation}
\lim_{j \to \infty} \int_{\Omega} (-\varphi) d\mu_j \geq \int_{\Omega} (-\varphi) d\mu. \tag{7}
\end{equation}
To show the equality in (7), we assume the contrary, i.e.,
\begin{equation}
\lim_{j \to \infty} \int_{\Omega} (-\varphi) d\mu_j > \int_{\Omega} (-\varphi) d\mu.
\end{equation}
Choose \( j_0 \) enough large such that \( \int_{\Omega} (-\varphi) d\mu_{j_0} > \int_{\Omega} (-\varphi) d\mu \), and a sequence \( \{\varphi_k\} \in \mathcal{E}_{0,m} \cap C(\Omega) \) such that \( \varphi_k \downarrow \varphi \). Then we might choose \( k_0 \) such that \( \int_{\Omega} (-\varphi_{k_0}) d\mu_{j_0} > \int_{\Omega} (-\varphi) d\mu \). It follows that
\[ \int_\Omega (-\varphi_{k_0}) d\mu = \lim_{j \to \infty} \int_\Omega (-\varphi_{k_0}) d\mu_j \geq \int_\Omega (-\varphi_{k_0}) d\mu_{j_0} \]

\[ > \int_\Omega (-\varphi) d\mu \geq \int_\Omega (-\varphi_{k_0}) d\mu, \]

which is a contradiction. \[\Box\]

If \( \{u_j\} \subset F_m(\Omega) \) converges to \( u \in F_m(\Omega) \) in \( L^1_{loc}(\Omega) \), then we can relate the limit measure of sequence \( \{H_m(u_j)\} \) in Theorem 13 to \( H_m(u) \) as follows.

**Corollary 1.** Assume that \( \{u_j\} \subset F_m(\Omega) \) such that

(1) \( u_j \) converges to \( u \in F_m(\Omega) \) in \( L^1_{loc}(\Omega) \),
(2) \( \{H_m(u_j)\} \) is \( m \)-subharmonically increasing,
(3) \( \sup_j H_m(u_j) < \infty \).

Then \( H_m(u_j) \) converges to a measure \( \mu \) in the weak*-topology such that \( \mu \succcurlyeq H_m(u) \). Moreover, \( \int_\Omega (-\varphi) H_m(u_j) \uparrow \int_\Omega (-\varphi) d\mu \) for each \( \varphi \in SH_m^{-}(\Omega) \).

**Proof.** By Theorem 13 it remains to show that \( \mu \succcurlyeq H_m(u) \). By the proof of Theorem 12, assumption (1) implies that \( \liminf_{j \to \infty} \int_\Omega (-\varphi) H_m(u_j) \geq \int_\Omega (-\varphi) H_m(u) \) for each \( \varphi \in SH_m^{-}(\Omega) \). \[\Box\]

The following theorem gives us a bridge between convergence in weak*-topology and the concept of maximal measures defined in Sect. 4.

**Theorem 14.** Let \( \{u_j\} \subset F_m(\Omega) \) such that

(1) \( u_j \) converges to \( u \in F_m(\Omega) \) in \( L^1_{loc}(\Omega) \),
(2) \( H_m(u) \) is a maximal measure,
(3) \( \lim_{j \to \infty} H_m(u_j)(\Omega) = H_m(u)(\Omega) \).

Then \( H_m(u_j) \) converges to \( H_m(u) \) in the weak*-topology.

**Proof.** Assumption (3) implies that there is a subsequence \( \{H_m(u_{j_k})\} \subset \{H_m(u_j)\} \) which converging to a measure \( \mu \) in the weak*-topology. Let \( \varphi \in E_{0,m}(\Omega) \cap C(\Omega) \) be given. As in the proof of Corollary 1, assumption (a) implies that \( \mu \succcurlyeq H_m(u) \). Moreover, by (3) we have \( \mu(\Omega) \leq \liminf_{j \to \infty} H_m(u_{j_k})(\Omega) \leq H_m(u)(\Omega) \). Thus, \( \mu(\Omega) = H_m(u)(\Omega) \). By assumption (2) we can conclude that \( \mu = H_m(u) \). \[\Box\]

**Open Question**

One might ask if there is a converse of Proposition 5. The answer is affirmative if \( n = m = 1 \) (see [3, Proposition 4.11]). In higher dimension, the answer is unknown.
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