

A local regularity of the complex Monge–Ampère equation

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Abstract We prove a local regularity (and a corresponding a priori estimate) for plurisubharmonic solutions of the nondegenerate complex Monge–Ampère equation assuming that their $W^{2,p}$ -norm is under control for some $p > n(n-1)$. This condition is optimal. We use in particular some methods developed by Trudinger and an estimate for the complex Monge–Ampère equation due to Kołodziej.

1 Introduction

The aim of this note is to prove the following a priori estimate for the complex Monge–Ampère equation:

Theorem *Assume that $p > n(n-1)$. Let $u \in W^{2,p}(\Omega)$ (that is partial derivatives of u up to the second order are in $L^p(\Omega)$), where Ω is a domain in \mathbb{C}^n , be a plurisubharmonic solution of*

$$\det(u_{z_j \bar{z}_k}) = \psi > 0. \quad (1)$$

Assume that $\psi \in C^{1,1}(\Omega)$ (that is $\psi \in C^1(\Omega)$ and the second partial derivatives of ψ are Lipschitz continuous). Then for $\Omega' \Subset \Omega$ we have

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$$\sup_{\Omega'} \Delta u \leq C,$$

where C is a constant depending only on $n, p, \text{dist}(\Omega', \partial\Omega), \inf_{\Omega} \psi, \|\psi\|_{C^{1,1}(\Omega)}$ and $\|\Delta u\|_{L^p(\Omega)}$.

By a complex version of the Evans–Krylov theory (see e.g. [5] or [11]), once one has an upper bound for the Laplacian (and thus for mixed complex second derivatives) then also a $C^{2,\alpha}$ -estimate follows. We thus get the following local regularity of plurisubharmonic solutions of (1)

$$u \in W_{loc}^{2,p} \text{ for some } p > n(n - 1), \quad \psi \in C^\infty \implies u \in C^\infty. \tag{2}$$

For $p > 2n(n - 1)$ this (and the theorem) is a consequence of a general real theory from [13] (see [4]). For $p > n^2$ a similar a priori estimate for C^3 -solutions (without a regularity result though) was recently shown in [7].

The main point about our result is that the condition $p > n(n - 1)$ is essentially optimal. The fact that it is false for $p < n(n - 1)$ follows from a complex counterpart of Pogorelov’s example [10] from [4]: the function

$$u(z) = (1 + |z_1|^2)|z'|^{2-2/n},$$

where $z' = (z_2, \dots, z_n)$, is in $W_{loc}^{2,p}$ if and only if $p < n(n - 1)$, plurisubharmonic in \mathbb{C}^n , and satisfies

$$\det(u_{z_j \bar{z}_k}) = c_n(1 + |z_1|^2)^{n-2} \in C^\infty(\mathbb{C}^n)$$

(c_n is a constant depending only on n) in the weak sense of [2].

The corresponding estimates and regularity for the real Monge–Ampère equation can be found in [14].

The main tool in the proof of Theorem will be the following estimate of Kołodziej [8] (see also [9]): if a plurisubharmonic u with $u \geq 0$ on $\partial\Omega$ solves (1) (with ψ satisfying only $\psi \geq 0$) then for $q > 1$ we have

$$\sup_{\Omega} (-u) \leq C(q, n, \text{diam}\Omega) \|\psi\|_{L^q(\Omega)}^{1/n}. \tag{3}$$

This result for $q = 2$ is due to Cheng and Yau (see [1,6]).

2 Proof of Theorem

By C_1, C_2, \dots we will denote possibly different positive constants depending only on the required quantities. Without loss of generality we may assume that $\Omega = B$ is the unit ball in \mathbb{C}^n and that u is defined in some neighborhood of \bar{B} . We will use the notation $u_j = u_{z_j}, u_{\bar{j}} = u_{\bar{z}_j}$ and $\Delta u = \sum_j u_{j\bar{j}}$. As usual, by $(u^{i\bar{j}})$ we will denote the inverse transposed of $(u_{i\bar{j}})$.

We will first prove Theorem assuming that u is in C^4 . Differentiating (1) w.r.t. z_p and \bar{z}_p we will get

$$u^{i\bar{j}}u_{i\bar{j}p} = (\log \psi)_p$$

and

$$u^{i\bar{j}}u_{i\bar{j}p\bar{p}} = (\log \psi)_{p\bar{p}} + u^{i\bar{l}}u^{k\bar{j}}u_{k\bar{l}\bar{p}}u_{i\bar{j}p}.$$

Therefore

$$u^{i\bar{j}}(\Delta u)_{i\bar{j}} \geq \Delta(\log \psi). \tag{4}$$

We will now use an idea from [12]. For some $\alpha, \beta \geq 2$ to be determined later set

$$w := \eta(\Delta u)^\alpha,$$

where

$$\eta(z) := (1 - |z|^2)^\beta$$

Then

$$w_i = \eta_i(\Delta u)^\alpha + \alpha\eta(\Delta u)^{\alpha-1}(\Delta u)_i$$

and

$$\begin{aligned} u^{i\bar{j}}w_{i\bar{j}} &= \alpha\eta(\Delta u)^{\alpha-1}u^{i\bar{j}}(\Delta u)_{i\bar{j}} + \alpha(\alpha - 1)\eta(\Delta u)^{\alpha-2}u^{i\bar{j}}(\Delta u)_i(\Delta u)_{\bar{j}} \\ &\quad + 2\alpha(\Delta u)^{\alpha-1}\operatorname{Re}\left(u^{i\bar{j}}\eta_i(\Delta u)_{\bar{j}}\right) + (\Delta u)^\alpha u^{i\bar{j}}\eta_{i\bar{j}}. \end{aligned}$$

By (4) and the Schwarz inequality for $t > 0$

$$\begin{aligned} u^{i\bar{j}}w_{i\bar{j}} &\geq \alpha\eta(\Delta u)^{\alpha-1}\Delta(\log \psi) + \alpha(\alpha - 1)\eta(\Delta u)^{\alpha-2}u^{i\bar{j}}(\Delta u)_i(\Delta u)_{\bar{j}} \\ &\quad - t\alpha(\Delta u)^{\alpha-1}u^{i\bar{j}}(\Delta u)_i(\Delta u)_{\bar{j}} - \frac{1}{t}\alpha(\Delta u)^{\alpha-1}u^{i\bar{j}}\eta_i\eta_{\bar{j}} + (\Delta u)^\alpha u^{i\bar{j}}\eta_{i\bar{j}}. \end{aligned}$$

Therefore with $t = (\alpha - 1)\eta/\Delta u$ we get

$$u^{i\bar{j}}w_{i\bar{j}} \geq \alpha\eta(\Delta u)^{\alpha-1}\Delta(\log \psi) + (\Delta u)^\alpha u^{i\bar{j}}\left(\eta_{i\bar{j}} - \frac{\alpha}{\alpha - 1}\frac{\eta_i\eta_{\bar{j}}}{\eta}\right).$$

We now have

$$\begin{aligned} \eta_i &= -\beta z_i \eta^{1-1/\beta} \\ \eta_{i\bar{j}} &= -\beta \delta_{i\bar{j}} \eta^{1-1/\beta} + \beta(\beta - 1)\bar{z}_i z_j \eta^{1-2/\beta} \end{aligned}$$

and thus

$$|\eta_{i\bar{j}}|, \quad \left| \frac{\eta_i \eta_{\bar{j}}}{\eta} \right| \leq C(\beta) \eta^{1-2/\beta}.$$

We will get

$$u^{i\bar{j}} w_{i\bar{j}} \geq -C_1(\Delta u)^{\alpha-1} - C_2 w^{1-2/\beta} (\Delta u)^{2\alpha/\beta} \sum_{i,j} |u^{i\bar{j}}|.$$

Fix q with $1 < q < p/(n(n - 1))$. Since $\|\Delta u\|_p$ (this way we will denote norms in $L^p(B)$) is under control, it follows that $\|u_{i\bar{j}}\|_p$ and $\|u^{i\bar{j}}\|_{p/(n-1)}$ are as well. It follows that for

$$\alpha = 1 + \frac{p}{qn}, \quad \beta = 2 \left(1 + \frac{qn}{p} \right)$$

we have

$$\left\| \left(u^{i\bar{j}} w_{i\bar{j}} \right)_- \right\|_{qn} \leq C_3 \left(1 + \left(\sup_B w \right)^{1-2/\beta} \right),$$

where $f_- := -\min(f, 0)$.

By [2] we can find continuous plurisubharmonic v vanishing on ∂B and such that

$$\det(v_{i\bar{j}}) = \left(\left(u^{i\bar{j}} w_{i\bar{j}} \right)_- \right)^n$$

(weakly). Essentially by an inequality between arithmetic and geometric means (see [3] how to extend it to the weak case) we have

$$\begin{aligned} u^{i\bar{j}} v_{i\bar{j}} &\geq n \left(\det \left(u^{i\bar{j}} \right) \right)^{1/n} \left(\det \left(v_{i\bar{j}} \right) \right)^{1/n} \\ &= n \psi^{-1/n} \left(u^{i\bar{j}} w_{i\bar{j}} \right)_- \\ &\geq -\frac{1}{C_4} u^{i\bar{j}} w_{i\bar{j}}. \end{aligned}$$

It follows that $w \leq -C_4 v$ and by Kołodziej’s inequality (3)

$$\begin{aligned} \sup_B w &\leq C_5 \|\det(v_{i\bar{j}})\|_q^{1/n} \\ &= C_5 \left\| \left(u^{i\bar{j}} w_{i\bar{j}} \right)_- \right\|_{qn} \\ &\leq C_6 \left(1 + \left(\sup_B w \right)^{1-2/\beta} \right). \end{aligned}$$

Therefore $w \leq C_7$ and the desired estimate follows if $u \in C^4$.

Now assume that the solution is just in $W^{2,p}$. Similarly to [2], instead of Δu we will consider for $\varepsilon > 0$ the following approximations to the Laplacian

$$T = T_\varepsilon u = \frac{n + 1}{\varepsilon^2} (u_\varepsilon - u),$$

where

$$u_\varepsilon(z) = \frac{1}{\lambda(B(z, \varepsilon))} \int_{B(z, \varepsilon)} u \, d\lambda$$

and λ denotes the Lebesgue measure in \mathbb{C}^n . Since $T_\varepsilon u \rightarrow \Delta u$ weakly as $\varepsilon \rightarrow 0$, it is enough to show a uniform upper bound for T independent of ε .

By [2] we have

$$u^{i\bar{j}} u_{\varepsilon, i\bar{j}} \geq n\psi^{-1/n} \left(\det(u_{\varepsilon, i\bar{j}}) \right)^{1/n} \geq n\psi^{-1/n} (\psi^{1/n})_\varepsilon$$

and thus, coupling this with $u^{i\bar{j}} u_{i\bar{j}} = n$, we obtain the following counterpart of (4)

$$u^{i\bar{j}} T_{i\bar{j}} \geq n\psi^{-1/n} T_\varepsilon (\psi^{1/n}) \geq -C_8.$$

Changing the definition of w to ηT^α (since u is plurisubharmonic, T is nonnegative, hence T^α is well defined) and repeating the previous computations we will get

$$u^{i\bar{j}} w_{i\bar{j}} \geq C_9 T^{\alpha-1} - C_{10} w^{1-2/\beta} T^{2\alpha/\beta} \sum_{i,j} \left| u^{i\bar{j}} \right|.$$

The rest of the proof is now the same as before.

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References

1. Bedford, E.: Survey of pluri-potential theory. In: Fornæss, J.E. (ed.) *Several Complex Variables. Proceedings of the Mittag-Leffler Institute*, vol. 1987–1988. Princeton University Press, New Jersey (1993)
2. Bedford, E., Taylor, B.A.: The Dirichlet problem for a complex Monge–Ampère equation. *Invent. Math.* **37**, 1–44 (1976)
3. Błocki, Z.: The complex Monge–Ampère operator in hyperconvex domains. *Ann. Scuola Norm. Sup. Pisa* **23**, 721–747 (1996)
4. Błocki, Z.: On the regularity of the complex Monge–Ampère operator. In: *Complex Geometric Analysis in Pohang, 1997*. *Contemp. Math.*, vol. 222, pp. 181–189. Amer. Math. Soc., Providence (1999)

5. Błocki, Z.: Interior regularity of the complex Monge–Ampère equation in convex domains. *Duke Math. J.* **105**, 167–181 (2000)
6. Cegrell, U., Persson, L.: The Dirichlet problem for the complex Monge–Ampère operator: stability in L^2 . *Michigan Math. J.* **39**, 145–151 (1992)
7. He, W.: On regularity of complex Monge–Ampère equations. arXiv:1002.4825v2
8. Kołodziej, S.: Some sufficient conditions for solvability of the Dirichlet problem for the complex Monge–Ampère operator. *Ann. Pol. Math.* **65**, 11–21 (1996)
9. Kołodziej, S.: The complex Monge–Ampère equation. *Acta Math.* **180**, 69–117 (1998)
10. Pogorelov, A.V.: The Dirichlet problem for the multidimensional analogue of the Monge–Ampère equation. *Dokl. Akad. Nauk SSSR* **201**, 790–793 (1971) [transl.: *Soviet Math. Dokl.* **12**, 1727–1731 (1971)]
11. Siu, Y.-T.: *Lectures on Hermitian–Einstein metrics for stable bundles and Kähler–Einstein metrics.* Birkhäuser, Basel (1987)
12. Trudinger, N.S.: Local estimates for subsolutions and supersolutions of general second order elliptic quasilinear equations. *Invent. Math.* **61**, 67–79 (1980)
13. Trudinger, N.S.: Regularity of solutions of fully nonlinear elliptic equations. *Boll. Un. Mat. Ital. A* (6) **3**, 421–430 (1984)
14. Urbas, J.: Regularity of generalized solutions of Monge–Ampère equations. *Math. Z.* **197**, 365–393 (1988)