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Computer-assisted proof of the existence of periodic orbits in delay differential equations

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Chapter 1

Introduction

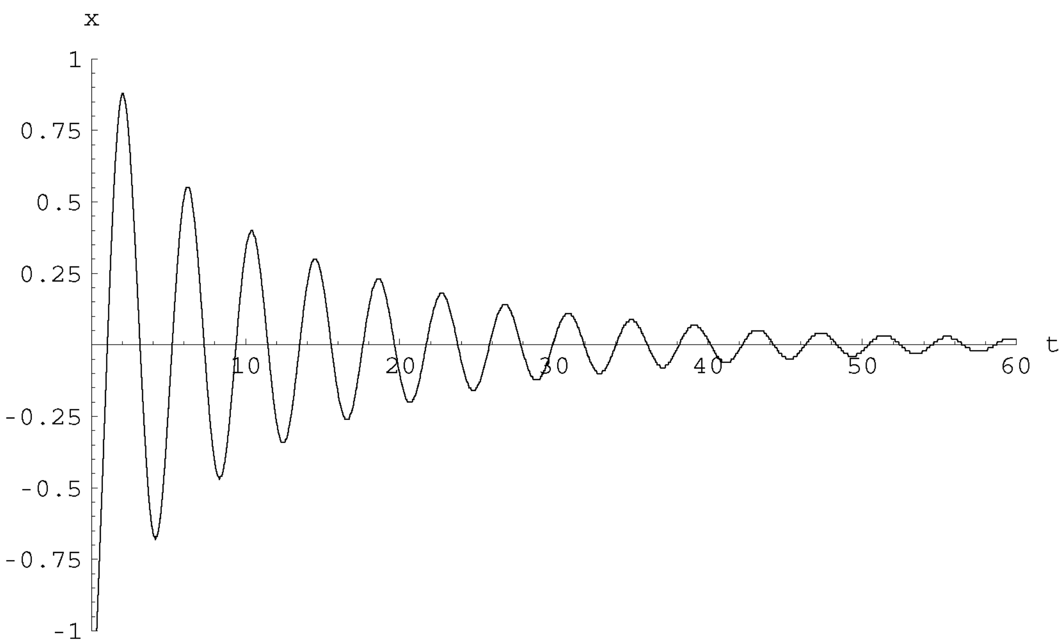
1.1 Overview

The equation:

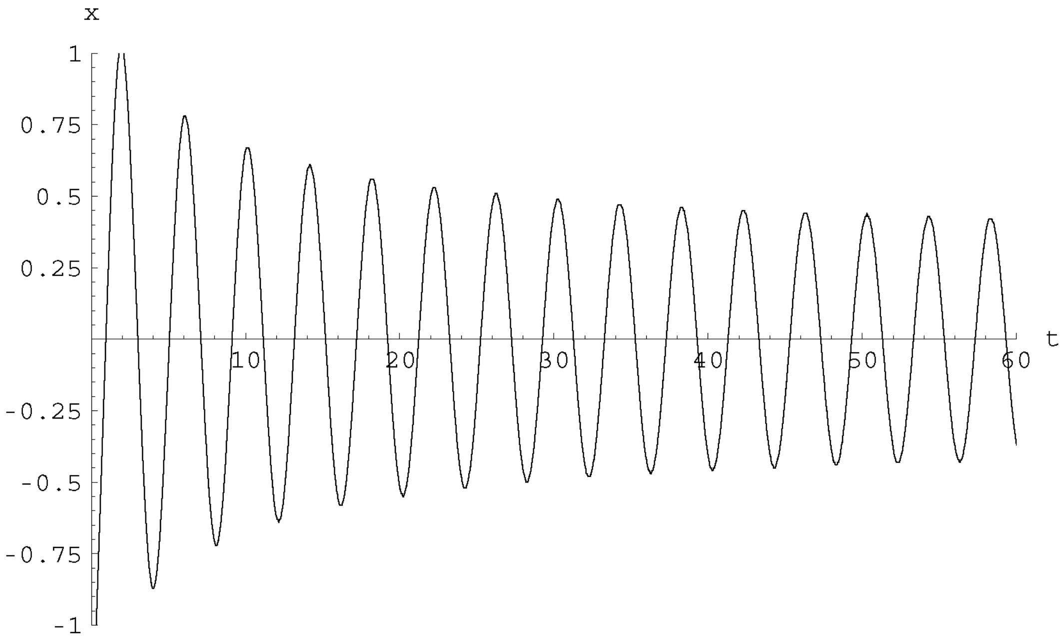
$$x'(t) = -K \sin(x(t-1)) \tag{1.1}$$

is an example of a nonlinear feedback delay differential equation (see [2]). It is used to model, for example, delay-lock loops in electronics (see [5]).

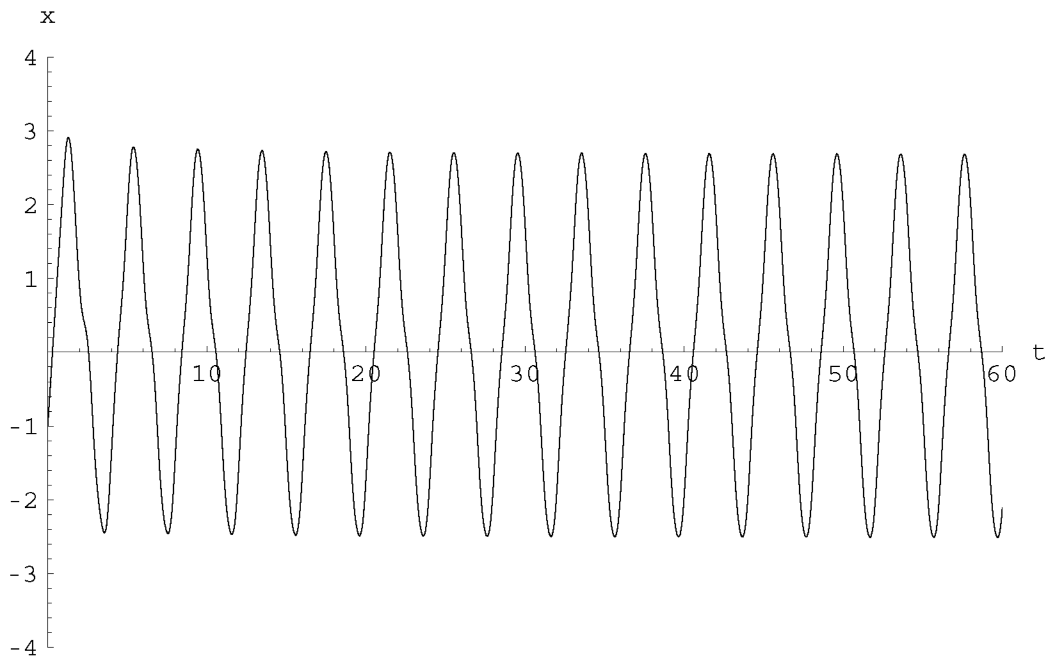
From numerical simulations it follows that for small $K < \frac{\pi}{2}$ the solutions tend to 0. The graph below is for $K = 1.45$ and the initial condition $x(t) = -1.5$ for $t \in [-1; 0]$:



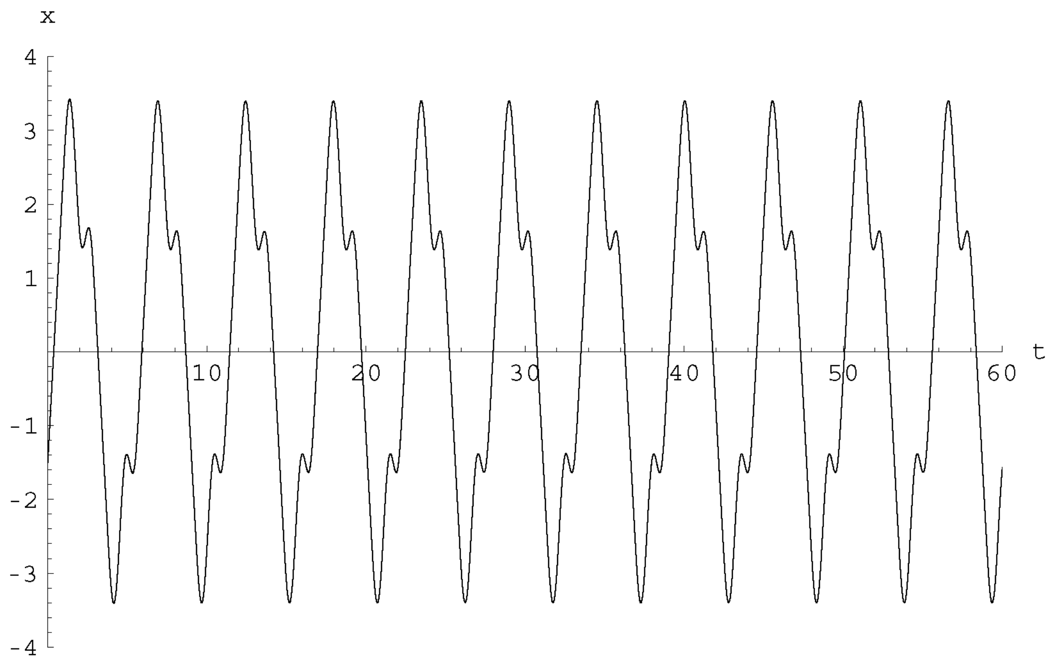
If we increase K, there is an attracting periodic orbit oscillating around 0 (of course, there are also orbits oscillating around $2k\pi$ for any k, because of the periodicity of the sinus). The graph below is for $K = 1.6$:



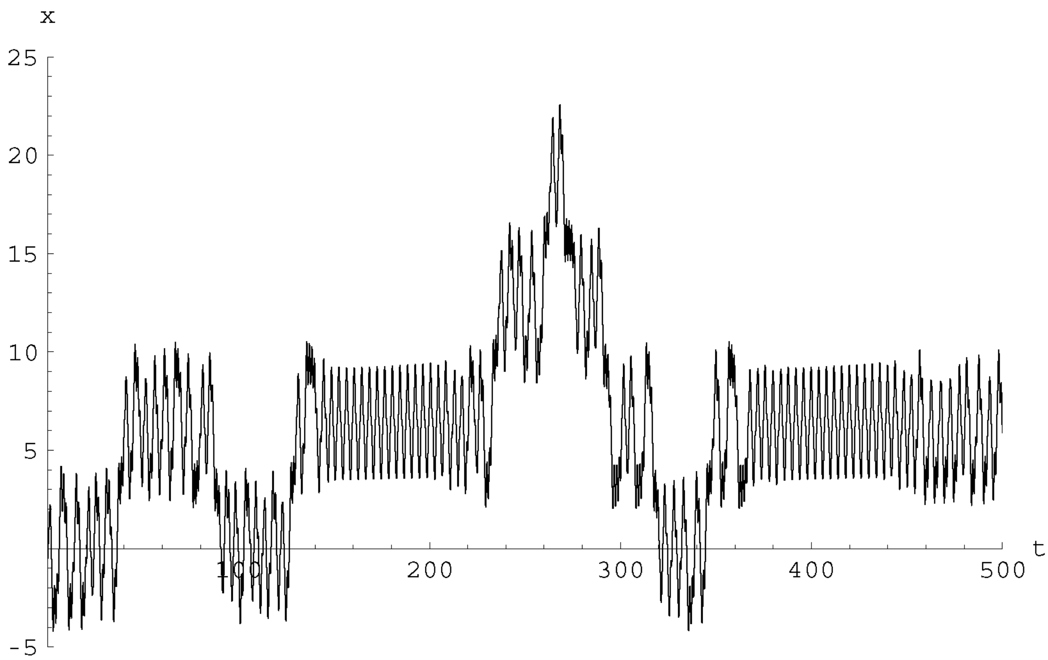
If we further increase K, but keep it smaller than 5.1, the amplitude of the orbit increases. The graph below is for $K = 4$ and the initial condition $x(t) = -1$ (notice the x axis' scale change):



For $K = 4$ we can also find another orbit that is numerically stable and attracting. This is an orbit with an amplitude greater than π, i.e. it enters into what for the corresponding ODE ($x'(t) = -K \sin x(t)$) would be the basins of attraction of the fixed points $\pm 2\pi$. Initial condition: $x(t) = -1.5$.



If we increase K above about 5.1, there seems to be chaos. This is a graph for $K = 5.13$ (with the initial condition $x(t) = -0.5$):



It looks that for $K = 5.13$ there are still orbits oscillating around $2k\pi$, but they are no longer attracting. The solutions jump by $\pm 2\pi$, from one of these orbits to another.

However, there are no rigorous proofs for such behaviors. In [4] there is a proof of the chaos for large K, but after changing the sinus to a function close to a piecewise linear function. Paper [5] analyzes the eigenvalues of the linearization and provides an informal argument for the bifurcation in $K = \frac{\pi}{2}$ and for the creation of an attracting orbit, but this is not a proof.

In this thesis we will show a method to rigorously prove an element of the dynamic – the existence of periodic orbits that are solutions of such equations. By a solution we mean a $\mathcal{C}^1$ function such that the equation is satisfied. We will use the method of self-consistent bounds that was introduced in the context of Kuramoto-Shivashinsky PDEs in [7], [8]. When applied to the boundary value problem for ODEs or DDEs, this method is similar to the Cesari method introduced in [1] but does not require one of the conditions – see Section 2.4 in [7] for a comparison.

In Chapter 3, we show the existence of periodic orbits for $K = 1.6$ and for $K = 5.13$ (the result for $K = 1.6$ has been already accepted for publication in [6]). We will rigorously establish the existence of an orbit in a very small neighborhood of an approximate solution, that was found using non-rigorous methods.

As a starting point for finding the approximate solution, for $K = 1.6$, I have taken the attracting orbit from the numerical solution. For $K = 5.13$, I have taken an interval were the numerical solution seems to be periodic. Later, I have refined these approximations – this will be described in more details in Section 3.2.

One of these orbits seems to be attracting while the other does not, but this is not a problem for the method of self-consistent bounds – it works in both cases (the existence of the second orbit is harder to prove, but this is because of a higher amplitude requiring better estimates).

In Chapter 4, we derive some conditions under which this method will work (see Theorem 6). In that chapter, we consider a general problem:

$$x'(t) = f\left(x(t-1)\right) \tag{1.2}$$

We assume that f is analytic on the whole $\mathbb{R}$ and that the derivatives grow at most exponentially:

$$f(x) = \sum_{k=0}^{\infty} f_k x^k$$

$$\exists \alpha_f : |f_k| < \frac{\alpha_f^k}{k!}$$

Of course, $f(x) = -K \sin x$ satisfies these assumptions, thus, it is a special case of the general equation.

In the remainder of the introduction, we briefly outline the method. Let us assume we have a periodic orbit. We will use the Fourier coefficients of the orbit to prove its existence. Hence, we will want to rescale the time to make the period 2π. We make the substitution: $\tilde{x}(t) = x\left(\frac{t}{\tau}\right)$, where τ is a parameter. If τ is equal to $\frac{2\pi}{T}$ (where T is the period of the solution in the original equation), we will have a 2π-periodic solution of the new equation:

$$\tilde{x}'(t) = \frac{1}{\tau}x'\left(\frac{t}{\tau}\right) = \frac{1}{\tau}f\left(x\left(\frac{t}{\tau} - 1\right)\right) =$$
$$= \frac{1}{\tau}f\left(x\left(\frac{t-\tau}{\tau}\right)\right) = \frac{1}{\tau}f(\tilde{x}(t-\tau))$$

By renaming $\tilde{x}$ to x, we obtain:

$$x'(t) := \frac{1}{\tau}f(x(t-\tau)) \tag{1.3}$$

The method proves that there exists a τ from a small interval such that there exists a 2π-periodic orbit in a small neighborhood of a specified function. Note that we do not obtain the exact value of the period (even if for both orbits for $f(x) = -K\sin x$, the τ is from a very small interval containing $\frac{\pi}{2}$, what suggests that the period is 4), but treat τ as a variable.

The first step of the proof is to write the equation as an equation on the Fourier coefficients. We will prove (see Theorem 1), that these equations are:

$$\forall n \in \mathbb{Z} : inc_n = \frac{1}{\tau}\sum_{k=0}^{\infty} f_k \left(c^{*k}\right)_n e^{-in\tau}$$

where $(c^{*k})_n$ means convoluting the sequence c with itself k times and then taking the n-th coefficient. The sequences are from a domain where the convolution is convergent. The operation of convolution and this notation is introduced in detail in Section 2.1.

Next, we will use the Galerkin projections to work in a finite dimension. The domain will be compact what will allow us to show that if every finite set of equations has a solution then there is a solution of the whole system (see Lemma 19).

The tool we will use to prove the solutions of the finite systems, is the local Brouwer degree. We will show that the degree of 0 is non-zero, what implies a solution. Below we give some basic facts about the degree.

1.2 Local Brouwer degree

The local Brouwer degree is introduced e.g. in [3, Chapter 12]. Let $X, Y \subset \mathbb{R}^n$ for some n. Let $f : X \to Y$ be continuous. Then, for U open, $\overline{U} \subset X$, $y_0 \in Y$ we will denote by $\deg(f, U, y_0)$ the local Brouwer degree of y_0 on the set U for the function f. The degree is well-defined if $x \in \partial U \Rightarrow f(x) \neq y_0$. A non-zero degree implies that $\exists x \in U : f(x) = y_0$.

There are many properties that help to compute the local Brouwer degree. The one that we will be using is the homotopy invariance and the degree of linear maps:

Lemma 1 ([3, Property 12.3 and Theorem 12.7]). *Let $X, Y \subset \mathbb{R}^n$, U open, $\overline{U} \subset X$ and $y_0 \in Y$. Let $H : [0; 1] \times \overline{U} \to Y$ be a homotopy such that for $h \in [0; 1], x \in \partial U : H(h; x) \neq y_0$. Let us denote $f(x) = H(0, x)$, $g(x) = H(1, x)$. Then* $\deg(f, U, y_0) = \deg(g, U, y_0)$

Lemma 2 ([3, corollary from Property 12.2]). *Let U be a neighborhood of zero in $\mathbb{R}^n$ and let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear invertible function. Then* $\deg(L, U, 0) = \operatorname{sgn} \det L$

Our f will be the finite system of equations, while the set U will be the neighborhood of the approximated solution where we expect the true solution to lie. To compute the degree we will use the homotopy invariance (Lemma 1) – we will deform the equations into a linear function. Thus, the degree of the system of equations will be equal to the degree of the linear function. Moreover, as this homotopy will meet the assumptions of Lemma 1, it will have no zeros on the boundary for $h = 1$, which implies that the linear function is invertible (non-invertible linear function have zeros on the boundaries of all bounded neighborhoods of zero). Thus, from Lemma 2, the degree is non-zero.

For a given U and y_0, we will call a homotopy that meets the assumptions of Lemma 1 admissible. It will be clear from the context what U and y_0 we are referring to.

Checking that the homotopy has on zeros no the boundary is the hardest part of the proof and is computer-assisted – we will need to check a lot of inequalities, and the computations will be done by a computer program.

1.3 The programs

This programs are written in C++. The programs for the proofs in Sections 3.3 and 3.4 can be downloaded from:

- http://www.im.uj.edu.pl/MikolajZalewski/dl/delay-sin.tgz
- http://www.im.uj.edu.pl/MikolajZalewski/dl/delay-sin2.tgz

The rounding-mode changing code required by the interval arithmetic is system-dependent and has been checked to work on PCs (both 32-bit and 64-bit) on both Windows (compiled with cygwin) and Linux (compiled with gcc). It should also work on SPARC and Mac OS X, although that has not been tested. Using other CPUs or compilers might require modifications to the rounding code.

The constants used in the theorems in Section 3.3 – K, β_1, β_2, $\Delta\tau$, $\hat{\tau}$ and $\hat{c}_n$ – are represented in the program as small intervals containing their values. Thus, the theorems are true for values as written in the thesis, even if they are not representable as IEEE floating point numbers.

In Section 3.4 of the thesis, in order to save some space, there are only ten digits after the decimal point of the $\hat{c}$ constants provided. The program stores 15 digits of these constants and uses IEEE doubles close to these numbers. The β_1, β_2 and $\Delta\tau$ are also approximated by IEEE doubles when the program is run. Thus, the constants for which the theorem in Section 3.4 is proved may differ up to 10^{-10} from the ones provided in the thesis.

Chapter 2

Basic information

2.1 Fourier coefficients

In this thesis, we will use the following notation: if $\{x_n\}_{n=-\infty}^{\infty}$ and $\{y_n\}_{n=-\infty}^{\infty}$ are sequences with complex values then by $x*y$ we will denote the convolution of the sequences:

$$x * y := \left\{ \sum_{k=-\infty}^{\infty} x_k y_{n-k} \right\}_{n=-\infty}^{\infty}$$

If the sums in the convolutions converge, the operation of convolution is associative.

We will also use the notation $(...)_n$ for the n-th coefficient of the sequence in brackets – e.g. $(x * y)_n = \sum_{k=-\infty}^{\infty} x_k y_{n-k}$. Also:

$$x^{*k} := \underbrace{x * \cdots * x}_{k \text{ times}} \text{ (for } k \geq 1)$$

$$(x^{*0})_n := \begin{cases} 1 \text{ if } n = 0 \\ 0 \text{ if } n \neq 0 \end{cases}$$

Of course, the sum in the definition of the convolution may be not convergent. We will limit our attention to a domain where the sum will be always absolutely convergent. The following lemma holds:

Lemma 3. *If x and y is such that $\exists \alpha \geq 2, \beta_1, \beta_2 > 0 : \forall n : |x_n| \leq \frac{\beta_1}{(|n|+1)^\alpha}, |y_n| \leq \frac{\beta_2}{(|n|+1)^\alpha}$, then $(x * y)_n$ is absolutely convergent for each n and $|(x * y)_n| \leq C \frac{\beta_1 \beta_2}{(|n|+1)^\alpha}$ where $C = \frac{2(2\alpha+1)}{\alpha-1}$*

Proof: Let us assume $n \geq 0$ and let us estimate $|(x * y)_n|$:

$$\left|\sum_{k=-\infty}^{\infty} x_k y_{n-k}\right| \le \left|\sum_{k=0}^{\infty} x_{-k} y_{n+k}\right| + \left|\sum_{k=1}^{n-1} x_k y_{n-k}\right| + \left|\sum_{k=0}^{\infty} x_{n+k} y_{-k}\right|$$

$$\begin{aligned} \left|\sum_{k=1}^{n-1} x_k y_{n-k}\right| &\le \beta_1\beta_2 \sum_{k=1}^{n-1} \tfrac{1}{(k+1)^\alpha} \tfrac{1}{(n-k+1)^\alpha} \\ &\le 2\beta_1\beta_2 \sum_{1\le k\le \frac{n}{2}} \tfrac{1}{(k+1)^\alpha} \tfrac{1}{(n-k+1)^\alpha} \\ &\le 2\beta_1\beta_2 \sum_{1\le k\le \frac{n}{2}} \tfrac{1}{(k+1)^\alpha} \tfrac{1}{((n/2)+1)^\alpha} \\ &\le 2\tfrac{\beta_1\beta_2 2^\alpha}{(n+2)^\alpha} \sum_{1\le k\le \frac{n}{2}} \tfrac{1}{(k+1)^\alpha} \\ &\le 2\tfrac{\beta_1\beta_2 2^\alpha}{(n+2)^\alpha} \sum_{k\ge 1} \tfrac{1}{(k+1)^\alpha} \end{aligned}$$

$$\begin{aligned} \left|\sum_{k=0}^{\infty} x_{-k} y_{n+k}\right| &\le \beta_1\beta_2 \sum_{k=0}^{\infty} \tfrac{1}{(k+1)^\alpha} \tfrac{1}{(n+k)^\alpha+1} \\ &\le \tfrac{\beta_1\beta_2}{(n+1)^\alpha} \sum_{k\ge 0} \tfrac{1}{(k+1)^\alpha} \end{aligned}$$

Analogously:

$$\left|\sum_{k=0}^{\infty} x_{n+k} y_{-k}\right| \le \frac{\beta_1\beta_2}{(n+1)^\alpha} \sum_{k\ge 0} \frac{1}{(k+1)^\alpha}$$

The sums $\sum_{k=k_0}^{\infty} \frac{1}{k^\alpha+1}$ can be estimated by integrals:

$$\sum_{k=0}^{\infty} \frac{1}{(k+1)^\alpha} \le 1 + \int_0^\infty \frac{1}{(x+1)^\alpha} dx = \frac{\alpha}{\alpha-1}$$

$$\sum_{k=1}^{\infty} \frac{1}{(k+1)^\alpha} \le \frac{1}{2^\alpha} + \int_1^\infty \frac{1}{(x+1)^\alpha} dx = \frac{\alpha+1}{2^\alpha(\alpha-1)}$$

From that we obtain:

$$\left|\sum_{k=-\infty}^{\infty} x_k y_{n-k}\right| \le \frac{\beta_1\beta_2}{(n+1)^\alpha}\left[2^{\alpha+1}\frac{\alpha+1}{2^\alpha(\alpha-1)} + 2\frac{\alpha}{\alpha-1}\right] \le \frac{\beta_1\beta_2}{(n+1)^\alpha}\frac{2(2\alpha+1)}{\alpha-1}$$

The result for $n < 0$ can be obtained by analogous estimations or by taking sequences $\tilde{x}$, $\tilde{y}$: $\tilde{x}_n := x_{-n}$, $\tilde{y}_n := y_{-n}$ and applying for them the result for $n > 0$.

□

With a few exceptions, we will use this lemma for $\alpha = 2$. For $\alpha = 2$ we have $C = 10$.

Observation 1. *If x is such that $\exists \alpha \ge 2, \beta > 0 : \forall n : |x_n| \le \frac{\beta}{(|n|+1)^\alpha}$ then $\left|(x^{*k})_n\right| \le C^{k-1}\frac{\beta^k}{(|n|+1)^\alpha}$ where $C = \frac{2(2\alpha+1)}{\alpha-1}$, $k \ge 1$*

Proof: By induction. For $k = 1$ it is true. For $k+1$ we have $\left|(x^{*(k+1)})_n\right| = \left|(x * x^{*k})_n\right| \leq C\frac{\beta}{(|n|+1)^\alpha} \cdot C^{k-1}\frac{\beta^k}{(|n|+1)^\alpha} = C^k\frac{\beta^{k+1}}{(|n|+1)^\alpha}$ □

Observation 2. *If x and y are such that $x_{-n} = \overline{x_n}$, $y_{-n} = \overline{y_n}$ then $(x*y)_{-n} = \overline{(x * y)_n}$*

Proof: $(x*y)_{-n} = \sum_{k=-\infty}^{\infty} x_k y_{-n-k} = \overline{\sum_{k=-\infty}^{\infty} x_{-k} y_{n-(-k)}} = \overline{\sum_{k_2=-\infty}^{\infty} x_{k_2} y_{n-k_2}} = \overline{(x * y)_n}$ □

As will be shown below, the condition $x_{-n} = \overline{x_n}$ implies that the function is real. Thus, a natural choice of domain is:

$$X_\beta := \left\{ x : \mathbb{Z} \to \mathbb{C} \middle| \forall n \in \mathbb{Z} : |x_n| \leq \frac{\beta}{(|n|+1)^2} \text{ and } x_n = \overline{x_{-n}} \right\}$$

For such a domain, we have $x \in X_{\beta_1}$ and $y \in X_{\beta_2}$ implies $x * y \in X_{C\beta_1\beta_2}$ (where C is from Lemma 3). It will be also useful to define a set of sequences from any X_β:

$$X := \bigcup_{\beta>0} X_\beta$$

We will limit ourselves to the sequences in X. For them, we have:

Lemma 4. *Let $c \in X$. Then $\sum_{n=-\infty}^{\infty} c_n e^{int}$ converges to a real-valued continuous function*

Proof: The functions $\sum_{n=-N}^{N} c_n e^{int}$ are continuous and real-valued as $c_n = \overline{c_{-n}} \Rightarrow c_n e^{int} + c_{-n}e^{-int} \in \mathbb{R}$. They converge uniformly because $\sum_{n=-\infty}^{\infty} |c_n e^{int}| \leq \sum_{n=-\infty}^{\infty} \frac{\beta}{(|n|+1)^2} < \infty$. Thus, the limit is also real-valued and continuous. □

Let us note that $c \in X$ does not guarantee that the function is $\mathcal{C}^1$. The convolution will appear in our formulas because of the following fact:

Lemma 5. *If $c \in X$ are the Fourier coefficients of $x(t)$, $d \in X$ are the coefficients of $y(t)$ then the Fourier coefficients of $x(t) \cdot y(t)$ are $c * d$. As a consequence, the coefficients of $x^n(t)$ are c^{*n}.*

Proof: From the Cauchy summation theorem:

$$\begin{aligned} &\left(\sum_{n=-\infty}^{\infty} c_n e^{int}\right) \cdot \left(\sum_{m=-\infty}^{\infty} d_m e^{imt}\right) \\ = \ &\sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \left(c_k e^{ikt}\right) \cdot \left(d_{n-k} e^{i(n-k)t}\right) = \\ = \ &\sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} c_k \cdot d_{n-k}\right) e^{int} \end{aligned}$$

Thus the coefficients of $x(t) \cdot y(t)$ are $c * d$. □

Now, we will show that proving the existence of a periodic orbit of a DDE of the form (1.3) can be done by solving an infinite system of equations on Fourier coefficients:

Theorem 1. *Let τ be fixed and $x(t) : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function with the Fourier coefficients $c \in X$. Then:*

(i) for each $n \in \mathbb{Z}$ the sum on the right-hand side of the following equation converges:

$$inc_n = \frac{1}{\tau} \sum_{k=0}^{\infty} f_k \left(c^{*k}\right)_n e^{-in\tau} \tag{2.1}$$

(ii) $x(t)$ is a 2π-periodic solution of (1.3) $\Leftrightarrow$ c satisfies the equations (2.1).

To prove the theorem, we will need some lemmas. Let us first introduce an auxiliary notation:

$$\tilde{f}(x) = \sum_{k=0}^{\infty} |f_k| x^k$$

Since the derivatives of f grows at most exponentially, $\tilde{f}(x)$ is well-defined on the whole $\mathbb{R}$.

Lemma 6. *If x is such that $|x_n| \leq \frac{\beta}{(|n|+1)^\alpha}$ for some α and β, then:*

$$\sum_{k=0}^{\infty} \left| f_k \left(c^{*k}\right)_n e^{-in\tau} \right| \leq \frac{1}{C_\alpha} \tilde{f}(C_\alpha \beta) \frac{1}{(|n|+1)^\alpha} \tag{2.2}$$

where $C_\alpha := \frac{2(2\alpha+1)}{\alpha-1}$

Proof:

$$\begin{aligned} \textstyle\sum_{k=0}^{\infty} \left| f_k \left(c^{*k}\right)_n e^{-in\tau} \right| &\leq \textstyle\sum_{k=0}^{\infty} |f_k| \left| \left(c^{*k}\right)_n \right| \leq \\ &\leq \textstyle\sum_{k=0}^{\infty} |f_k| \frac{C_\alpha^{k-1} \beta^k}{(|n|+1)^\alpha} = \\ &= \tfrac{1}{C_\alpha} \tilde{f}(C_\alpha \beta) \frac{1}{(|n|+1)^\alpha} \end{aligned}$$

□

As already mentioned, $c \in X$ does not imply that $x(t)$ is $\mathcal{C}^1$. However, if c is the solution of equation (2.1), we have the following lemma that will allow us to show that $x(t)$ is $\mathcal{C}^1$ (using this method, one can show that $x(t)$ is $\mathcal{C}^\infty$, but we don't need it):

Lemma 7. *If c satisfies equation (2.1) and $\beta > 0, \alpha \geq 2$ are such that $\forall n : |c_n| \leq \frac{\beta}{(|n|+1)^\alpha}$ then $\exists \beta' : \forall n : |c_n| \leq \frac{\beta'}{(|n|+1)^{\alpha+1}}$*

Proof: From Lemma 6, we have that there exists a C' (independent of n) such that $|\text{LHS}| \leq \frac{C'}{(|n|+1)^\alpha}$. On the other hand, $|\text{RHS}| = |inc_n| = |n|\,|c_n|$. The two sides of the equation are equal, hence we obtain:

$$\frac{C'}{(|n|+1)^\alpha} \geq |n|\,|c_n|$$

Thus $\beta' = 2C'$ satisfies the assertion for $n \neq 0$. If it is not satisfied for $n = 0$, we can increase β'. □

Increasing α is important, as we have:

Lemma 8. *Let c be a sequence of complex values satisfying $|c_n| \leq \frac{\beta}{(|n|+1)^3}$ for some β and $c_n = \overline{c_{-n}}$. Then, the sequence c is a sequence of Fourier coefficients of a real-valued $\mathcal{C}^1$ function $x(t)$.*

Proof: The sequences $\sum_{k=-n}^{n} c_k e^{ikt}$ and $\sum_{k=-n}^{n} ikc_k e^{ikt}$ are real-valued, the second is the derivative of the first one, and are uniformly convergent as $n \to \infty$. Hence, both converge to continuous functions and the second is the derivative of the first one. Thus, $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt}$ is $\mathcal{C}^1$. □

Proof of the Theorem 1:

Ad (i): The convolutions are convergent because $x \in X$. From Lemma 6, we have that the sums are absolutely convergent.

Ad (ii): Implication $\Rightarrow$: It is enough to show that $\{inc_n\}_{n=-\infty}^{\infty}$ are the Fourier coefficients of $x'(t)$, while $\frac{1}{\tau}\sum_{k=0}^{\infty} f_k \left(c^{*k}\right)_n e^{-in\tau}$ are the Fourier coefficients of $\frac{1}{\tau} f(x(t-\tau))$.

The first part can be obtained by integrating $\int_0^{2\pi} x'(t)e^{-int}dt$ by parts:

$$\int_0^{2\pi} x'(t)e^{-int}dt = x(t)e^{-int}\Big|_0^{2\pi} + in\int_0^{2\pi} x(t)e^{-int}dt = inc_n$$

As for the second part, we have that $\frac{1}{\tau}\sum_{k=0}^{N} f_k \left(x(t-\tau)\right)^k \to \frac{1}{\tau} f(x(t-\tau))$ as $N \to \infty$. The Fourier coefficients of $x(t-\tau)$ are equal to $d := \{c_n e^{-in\tau}\}_{n=-\infty}^{\infty}$. Thus, the coefficients of $x(t-\tau)^{2k+1}$ are $d^{*(2k+1)}$ which is equal to $c^{*(2k+1)}e^{-in\tau}$.

Hence, we have that the Fourier coefficients of $\frac{1}{\tau}\sum_{k=0}^{N} f_k \left(x(t-\tau)\right)^k$ are equal to $\left\{\frac{1}{\tau}\sum_{k=0}^{N} f_k \left(c^{*k}\right)_n e^{-in\tau}\right\}_{n=-\infty}^{\infty}$. It has been shown that this sequence in convergent as $N \to \infty$, thus the n-th coefficient of $\frac{1}{\tau} f(x(t-\tau))$ is $\frac{1}{\tau}\sum_{k=0}^{\infty} f_k \left(c^{*k}\right)_n e^{-in\tau}$. This ends the proof of this case.

Implication $\Leftarrow$: From Lemmas 7 and 8 we obtain that $x(t)$ is a $\mathcal{C}^1$ function. We know that $\frac{1}{\tau}\sum_{k=0}^{\infty} f_k\left(c^{*k}\right)_n e^{-in\tau}$ are the Fourier coefficients of $\frac{1}{\tau}f(x(t-\tau))$. They are equal to inc_n – the Fourier coefficients of $x'(t)$. So both $\frac{1}{\tau}f(x(t-\tau))$ and $x'(t)$ are continuous and 2π-periodic functions with equal Fourier coefficients. Hence, the functions themselves are equal and equation (1.3) is satisfied. □

Thus, every solutions whose Fourier coefficients are in X corresponds to a solution of the equations (2.1) on the Fourier coefficients. It is easy to show that these are all the solutions:

Lemma 9. *If $x(t)$ is a 2π-periodic solution of (1.3) then its Fourier coefficients are well-defined and are in X.*

Proof: By definition, a solution is $\mathcal{C}^1$. Thus, the LHS of (1.3) is $\mathcal{C}^1$, the RHS is also $\mathcal{C}^1$, and the solution must be $\mathcal{C}^2$. Doing this reasoning again, we obtain that the solution is $\mathcal{C}^3$. This gives that $x(t)$ and $x''(t)$ are $\mathcal{C}^1$, thus theirs Fourier coefficients are well-defined and absolutely convergent. If we denote the coefficients of $x(t)$ by c, the coefficients of $x''(t)$ are $\{-n^2c_n\}_{n=-\infty}^{\infty}$.

Let us assume $c \notin X$. Then $\forall n \in \mathbb{N} : \exists k_n : |c_{k_n}| > \frac{n}{(k_n+1)^2} \geq \frac{1}{(k_n+1)^2}$. As any element in k_n can be repeated only a finite number of times, after removing the repetitions we still have an infinite sequence. Thus, let us assume that k_n has no repetitions. But then, we can show that the Fourier series for $x''(t)$ is not absolutely convergent:

$$\left|\sum_{k=0}^{\infty} -n^2 c_n e^{int}\right| \geq \sum_{n=0}^{\infty} k_n^2 |c_{k_n}| \geq \sum_{n=0}^{\infty} k_n^2 \frac{1}{(k_n+1)^2} = \infty$$

This contradiction show that $c \in X$.

□

Let us define a function $F : \mathbb{R} \times X \to \mathbb{C}^{\mathbb{Z}}$ (with the product topology on $\mathbb{C}^{\mathbb{Z}}$) corresponding to equation (2.1):

$$F(\tau, c) := \left\{ in\tau e^{in\tau} c_n - \sum_{k=0}^{\infty} f_k\left(c^{*k}\right)_n \right\}_{n=-\infty}^{\infty}$$

Let us note that for $f(x) = -K\sin x$ this function has the form:

$$F(\tau, c) := \left\{ in\tau e^{in\tau} c_n + K\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\left(c^{*(2k+1)}\right)_n \right\}_{n=-\infty}^{\infty}$$

Of course having $F(\tau, c) = 0$ is equivalent to the fact that τ, c satisfies equation (2.1). In Chapter 3 we will have $f(x) = -K\sin x$ and, for two values of K, show that F has a nontrivial zero. Let us note that $F(\tau, c)_n = \overline{F(\tau, c)_{-n}}$.

As we will use a topological tool, we will need F to be continuous. First, let us note that the convolution is not continuous on the whole X (as written above, we use the product topology on X), but we have:

Lemma 10. *The operation of convolution is continuous on each X_β*

Proof: Let us fix some $x^0, y^0 \in X_\beta$ and some δ. Let N_δ be large enough that for $|n| > N_\delta : x, y, x^0, y^0 \in X_\beta \Rightarrow |x_n - (x^0)_n|, |y_n - (y^0)_n| < \delta$. Let us take a neighborhood U of (x^0, y^0) such that for each $(x, y) \in U$ we have $\forall |n| \le N_\delta : |x_n - (x^0)_n|, |y_n - (y^0)_n| < \delta$. Then, we will have:

$$\left|\left(x^0 * y^0 - x * y\right)_n\right| \le \left|\left(x^0 * (y^0 - y)\right)_n\right| + \left|\left((x^0 - x) * y\right)_n\right|$$

$$\left|\left(x^0 * (y^0 - y)\right)_n\right| \le \sum_{j=-\infty}^{\infty} \left|\left(x^0\right)_j\right| \cdot \left|\left(y^0\right)_{n-j} - y_{n-j}\right| \le \beta\delta \sum_{j=-\infty}^{\infty} \frac{1}{(|j|+1)^2}$$

This tends to zero as $\delta \to 0$. After an analogous estimation of $\left|\left((x^0 - x) * y\right)_n\right|$, we have that the convolution is continuous. □

Lemma 11. *The function F is continuous on each X_β.*

Proof: The convolutions are continuous on each X_β, and the result of a convolution lays in some $X_{\beta'}$ for some β'. Thus, $K \sum_{k=0}^{N} f_k \left(c^{*k}\right)_n$ is continuous for each N. From estimates as in Lemma 6, we have that for each coefficient this series converges uniformly as $N \to \infty$, hence the limit $K \sum_{k=0}^{\infty} f_k \left(c^{*k}\right)_n$ is continuous.

The term $in\tau e^{in\tau} c_n$ is also continuous, so F is continuous.

□

2.2 Estimates

To prove the results, we will need some estimates in some neighborhoods of an approximate solution $\hat{c}$. In this section, we will only assume about $\hat{c}$ that almost all coefficients are equal to zero. By Y_l we will denote the space of the possible values of $\hat{c}$ – the set of sequences such that at most the elements $-l, \ldots, l$ are non-zero:

$$Y_l := \{c \in X : \forall n : |n| > l \Rightarrow c_n = 0\}$$

We will use two kinds of sets: $\hat{c} + X_{\beta_2}$ that will be used to obtain finer estimates (as we can take β_2 small) and X_{β_1} (where β_1 will be large enough to contain the whole set $\hat{c} + X_{\beta_2}$, thus cannot be arbitrarily small) for some more rough but simpler ones.

First, let us note two simple properties:

Observation 3. *If $c \in Y_l$ then $c^{*k} \in Y_{kl}$*

Lemma 12. *If $x, y \in X$ then*

$$(x+y)^{*n} = \sum_{k=0}^{n} \binom{n}{k} x^{*k} * y^{*(n-k)}$$

Proof: Let the sequence x correspond to a function $f_x(t)$, and let y correspond to a function $f_y(t)$. Then, the RHS corresponds to the function $(f_x(t) + f_y(t))^n$, while the LHS corresponds to $\sum_{k=0}^{n} \binom{n}{k} f_x(t)^k f_y(t)^{n-k}$. These functions are equal, so their Fourier coefficients are equal as well. □

To estimate $\frac{K}{p!}\left(x^{*p}\right)_n$ – an element of the sum in the equation (2.1) – for $x \in X_{\beta_1}$, it is enough to use Lemma 3. However, for the sets of the form $\hat{c} + X_{\beta_2}$, we will use a more sophisticated estimate:

Lemma 13. *Let $\hat{c} \in Y_l$, $p > 1$, $\beta > 0$. Then for any $x \in X_\beta$:*

$$|f_p \cdot (\hat{c}+x)^{*p}_n| \leq |f_p| \left(|\hat{c}^{*p}_n| + \sum_{k=0}^{p-1} \binom{p}{k} \sum_{j=-lk}^{lk} \left|\hat{c}^{*k}_j\right| C^{p-k-1} \frac{\beta^{p-k}}{(|n-j|+1)^2} \right)$$

Where $C = 10$

Proof: From Lemmas 3 and 12 we have:

$$\begin{aligned} |f_p \cdot (x+\hat{c})^{*p}_n| &\leq |f_p| \textstyle\sum_{k=0}^{p} \binom{p}{k} \left|\left(\hat{c}^{*k} * x^{*(p-k)}\right)_n\right| \\ &\leq |f_p| \textstyle\sum_{k=0}^{p} \binom{p}{k} \sum_{j=-\infty}^{\infty} \left|\hat{c}^{*k}_j\right| \left|x^{*(p-k)}_{n-j}\right| \\ &\leq |f_p| \left(|\hat{c}^{*p}_n| + \textstyle\sum_{k=0}^{p-1} \binom{p}{k} \sum_{j=-\infty}^{\infty} \left|\hat{c}^{*k}_j\right| C^{p-k-1} \frac{\beta^{p-k}}{(|n-j|+1)^2} \right) \end{aligned}$$

We have $\hat{c} \in Y_l$, so $\hat{c}^{*k}_j = 0$ for $j > kl$ what ends the proof. □

When using this lemma, it can take some time to compute the estimate. That is why, for the second orbit, we will also use a simpler estimate that is faster to compute:

Lemma 14. *Let* $\hat{c} \in Y_l$, $p > 1$, $\beta > 0$. *Then, for any* $x \in X_\beta$*:*

$$|f_p \cdot (\hat{c} + x)^{*p}_n| \leq |f_p| \sum_{k=0}^{p} \left(\begin{array}{c} p \\ k \end{array} \right) C^{p-k} \frac{\hat{\beta}_k \beta^{p-k}}{(|n|+1)^2}$$

Where $C = 10$ *and* $\hat{\beta}_k$ *is such that* $\hat{c}^{*k} \in X_{\hat{\beta}_k}$

Proof: The proof is similar to Lemma 13, but we use Lemma 3 to estimate the convolution of $\hat{c}^{*k}$ and $x^{*(p-k)}$ instead of estimating it as a sum:

$$\begin{aligned} |f_p(\hat{c} + x)^{*p}_n| &= |f_p| \cdot \left| \sum_{k=0}^{p} \left(\begin{array}{c} p \\ k \end{array} \right) \hat{c}^{*k} * x^{*(p-k)} \right| \\ &\leq |f_p| \sum_{k=0}^{p} \left(\begin{array}{c} p \\ k \end{array} \right) \left| \hat{c}^{*k} * x^{*(p-k)} \right| \\ &\leq |f_p| \sum_{k=0}^{p} \left(\begin{array}{c} p \\ k \end{array} \right) C^{p-k} \frac{\hat{\beta}_k \beta^{p-k}}{(|n|+1)^2} \end{aligned}$$

□

Of course, we can use the previous lemmas only for a finite number of terms. To estimate the tail, we will use a neighborhood of the second type, apply Lemma 3 and sum the geometric sequence.

In the next two lemmas, we do this for $f(x) = -K \sin x$. In Chapter 4, we will apply this idea to any $f(x)$, but the estimates will be weaker.

Lemma 15. *Let* N *be odd,* $\beta \in [0; \frac{N}{10}]$. *If* $x \in X_\beta$ *then*

$$\left| \left(K \sum_{k=\frac{N-1}{2}}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{*(2k+1)} \right)_n \right| \leq \frac{K \cdot (C\beta)^{N-1}}{N!(1 - \left(\frac{C\beta}{N}\right)^2)} \cdot \frac{\beta}{(|n|+1)^2}$$

Where $C = 10$.

Proof: Using Lemma 3:

$$\begin{aligned} \left| \left(K \sum_{k=\frac{N-1}{2}}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{*(2k+1)} \right)_n \right| &\leq \sum_{k=(N-1)/2}^{\infty} \frac{K}{N! N^{2k-(N-1)}} \left| \left(x^{*(2k+1)} \right)_n \right| \\ &\leq \frac{K}{N!} \sum_{k=(N-1)/2}^{\infty} \frac{(C\beta)^{2k}}{N^{2k-(N-1)}} \cdot \frac{\beta}{(|n|+1)^2} \\ &= \frac{K(C\beta)^{N-1}}{N!} \left(\sum_{k=0}^{\infty} \left(\frac{C^2\beta^2}{N^2} \right)^k \right) \cdot \frac{\beta}{(|n|+1)^2} \\ &= \frac{K(C\beta)^{N-1}}{N!\left(1 - \left(\frac{C\beta}{N}\right)^2\right)} \cdot \frac{\beta}{(|n|+1)^2} \end{aligned}$$

The geometric sequence is convergent because $\beta < \frac{N}{10} \Rightarrow \left(\frac{C\beta}{N}\right)^2 < 1$. □

To prove the second orbit, this lemma will also be refined to use the fact that for the first convolutions we can compute $\hat{\beta}_k$ like in Lemma 14, what is better than the worst-case estimates from Lemma 3.

Lemma 16. *Let N be odd, $\beta \in [0; \frac{N}{10}]$. If $x + \hat{c} \in \hat{c} + X_{\beta_2} \subset X_{\beta_1}$ then*

$$\left| \left(K \sum_{k=\frac{N-1}{2}}^{\infty} \frac{(-1)^k}{(2k+1)!} (x+\hat{c})^{*(2k+1)} \right)_n \right| \leq \frac{K \cdot C \sum_{k=0}^{N} \begin{pmatrix} p \\ k \end{pmatrix} \hat{\beta}_k \left(C\beta_2\right)^{N-k}}{N! \left(1 - \left(\frac{C\beta_1}{N}\right)^2\right)} \cdot \frac{1}{\left(|n|+1\right)^2}$$

Where $C = 10$ and $\hat{\beta}_k$ is such that $\hat{c}^{*k} \in X_{\hat{\beta}_k}$

Proof: The same as in Lemma 15 but we estimate $\left((x+\hat{c})^{*N}\right)_n$ by $\frac{1}{(|n|+1)^2} \sum_{k=0}^{p} \begin{pmatrix} p \\ k \end{pmatrix} \hat{\beta}_k \left(C\beta_2\right)^{p-k}$ (as in Lemma 14) instead of $\frac{1}{(|n|+1)^2} C^{N-1} \beta^{N-2}$ (as the latter gives a worse approximation) □

From Lemma 13, we can obtain an estimate that after multiplication by $(|n|+1)^2$ is independent of n. That will be used to estimate the term in all inequalities for high n's with one formula:

Lemma 17. *Let $\beta > 0$, $\hat{c} \in Y_l$, $p > 1$, $N > pl$. Then for each $x \in X_\beta$, $n > N$ we have:*

$$\left| \frac{K}{p!} (x+\hat{c})_n^{(*p)} \right| \leq \frac{K}{p!} \left(\sum_{k=0}^{p-1} \begin{pmatrix} p \\ k \end{pmatrix} \sum_{j=-lk}^{lk} \left| \hat{c}_j^{*k} \right| (C\beta)^{p-k-1} \frac{(N+1)^2}{(N-|j|+1)^2} \right) \frac{\beta}{(|n|+1)^2}$$

Proof: Note that $n > pl \Rightarrow \hat{c}_n^{*p} = 0$ and $\frac{\beta}{(|n-j|+1)^2} = \frac{(n+1)^2}{(|n-j|+1)^2} \cdot \frac{\beta}{(n+1)^2} \leq \frac{(N+1)^2}{(N-|j|+1)^2} \cdot \frac{\beta}{(n+1)^2}$ as $N > j$ and apply Lemma 13 □

For the first terms, we will need to have a better estimate than in Lemma 13 so we will regroup the terms (to understand why this regrouping helps, let us compare two estimates: $|1.1 \cdot x - 1 \cdot x| \leq 1.1|x| + 1|x| = 2.1|x|$ and $|1.1 \cdot x - 1 \cdot x| = |(1.1-1)x| = 0.1|x|$).

Lemma 18. *Let $x \in X_\beta$, $\hat{c} \in Y_l$, $N \in \mathbb{N}$.*

$$\begin{aligned} &in\tau e^{in\tau}(\hat{c}_n + x_n) + \textstyle\sum_{k=0}^{N} f_k(\hat{c}+x)_n^{*k} = \\ &= in\tau e^{in\tau}\hat{c}_n + \textstyle\sum_{k=0}^{N} f_k \cdot \left(\hat{c}^{*k}\right)_n + in\tau e^{in\tau} x_n + \textstyle\sum_{p=1}^{N} \left(\gamma_p * x^{*p}\right)_n \end{aligned}$$

Where γ_p is a sequence: $\gamma_{p,j} = \sum_{k=p}^{N} \begin{pmatrix} k \\ p \end{pmatrix} f_k \left(\hat{c}^{*(k-p)}\right)_j$.

Proof:

$$\begin{aligned}
&in\tau e^{in\tau}(\hat{c}_n + x_n) + \textstyle\sum_{k=0}^{N} f_k(\hat{c}+x)_n^{*k} = \\
&= \quad in\tau e^{in\tau}(\hat{c}_n + x_n) + \textstyle\sum_{k=0}^{N} f_k \sum_{p=0}^{k} \begin{pmatrix} k \\ p \end{pmatrix} \left(\hat{c}^{*(k-p)} * x^{*p}\right)_n = \\
&= \quad in\tau e^{in\tau}(\hat{c}_n + x_n) + \textstyle\sum_{k=0}^{N} f_k \left(\hat{c}^{*k}\right)_n + \sum_{p=1}^{N} \sum_{k=p}^{N} f_k \begin{pmatrix} k \\ p \end{pmatrix} \left(\hat{c}^{*(k-p)} * x^{*p}\right)_n = \\
&= \quad in\tau e^{in\tau}\hat{c}_n + \textstyle\sum_{k=0}^{N} f_k \left(\hat{c}^{*k}\right)_n + in\tau e^{in\tau} x_n \sum_{p=1}^{N} \left(\gamma_p * x^{*p}\right)_n
\end{aligned}$$

□

Let us note that $\gamma_k \in Y_{(N-1)l}$ (from Observation 3) so it can be represented in a computer.

In Chapter 3 we will have $f(x) = -K \sin x$ but will apply this lemma to $-K\left[(\sin x) - x\right]$ (i.e. not to the series $K \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(\hat{c}+x\right)^{*(2k+1)}$ but $K \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(\hat{c}+x\right)^{*(2k+1)}$).

For this function, for $N = 7$, we have $\gamma_{p,j} = -\frac{K}{3!} \begin{pmatrix} 3 \\ p \end{pmatrix} \left(\hat{c}^{*(3-p)}\right)_j + \frac{K}{5!} \begin{pmatrix} 5 \\ p \end{pmatrix} \left(\hat{c}^{*(5-p)}\right)_j - \frac{K}{7!} \begin{pmatrix} 7 \\ p \end{pmatrix} \left(\hat{c}^{*(7-p)}\right)_j$ where we assume $\begin{pmatrix} n \\ k \end{pmatrix} = 0$ for $k > n$. For $N = 15$, we have $\gamma_{p,j} = \sum_{k=1}^{7} (-1)^k \frac{K}{(2k+1)!} \begin{pmatrix} 2k+1 \\ p \end{pmatrix} \left(\hat{c}^{*(2k+1-p)}\right)_j$.

For these gammas, we have:

$$\begin{aligned}
&in\tau e^{in\tau}(\hat{c}_n + x_n) + K \textstyle\sum_{k=0}^{N} \frac{(-1)^k}{(2k+1)!} \left(\hat{c}+x\right)^{*(2k+1)} = \\
&= in\tau e^{in\tau}\hat{c}_n + \textstyle\sum_{k=0}^{N} f_k \frac{(-1)^k}{(2k+1)!} \left(\hat{c}^{*(2k+1)}\right)_n + \left(in\tau e^{in\tau} + K\right) x_n + \sum_{p=1}^{N} \left(\gamma_p * x^{*p}\right)_n
\end{aligned}$$

Chapter 3

Proving the existence of orbits

3.1 Outline of the method

As concluded in Section 2.1, our goal is to show that the function F has a zero near a numerical approximation of the orbit. We will prove two such solutions for $f(x) = -K\sin x$. One will be for $K = 1.6$ and one for $K = 5.13$.

We will denote one of the approximated orbits by $\hat{c}$ ($\hat{c} \in Y_l$ for some l) and will outline the method. We want to work in a finite dimension. This can be done by using the Galerkin projections and the compactness of X_β. For each $l \in \mathbb{N}$ let us define the Galerkin projection P_l and immersion Q_l:

$$P_l : \mathbb{C}^{\mathbb{Z}} \ni c \to (c_0, \ldots, c_l) \in \mathbb{R} \times \mathbb{C}^l$$

$$Q_l : \mathbb{R} \times \mathbb{C}^l \ni (c_0, \ldots, c_l) \to (\ldots, 0, 0, \overline{c_{-l}}, \ldots, c_0, \ldots, c_l, 0, 0, \ldots) \in \mathbb{C}^{\mathbb{Z}}$$

Let us note that we need only the non-negative terms in the finite space as we are interested in elements of X_β, thus the negative terms can be obtained by conjugation. Also, as $c_0 = \overline{c_{-0}}$, we have $c_0 \in \mathbb{R}$.

Lemma 19. *Let $\beta > 0$, $l_0 > 0$, $\underline{\tau}, \overline{\tau} \in \mathbb{R}$ be fixed. If for each $l > l_0$ there is a $c^l \in X_\beta$ and $\tau_l \in [\underline{\tau}, \overline{\tau}]$ such that $P_l F(\tau_l, c^l) = 0$ then there exists $(c^0, \hat{\tau}) \in X_\beta \times [\underline{\tau}, \overline{\tau}]$ such that $F(\hat{\tau}, c^0) = 0$.*

Moreover, if all the c^l are in a closed set D then $c^0 \in D$.

Proof: From the compactness of $[\underline{\tau}, \overline{\tau}] \times X_\beta$, there exists a subsequence l_k such that c^{l_k} converges to a limit c^0 and τ_{l_k} converges to $\hat{\tau}$. Let us fix $n \geq 0$ and note that for $l > n$: if $P_l(F(\tau, c)) = 0$ then $F(\tau, c)_n = 0$. Thus, from the continuity of F we have $F(\hat{\tau}, c^0)_n = \lim_{k\to\infty} F(\tau_{l_k}, c^{l_k})_n = \lim_{k\to\infty} P_{l_k} F(\tau_{l_k}, c^{l_k})_n = 0$. For $n < 0$ we have $F(\hat{\tau}, c^0)_n = \lim_{k\to\infty} F(\tau_{l_k}, c^{l_k})_n = \overline{\lim_{k\to\infty} F(\tau_{l_k}, c^{l_k})_{-n}} = 0$. Thus $F(\hat{\tau}, c^0) = 0$.

The last assertion follows from the fact that c^0 is then a limit of a sequence in the closed set D. □

We will search for the c^l in Y_l. Thus, we will be solving the system of equations $F(\tau, Q_l(c^l))_n = 0$ for $n = 0, \ldots, l$.

The equation for $n = 0$ is real while the others are complex, thus we have $2l+1$ real equations. There are $2l+2$ real variables (τ and c^l_0 are real, the other c^l_i are complex), thus if there is a zero, we can expect a 1-dimensional manifold of zeros. This manifold can be easily identified – if $x(\cdot)$ is a solution then $x(\cdot + \phi)$ is also a solution. This means that if $P_l F\left(\tau, Q_l\left(c_0, c_1, \ldots, c_l\right)\right) = 0$ then $P_l F\left(\tau, Q_l\left(c_0, e^{i\phi}c_1, \ldots, e^{il\phi}c_l\right)\right) = 0$, as can be easy checked.

To use the local Brouwer degree, we need the dimension of the image and the dimension of the domain to be equal. Also, we cannot compute the degree if we have a manifold of zeros crossing the boundary of every neighborhood small enough. To avoid both these problems, we will restrict ourselves to a $2l+1$-dimensional subdomain, where the zero will be isolated. Of course, if we find a zero in the smaller domain then there exist a zero in the larger one. As we will see, for the first orbit the restriction will be done by assuming $c^l_1 - \hat{c}_1 \in \mathbb{R}$, while for the second, the domain will be the image of a $\mathbb{R}^{(2l+1)\times(2l+2)}$ matrix.

In this smaller space, we will compute the local Brouwer degree of 0 in the neighborhood of $P_l(\hat{c})$. Let us denote by $F_l(\tau, Q_l(c^l))$ the $P_l F(\tau, Q_l(c^l))$ with the restricted domain (exact definition will be provided in the Sections 3.3 and 3.4 as they will be different for both orbits – as mentioned, the restrictions are different). We will show that for some β_2, $\deg(F_l, P_l(\hat{c} + X_{\beta_2}), 0) \neq 0$ for l large enough.

For both orbits, we will use a family of homotopies that will transform each F_l (for l large enough) into a linear function. If these homotopies are admissible (i.e. satisfies the assumptions of Lemma 1 for $U = P_l(\hat{c}+X_{\beta_2})$ and $y_0 = 0$) then the linear functions must be invertible (if not, they would have zeros on the boundary for $h = 1$). Thus, the determinant and the Brouwer degree is non-zero and we have a solution for each F_l (for l large enough). Thus, from Lemma 19 we obtain a solution of the whole system.

Each of these homotopies will be defined as a composition of two homotopies. Checking that the second homotopy is admissible will be simple. However, proving that the first homotopy is admissible is the hardest part of the proof and is slightly different for the two orbits – because of the larger K and larger amplitude of the second orbit, we will have to use better estimates. Both proofs consist of checking a large number of inequalities to show that a zero cannot be obtained on the boundary. That part of the proof is computer-assisted.

3.2 Finding the approximate solutions

As outlined, we will prove the existence of a true solution in a neighborhood of an approximated solution.

To find such a solution, I have used the same program as to generate the data for the graphs in the introduction (Dynamics Solver). The program produced a file with the values of the numerical solution $x_k := x(k\Delta t)$ for $\Delta t := 0.05$. From these values I have computed Riemann's sum for an interval of length 4:

$$\hat{a}_n := \sum_{k=T_0}^{T_0+\frac{4}{\Delta t}} x_k sin(nk\Delta t)\Delta t$$

$$\hat{b}_n := \sum_{k=T_0}^{T_0+\frac{4}{\Delta t}} x_k cos(nk\Delta t)\Delta t$$

The T_0 for the second orbit was chosen to point to an interval where the solution seemed close to a periodic orbit. For the first solution, the T_0 was not very important – I have only chosen it large, so that the solution is already attracted to the periodic orbit.

These coefficients were further refined by using them as a starting point for solving a finite subset of equations ($P_l F(\tau, Q_l(c)) = 0$) using the secant method in Mathematica. To avoid having more variables than equations, an equation $b_1 - \hat{b}_1 = 0$ was added. The obtained solution was then used as the starting point for solving the equations with a larger l, until I reached $l = 100$.

Some first N elements of this solution are the $\hat{c}$ approximate solution. Other elements were used as a first guess to find the β_2 – the size of the neighborhood where the true solution will lie. I computed a β such that $\forall N < n \leq 100 : |c_n| \leq \frac{\beta}{(n+1)^2}$. This is a minimal β for which we can prove the existence of that solution in $\hat{c} + X_\beta$. Due to other estimates, the β had to be increased – the β_2 is 1.3β for the first orbit and 1.4β for the second.

3.3 The orbit for $K = 1.6$

For $K = 1.6$, our approximated orbit is:

n	$\hat{c}_n$
0	0
1	$-0.1521000000 - 0.1163508047i$
2	0
3	$0.0001123121 - 0.0002746107i$
4	0
5	$-0.0000008173 - 0.0000001014i$

For $n > 5$ the $\hat{c}_n$ is zero, for $n < 0$ we have $\hat{c}_n = \overline{\hat{c}_{-n}}$. Thus $\hat{c} \in Y_5$. The variable τ in the approximation is: $\hat{\tau} = 1.570796$.

Let us define the sets and boundaries on which we will work:

$$X_1 := X_{\beta_1}$$

$$X_2 := \hat{c} + X_{\beta_2}$$

$$X_3 := \{y \in X_2 : y_1 - \hat{c}_1 \in \mathbb{R}\}$$

$$\underline{\tau} := \hat{\tau} - \Delta\tau$$

$$\overline{\tau} := \hat{\tau} + \Delta\tau$$

where $\Delta\tau = 0.000001$, $\beta_2 = 0.0000002438$, $\beta_1 = 0.766763$.

We will prove that a solution exists in the set $[\underline{\tau}; \overline{\tau}] \times X_3$. The set X_1 is the bigger but simpler neighborhood, mentioned at the beginning of Section 2.2 – the β_1, β_2 are such that $X_2 \subset X_1$. Obviously, we have $X_3 \subset X_2$. The condition in the definition of X_3 was added because we need to make the dimension of the domain and the image equal, as explained in Section 3.1.

We will need a homotopy for each l, but these homotopies will be Galerkin projections of a homotopy defined on the whole space $[\underline{\tau}; \overline{\tau}] \times X_3 \to \mathbb{C}^{\mathbb{Z}}$. We will define only one function acting on the whole space, prove that it behaves correctly on the border and then use its Galerkin projections to prove the result.

As mentioned, we will do it in two steps. Before introducing the first homotopy, we will need some auxiliary notations. Let us denote:

$$t_n(\tau) := in\tau e^{in\tau}$$

$$L_n(\tau) := t'_n(\hat{\tau})(\tau - \hat{\tau}) = (ine^{in\hat{\tau}} - n^2 e^{in\hat{\tau}})(\tau - \hat{\tau})$$

$$r_n(\tau) := t_n(\tau) - t_n(\hat{\tau}) - L_n(\tau)$$

The first homotopy $H : [0; 1] \times [\underline{\tau}; \overline{\tau}] \times X_3 \to \mathbb{C}^{\mathbb{Z}}$ will be a linear deformation of F into a function G close to the linearization of F:

$$H(h, \tau, x) := hF(\tau, \hat{c} + x) + (1 - h)G(\tau, x)$$

where G on the n-th coefficient is equal to:

$$G(\tau, x)_n := \begin{cases} (t_n(\tau) + K)\, x_n \text{ for } n \neq \pm 1 \\ (t_n(\hat{\tau}) + K + \gamma_{1,0} + \gamma_{1,2n})\, x_n + L_n(\tau)\hat{c}_n \text{ for } n = \pm 1 \end{cases}$$

Where $\gamma_{p,j} = -\frac{K}{3!}\begin{pmatrix} 3 \\ p \end{pmatrix}\left(c^{*(3-p)}\right)_j + \frac{K}{5!}\begin{pmatrix} 5 \\ p \end{pmatrix}\left(c^{*(5-p)}\right)_j - \frac{K}{7!}\begin{pmatrix} 7 \\ p \end{pmatrix}\left(c^{*(7-p)}\right)_j$ is from Lemma 18.

G is not a linear function, as it is not linear with respect to τ. It also does not contain all the linear terms with respect to x_k. However, it contains the most important terms – the rest will be shown to be small compared to them. Let us note that for $n = \pm 1$ we have $x_{n-2n} = \overline{x_n} = x_n$ (because, from the definition of X_3: $x_{\pm 1} \in \mathbb{R}$), so the term $\gamma_{1,2n} x_{n-2n}$ in F is linear with respect to x_n.

Later we will use a simple homotopy to transform G into a linear function.

One can write explicit formulas for H. For $n \neq \pm 1$ we have:

$$\begin{aligned} H(h, \tau, x)_n &:= (t_n(\tau) + K)\, x_n + \\ &+ h\left((t_n(\tau) + K)\hat{c}_n + \textstyle\sum_{k=1}^{\infty} \frac{K}{(2k+1)!}\left((x + \hat{c})^{*(2k+1)}\right)_n\right) \end{aligned}$$

And for $n = \pm 1$:

$$\begin{aligned} H(t, \tau, x)_n \quad := \quad & (t_n(\hat{\tau}) + K + \gamma_{1,0} + \gamma_{1,2n})\, x_n + L_n(\tau)\hat{c}_n + \\ & + h\left((t_n(\hat{\tau}) + K)\, \hat{c}_n - \tfrac{K}{3!}\left(\hat{c}^{*3}\right)_n + \tfrac{K}{5!}\left(\hat{c}^{*5}\right)_n - \tfrac{K}{7!}\left(\hat{c}^{*7}\right)_n\right) + \\ & + h\textstyle\sum_{-6\cdot 5 \leq j \leq 6\cdot 5, j \neq 0, 2n} \gamma_{1,j} x_{n-j} + h\sum_{p=2}^{7}\sum_{j=-6\cdot 5}^{6\cdot 5} \gamma_{p,j} x_{n-j}^{*p} + \\ & + h r_n(\tau)\hat{c}_n + h(t_n(\tau) - t_n(\hat{\tau}))x_n + hR(\hat{c} + x)_n \end{aligned}$$

Where R is a short notation for:

$$R(y) := \sum_{k=4}^{\infty} \frac{K}{(2k+1)!} y^{*(2k+1)}$$

As already mentioned, the hardest point of the proof is showing that the homotopy H is admissible:

Theorem 2. *Let $x \in X_3$, $\tau \in [\underline{\tau}, \overline{\tau}]$. Let x be such that $\exists n : |x_n| = \frac{\beta_2}{(|n|+1)^2}$ or let $\tau \in \{\underline{\tau}, \overline{\tau}\}$. Then $\forall h \in [0; 1] : H(h, \tau, x) \neq 0$.*

Proof: The proof is computer assisted – the calculations are done by the program. The computation is as follow.

First, let us note that if $|x_n| = \frac{\beta_2}{(|n|+1)^2}$ for $n < 0$, then also $|x_{-n}| = |\overline{x_n}| = \frac{\beta_2}{(|-n|+1)^2}$, where $-n > 0$. Thus, it is enough to consider this condition for $n \geq 0$ (and the case $\tau \in \{\underline{\tau}, \overline{\tau}\}$). We do it in two steps:

1. Let us assume that $|x_n| = \frac{\beta_2}{(|n|+1)^2}$ for some $n \geq 0$, $n \neq 1$. We will show that $H(h,\tau,x)_n \neq 0$ (which obviously implies $H(h,\tau,x) \neq 0$). We have:

$$\begin{aligned} &|H(h,\tau,x)_n| = \\ &= \left|(t_n(\tau)+K)\,x_n + h\left((t_n(\tau)+K)\hat{c}_n + K\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k+1)!}\left((x+\hat{c})^{*(2k+1)}\right)_n\right)\right| \geq \\ &\geq |(t_n(\tau)+K)\,x_n| - \left|\left((t_n(\tau)+K)\hat{c}_n + K\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k+1)!}\left((x+\hat{c})^{*(2k+1)}\right)_n\right)\right| \end{aligned}$$

Thus, it is enough to show:

$$|(t_n(\tau)+K)\,x_n| \quad > \quad \left|(t_n(\tau)+K)\hat{c}_n + K\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k+1)!}\left((x+\hat{c})^{*(2k+1)}\right)_n\right|$$

To prove it, we will use the estimates from the previous chapter to compute a lower bound of the LHS and an upper bound of the RHS. As it will require a lot of computations, we will use the program to compute the results. To overcome the problem of computers being able to represent naturally only a finite subset of $\mathbb{R}$, we will use the interval arithmetics. It will compute some small intervals where the mathematically strict results lies. This will be enough to be able to prove the inequality – if the left bound of LHS estimate interval will be bigger than the right bound of the RHS estimate interval, then the inequality will be proved.

The LHS will be estimated by:

$$\begin{aligned} |(t_n(\tau)+K)\,x_n| &\geq ||t_n(\tau)| - K|\,\frac{\beta_2}{(|n|+1)^2} = \\ &= |n\tau - K|\,\frac{\beta_2}{(|n|+1)^2} \end{aligned} \tag{3.1}$$

τ can be any number from the interval $[\underline{\tau};\overline{\tau}]$. We can take advantage of the interval arithmetics and substitute for τ the whole interval. Hence, for any specified n, this estimate is computable by a program. We will use it for every $n < 225$ to obtain an interval containing a mathematically strict lower bound for the LHS.

Of course, our program cannot compute the bounds for each $n \in \mathbb{N}$ separately, thus for $n \geq 225$, we want to prove all the inequalities in some finite computations. Thus, we have two cases:

a. For $n \geq 225$:

Let us multiply the estimate for LHS by $(n+1)^2$. We have

$$|(t_n(\tau)+K)\,x_n|\,(n+1)^2 \geq |n\tau - K|\,\beta_2 \geq (225\tau - K)\beta_2$$

what, after substituting $[\underline{\tau};\overline{\tau}]$ for τ, gives us an estimate independent of n.

On the RHS, we have $(t_n(\tau)+K)\hat{c}_n = 0$ because $\hat{c}_n = 0$ for $n > 225$. Thus, we have only the sum $K\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k+1)!}\left((x+\hat{c})^{*(2k+1)}\right)_n$. We have:

$$\left|K\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k+1)!}\left((x+\hat{c})^{*(2k+1)}\right)_n\right| \leq$$
$$\leq \sum_{k=1}^{21}\frac{K}{(2k+1)!}\left|\left((x+\hat{c})^{*(2k+1)}\right)_n\right| + \left|K\sum_{k=22}^{\infty}\frac{(-1)^k}{(2k+1)!}\left((x+\hat{c})^{*(2k+1)}\right)_n\right|$$

The terms for $k = 1,\ldots,21$ are estimated from Lemma 17 (with $N = 255$). For the sum for $k \geq 22$, we use Lemma 15.

Both Lemma 17 and Lemma 15 give us estimates that after a multiplication by $(n+1)^2$ are independent of n. Of course, $|\text{LHS}| > |\text{RHS}|$ is equivalent to $|\text{LHS}|(n+1)^2 > |\text{RHS}|(n+1)^2$, and the latter can be checked for every $n > 255$ by the program by checking just one inequality.

The program checks that inequality, and it is satisfied.

b. For $n < 225$:

In this case, the program computes a separate estimate for each $n \in \{0,2,3,4,\ldots,224\}$ and checks that the inequalities are satisfied. We will need to use some more sophisticated estimates.

Let us use Lemma 18 to group the terms of the RHS:

$$\begin{aligned}
&\left|(t_n(\tau)+K)\hat{c}_n + K\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k+1)!}\left((x+\hat{c})^{*(2k+1)}\right)_n\right| = \\
&= \ \left|\left((t_n(\tau)+K)\,\hat{c}_n - \tfrac{K}{3!}\left(\hat{c}^{*3}\right)_n + \tfrac{K}{5!}\left(\hat{c}^{*5}\right)_n - \tfrac{K}{7!}\left(\hat{c}^{*7}\right)_n\right) + \right. \\
&\qquad \left. \sum_{p=1}^{7}\sum_{j=-6l}^{6l}\gamma_{p,j}x^{*k}_{n-j} + R(\hat{c}+x)\right| \leq \\
&\leq \ \left|(t_n(\tau)+K)\,\hat{c}_n - \tfrac{K}{3!}\left(\hat{c}^{*3}\right)_n + \tfrac{K}{5!}\left(\hat{c}^{*5}\right)_n - \tfrac{K}{7!}\left(\hat{c}^{*7}\right)_n\right| + \\
&\qquad + \left|\sum_{p=1}^{7}\sum_{j=-6l}^{6l}\gamma_{p,j}x^{*k}_{n-j}\right| + |R(\hat{c}+x)| \leq \\
&\leq \ |t_n(\tau)-t_n(\hat{\tau})|\,|\hat{c}_n| + \\
&\qquad + \left|(t_n(\hat{\tau})+K)\,\hat{c}_n - \tfrac{K}{3!}\left(\hat{c}^{*3}\right)_n + \tfrac{K}{5!}\left(\hat{c}^{*5}\right)_n - \tfrac{K}{7!}\left(\hat{c}^{*7}\right)_n\right| + \\
&\qquad + \sum_{p=1}^{7}\sum_{j=-6l}^{6l}|\gamma_{p,j}|\,\frac{\beta_2(10\beta_2)^{p-1}}{(|n-j|+1)^2} \\
&\qquad + \sum_{k=4}^{21}\frac{K}{(2k+1)!}\left|\left((x+\hat{c})^{*(2k+1)}\right)_n\right| + \left|K\sum_{k=22}^{\infty}\frac{(-1)^k}{(2k+1)!}\left((x+\hat{c})^{*(2k+1)}\right)_n\right|
\end{aligned}$$

For the first term, we will use the estimate:

$$\begin{aligned}
|(t_n(\tau)-t_n(\hat{\tau}))| \ &\leq \ \max_{\tau\in[\underline{\tau},\overline{\tau}]}|f'(\tau)(\tau-\hat{\tau})| \leq \\
&\leq \ \max_{\tau\in[\underline{\tau},\overline{\tau}]}|ine^{in\tau} - n^2\tau e^{in\tau}|\cdot|\Delta\tau| \leq \\
&\leq \ \max_{\tau\in[\underline{\tau},\overline{\tau}]}(n+n^2\tau)|\Delta\tau|
\end{aligned} \tag{3.2}$$

To compute the maximum, it is enough to substitute $[\underline{\tau},\overline{\tau}]$ as τ and use the interval arithmetics – if we compute this expression for all possible τ,

Table 3.1: Estimates of the LHS and the RHS for small n

n	LHS	RHS
0	$4.1792 \cdot 10^{-7}$	$0.1687 \cdot 10^{-7}$
2	$0.4474 \cdot 10^{-7}$	$0.0958 \cdot 10^{-7}$
3	$0.5080 \cdot 10^{-7}$	$0.0827 \cdot 10^{-7}$
4	$0.4893 \cdot 10^{-7}$	$0.0160 \cdot 10^{-7}$
5	$0.4537 \cdot 10^{-7}$	$0.0118 \cdot 10^{-7}$
6	$0.4171 \cdot 10^{-7}$	$0.0070 \cdot 10^{-7}$
7	$0.3834 \cdot 10^{-7}$	$0.3465 \cdot 10^{-7}$
8	$0.3536 \cdot 10^{-7}$	$0.0040 \cdot 10^{-7}$
9	$0.3274 \cdot 10^{-7}$	$0.0036 \cdot 10^{-7}$

The estimates obtained by the program. The RHS is closest to the LHS for $n = 7$, because we have arbitrarily chosen $\hat{c}_7 = 0$. That makes the term $\left|(t_n(\tau) + K)\,\hat{c}_n - \frac{K}{3!}\left(\hat{c}^{*3}\right)_n + \frac{K}{5!}\left(\hat{c}^{*5}\right)_n - \frac{K}{7!}\left(\hat{c}^{*7}\right)_n\right|$ for $n = 7$ approximately equal $0.3411 \cdot 10^{-7}$.

then the resulting interval will also contain the value of the expression for the maximal τ. Thus, the mathematically strict result will be in the result interval.

The fact that this estimate grows quickly with n is irrelevant, as for $n > 5$ we have $\hat{c}_n = 0$.

The sum $\left|(t_n(\hat{\tau}) + K)\,\hat{c}_n - \frac{K}{3!}\left(\hat{c}^{*3}\right)_n + \frac{K}{5!}\left(\hat{c}^{*5}\right)_n - \frac{K}{7!}\left(\hat{c}^{*7}\right)_n\right|$ can be directly computed and is small as it is the numerical solution that is close to zero.

The $\sum_{p=1}^{7}\sum_{j=-6l}^{6l}\left|\gamma_{p,j}\right|\frac{\beta_2(10\beta_2)^{p-1}}{(|n-j|+1)^2}$ can be directly computed and is small for $p > 1$ because $10\beta_2$ is small. The terms for $p = 1$ happen to be small and they are not a problem for the inequality to hold (in the orbit for $K = 5.13$, they will not be small and we will need to change the coordinates to diagonalize the linear part). We estimate $\frac{K}{(2k+1)!}\left|\left((x+\hat{c})^{*(2k+1)}\right)_n\right|$ for $k \in \{4, .., 21\}$ from Lemma 13. To estimate $\left|K\sum_{k=22}^{\infty}\frac{(-1)^k}{(2k+1)!}\left((x+\hat{c})^{*(2k+1)}\right)_n\right|$, we use Lemma 15. In both estimates we have $\frac{1}{(2k+1)!}$ with k large enough to make the result small.

Thus, we see that unless $\gamma_{1,\cdot}$ are big, the estimate of the RHS upper bound should be small. The program checks that it is so – for each $n \in \{0, 2, 3, 4, .., 224\}$ it computes this upper bound of the RHS and the lower bound of the LHS from equation (3.1), compares them and finds that the inequality holds. This ends the case for $n \neq 1$.

2. We have two cases left: $\tau \in \{\underline{\tau}, \overline{\tau}\}$ and $|x_1| = \frac{\beta_2}{4}$. We have two variables left but only one equation – the equation for $n = 1$. However, this is a complex equation and the variables are real, so we will be able to show that in both cases $H(\tau, x) \neq 0$.

Let us denote:

$$L_\tau := L_1(\tau)\hat{c}_1$$

$$L_x := \left(t_1(\hat{\tau}) + K + \gamma_{1,0} + \gamma_{1,2}\right) x_1$$

$$\begin{aligned} N \;:=\; & \left(t_1(\hat{\tau}) + K\right)\hat{c}_1 - \tfrac{K}{3!}\left(\hat{c}^{*3}\right)_1 + \tfrac{K}{5!}\left(\hat{c}^{*5}\right)_1 - \tfrac{K}{7!}\left(\hat{c}^{*7}\right)_1 + \\ & + \textstyle\sum_{-6l \leq j \leq 6l, j \neq 0,2} \gamma_{p,j} x^{*k}_{1-j} + \sum_{p=2}^{7} \sum_{j=-6l}^{6l} \gamma_{p,j} x^{*k}_{1-j} + \\ & + R(c) + r_1(\tau)\hat{c}_1 + (t_1(\tau) - t_1(\hat{\tau}))x_1 \end{aligned}$$

The L_τ, L_x, N depend on τ, x, but to make the notation short, we skip it. They are chosen such that $G(\tau, x)_1 = L_\tau + L_x$, $F(\tau, x)_1 = L_\tau + L_x + N$ and $H(h, \tau, x)_1 = L_\tau + L_x + hN$.

If $\tau \in \{\underline{\tau}, \overline{\tau}\}$, then we will show that:

$$|L_\tau| > |L_x| + |N| \tag{3.3}$$

which will give $H(h, \tau, x)_1 \neq 0$.

The absolute value $|L_\tau|$ can be computed:

$$|L_1(\tau)\hat{c}_1| = \left.\ |in - n^2| \cdot \Delta\tau \cdot |\hat{c}_n|\ \right|_{n=1} = \sqrt{2}\Delta\tau \cdot |\hat{c}_1|$$

We estimate the terms in $|N|$ like for $n \neq 1$, with the exception of the new terms: $|(t_1(\tau) - t_1(\hat{\tau}))x_1|$ and $|r_1(\tau)\hat{c}_1|$. For the first one, we use inequality (3.2). For the second, we use the estimate:

$$\begin{aligned} |r_1(\tau)\hat{c}_1| \;&\leq\; \max_{\tau \in [\underline{\tau}; \overline{\tau}]} \left|\tfrac{1}{2} t_1''(\tau)(\tau - \hat{\tau})^2\right| \cdot |\hat{c}_1| = \\ &=\; \tfrac{|\hat{c}_1|}{2} \max_{\tau \in [\underline{\tau}; \overline{\tau}]} \left|-\left(2n^2 + in^3\tau\right) e^{in\tau}\right| = \\ &=\; \tfrac{|\hat{c}_1|}{2} \max_{\tau \in [\underline{\tau}; \overline{\tau}]} |2 + i\tau| \end{aligned}$$

Like for equation (3.2), we compute the maximum by substituting $[\underline{\tau}; \overline{\tau}]$ for τ and computing all the possible values.

Having all these estimates, our program checks that the inequality (3.3) is satisfied.

In the case $|x_1| = \frac{\beta_2}{4}$, the inequality $|L_x| > |L_\tau| + |N|$ is obviously false. To prove that there are no zeros, we will need to use the fact that $\tau \in \mathbb{R}$ and $x_1 \in \mathbb{R}$. On Figure 1, we sketch how the sets of possible values of L_τ and $L_x + N$ look like on the complex plane. We see that they should not

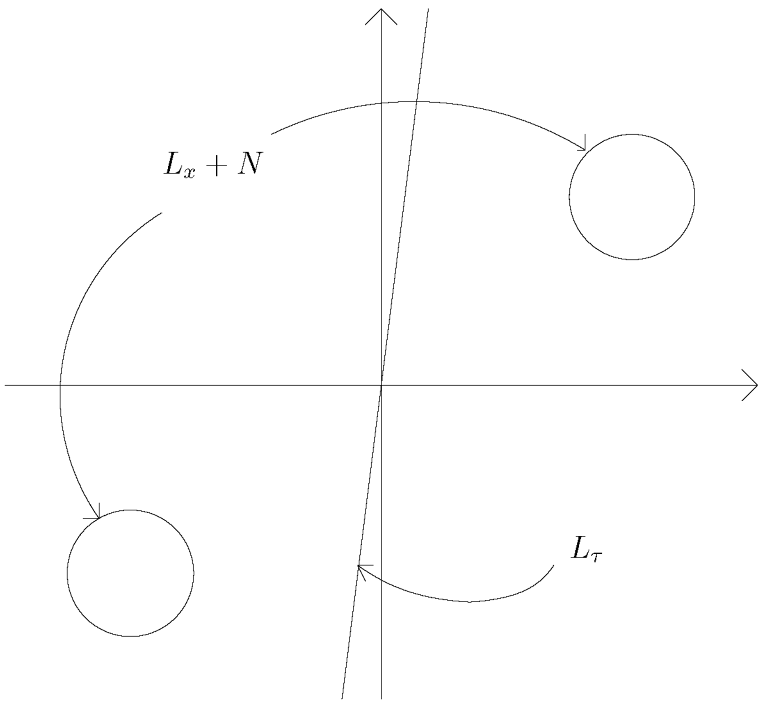


Figure 3.1: The possible values of L_τ and $L_x + N$

intersect, i.e. $0 \notin L_\tau - (L_x + N)$. For an estimate that is symmetric with respect to 0, this is equivalent to $0 \notin L_\tau + (L_x + N)$.

Formally, we will show that $\{\tan[\arg(L_\tau)] : (\tau, x) \in [\underline{\tau}, \overline{\tau}] \times X_3, x_1 = \pm\frac{\beta_2}{4}\} \cap \{\tan[\arg(L_x + N)] : (\tau, x) \in [\underline{\tau}, \overline{\tau}] \times X_3, x_1 = \pm\frac{\beta_2}{4}\} = \emptyset$ (where arg is the complex number argument). This implies that if we take an a from the first set and a b from the second, then $a + b \neq 0$. Thus $0 \neq L_\tau + L_x + N = H(h, \tau, x)_1$.

The $\arg(L_\tau)$ is easy to compute as this is $\tan\arg\left(\left(ie^{i\hat{\tau}} - e^{i\hat{\tau}}\right)\hat{c}_1\right)$, and $\tan\arg z$ for a complex number z can be computed as $\frac{\Im z}{\Re z}$. To estimate the other set, we will use the estimate for N from the previous point. Let us denote by λ the right end of the interval containing the upper bound for N. Then we have $|N| \leq \lambda$. Thus, $L_x + N \in \pm\left(t_n(\hat{\tau}) + K + \gamma_{1,0} + \gamma_{1,2}\right)\frac{\beta_2}{4} + [-\lambda; \lambda] + [-\lambda; \lambda]i$ and using the interval arithmetics we can find $\tan\arg(L_x + N)$.

The program checks that these two sets are disjoint, and this ends the proof of the theorem.

$\square$

Now, let us define the second homotopy:

$$H^L(h,\tau,x) := \begin{cases} (t_n(\hat{\tau}+h(\tau-\hat{\tau}))+K)\,x_n \text{ for } n\neq\pm1 \\ (t_n(\hat{\tau})+K+\gamma_{1,0}+\gamma_{1,2n})\,x_n + L_n(\tau)c_n \text{ for } n=\pm1 \end{cases}$$

It deforms G into:

$$G^L(h,\tau,x) := \begin{cases} (t_n(\hat{\tau})+K)\,x_n \text{ for } n\neq\pm1 \\ (t_n(\hat{\tau})+K+\gamma_{1,0}+\gamma_{1,2n})\,x_n + L_n(\tau)c_n \text{ for } n=\pm1 \end{cases}$$

which is a linear function.

Lemma 20. *If* $(\tau,x)\in[\underline{\tau},\overline{\tau}]\times X_3$ *such that* $\tau\in\{\underline{\tau},\overline{\tau}\}$ *or* $\exists n: |x_n| = \frac{\beta_2}{(|n|+1)^2}$ *then* $H^L(h,x,\tau)\neq 0$

Proof: Let us assume there exist h,x,τ such that $H^L(h,x,\tau)=0$.

Let $n\neq\pm1$. The $\hat{\tau}+h(\tau-\hat{\tau})\in[\underline{\tau};\overline{\tau}]$, thus $|t_n(\hat{\tau}+h(\tau-\hat{\tau}))+K|$ can be estimated as in equation (3.1). For each such value we have proven that it is strictly greater than an RHS≥ 0. Hence $|t_n(\hat{\tau}+h(\tau-\hat{\tau}))+K|>0$ and $H^L(h,\tau,x)=0 \Rightarrow x_n=0$.

But if $x_n=0$ for each $n\neq\pm1$ then $H^L(h,\tau,x)=G(\tau,x)$. Thus, $0=G(\tau,x)=H(0,\tau,x)$, but H has no zeros on the boundary. This contradiction ends the proof. □

Observation 4. *The Galerkin projection of* H*, i.e.:* $H_l(\tau,y_0,\ldots,y_l) := P_lH(\tau,Q_l(y_0,\ldots,y_l))$ *(for* $(y_0,...,y_l)\in\mathbb{R}\times\mathbb{R}\times\mathbb{C}^{l-1}$*) is a homotopy from* $F^l(\tau,c+\cdot)$ *to the projection of* G*. Analogously, the projection of* H^L *is a homotopy from* G *to* G^L*. There are no zeros on the borders of* $[\underline{\tau};\overline{\tau}]\times P_lX_3$ *for these homotopies.*

Proof: We have $(\tau,x)\in\partial[\underline{\tau};\overline{\tau}]\times P_lX_3$ if $|\tau-\hat{\tau}|=\Delta\tau$ or $|x_n|=\frac{\beta_2}{(n+1)^2}$ for $n\in\{0,\ldots,l\}$. From the proof of the theorem, in the first case we have $0\neq H(\tau,Q_l(y_0,\ldots,y_l))_1 = H_l(\tau,y_0,\ldots,y_l)_1$, in the second we have $0\neq H(\tau,Q_l(y_0,\ldots,y_l))_n = H_l(\tau,y_0,\ldots,y_l)_n$. Thus, in both cases $H_l(\tau,y_0,\ldots,y_l)\neq 0$.

Analogously for the projections of H^L. □

Observation 5. *The degree* $\deg(F_l,[\underline{\tau},\overline{\tau}]\times P_l(X_3),0)$ *is well-defined.*

Proof: The homotopy H_l for $h=1$ have no zeros on the boundary and this is F_l. □

Lemma 21. $\deg(F_l,[\underline{\tau},\overline{\tau}]\times P_l(X_3),0)\neq 0$

Proof: We know that the degree of F_l is equal to the degree of the Galerkin projection of G^L – let us denote it by G_l^L. Function G_l^L is linear. If the determinant were zero then there would be a zero on the boundary of any bounded neighborhood of $(\hat{\tau}, 0)$, and the homotopy would not be admissible. From Lemma 2 the degree is ± 1 and thus is not zero. □

Theorem 3. *Equation (1.3) for* $f(x) = -1.6 \sin x$ *has a periodic solution for some* $\tau \in [\underline{\tau}, \overline{\tau}]$ *whose Fourier coefficients are in the set* X_3.

Proof: From Lemma 21 we know that the local Brouwer degree is nonzero thus each F_l has a zero in X_3. From Lemma 19 we have that F has a zero in X_3. From Theorem 1 we obtain that in X_3 there is a solution of the equation. □

3.4 The orbit for $K = 5.13$

The coefficients of this orbit converge more slowly to 0 than in the previous one, so we need more coefficients from the numerical approximation. The approximated orbit $\hat{c}$ is: (the table show 10 digits of the 15 digit numbers used in the program)

n	$\hat{c}_n$	n	$\hat{c}_n$
0	0	17	$0.0000440480 + 0.0000155909i$
1	$-1.2478700000 + 0.2075161511i$	18	0
2	0	19	$0.0000159735 + 0.0000001695i$
3	$-0.1856616804 + 0.1000731219i$	20	0
4	0	21	$-0.0000052439 + 0.0000017317i$
5	$0.0344005706 - 0.0371601212i$	22	0
6	0	23	$-0.0000015354 + 0.0000011637i$
7	$0.0056400266 - 0.0127224762i$	24	0
8	0	25	$0.0000003783 - 0.0000005617i$
9	$-0.0003626201 + 0.0041240920i$	26	0
10	0	27	$0.0000000623 - 0.0000002313i$
11	$0.0003100522 + 0.0012567745i$	28	0
12	0	29	$0.0000000056 + 0.0000000850i$
13	$-0.0002263587 - 0.0003520254i$	30	0
14	0	31	$0.0000000117 + 0.0000000281i$
15	$-0.0001086526 - 0.0000860395i$	32	0
16	0	33	$-0.0000000072 - 0.000000008i$

$$\hat{\tau} = 1.570796326794899$$

For $n > 33$ the $\hat{c}_n$ is zero, for $n < 0$ we have $\hat{c}_n = \overline{\hat{c}_{-n}}$.
The bounds are:

$$\beta_1 = 10.1301876135$$
$$\beta_2 = 0.0000122547$$
$$\Delta\tau = 0.0000005$$

Like for the first orbit:

$$X_1 := X_{\beta_1}$$
$$X_2 := \hat{c} + X_{\beta_2}$$
$$\underline{\tau} := \hat{\tau} - \Delta\tau$$
$$\overline{\tau} := \hat{\tau} + \Delta\tau$$

β_1, β_2 and $\hat{c}$ are such that we have $X_2 \subset X_1$.

We will also have a set similar to the set X_3 for the first orbit. As we know, we need to reduce the dimension by 1, but for this orbit we also need to change the coordinates in low dimensions. Geometrically, we will use a different set on which we compute the local Brouwer degree – it will not be a cuboid parallel to the axis (like it was for the first orbit) but a paralleliped. We will choose such a paralleliped that proving that there are no zeros on its boundary will be easier.

We will parameterize both τ and x_i for $0 \le i \le 10$ – we will use a matrix $L \in \mathbb{R}^{22\times 21}$ to transform some new real variables $q_0, \ldots, q_{20}$ into $\tau - \hat{\tau}, x_0, \ldots, x_{10}$ (τ and x_0 is real, other x_n are complex thus, we have 22 real dimensions):

$$\begin{bmatrix} \tau - \hat{\tau} \\ x_0 \\ \vdots \\ x_{10} \end{bmatrix} = L \begin{bmatrix} q_0 \\ \vdots \\ q_{20} \end{bmatrix}$$

The matrix L can be found in the source code of the program. Later in the proof, we will describe where it came from – it will be a numerical pseudo-inverse of the linear part of the equations. However, we do not need any assumptions about the matrix being from a special class – the theoretical part of the proof can be done for any matrix. Of course, for most matrices the inequalities will not be satisfied.

We will denote the elements of L as l_{ij} ($L := [l_{ij}]_{i=0\ldots21, j=0\ldots20}$). The exact transformations are as follows:

$$\tau = \hat{\tau} + \sum_{j=0}^{20} l_{1j} q_j \tag{3.4}$$

$$x_0 = \sum_{j=0}^{20} l_{0j} q_j \tag{3.5}$$

$$x_k = \sum_{j=0}^{20} l_{(2k)j} q_j + i \sum_{j=0}^{20} l_{(2k+1)j} q_j \text{ for } 1 \le k \le 10 \tag{3.6}$$

For this orbit, the shape of the set can no longer be described as a Cartesian product of a set in τ and a set in x. Because of that, apart from the already defined projection and immersion P_l and Q_l, it will be helpful to have a projection and immersion that acts on both τ and x:

$$\tilde{P}_l : \mathbb{R} \times \mathbb{C}^{\mathbb{Z}} \ni (\tau, c) \to (\tau, c_0, \ldots, c_l) \in \mathbb{R} \times \mathbb{R} \times \mathbb{C}^l$$

$$\tilde{Q}_l : \mathbb{R}\times\mathbb{R}\times\mathbb{C}^l \ni (\tau, c_0, \ldots, c_l) \to (\tau, (\ldots, 0, \overline{c_{-l}}, \ldots, c_0, \ldots, c_l, 0, \ldots)) \in \mathbb{R}\times\mathbb{C}^{\mathbb{Z}}$$

For any $l > 10$, we will prove a zero of a function:

$$F_l := P_l F \left(\tilde{Q}_l \left(L(q_0, \ldots, q_{20}), x_{11}, \ldots, x_l \right) \right)$$

This implies a zero of $P_l F$, so we will be able to use Lemma 19. Using L reduces the number of dimensions by one, so we have $2l + 1$ real dimensions both in the domain as well as in the image and we can use the local Brouwer degree.

We will be computing the Brouwer degree on the Galerkin projections of $\tilde{X}_3$:

$$\tilde{X}_3 := \{(\tau, x) \subset [\underline{\tau}; \overline{\tau}] \times X_2 : \\ \exists q_0, \ldots, q_{20} : (\tau - \hat{\tau}, x_0, \ldots, x_{10}) = L(q_0, \ldots, q_{20}) \text{ and } \forall i : |q_i| \le \tilde{q}_i\}$$

Where the bound $\tilde{q}_i$ are (in the proof it will be explained where these numbers came from):

i	$\tilde{q}_i$	i	$\tilde{q}_i$	i	$\tilde{q}_i$
0	$2.013374342e-09$	7	$3.893390219e-09$	14	$6.668582623e-08$
1	$1.266037525e-07$	8	$4.152544154e-09$	15	$2.148774353e-08$
2	$1.265112042e-07$	9	$1.052777396e-07$	16	$2.142484828e-08$
3	$5.682925877e-09$	10	$1.052542006e-07$	17	$2.772228444e-07$
4	$5.477650851e-09$	11	$2.927459967e-08$	18	$2.772612862e-07$
5	$1.065338515e-07$	12	$2.916634818e-08$	19	$2.206303501e-07$
6	$1.064380049e-07$	13	$6.685333515e-08$	20	$2.204931365e-07$

It may seem that the Galerkin projection of $\tilde{X}_3$ can have a more complicated shape than a paralleliped, as we have two kind of constraints on $x_0, \ldots, x_{10}$: $|q_i| \leq \tilde{q}_i$ and $x_n \leq \frac{\beta_2}{(|n|+1)^2}$ (because $x \in X_2$). However, one can show that the $\tilde{q}_i$ are such that:

$$(\forall i : |q_i| \leq \tilde{q}_i) \Rightarrow \forall |n| \leq 10 : x_n \leq \frac{\beta_2}{(|n|+1)^2} \text{ and } \tau \in [\underline{\tau}; \overline{\tau}] \tag{3.7}$$

We check in the program that this holds – we have the inequalities:

$$|x_0| = \left| \sum_{j=0}^{20} l_{0j} q_j \right| \leq \sum_{j=0}^{20} |l_{0j}| \, \tilde{q}_j$$

$$|\tau - \hat{\tau}| \leq \sum_{j=0}^{20} |l_{1j}| \, \tilde{q}_j$$

$$|x_n| \leq \left| \sum_{j=0}^{20} \left| l_{(2n)j} \right| \tilde{q}_j + i \sum_{j=0}^{20} \left| l_{(2n+1)j} \right| \tilde{q}_j \right|$$

where the sums on the right hand sides are computed by the program and are checked to be smaller than $\frac{\beta_2}{(|n|+1)^2}$ or $\Delta\tau$.

Thus, the Galerkin projection is a paralleliped and its boundary can be easily described – $|q_i| = \tilde{q}_i$ for some $0 \leq i \leq 20$ or $x_n = \frac{\beta_2}{(|n|+1)^2}$ for some $10 < n \leq l$. The fact that $\tilde{X}_3 \subset [\underline{\tau}; \overline{\tau}] \times X_2$ means that, despite using a paralleliped, we can still use all the estimates from Section 2.2 for X_2.

We should also note that $\forall i : \tilde{q}_i < \beta_2$ – we will use this bound in the proof (the method does not need necessarily β_2. We just need a rough bound for all $\tilde{q}_i$, and β_2 was a good candidate).

Like for the first orbit, we will use two homotopies with the first one deforming F_l into a G_l ($l \geq 10$):

$$G_l : \mathbb{R}^{21} \times \mathbb{C}^{l-10} \longrightarrow \mathbb{R} \times \mathbb{C}^l$$

$$G_l(q_0, \ldots, q_{20}, x_{11}, \ldots, x_l) := \begin{bmatrix} q_0 \\ q_1 + iq_2 \\ q_3 + iq_4 \\ \vdots \\ q_{19} + iq_{20} \\ (t_{11}(\tau) + K)\, x_{11} \\ \vdots \\ (t_l(\tau) + K)\, x_l \end{bmatrix}$$

where τ on the RHS is computed from $L(q_0,\ldots,q_{20})$ and $t_n(\tau)$ is defined in Section 3.3 as $in\tau e^{in\tau}$.

The homotopy H_l will be a linear deformation:

$$H_l(h,q_0,\ldots,q_{20},x_{11},\ldots,x_l) := \\ hF_l\left(q_0,\ldots,q_{20},x_{11},\ldots,x_l\right)+(1-h)G_l\left(q_0,\ldots,q_{20},x_{11},\ldots,x_l\right)$$

For this orbit as well, the hardest part is to prove that the homotopy is admissible:

Theorem 4. *Let $l>10$, $(L(q_0,\ldots,q_{20}),x_{11},\ldots,x_l)\in\tilde{P}_l(\tilde{X}_3)$ and let $\exists k\leq 20:|q_k|=\tilde{q_k}$ or $\exists 10<n\leq l:|x_n|=\frac{1}{(|n|+1)^2}$. Then $\forall h\in[0;1]:$ $H_l(h,q_0,\ldots,q_{20},x_{11},\ldots,x_l)\neq 0$*

Proof: Similarly to the first orbit, proving this theorem will require a lot of computations that will be done by a computer program using strict interval arithmetics.

Let us define: $(\tau,x_1,\ldots,x_{10}) := L(q_0,\ldots,q_{20})+(\hat{\tau},0,\ldots,0)$ and $x := Q_l(x_1,\ldots,x_l)$. We have $\tilde{X}_3\subset[\underline{\tau};\overline{\tau}]\times X_2$, thus $\tau\in[\underline{\tau};\overline{\tau}]$ and $x\in X_2$.

1. Let us assume that $|x_n|=\frac{1}{(|n|+1)^2}$ for some $n>10$.

This case is very similar to the proof of the first orbit – only some estimates and constants will be different.

The value of H_l on the n-th coordinate is:

$$H_l\left(h,q_0,\ldots,q_{20},x_{11},\ldots,x_l\right)_n=\left(t_n(\tau)+K\right)x_n+ \\ +h\left(\left(t_n(\tau)+K\right)\hat{c}_n+K\textstyle\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k+1)!}\left((x+\hat{c})^{*(2k+1)}\right)_n\right)$$

As before, it is enough to show such an inequality:

$$\left|\left(t_n(\tau)+K\right)x_n\right|>\left|\left(t_n(\tau)+K\right)\hat{c}_n+K\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k+1)!}\left((x+\hat{c})^{*(2k+1)}\right)_n\right|$$

as this implies that $H_l\left(h,q_0,\ldots,q_{20},x_{11},\ldots,x_l\right)\neq 0$. In equation (3.1) we have an estimate for the LHS:

$$\left|\left(t_n(\tau)+K\right)x_n\right|\geq|n\tau-K|\frac{\beta_2}{(|n|+1)^2}$$

where we need to substitute the whole interval $[\underline{\tau};\overline{\tau}]$ for τ.

To get an upper bound for the RHS, we will have two cases. We will check the small n's one by one, while for the high n's, we will prove them all by checking just one inequality – after multiplication by $(n+1)^2$ the estimates for the LHS and RHS will be independent of n. For this orbit, the cut-off will be for $n = 3465$.

a. For $n \geq 3465$:

We have:

$$(n+1)^2 \left|(t_n(\tau) + K)\, x_n\right| \geq |n\tau - K|\, \beta_2 \geq (3465\underline{\tau} - K)\, \beta_2$$

Let us note that $(t_n(\tau) + K)\, \hat{c}_n = 0$ as $n > 33$ thus to estimate the RHS we need only to estimate the sum $\left|K \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)!} \left((x+\hat{c})^{*(2k+1)}\right)_n\right|$. For $k = 1, \ldots, 11$ we estimate each term $\left|K \frac{(-1)^k}{(2k+1)!} \left((x+\hat{c})^{*(2k+1)}\right)_n\right|$ using Lemma 17. For $k = 12, \ldots, 51$ we use Lemma 14. The whole sum $\left|K \sum_{k=52}^{\infty} \frac{(-1)^k}{(2k+1)!} \left((x+\hat{c})^{*(2k+1)}\right)_n\right|$ is estimated from Lemma 16. All these estimates, after multiplication by $(n+1)^2$, are independent of n.

The program checks that the inequality $(n+1)^2$LHS $> (n+1)^2$RHS holds, what proves LHS $>$ RHS for all $n \geq 3465$.

b. For $10 < n < 3465$:

The programs checks each n individually and uses estimates for the RHS depending on n. We will first use Lemma 18 with $N = 15$ to regroup the terms:

$$\begin{aligned}
&\left|(t_n(\tau) + K)\hat{c}_n + K \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)!} \left((x+\hat{c})^{*(2k+1)}\right)_n\right| = \\
= \quad & \Big| t_n(\tau)\hat{c}_n + K \sum_{k=0}^{7} \frac{(-1)^k}{(2k+1)!} \left(\hat{c}^{*(2k+1)}\right)_n + \\
& \sum_{p=1}^{15} \sum_{j=-14l}^{14l} \gamma_{p,j} x_{n-j}^{*p} + K \sum_{k=8}^{\infty} \frac{(-1)^k}{(2k+1)!} \left((x+\hat{c})^{*(2k+1)}\right)_n \Big| \leq \\
\leq \quad & \left| t_n(\tau)\hat{c}_n + K \sum_{k=0}^{7} \frac{(-1)^k}{(2k+1)!} \left(\hat{c}^{*(2k+1)}\right)_n \right| + \\
& \sum_{p=1}^{15} \sum_{j=-14l}^{14l} \left|\gamma_{p,j} x_{n-j}^{*p}\right| + \left|K \sum_{k=8}^{\infty} \frac{(-1)^k}{(2k+1)!} \left((x+\hat{c})^{*(2k+1)}\right)_n\right| \leq \\
\leq \quad & \left|\left(t_n(\tau) - t_n(\hat{\tau})\right) \hat{c}_n\right| + \\
& \left| t_n(\hat{\tau})\hat{c}_n + K \sum_{k=0}^{7} \frac{(-1)^k}{(2k+1)!} \left(\hat{c}^{*(2k+1)}\right)_n \right| + \\
& \sum_{p=1}^{15} \sum_{j=-14l}^{14l} \left|\gamma_{p,j}\right| \frac{\beta_2 (10\beta_2)^{p-1}}{(|n-j|+1)^2} + \\
& + \sum_{k=8}^{52} \frac{K}{(2k+1)!} \left|\left((x+\hat{c})^{*(2k+1)}\right)_n\right| + \left|K \sum_{k=53}^{\infty} \frac{(-1)^k}{(2k+1)!} \left((x+\hat{c})^{*(2k+1)}\right)_n\right|
\end{aligned}$$

For the first term of the RHS, we will use the estimate from equation (3.2). The second and the third can be computed directly. In the fourth we estimate each element of the sum separately and use Lemma 13 for $k \in \{8, \ldots, 11\}$ and Lemma 14 for $k \in \{12, \ldots, 52\}$. For the last term we use Lemma 16.

For each $n = 11, \ldots, 3464$ the program computes the lower bound for the LHS and the upper bound for the RHS and checks that the inequality holds. This ends the case of $n > 10$.

2. Let us assume that $|q_k| = \tilde{q}_k$ for some $k \in \{0, \dots, 20\}$.
Let $n := \left\lceil \frac{k}{2} \right\rceil$. We will show that $H_l\,(h, q_0, \dots, q_{20}, x_{11}, \dots, x_l)_n \neq 0$.
Let us expand the n-th coefficient of F_l:

$$\begin{aligned}
& F_l\,(h, q_0, \dots, q_{20}, x_{11}, \dots, x_l)_n = \\
= \;& t_n(\tau)\,(x_n + \hat{c}_n) + K \textstyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left((x + \hat{c})^{*(2k+1)}\right)_n = \\
= \;& (t_n(\tau) + K)\, x_n + t_n(\tau)\hat{c}_n + K \textstyle\sum_{k=0}^{7} \frac{(-1)^k}{(2k+1)!} \left(\hat{c}^{*(2k+1)}\right)_n + \\
& + \textstyle\sum_{p=1}^{15} \sum_{j=-14l}^{14l} \gamma_{p,j} x^{*p}_{n-j} + K \sum_{k=8}^{\infty} \frac{(-1)^k}{(2k+1)!} \left((x + \hat{c})^{*(2k+1)}\right)_n \\
= \;& (t_n(\hat{\tau}) + K)\, x_n + (t_n(\tau) - t_n(\hat{\tau}))\hat{c}_n + \\
& + (t_n(\tau) - t_n(\hat{\tau}))x_n + t_n(\hat{\tau})\hat{c}_n + K \textstyle\sum_{k=0}^{7} \frac{(-1)^k}{(2k+1)!} \left(\hat{c}^{*(2k+1)}\right)_n + \\
& + \textstyle\sum_{j=-14l}^{14l} \gamma_{1,j} x_{n-j} + \\
& + \textstyle\sum_{p=2}^{15} \sum_{j=-14l}^{14l} \gamma_{p,j} x^{*p}_{n-j} + K \sum_{k=8}^{\infty} \frac{(-1)^k}{(2k+1)!} \left((x + \hat{c})^{*(2k+1)}\right)_n
\end{aligned}$$

Let us recall the definition of $L_n(\tau) := \left(ine_{in\hat{\tau}} - n^2\hat{\tau}e^{in\hat{\tau}}\right)(\tau - \hat{\tau})$ and note that $(t_n(\tau) - t_n(\hat{\tau}))\hat{c}_n = L_n(\tau)\hat{c}_n + \frac{1}{2}t''_n(\tilde{\tau})(\tau - \hat{\tau})^2\hat{c}_n$ for some $\tilde{\tau} \in [\hat{\tau}; \tau] \subset [\underline{\tau}; \overline{\tau}]$.
Thus, we have:

$$\begin{aligned}
& F_l\,(h, q_0, \dots, q_{20}, x_{11}, \dots, x_l)_n = \\
= \;& (t_n(\hat{\tau}) + K)\, x_n + L_n(\tau)\hat{c}_n + \textstyle\sum_{j=-14l}^{14l} \gamma_{1,j} x_{n-j} + \\
& + \tfrac{1}{2}t''_n(\tilde{\tau})(\tau - \hat{\tau})^2\hat{c}_n + (t_n(\tau) - t_n(\hat{\tau}))x_n + t_n(\hat{\tau})\hat{c}_n + K \textstyle\sum_{k=0}^{7} \frac{(-1)^k}{(2k+1)!} \left(\hat{c}^{*(2k+1)}\right)_n + \\
& + \textstyle\sum_{p=2}^{15} \sum_{j=-14l}^{14l} \gamma_{p,j} x^{*p}_{n-j} + K \sum_{k=8}^{\infty} \frac{(-1)^k}{(2k+1)!} \left((x + \hat{c})^{*(2k+1)}\right)_n
\end{aligned}$$

In this case, we will also estimate the n-th coefficient of H_l to show that q_i dominates the other terms. However the expansion will be slightly more complicated. Let us denote:

$$G_n := G_l(q_0, \dots, q_{20}, x_{11}, \dots, x_l)_n = \begin{cases} q_0 & \text{for } n = 0 \\ q_{2n-1} + iq_{2n} & \text{for } n > 0 \end{cases}$$

We have:

$$
\begin{aligned}
&\left|H_l\left(h,q_0,\ldots,q_{20},x_{11},\ldots,x_l\right)_n\right| = \\
&= \quad \left|(1-h)G_n + hF_l\left(q_0,\ldots,q_{20},x_{11},\ldots,x_l\right)_n\right| \geq \\
&\geq \quad \left|(1-h)G_n + h\left(t_n(\hat\tau)+K\right)x_n + hL_n(\tau)\hat c_n + h\sum_{j=-14l}^{14l}\gamma_{1,j}x_{n-j}\right| - \\
&\quad -h\left|\tfrac{1}{2}t_n''(\tilde\tau)(\tau-\hat\tau)^2\hat c_n + (t_n(\tau)-t_n(\hat\tau))x_n + t_n(\hat\tau)\hat c_n + K\sum_{k=0}^{7}\tfrac{(-1)^k}{(2k+1)!}\left(\hat c^{*(2k+1)}\right)_n + \right. \\
&\quad \left. +\sum_{p=2}^{15}\sum_{j=-14l}^{14l}\gamma_{p,j}x_{n-j}^{*p} + K\sum_{k=8}^{\infty}\tfrac{(-1)^k}{(2k+1)!}\left((x+\hat c)^{*(2k+1)}\right)_n\right| \geq \\
&\geq \quad \left|(1-h)G_n + h\left(t_n(\hat\tau)+K\right)x_n + hL_n(\tau)\hat c_n + h\sum_{j=-10}^{10}\gamma_{1,n-j}x_j\right| - \\
&\quad -\sum_{j\in\{-14l,\ldots,14l\},n-j<-10\vee n-j>10}\left|\gamma_{1,j}\right|\tfrac{\beta_2}{(|n-j|+1)^2} - \\
&\quad -\max_{\tilde\tau\in[\underline\tau,\overline\tau]}\tfrac{1}{2}\left|t_n''(\tilde\tau)\hat c_n\right|\Delta\tau^2 - \left|(t_n(\tau)-t_n(\hat\tau))x_n\right| - \\
&\quad -\left|t_n(\hat\tau)\hat c_n + K\sum_{k=0}^{7}\tfrac{(-1)^k}{(2k+1)!}\left(\hat c^{*(2k+1)}\right)_n\right| - \\
&\quad -\sum_{p=2}^{15}\sum_{j=-14l}^{14l}\left|\gamma_{p,j}\right|\tfrac{\beta_2(10\beta_2)^{p-1}}{(|n-j|+1)^2} - \sum_{k=8}^{52}\tfrac{K}{(2k+1)!}\left|\left((x+\hat c)^{*(2k+1)}\right)_n\right| - \\
&\quad -\left|K\sum_{k=53}^{\infty}\tfrac{(-1)^k}{(2k+1)!}\left((x+\hat c)^{*(2k+1)}\right)_n\right|
\end{aligned}
$$

The third term can be computed directly using the interval arithmetics by substituting $[\underline{\tau};\overline{\tau}]$ for $\tilde\tau$. The fourth is less or equal to $\max_{\tilde\tau\in[\underline\tau,\overline\tau]}\left|t_n'(\tilde\tau)\right|\Delta\tau\frac{\beta_2}{(n+1)^2}$ what can also be computed directly.

The second terms and the terms five to eight can be computed like for $n>10$ – the second, fifth and sixth is directly computable, the seventh using Lemma 13 for $k\in\{8,\ldots,11\}$ and Lemma 14 for $k\in\{12,\ldots,52\}$. For the eighth we use Lemma 16.

The program computes the result as an interval where the true result lies. If we take the right bound of this interval (let us denote it by δ), we get a mathematically strict inequality:

$$
\begin{aligned}
&\left|H_l\left(h,q_0,\ldots,q_{20},x_{11},\ldots,x_l\right)_n\right| \geq \\
&\geq \quad \left|(1-h)G_n + h\left(t_n(\hat\tau)+K\right)x_n + hL_n(\tau)\hat c_n + h\sum_{j=-10}^{10}\gamma_{1,n-j}x_j\right| - \delta = \\
&= \quad \left|(1-h)G_n + h\left(\left(t_n(\hat\tau)+K\right)x_n + \hat c_n\left(ine_{in\hat\tau} - n^2\hat\tau e^{in\hat\tau}\right)(\tau-\hat\tau) + \sum_{j=-10}^{10}\gamma_{1,n-j}x_j\right)\right| - \\
&\quad -\delta
\end{aligned}
\tag{3.8}
$$

If we use the property $x_{-n}=\overline{x_n}$ the the term in the bracket depends only on $\tau,x_0,\ldots,x_{10}$. These are defined as images of $q_0,\ldots,q_{20}$ through L, so we can substitute the q_i for them using equations (3.4-3.6).

As the original expression in the bracket is an $\mathbb{R}$-linear function with respect to $(\tau-\hat\tau),x_0,\ldots,x_{10}$, so after the linear substitution we will obtain a linear expression with respect to $q_0,\ldots,q_{20}$. Thus, for some $\alpha_i\in\mathbb{C}$:

$$\begin{array}{rl} & |H_l\,(h, q_0, \ldots, q_{20}, x_{11}, \ldots, x_l)_n| \geq \\ \geq & \left|(1-h)G_n + h\sum_{i=0}^{20} \alpha_i q_i\right| - \delta \end{array}$$

For $n = 0$ we have $\gamma_{1,n+j}x_{-j} + \gamma_{1,n-j}x_j = \overline{\gamma_{1,-j}x_j} + \gamma_{1,-j}x_j \in \mathbb{R}$ thus $\sum_{j=-10}^{10} \gamma_{1,n-j}x_j \in \mathbb{R}$. We also have $t_0(\hat{\tau}) \in \mathbb{R}$, $L_0(\tau) \in \mathbb{R}$ and $\hat{c}_0 \in \mathbb{R}$. Thus, for $n = 0$ the expression in the bracket is real, and we have $\alpha_i \in \mathbb{R}$.

The matrix L has been chosen for the α_i to be in a special form. Let us denote by T the linear transformation in the bracket of equations (3.8) for all $n \in \{0, \ldots, 10\}$:

$$T(\tau, x_0, \ldots, x_{10}) := \left\{ (t_n(\hat{\tau}) + K)\, x_n + \hat{c}_n \left(ine^{in\hat{\tau}} - n^2\hat{\tau}e^{in\hat{\tau}}\right)(\tau - \hat{\tau}) + \sum_{j=-10}^{10} \gamma_{1,n-j}x_j \right\}_{n=0}^{10}$$

The T transforms $\mathbb{R}\times\mathbb{R}\times\mathbb{C}^{10}$ into $\mathbb{R}\times\mathbb{C}^{10}$. This is equivalent to a $\mathbb{R}^{21\times 22}$ matrix. Doing the variable substitution L is equivalent to multiplying from the right hand side by L – the α_i are equal to the n-th row of the matrix of TL (when treated as a $\mathbb{R}^{21} \longrightarrow \mathbb{R}\times\mathbb{C}^{10}$ function)

The matrix L has been chosen as the right pseudo-inverse of T – we would like to have $TL = I$, where $I \in \mathbb{R}^{21\times 21}$ is the identity. However, we do not need a strict equality. The L can be computed using a non-strict method (I have used a standard Mathematica function) – the proof does not have any assumptions about L having a special property. However, if TL is far away from the identity, the estimates will be bad (and the inequalities may fail to be true).

Let us consider two subcases:

a. $k > 0$

Then $G_n = q_{2n-1} + iq_{2n}$. As L is close to the pseudo-inverse, we can expect $\alpha_{2n-1} \approx 1$, $\alpha_{2n} \approx i$ and $\alpha_j \approx 0$ for all other j.

$$\begin{array}{rl} & |H_l\,(h, q_0, \ldots, q_{20}, x_{11}, \ldots, x_l)_n| \geq \\ \geq & \left|(1-h)q_{2n-1} + (1-h)iq_{2n} + h\sum_{i=0}^{20} \alpha_i q_i\right| - \delta \geq \\ \geq & |(1-h)q_{2n-1} + (1-h)iq_{2n} + h\alpha_{2n-1}q_{2n-1} + h\alpha_{2n}q_{2n}| - \\ & - \sum_{i\in\{0,\ldots,20\}, i\neq 2n-1,2n} |\alpha_i|\,\beta_2 - \delta \end{array}$$

Note that we have used the fact that $|q_i| \leq \tilde{q}_i \leq \beta_2$.

If $k = 2n - 1$ then let us denote $\alpha := \Re\alpha_{2n-1}$, $\tilde{\alpha} := \Re\alpha_{2n}$ and use the fact that an absolute value of a number is greater or equal to the absolute value of its real part.

If $k = 2n$ then let us denote $\alpha := \Im\alpha_{2n}$, $\tilde{\alpha} := \Im\alpha_{2n-1}$ and use the fact that an absolute value of a number is greater or equal to the absolute value of its imaginary part.

In both cases we will obtain:

$$\begin{aligned} &|H_l(h, q_0, \dots, q_{20}, x_{11}, \dots, x_l)_n| \geq \\ &\geq \; |(1-h)q_k + h\alpha q_k| - |\tilde{\alpha}|\beta_2 - \textstyle\sum_{i\in\{0,\dots,20\}, i\neq 2n-1,2n} |\alpha_i|\,\beta_2 - \delta \geq \\ &\geq \; |q_k| - \Big(|1-\alpha|\beta_2 + |\tilde{\alpha}|\beta_2 + \textstyle\sum_{i\in\{0,\dots,20\}, i\neq 2n-1,2n} \left(|\Re\alpha_i| + |\Im\alpha_i|\right)\beta_2 + \delta\Big) \end{aligned}$$

We use the inequality $|c| \leq |\Re c| + |\Im c|$ because the program uses real matrices and, from that point of view, it was the natural way to compute it. The term in the bracket is computed using the interval arithmetics and we obtain another constant δ_2 such that:

$$|H_l(h, q_0, \dots, q_{20}, x_{11}, \dots, x_l)_n| \geq |q_k| - \delta_2$$

b. For $k = 0$.

This case is even simpler than the previous, as we have $G_0 = q_0$, $\alpha_0 \in \mathbb{R}$ and we expect $\alpha_0 \approx 1$ and $\alpha_j \approx 0$ for all other j.

Similarly to the previous case we obtain:

$$\begin{aligned} &|H_l(h, q_0, \dots, q_{20}, x_{11}, \dots, x_l)_n| \geq \\ &\geq \; \left|(1-h)q_0 - h\textstyle\sum_{i=0}^{20} \alpha_i q_i\right| - \delta \geq \\ &\geq \; |(1-h)q_0 + h\alpha_0 q_0| - \textstyle\sum_{i\in\{1,\dots,20\}} |\alpha_i|\,\beta_2 - \delta \geq \\ &\geq \; |q_0| - |1-\alpha|\beta_2 - \textstyle\sum_{i\in\{1,\dots,20\}} \left(|\Re\alpha_i| + |\Im\alpha_i|\right)\beta_2 - \delta \geq \\ &\geq \; |q_0| - \delta_2 \end{aligned}$$

For some δ_2 that can be computed.

Thus in both cases we obtain $|H_l(h, q_0, \dots, q_{20}, x_{11}, \dots, x_l)_n| \geq |q_k| - \delta_2 = \tilde{q}_k - \delta_2$. But the $\tilde{q}_k$ have been chosen as $1.000001\delta_2$, so the inequality $H_l \neq 0$ is obviously satisfied. What could have failed if the $\tilde{q}_i$ were too large is condition (3.7). But the program checks that it is satisfied.

This completes the proof.

□

Table 3.2: The estimates of of $\max|x_n|$ and $\max|\tau - \hat{\tau}|$ for the computed $\tilde{q}_i$

n	$\max\lvert x_n\rvert$	$\frac{\beta_2}{(n+1)^2}$
0	$0.96605 \cdot 10^{-7}$	$122.547 \cdot 10^{-7}$
1	$0.66554 \cdot 10^{-7}$	$30.6367 \cdot 10^{-7}$
2	$1.60240 \cdot 10^{-7}$	$13.6163 \cdot 10^{-7}$
3	$0.93587 \cdot 10^{-7}$	$7.65919 \cdot 10^{-7}$
4	$0.92699 \cdot 10^{-7}$	$4.90188 \cdot 10^{-7}$
5	$0.95484 \cdot 10^{-7}$	$3.40408 \cdot 10^{-7}$
6	$0.24202 \cdot 10^{-7}$	$2.50096 \cdot 10^{-7}$
7	$0.37490 \cdot 10^{-7}$	$1.91480 \cdot 10^{-7}$
8	$0.06137 \cdot 10^{-7}$	$1.51293 \cdot 10^{-7}$
9	$0.20605 \cdot 10^{-7}$	$1.22547 \cdot 10^{-7}$
10	$0.22103 \cdot 10^{-7}$	$1.01279 \cdot 10^{-7}$

$\max\lvert\tau - \hat{\tau}\rvert$	$\Delta\tau$
$3.26242 \cdot 10^{-7}$	$5.00000 \cdot 10^{-7}$

Now, let us define the second homotopy:

$$H_l^L(h, q_0, \ldots, q_{20}, x_{11}, \ldots, x_l) := \begin{bmatrix} q_0 \\ q_1 + iq_2 \\ q_3 + iq_4 \\ \vdots \\ q_{19} + iq_{20} \\ \left(t_{11}\left(\hat{\tau} + h\left(\hat{\tau} - \tau\right)\right) + K\right) x_{11} \\ \vdots \\ \left(t_l\left(\hat{\tau} + h\left(\hat{\tau} - \tau\right)\right) + K\right) x_l \end{bmatrix}$$

This deforms G_l into a linear G_l^L:

$$G_l^L : \mathbb{R}^{21} \times \mathbb{C}^{l-10} \longrightarrow \mathbb{R} \times \mathbb{C}^l$$

$$G_l^L(q_0,\ldots,q_{20},x_{11},\ldots,x_l) := \begin{bmatrix} q_0 \\ q_1+iq_2 \\ q_3+iq_4 \\ \vdots \\ q_{19}+iq_{20} \\ (t_{11}(\hat\tau)+K)\,x_{11} \\ \vdots \\ (t_l(\hat\tau)+K)\,x_l \end{bmatrix}$$

Lemma 22. *Let $l > 10$, $(q_0,\ldots,q_{20},x_{11},\ldots,x_l) \in P_l(\tilde X_3)$ and let $\exists k \le 20 : |q_k| = \tilde q_k$ or $\exists 10 < n \le l : |x_n| = \frac{1}{(|n|+1)^2}$. Then $\forall h \in [0;1]$: $H_l^L(h,q_0,\ldots,q_{20},x_{11},\ldots,x_l) \neq 0$*

Proof: If $|q_k| = \tilde q_k$ then let $n := \left\lceil \frac{k}{2} \right\rceil$. We have $H_l^L(h,q_0,\ldots,q_{20},x_{11},\ldots,x_l)_n = G(q_0,\ldots,q_{20},x_{11},\ldots,x_l)_n = H_l(0,q_0,\ldots,q_{20},x_{11},\ldots,x_l)_n$ and in the proof of last lemma we have shown that for $|q_k| = \tilde q_k$ it is not zero.

If $|x_n| = \frac{\beta_2}{(n+1)^2}$ then we have $H_l^L(h,q_0,\ldots,q_{20},x_{11},\ldots,x_l)_n = (t_n(\hat\tau + h(\hat\tau - \tau)) + K)\,x_n = (t_n(\tilde\tau)+K)\,x_n$ for some $\tilde\tau \in [\underline{\tau};\overline{\tau}]$, and in the previous proof, we have shown the absolute value of such expression to be strictly bigger than a non-negative RHS. □

Observation 6. *The degree* $\deg(F_l, \tilde P_l(\tilde X_3), 0)$ *is well-defined and non-zero.*

Proof: If it well-defined because $H_l(1,q_0,\ldots,q_{20},x_{11},\ldots,x_l) \neq 0$ for $q_0,\ldots,q_{20},x_{11},\ldots,x_l$ on the boundary. The degree of H_l is equal to the degree of the linear function G_l^L. Its degree is also well-defined because $H_l^L(0,q_0,\ldots,q_{20},x_{11},\ldots,x_l)$ has no zeros on the border. Thus, the determinant of G_l^L must be non-zero and from Lemma 2 the degree is non-zero. □

Theorem 5. *Equation (1.3) for $f(x) = -5.13\sin x$ has a periodic orbit for some $\tau \in [\underline{\tau},\overline{\tau}]$ those Fourier coefficients x are such that $(\tau,x) \in \tilde X_3$.*

Proof: From Observation 6 we have that the Brouwer local degree is non-zero, thus there is a zero of F_l in $\tilde P_l(\tilde X_3)$. This happens for each $l > 10$, thus from Lemma 19 we have a zero of F and from Theorem 1 we have a solution of the DDE.

□

Chapter 4

On the applicability of the method

4.1 Abstract result

In this chapter we will prove that the method can be applied for a class of equations and orbits. This will be an abstract result – we will not try to check if the proof can be made within a reasonable computing time. However, the results from Chapter 3 show that the method can be successfully used.

For simplicity, we will assume that we have an ideal computer with an infinite precision (i.e. it can represent all real numbers including irrationals). However, the inequalities we check are sharp (i.e. the intervals our ideal program will compute for the LHS and RHS will be separated by a gap of an $\epsilon > 0$), and the computations are finite. Thus, we can expect that on a real computer we can use a precision high enough for the inequality still to be true, even if with a gap of $\frac{\epsilon}{2}$.

Let us recall the general form of equation (1.3):

$$x'(t) := \frac{1}{\tau} f(x(t - \tau))$$

Where:

$$f(x) = \sum_{k=0}^{\infty} f_k x^k$$

$$|f_k| < \frac{\alpha_f}{k!}$$

Let us assume that there is a periodic solution of the equation. For some $\hat{\tau} > 0$ it is 2π-periodic. From Lemma 9 and Theorem 1 we know that its

Fourier coefficients are in X_β for some β and that they satisfy equation (2.1). Let us denote these Fourier coefficients by c. Let β_c be such that $c \in X_{\beta_c}$

We will assume that we have found an approximation of this solution. Then, using e.g. the Newton method, we can refine it. We can expect to be able to find an approximation of any arbitrary precision for an arbitrary (but finite) number of terms.

Again, to simplify the calculations, we will assume that we can find the exact values for any finite number of terms. By $\hat{c}^N$ we will denote such a finite subsequence of c:

$$\left(\hat{c}^N\right)_n := \begin{cases} c_n & \text{for } -N \le n \le N \\ 0 & \text{otherwise} \end{cases}$$

We will derive some conditions for c and f that will guarantee that the method works (Theorem 6) – that for some arbitrary N_m and β_m one can write a program that proves the existence of an orbit in $c^N + X_{\beta_2}$, for some $N > N_m$, $\beta_2 < \beta_m$.

To have such a program, we will need to have explicit formulas for most of the constants in this chapter. These formulas should be possible to compute with some finite computations (e.g. no infinite sum or integrals). We will call such constants computable by a program or computable.

N and β_2 from Theorem 6 will also be computable by a program what allows for an even stronger interpretation of the result – one can (theoretically, on an ideal computer) write a better program than the ones in Chapter 3 – a program that, if provided with approximated solutions of arbitrary length, can by itself find some β_2 small enough and N large enough such that the method proves the existence of the orbit in an X_{β_2} neighborhood of $\hat{c}^N$.

There is a problem that to compute the N we will need a $\beta_{\hat{c}}$ such that $\hat{c}^N \in X_{\beta_{\hat{c}}}$. To compute it directly, we would need to know N before-hand which is, of course, impossible. The β_c would be a valid $\beta_{\hat{c}}$ for any N, but β_c is not computable. We know that β_c exists and from Lemma 7 we even know that for some C': $\forall n : |c_n| < \frac{C'}{(n+1)^3}$. Thus, the β_c can be computed from a finite number of terms – for some N_0 the $\max_{n=0,\ldots,N_0} |c_n|(n+1)^2$ is a valid β_c. But the N_0 can be arbitrarily large (even if in practice we expect it to be small), which makes it not computable.

However, if we want to prove the existence of a program proving the existence of a solution, the program can have some parameters in its code. We can require an N_0 and a $\beta_{\hat{c}}$ to be among them. The program checks that $\hat{c}^{N_0} \in X_{\beta_{\hat{c}}}$, computes the N from the theorem with this $\beta_{\hat{c}}$ and checks that $N = N_0$. If all this is satisfied, the program has proved the existence of a solution.

If we want to have the better program – the program automatically proving the existence of a solution, provided arbitrarily good approximations – we can also overcome this problem. We can start with computing a $\beta_{\hat{c}}$ such that e.g. $\hat{c}^{10} \in X_{\beta_{\hat{c}}}$. For this $\beta_{\hat{c}}$, we can compute N from the theorem. For this N we check if $\hat{c}^N \in X_{\beta_{\hat{c}}}$. If it is not, then we take a larger $\beta_{\hat{c}}$ and, for it, compute another N. We repeat it until $\hat{c}^N \in X_{\beta_{\hat{c}}}$. Because $\beta_{\hat{c}} \leq \beta_c$ and β_c can always be reached after a finite number of steps, this procedure will stop. As the last N was computed using a $\beta_{\hat{c}}$ such that $\hat{c}^N \in X_{\beta_{\hat{c}}}$, the program has rigorously proved the existence of an orbit in $c^N + X_{\beta_2}$. It may happen that $c \notin X_{\beta_{\hat{c}}}$. That could mean (if β_2 is small enough) that we have proved the existence of another solution that was near c. This is a generic problem of an automated search for solutions – if two solutions are close to each other then the Newton method can also be approximating any of them.

Thus, in the rest of the chapter we can treat the $\beta_{\hat{c}}$ as known.

Let us recall some auxiliary notation defined in previous chapters that will be used below:

$$\tilde{f}(x) = \sum_{k=0}^{\infty} |f_k| x^k$$

$$t_n(\tau) = in\tau e^{in\tau}$$

$$L_n(\tau) = t_n'(\hat{\tau})(\tau - \hat{\tau}) = (ine^{in\hat{\tau}} - n^2 e^{in\hat{\tau}})(\tau - \hat{\tau})$$

$$r_n(\tau) = t_n(\tau) - t_n(\hat{\tau}) - L_n(\tau)$$

and prove some lemmas:

Lemma 23. *Let I be a compact interval. The sum in the definition of $\tilde{f}(x)$:*

$$\sum_{k=0}^{\infty} |f_k| x^k$$

is finite and a uniform upper bound of it for $x \in I$ can be computed by a program.

Proof: Let $x \in I$, $M = \max I$.

$$\sum_{k=0}^{\infty} |f_k| x^k \leq \sum_{k=0}^{\infty} \frac{\alpha_f^k}{k!} x^k \leq \sum_{k=0}^{\infty} \frac{(x\alpha_f)^k}{k!}$$

Let $k_0 > M\alpha_f$. Then:

$$\sum_{k=0}^{\infty} |f_k| x^k \leq \sum_{k=0}^{k_0-1} \frac{(M\alpha_f)^k}{k!} + \frac{(M\alpha_f)^{k_0}}{k_0!} \sum_{k=k_0}^{\infty} \frac{(M\alpha_f)^{k-k_0}}{k_0^{k-k_0}} = \\ = \sum_{k=0}^{k_0-1} \frac{(M\alpha_f)^k}{k!} + \frac{(M\alpha_f)^{k_0}}{k_0!} \frac{1}{1-\frac{M\alpha_f}{k_0}}$$

and the last formula can be computed by a program. □

Lemma 24. *There exist a constant K_{24} (computable by a program) such that for each: $N \in \mathbb{N}$, $\frac{K_{24}}{N} \le \beta_2 < \frac{\hat{\tau}}{2}$, $n > N$, $x \in X_{\beta_2}$ and $\tau \in [\hat{\tau} - \beta_2; \hat{\tau} + \beta_2]$, we have:*

$$|t_n(\tau)| \frac{\beta_2}{(n+1)^2} > \left| \sum_{k=0}^{\infty} f_k \left((x + \hat{c}^N)^{*k} \right)_n \right|$$

Before proceeding to the proof, let us note that the condition $\beta_2 \ge \frac{K_{24}}{N}$ is the key of the lemma. It shows how decreasing the β_2 increases the threshold above which the uniform estimate works. Other assumptions are technical.

Proof: For the LHS we have the estimate:

$$|t_n(\tau)| \frac{\beta_2}{(n+1)^2} = \frac{n\tau\beta_2}{(n+1)^2} \ge \frac{\frac{1}{2}n\hat{\tau}\beta_2}{(n+1)^2}$$

For the RHS we have:

$$\begin{aligned} \left|\sum_{k=0}^{\infty} f_k \left((x+\hat{c}^N)^{*k}\right)_n\right| &\le \sum_{k=0}^{\infty} |f_k| \left|\left((x+\hat{c}^N)^{*k}\right)_n\right| \le \\ \le \ \sum_{k=0}^{\infty} |f_k| \frac{C^{k-1}(\beta_{\hat{c}}+\beta_2)^k}{(n+1)^2} &< \frac{1}{C(n+1)^2} \tilde{f}(C(\beta_{\hat{c}} + \tfrac{\hat{\tau}}{2})) \end{aligned}$$

The inequality LHS > RHS is satisfied if:

$$\beta_2 \ge \frac{\tilde{f}(C(\beta_{\hat{c}} + \frac{\hat{\tau}}{2}))}{C\frac{1}{2}n\hat{\tau}} = \frac{2\tilde{f}(C(\beta_{\hat{c}} + \frac{\hat{\tau}}{2}))}{C\hat{\tau}} \cdot \frac{1}{n}$$

Thus, if we take a computable constant $K_{24} := \frac{2\tilde{f}(C(\beta_{\hat{c}}+\frac{\hat{\tau}}{2}))}{C\hat{\tau}}$, the inequality will be always satisfied.

□

Lemma 25. *For all $N > 0$ and $\beta_2 := \frac{K_{24}}{N}$ (where K_{24} is the constant from Lemma 24) there exist a computable constant $K(N)$, such that $K(N) \to 0$ ($N \to \infty$) and for each $n \le N$:*

$$\left| t_n(\hat{\tau}) \left(\hat{c}^N\right)_n - \sum_{k=0}^{N} f_k \left(\hat{c}^N\right)_n^{*k} \right| \le N^{-\frac{1}{2}} \frac{\beta_2 K(N)}{(n+1)^{1.3}}$$

Proof: Let us define:

$$K(N) := \left| t_n(\hat{\tau}) \left(\hat{c}^N\right)_n - \sum_{k=0}^{N} f_k \left(\hat{c}^N\right)_n^{*k} \right| \frac{(n+1)^{1.3}}{N^{-\frac{1}{2}}\beta_2}$$

Such $K(N)$ is computable and obviously satisfies the inequality. What we need to show is that $K(N) \to 0$ ($N \to \infty$). We need only a theoretical result, thus we do not need the formulas to be computable. Hence, we can use β_c (which is such that $c \in X_{\beta_c}$) and not only $\beta_{\hat{c}}$.

We want to prove an asymptotic behavior, thus we can assume $N > \alpha_f C\beta_c$. Let us define:

$$\Delta c := \hat{c}^N - c$$

From Lemma 7, there exist a C_4 such that:

$$|c_n| \leq \frac{C_4}{(|n|+1)^4}$$

Thus, we have that:

$$\Delta c \in X_{\frac{C_4}{(N+1)^2}}$$

Let $\beta_\Delta := \frac{C_4}{(N+1)^2}$. Let us note that $t_n(\hat{\tau})c_n - \sum_{k=0}^{\infty} f_k c_n^{*k} = 0$, as c is the solution of such equations, and let us compute:

$$\begin{aligned}
&\left|t_n(\hat{\tau})\left(\hat{c}^N\right)_n - \textstyle\sum_{k=0}^{N} f_k \left(\hat{c}^N\right)_n^{*k}\right| = \left|t_n(\hat{\tau})c_n - \textstyle\sum_{k=0}^{N} f_k \left(\hat{c}^N\right)_n^{*k}\right| \leq \\
&\leq \left|t_n(\hat{\tau})c_n - \textstyle\sum_{k=0}^{N} f_k (c+\Delta c)_n^{*k} - \sum_{k=N+1}^{\infty} f_k c_n^{*k}\right| + \left|\textstyle\sum_{k=N+1}^{\infty} f_k c_n^{*k}\right| \leq \\
&\leq \left|t_n(\hat{\tau})c_n - \textstyle\sum_{k=0}^{\infty} f_k c_n^{*k}\right| + \left|\textstyle\sum_{k=1}^{N} f_k \sum_{p=1}^{k} \begin{pmatrix} k \\ p \end{pmatrix} \left(c^{*(k-p)} * \Delta c^{*p}\right)_n\right| + \\
&\quad + \textstyle\sum_{k=N+1}^{\infty} |f_k| \frac{C^{k-1}\beta_c^k}{(n+1)^2} \leq \\
&\leq \textstyle\sum_{k=1}^{N} |f_k| \sum_{p=1}^{k} \begin{pmatrix} k \\ p \end{pmatrix} \frac{C^{k-1}\beta_c^{k-p}\beta_\Delta^p}{(n+1)^2} + \sum_{k=N+1}^{\infty} |f_k| \frac{C^{k-1}\beta_c^k}{(n+1)^2} \leq \\
&\leq \tfrac{1}{C} \textstyle\sum_{k=0}^{N} |f_k| \frac{C^k(\beta_c+\beta_\Delta)^k}{(n+1)^2} - \frac{1}{C}\sum_{k=0}^{N} |f_k| \frac{C^k\beta_c^k}{(n+1)^2} + \frac{1}{C}\sum_{k=N+1}^{\infty} |f_k| \frac{C^k\beta_c^k}{(n+1)^2} \leq \\
&\leq \tfrac{1}{C} \textstyle\sum_{k=0}^{\infty} |f_k| \frac{C^k(\beta_c+\beta_\Delta)^k}{(n+1)^2} - \frac{1}{C}\sum_{k=0}^{\infty} |f_k| \frac{C^k\beta_c^k}{(n+1)^2} + \frac{1}{C}\sum_{k=N+1}^{\infty} |f_k| \frac{C^k\beta_c^k}{(n+1)^2} = \\
&= \tfrac{1}{C} \frac{\tilde{f}(C(\beta_c+\beta_\Delta)) - \tilde{f}(C\beta_c)}{(n+1)^2} + \frac{1}{C}\textstyle\sum_{k=N+1}^{\infty} |f_k| \frac{C^k\beta_c^k}{(n+1)^2}
\end{aligned}$$

The last inequality is true because for each k: $|f_k|\frac{C^k(\beta_c+\beta_\Delta)^k}{(n+1)^2} - |f_k|\frac{C^k\beta_c^k}{(n+1)^2} \geq 0$. Function $\tilde{f}'$ has all the derivatives in 0 positive, thus it is growing, and we have:

$$\begin{aligned}
\tilde{f}(C(\beta_c+\beta_\Delta)) - \tilde{f}(C\beta_c) &\leq \max_{h\in[0;\beta_\Delta]} \tilde{f}'(C(\beta_c+x))h \leq \\
&\leq \tilde{f}'(C(\beta_c+\beta_\Delta))\beta_\Delta \leq \\
&\leq \tilde{f}'(C(\beta_c+C_4))\frac{C_4}{(N+1)^2}
\end{aligned}$$

Let us denote: $K_1 := \tilde{f}'(C(\beta_c + C_4)C_4)$.

We have $N > \alpha_f C\beta_c$, thus:

$$N^2\sum_{k=N+1}^{\infty}|f_k|C^k\beta_c^k \le N^2\sum_{k=N+1}^{\infty}\frac{(\alpha_f C\beta_c)^k}{k!} \le \frac{(\alpha_f C\beta_c)^{N+1}}{(N-1)!}\sum_{m=0}^{\infty}\frac{(\alpha_f C\beta_c)^m}{(N+1)^m} = \\ = \frac{(\alpha_f C\beta_c)^{N+1}}{(N-1)!}\cdot\frac{1}{1-\frac{\alpha_f C\beta_c}{N+1}}$$

Let us denote the last expression by K_2. We have:

$$\left|t_n(\hat{\tau})\left(\hat{c}^N\right)_n - \sum_{k=0}^{N} f_k\left(\hat{c}^N\right)_n^{*k}\right| \le \frac{1}{C}N^{-2}\frac{K_1+K_2}{(n+1)^2} \le \frac{N^{-\frac{1}{2}}\beta_2}{(n+1)^{1.3}}\frac{K_1+K_2}{\sqrt{N}CK_{24}}$$

Thus, we have $0 \le K(N) \le \frac{K_1+K_2}{\sqrt{N}CK_{24}}$, which implies $K(N) \to 0$, as $N \to \infty$.

□

Lemma 26. *There exist a computable constant N_0 such that for all $N > N_0$ and $\beta_2 := \frac{K_{24}}{N}$ there exist a computable constant $K(N)$, such that $K(N) \to 0$ $(N \to \infty)$ and for all $x \in X_{\beta_2}$, $n \in \mathbb{N}$:*

$$\left|\sum_{k=N+1}^{\infty} f_k(\hat{c}^N + x)_n^{*k}\right| \le N^{-\frac{1}{2}}\frac{\beta_2 K(N)}{(n+1)^{1.3}}$$

Proof: Let $N_0 := \max\{2\alpha_f C(\beta_c+1), K_{24}\}$. Then $\beta_2 \le 1$ and $\hat{c}^N + x \in X_{\beta_{\hat{c}}+1}$ (as $\hat{c}^N \in X_{\beta_{\hat{c}}}$). Let us estimate:

$$\left|\sum_{k=N+1}^{\infty} f_k(\hat{c}^N + x)^{*k}\right| \le \sum_{k=N+1}^{\infty}|f_k|\frac{C^{k-1}(\beta_{\hat{c}}+1)^k}{(n+1)^2} \le \frac{1}{C(n+1)^2}\sum_{k=N+1}^{\infty}\frac{\left(\alpha_f C\left(\beta_{\hat{c}}+1\right)\right)^k}{k!}$$

We have $N > N_0 = 2\alpha_f C(\beta_{\hat{c}}+1)$, thus:

$$\begin{aligned} & N^2 2^N\cdot\frac{1}{C(n+1)^2}\sum_{k=N+1}^{\infty}\frac{\left(\alpha_f C(\beta_{\hat{c}}+1)\right)^k}{k!} \le \\ \le\ & \frac{N^2 2^N}{C(n+1)^2}\left(\frac{\left(\alpha_f C(\beta_{\hat{c}}+1)\right)^{N+1}}{(N+1)!}\sum_{k=N+1}^{\infty}\frac{\left(\alpha_f C(\beta_{\hat{c}}+1)\right)^{k-N-1}}{(N+1)^{k-N-1}}\right) \le \\ \le\ & \frac{1}{(n+1)^2}\left(\frac{\left(2\alpha_f C(\beta_{\hat{c}}+1)\right)^{N+1}}{C(N-1)!}\cdot\frac{1}{1-\frac{\left(\alpha_f C(\beta_{\hat{c}}+1)\right)}{N+1}}\right) \le \\ \le\ & \frac{1}{(n+1)^2}\left(\frac{\left(2\alpha_f C(\beta_{\hat{c}}+1)\right)^{N_0+2}}{CN_0!}\cdot\frac{1}{1-\frac{1}{2}}\right) \end{aligned}$$

where the $K_1 := \frac{2\left(2\alpha_f C(\beta_{\hat{c}}+1)\right)^{N_0+2}}{CN_0!}$ is computable. Thus, we have:

$$\left|\sum_{k=N+1}^{\infty} f_k(\hat{c}^N + x)^{*k}\right| \leq N^{-2}2^{-N}\frac{K_1}{(n+1)^2} \leq N^{-\frac{1}{2}}\left(\frac{K_1}{K_{24}\sqrt{N}2^N}\right)\frac{\beta_2}{(n+1)^{1.3}}$$

Thus, taking $K(N) := \frac{K_1}{K_{24}\sqrt{N}2^N}$ satisfies the assertion.

□

Lemma 27. *For each $N > 0$ and $\beta_2 := \frac{K_{24}}{N}$ there exist a computable constant $K(N)$, such that $K(N) \to 0$ ($N \to \infty$) and for all $x \in X_{\beta_2}$ and $n \in \mathbb{N}$:*

$$\left|\sum_{p=2}^{N} (\gamma_p * x^{*p})_n\right| \leq N^{-\frac{1}{2}}\frac{\beta_2 K(N)}{(n+1)^{1.3}}$$

where γ_p is from Lemma 18 and is equal to $\sum_{k=p}^{N} f_k \begin{pmatrix} k \\ p \end{pmatrix} \left((\hat{c}^N)^{*(k-p)}\right)$.

Proof: We have:

$$|\gamma_{p,n}| \leq \sum_{k=p}^{N} |f_k| \begin{pmatrix} k \\ p \end{pmatrix} \left|(\hat{c}^N)_n^{*(k-p)}\right| \leq \sum_{k=p}^{N} \frac{\alpha_f^k}{p!(k-p)!} \cdot \frac{(C\beta_{\hat{c}})^{k-p}}{C(n+1)^2} \leq$$
$$\leq \frac{\alpha_f^p}{p!} \sum_{m=0}^{N-p} \frac{\alpha_f^m}{m!} \frac{(C\beta_{\hat{c}})^m}{C(n+1)^2} \leq \frac{\alpha_f^p}{p! C(n+1)^2} \sum_{m=0}^{\infty} \frac{(C\beta_{\hat{c}}\alpha_f)^m}{m!}$$

Analogously as in Lemma 23, the sum $\sum_{m=0}^{\infty} \frac{(C\beta_{\hat{c}}\alpha_f)^m}{m!}$ is finite and an upper bound of it (that we will denote by K_1) can be found by a program:

$$|\gamma_{p,n}| \leq \frac{K_1\alpha_f^p}{Cp!} \cdot \frac{1}{(n+1)^2} \tag{4.1}$$

Thus $\gamma_p \in X_{\frac{K_1\alpha_f^p}{Cp!}}$, and:

$$\left|\sum_{p=2}^{N} (\gamma_p * x^{*p})_n\right| \leq \sum_{p=2}^{N} \frac{C^p\beta_2^p}{(n+1)^2}\frac{K_1\alpha_f^p}{Cp!} \leq \frac{(C\alpha_f)^2 K_1\beta_2^2}{2!C(n+1)^2}\sum_{p=0}^{N-2}\frac{(C\beta_2\alpha_f)^p}{p!} \leq$$
$$\leq \frac{(C\alpha_f)^2 K_1\beta_2^2}{2!C(n+1)^2}\sum_{p=0}^{\infty}\frac{(C\beta_2\alpha_f)^p}{p!}$$

Again, like in Lemma 23, the sum is finite and we can compute an upper bound K_2. Let $K_3 := \frac{(C\alpha_f)^2 K_1 K_2}{2C}$ and we have:

$$\left|\sum_{p=2}^{N} (\gamma_p * x^{*p})_n\right| \leq \frac{K_3\beta_2^2}{(n+1)^2} \leq N^{-\frac{1}{2}}\left(K_3K_{24}N^{-\frac{1}{2}}\right)\frac{\beta_2}{(n+1)^{1.3}}$$

Taking $K(N) := K_3 K_{24} N^{-\frac{1}{2}}$ satisfies the assertion.

□

Lemma 28. *For each $N > 0$ and $\beta_2 := \frac{K_{24}}{N}$ there exist a constant $K(N)$ (computable by a computer), such that $K(N) \to 0$ ($N \to \infty$) and for all $0 \leq n \leq N$:*

$$\left| \sum_{|k|>N} \gamma_{1,n-k} x_k \right| \leq N^{-\frac{1}{2}} \frac{K(N)\beta_2}{(n+1)^{1.3}}$$

Proof: From equation (4.1), we have that:

$$|\gamma_{1,n}| \leq \frac{K_\gamma}{(|n|+1)^2}$$

for some K_γ computable by a computer.

$$\begin{aligned}
\left|\sum_{|k|>N} \gamma_{1,n-k} x_k\right| &\leq \left|\sum_{k=N+1}^{\infty} \gamma_{1,n-k} x_k\right| + \left|\sum_{k=-\infty}^{-N-1} \gamma_{1,n-k} x_k\right| \leq \\
&\leq \left(\left|\sum_{k=N+1}^{\infty} \gamma_{1,n-k}\right| + \left|\sum_{k=-\infty}^{-N-1} \gamma_{1,n-k}\right|\right) \frac{\beta_2}{(N+1)^2} \leq \\
&\leq \left(\sum_{k=-\infty}^{n-N-1} \frac{K_\gamma}{(|k|+1)^2} + \sum_{k=n+N+1}^{\infty} \frac{K_\gamma}{(k+1)^2}\right) N^{-\frac{1}{2}} \frac{\beta_2}{(N+1)^{1.5}} \leq \\
&\leq 2N^{-0.2} \left(\sum_{k=1}^{\infty} \frac{K_\gamma}{(k+1)^2}\right) N^{-\frac{1}{2}} \frac{\beta_2}{(n+1)^{1.3}}
\end{aligned}$$

The sum $\sum_{k=1}^{\infty} \frac{K_\gamma}{(k+1)^2}$ can be estimated by:

$$\sum_{k=1}^{\infty} \frac{K_\gamma}{(k+1)^2} \leq K_\gamma \int_{x=1}^{\infty} \frac{1}{x^2} dx = K_\gamma$$

Thus, by taking $K(N) = 2N^{-0.2} K_\gamma$, we obtain the assertion.

□

Lemma 29. *For each $N > 0$ and $\Delta\tau := \frac{K_{24}}{N}$ there exist a constant $K(N)$ (computable by a computer), such that $K(N) \to 0$ ($N \to \infty$) and for all $\tau \in [\hat{\tau} - \Delta\tau; \hat{\tau} + \Delta\tau]$, $n \in \mathbb{N}$:*

$$\left| r_n(\tau) \left(\hat{c}^N\right)_n \right| \leq N^{-\frac{1}{2}} \frac{K(N)\Delta\tau}{(n+1)^{1.3}}$$

Proof: Let us take:

$$K(N) := \frac{\left| r_n(\tau) \left(\hat{c}^N\right)_n \right| (n+1)^{1.3}}{N^{-\frac{1}{2}} \Delta\tau}$$

This is computable and the inequality is obviously satisfied. What we need to prove is that $K(N) \to 0$ $(N \to \infty)$.

From Lemma 7 we know that there exist a C_5 such that:

$$\left|\left(\hat{c}^N\right)_n\right| \leq \frac{C_5}{(n+1)^5}$$

We have that $r_n(\tau) = \frac{1}{2}t''_n(\tilde{\tau})\Delta\tau^2$ for some $\tilde{\tau} \in [\hat{\tau} - \Delta\tau; \hat{\tau} + \Delta\tau]$, thus:

$$\begin{aligned} \left|r(n)\left(\hat{c}^N\right)_n\right| &\leq \left|\tfrac{1}{2}t''_n(\tilde{\tau})\right|\tfrac{C_5\Delta\tau^2}{(n+1)^5} \leq \left|\tfrac{1}{2}\left(-in^3 - 2n^2\tilde{\tau}\right)e^{in\tilde{\tau}}\right|\tfrac{C_5\Delta\tau^2}{(n+1)^5} \leq \\ &\leq \tfrac{n^3(\tilde{\tau}+1)}{(n+1)^3}\tfrac{K_{24}}{N}\tfrac{C_5\Delta\tau}{(n+1)^2} \leq N^{-\frac{1}{2}}\left((\hat{\tau}+1)K_{24}C_5N^{-\frac{1}{2}}\right)\tfrac{\Delta\tau}{(n+1)^{1.3}} \end{aligned}$$

Thus, we have that $0 \leq K(N) \leq (\hat{\tau}+1)K_{24}C_5N^{-\frac{1}{2}}$, thus $K(N) \to 0$ $(N \to \infty)$.

□

For $N \in \mathbb{N}$ let us define:

$$T_N : \mathbb{R} \times \mathbb{R} \times \mathbb{C}^N \longrightarrow \mathbb{R} \times \mathbb{C}^N$$

$$T_N(\tau - \hat{\tau}, x_0, \ldots, x_N) :=$$

$$\left\{(in - n^2\hat{\tau})e^{in\hat{\tau}}c_n(\tau - \hat{\tau}) + in\hat{\tau}e^{in\hat{\tau}}x_n - \sum_{k=-N}^{-1}\gamma_{1,n-k}\overline{x_{-k}} - \sum_{k=0}^{N}\gamma_{1,n-k}x_k\right\}_{n=0}^{N}$$

where $\gamma_{1,n} := \sum_{k=1}^{N} kf_k\left(\left(\hat{c}^N\right)^{*(k-1)}\right)_n$ is from Lemma 18. Let us note that T_N is an $\mathbb{R}$-linear function. Hence, we will also consider T_N as a $\mathbb{R}^{2N+2} \to \mathbb{R}^{2N+1}$ function.

Let us also note that the sequence γ_1 consists of the Fourier coefficients of $f'(x(t))$ (where $x(t)$ is the solution those coefficients are c. This follows from using Lemma 18 and taking the limit $N \to \infty$). Thus, any condition on the family $\{T_N\}_{N\in\mathbb{N}}$ is in fact a condition on $f'(x(t))$ and $\hat{\tau}$ (the latter being defined by the period of the solution).

Like in Section 3.4, we will use pseudo-inverses of T_N (which is the linear part of the equations) to change the variables. Again, we will assume that we have an ideal computer that computes true pseudo-inverses, not approximations. We will call $\left\{L_N : \mathbb{R}^{2N+1} \to \mathbb{R}^{2N+2}\right\}_{N=N_0}^{\infty}$ a family of pseudo-inverses if for each N: $T_N \circ L_N = I_{2N+1}$, where I_{2N+1} is the identity on $\mathbb{R}^{2N+1}$.

For such a family of pseudo-inverses, we will define the homotopy. Like in Section 3.3 we will do it on the infinite-dimensional space:

$$\begin{aligned} &H^N(h, q_0, \ldots, q_{2N}, x_{N+1}, \ldots) := \\ &\quad hF(L_N(q_0, \ldots, q_{2N}), x_{N+1}, x_{N+2}, \ldots) + (1-h)G^N(q_0, \ldots, q_{2N}, x_{N+1}, \ldots) \end{aligned}$$

where G^N is:

$$G^N(q_0,\ldots,q_{2N},x_{N+1},\ldots) := \begin{bmatrix} q_0 \\ q_1 + iq_2 \\ \vdots \\ q_{2N-1} + iq_{2N} \\ t_{N+1}(\tau)x_{N+1} \\ t_{N+2}(\tau)x_{N+2} \\ \vdots \end{bmatrix}$$

We will prove that, under some conditions on $\{L_N\}$, for some N big enough our estimates will be good enough to prove that there are no zeros on the boundary of a certain, arbitrarily small neighborhood.

Before proceeding to the theorem, let us define two diagonal martices. Let δ_{ij} be the Kronecker delta.

$$D_1 := \left\{\delta_{ij}(\lfloor 0.5i\rfloor + 1)^2\right\}_{i=0\ldots 2N+1, j=0\ldots 2N+1}$$

$$D_2 := \left\{\delta_{ij}\frac{1}{(\lceil 0.5i\rceil + 1)^{1.3}}\right\}_{i=0\ldots 2N, j=0\ldots 2N}$$

The assumption about L_N will be that:

$$\forall N : \|D_1 L_N D_2\|_\infty \leq D \tag{4.2}$$

where $\|\cdot\|_\infty$ is the matrix norm induced by the supremum norm.

This means that if we find some $\tilde{q}_i$ with $\tilde{q}_i \leq \frac{C(N)}{(\lceil 0.5i\rceil+1)^{1.3}}$ then this implies that $x_n \leq \frac{C(N)D}{(n+1)^2}$, where the factor D does not depend on N. This will be enough for the method to work.

We will discuss this assumption in more detail in Section 4.2.

Theorem 6. *Let $c \in X$ and $\hat{\tau} > 0$ be a solution of equation (2.1). Let $N_m > 0$, $\beta_m > 0$ fixed.*

Then, if there exists a family of pseudo-inverses $\{L_N\}_{N=N_m}^\infty$ such that there exist a constant D such that:

$$\forall N : \|D_1 L_N D_2\|_\infty \leq D$$

then there exist constants $N > N_m$, $\beta_2 < \beta_m$ and $\tilde{q_0},\ldots,\tilde{q_{2N}}$ (all computable by a program), such that the homotopy H^N has no zeros on the boundary of the set:

$$\tilde{X}_3 := \left\{(q_0,\ldots,q_{2N},x_{N+1},\ldots) : \forall 0 \leq i \leq 2N : |q_i| \leq \tilde{q}_i \forall n > N : |x_n| \leq \frac{\beta_2}{(n+1)^2}\right\}$$

Moreover, on the set $\tilde{X}_3$ we have that $|\hat{\tau} - \tau| < \beta_2$, where $\tau := \hat{\tau} + L_N(q_0, \ldots, q_{2N}, x_{N+1}, \ldots, x_l)_1$

Proof:

I. Defining the constants

Let N_0 be the maximum of N_m, K_{24} (from Lemma 24), $\frac{2K_{24}}{\hat{\tau}}$, $\frac{K_{24}}{\beta_m}$ and N_0 from Lemma 26. First, let us do some estimates valid for any $N > N_0$. Later we will fix a specific value of N.

Let $\beta_2 := \frac{K_{24}}{N}$ (where K_{24} is from Lemma 24). Let us note that $\beta_2 < \min\left\{1, \frac{\hat{\tau}}{2}, \beta_m\right\}$.

To define the $\tilde{q}_i$, we will proceed like in Section 3.4 – we will choose them so that an inequality between an estimate of the LHS and an estimate of the RHS is true. Later we will need to check that the assumptions under which the estimates were made are true. However, the assumptions in this case will be stronger – for one estimate it will not be enough that $|x_n| < \frac{\beta_2}{(n+1)^2}$, but we will need that for $0 \le n \le N : |x_n| < N^{-\frac{1}{2}} \frac{\beta_2}{(n+1)^2}$.

Thus, let us take some $\tau \in [\hat{\tau} - \Delta\tau; \hat{\tau} + \Delta\tau]$ for $\Delta\tau = N^{-\frac{1}{2}}\beta_2$ and $x \in X_{\beta_2}$ such that $\exists q_0, \ldots, q_{2N} : (\tau - \hat{\tau}, x_0, \ldots, x_N) := L_N(q_0, \ldots, q_{2N})$, $\forall 0 \le n \le N : |x_n| < N^{-\frac{1}{2}} \frac{\beta_2}{(n+1)^2}$.

To compute $\tilde{q}_i$, let us take $n := \left\lceil \frac{i}{2} \right\rceil$. Let us denote $G^N_n := G^N(q_0, \ldots, q_{2N}, x_{N+1}, \ldots)_n$ and estimate:

$$
\begin{aligned}
&\left|H^N(h, q_0, \ldots, q_{2N}, x_{N+1}, \ldots)_n\right| = \\
&= \left|(1-h)G^N_n + h\left(t_n(\tau)\left(\left(\hat{c}^N\right)_n + x_n\right) - \sum_{k=0}^{\infty} f_k\left(\hat{c}^N + x\right)^{*k}_n\right)\right| \ge \\
&\ge \left|(1-h)G^N_n + h\left(t_n(\tau)\left(\left(\hat{c}^N\right)_n + x_n\right) - \sum_{k=0}^{N} f_k\left(\hat{c}^N + x\right)^{*k}_n\right)\right| - \\
&\quad - \left|\sum_{k=N+1}^{\infty} f_k\left(\hat{c}^N + x\right)^{*k}_n\right| \ge \\
&\ge \left|(1-h)G^N_n + h\left(t_n(\hat{\tau})x_n + \left(\hat{c}^N\right)_n L_n(\tau) - \sum_{k=-N}^{N} \gamma_{1,n-k}x_k\right)\right| - \\
&\quad - \left|((t_n(\tau) - t_n(\hat{\tau}))x_n\right| - \left|r_n(\tau)\left(\hat{c}^N\right)_n\right| - \left|t_n(\hat{\tau})\left(\hat{c}^N\right)_n - \sum_{k=0}^{\infty} f_k\left(\hat{c}^N\right)^{*p}_n\right| - \\
&\quad - \left|\sum_{|k|>N} \gamma_{1,n-k}x_k\right| - \left|\sum_{p=2}^{N}(\gamma_p * x^{*p})_n\right| - \left|\sum_{k=N+1}^{\infty} f_k\left(\hat{c}^N + x\right)^{*k}_n\right|
\end{aligned}
$$

The second term can be estimated by:

$$
\begin{aligned}
&\left|((t_n(\tau) - t_n(\hat{\tau}))x_n\right| \le \max_{\tilde{\tau}\in[\hat{\tau}-\Delta\tau;\hat{\tau}+\Delta\tau]} \left|f'_n(\tilde{\tau})\right| \Delta\tau \frac{N^{-\frac{1}{2}}\beta_2}{(n+1)^2} \le \\
&\le \max_{\tilde{\tau}\in[\hat{\tau}-\Delta\tau;\hat{\tau}+\Delta\tau]} \left|(in - n^2\tilde{\tau})e^{in\tilde{\tau}}\right| N^{-\frac{1}{2}}\beta_2 \frac{N^{-\frac{1}{2}}\beta_2}{(n+1)^2} \le \\
&\le n^2(\hat{\tau}+1)\frac{K_{24}}{N^2}\frac{\beta_2}{(n+1)^2} \le N^{-\frac{1}{2}}\left(K_{24}\left(\hat{\tau}+1\right)N^{-\frac{1}{6}}\right)\frac{\beta_2}{(n+1)^{1.3}}
\end{aligned}
$$

Thus, this is an estimation by $N^{-\frac{1}{2}}\frac{K_1(N)\beta_2}{(n+1)^{1.3}}$ where $K_1(N)\to 0$ $(N\to\infty)$.

For each of the terms 3-6 we have a lemma that allows us to estimate the upper bound of the term. These estimates also have a form $N^{-\frac{1}{2}}\frac{K(N)\beta_2}{(n+1)^{1.3}}$, where $K(N)\to 0$ $(N\to\infty)$. Thus, the whole can be estimated like that:

$$\begin{aligned}&\left|H^N(h,q_0,\dots,q_{2N},x_{N+1},...)_n\right|\geq\\ \geq\ &\left|(1-h)G_n^N+h\left(t_n(\hat\tau)x_n+\left(\hat c^N\right)_n L_n(\tau)-\textstyle\sum_{k=-N}^{N}\gamma_{1,n-k}x_k\right)\right|-N^{-\frac{1}{2}}\frac{K(N)\beta_2}{(n+1)^{1.3}}=\\ =\ &\left|(1-h)G_n^N+hT_N(\tau-\hat\tau,x_0,\dots,x_N)_n\right|-N^{-\frac{1}{2}}\frac{K(N)\beta_2}{(n+1)^{1.3}}\end{aligned}$$

The $(\tau-\hat\tau,x_0,\dots,x_N)$ are an image of $(q_0,\dots,q_{2N})$ through L_N, thus $T_N(\tau-\hat\tau,x_0,\dots,x_N)=T_N\circ L_N(q_0,\dots,q_{2N})=(q_0,\dots,q_{20})$. This means that on the n-th coefficient: $T_N(\tau-\hat\tau,x_0,\dots,x_N)_n=G_n^N$. Thus:

$$\left|H^N(h,q_0,\dots,q_{2N},x_{N+1},...)_n\right|\geq\left|G_n^N\right|-N^{-\frac{1}{2}}\frac{K(N)\beta_2}{(n+1)^{1.3}}\tag{4.3}$$

Let us define $\tilde q_i:=2N^{-\frac{1}{2}}\frac{K(N)\beta_2}{(n+1)^{1.3}}$.

For such $\{\tilde q_i\}_{i=0}^{2N}$, we will show that we can choose N so that we have:

$$\forall 0\leq i\leq 2N:|q_i|\leq\tilde q_i\Rightarrow\forall 0\leq n\leq N:|x_n|\leq N^{-\frac{1}{2}}\frac{\beta_2}{(n+1)^2},|\tau-\hat\tau|\leq N^{-\frac{1}{2}}\beta_2$$

We have that:

$$\begin{aligned}|x_0|&=\left|\left(L_N\begin{bmatrix}q_0\\ \vdots\\ q_{2N}\end{bmatrix}\right)_0\right|\leq\left|\left(L_N\begin{bmatrix}\vdots\\ 2N^{-\frac{1}{2}}\frac{K(N)\beta_2}{(n+1)^{1.3}}\\ \vdots\end{bmatrix}\right)_0\right|=\frac{2K(N)\beta_2}{\sqrt N}\left|\left(L_ND_2\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}\right)_0\right|=\\ &=\frac{2K(N)\beta_2}{\sqrt N}\left|\left(D_1L_ND_2\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}\right)_0\right|\leq N^{-\frac{1}{2}}\beta_2 2K(N)D\leq N^{-\frac{1}{2}}\beta_2 4K(N)D\end{aligned}$$

$$\begin{aligned}|\tau-\hat\tau|&=\left|\left(L_N\begin{bmatrix}q_0\\ \vdots\\ q_{2N}\end{bmatrix}\right)_1\right|\leq\left|\left(L_N\begin{bmatrix}\vdots\\ 2N^{-\frac{1}{2}}\frac{K(N)\beta_2}{(n+1)^{1.3}}\\ \vdots\end{bmatrix}\right)_1\right|=\frac{2K(N)\beta_2}{\sqrt N}\left|\left(L_ND_2\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}\right)_1\right|=\\ &=\frac{2K(N)\beta_2}{\sqrt N}\left|\left(D_1L_ND_2\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}\right)_1\right|\leq N^{-\frac{1}{2}}\beta_2 2K(N)D\leq N^{-\frac{1}{2}}\beta_2 4K(N)D\end{aligned}$$

$$\begin{aligned}
(n+1)^2|x_n| &= (n+1)^2\left|\left(L_N\begin{bmatrix} q_0\\ \vdots \\ q_{2N}\end{bmatrix}\right)_{2n} + i\left(L_N\begin{bmatrix} q_0\\ \vdots \\ q_{2N}\end{bmatrix}\right)_{2n+1}\right| \le \\
&\le (n+1)^2\left(\left|\left(L_N\begin{bmatrix} \vdots \\ 2N^{-\frac{1}{2}}\frac{K(N)\beta_2}{(n+1)^{1.3}} \\ \vdots \end{bmatrix}\right)_{2n}\right| + \left|\left(L_N\begin{bmatrix} \vdots \\ 2N^{-\frac{1}{2}}\frac{K(N)\beta_2}{(n+1)^{1.3}} \\ \vdots \end{bmatrix}\right)_{2n+1}\right|\right) = \\
&= \tfrac{2K(N)\beta_2}{\sqrt{N}}\left(\left|\left(D_1L_ND_2\begin{bmatrix} 1\\ \vdots \\ 1\end{bmatrix}\right)_{2n}\right| + \left|\left(D_1L_ND_2\begin{bmatrix} 1\\ \vdots \\ 1\end{bmatrix}\right)_{2n+1}\right|\right) \le \\
&\le N^{-\frac{1}{2}}\beta_2 2K(N)(D+D) = N^{-\frac{1}{2}}\beta_2 4K(N)D
\end{aligned}$$

Thus, we have:

$$\begin{aligned} |x_n| &\le N^{-\frac{1}{2}}(4K(N)D)\tfrac{\beta_2}{(n+1)^2} \\ |\tau-\hat{\tau}| &\le N^{-\frac{1}{2}}(4K(N)D)\beta_2 \end{aligned} \tag{4.4}$$

As $K(N) \to 0$ $(N \to \infty)$, there exist an N greater than N_0 such that $4K(N)D < 1$. We give no explicit formula for the N, but our theoretical program can check all the possible values of N starting from N_0+1 until it finds one that satisfies $K(N) < \frac{1}{4D}$. This will be the N from the assertion.

II. Proving that there are no zeros on the boundary

Let us take some $(q_0, \ldots, q_{2N}, x_{N+1}, \ldots) \in \tilde{X}_3$. Let $(\tau, x_0, \ldots, x_N) := L_N(q_0, \ldots, q_{2N}) + (\hat{\tau}, 0, \ldots, 0)$.

Let us note that N was chosen such that, for $0 \le n \le N$, from (4.4) we have:

$$|x_n| \le N^{-\frac{1}{2}}\frac{\beta_2}{(n+1)^2} \tag{4.5}$$

$$|\Delta\tau| \le N^{-\frac{1}{2}}\beta_2 \tag{4.6}$$

Thus $x \in X_{\beta_2}$.

1. Let $|x_n| = \frac{\beta_2}{(n+1)^2}$, for $n > N$

We have:

$$\begin{aligned}
&\left|H^N(h, q_0, \ldots, q_{2N}, x_{N+1}, \ldots)_n\right| = \\
&= \left|t_n(\tau)x_n + t_n(\tau)\left(\hat{c}^N\right)_n - \textstyle\sum_{k=0}^{\infty} f_k\left((x+\hat{c}^l)^{*k}\right)_n\right| = \\
&= \left|t_n(\tau)x_n - \textstyle\sum_{k=0}^{\infty} f_k\left((x+\hat{c}^l)^{*k}\right)_n\right| \ge \\
&\ge |t_n(\tau)|\tfrac{\beta_2}{(n+1)^2} - \left|\textstyle\sum_{k=0}^{\infty} f_k\left((x+\hat{c}^l)^{*k}\right)_n\right|
\end{aligned}$$

As $N > N_0$, we can use Lemma 24 and obtain that this is greater from zero, thus $H^N(h, q_0, \ldots, q_{2N}, x_{N+1}, \ldots) \ne 0$.

2. $q_i = \tilde{q}_i$ for some $0 \leq i \leq 2N$

Let $n := \left\lceil \frac{i}{2} \right\rceil$. From equations (4.5-4.6) we have that the assumptions under which estimation (4.3) was obtained are satisfied. Thus, the estimation holds:

$$\left| H^N(h, q_0, \ldots, q_{2N}, x_{N+1}, \ldots)_n \right| \geq \left| G_n^N \right| - N^{-\frac{1}{2}} \frac{K(N)\beta_2}{(n+1)^{1.3}}$$

For $i = 0$ or i odd, the absolute value of the real part of G_n^N is equal to $\tilde{q}_i$. For $i > 0$ and even, the absolute value of the imaginary part of G_n^N is equal to $\tilde{q}_i$. The absolute value of a complex number is greater than the absolute value of the real or the imaginary part, thus $|G_n^N| \geq \tilde{q}_i$. This implies $|G_n^N| > N^{-\frac{1}{2}} \frac{K(N)\beta_2}{(n+1)^{1.3}}$, hence $H^N(h, q_0, \ldots, q_{2N}, x_{N+1}, \ldots)_n \neq 0$

This ends the proof. □

If this homotopy is admissible then we have a solution of the equation. To show that, we will (as usually) define a second homotopy:

$$H_N^L(h, q_0, \ldots, q_{2N}, x_{N_1}, \ldots) := \begin{bmatrix} q_0 \\ q_1 + iq_2 \\ \vdots \\ q_{2N-1} + iq_{2N} \\ t_{N+1}(h\tau + (1-h)\hat{\tau})x_{N+1} \\ t_{N+2}(h\tau + (1-h)\hat{\tau})x_{N+2} \\ \vdots \end{bmatrix}$$

Where the τ on the RHS is defined as $L_N(q_1, \ldots, q_{2N})_1$.

Lemma 30. *Let $N > 0$. Let $\tilde{X}_3 := \{(q_0, \ldots, q_{2N}, x_{N+1}, \ldots) : \forall 0 \leq i \leq 2N : |q_i| \leq \tilde{q}_i \forall n > N : |x_n| \leq \frac{\beta_2}{(n+1)^2}\}$ for some $\tilde{q}_i > 0$, $0 < \beta_2 < \frac{\hat{\tau}}{2}$. Let $\tilde{X}_3$ and L_N be such that $|\tau - \hat{\tau}| \leq \beta_2$ for elements of $\tilde{X}_3$. Let $(q_0, \ldots, q_{2N}, x_{N+1}, \ldots) \in \tilde{X}_3$ such that $\exists 0 \leq i \leq 2N : |q_i| = \tilde{q}_i$ or $\exists n > N : |x_n| = \frac{\beta_2}{(n+1)^2}$. Then $\forall h \in [0;1] : H_N^L(h, q_0, \ldots, q_{2N}, x_{N+1}, \ldots) \neq 0$.*

Proof: If $|q_i| = \tilde{q}_i$ then for $n := \left\lceil \frac{i}{2} \right\rceil$ we have on the n-th coefficient: $\forall h \in [0;1] : H_N^L(h, q_0, \ldots, q_{2N}, x_{N+1}, \ldots)_n \neq 0$.

If $|x_n| = \frac{\beta_2}{(n+1)^2}$ then $|H_N^L(h, q_0, \ldots, q_{2N}, x_{N+1}, \ldots)_n| = n\tau \geq \frac{1}{2} n\hat{\tau} > 0$ and $H_N^L(h, q_0, \ldots, q_{2N}, x_{N+1}, \ldots) \neq 0$.

□

Let us define:

$$F_l(q_0, \ldots, q_{2N}, x_{N+1}, \ldots, x_l) := P_l F(\tilde{Q}_l(L_N(q_0, \ldots, q_{2N}), x_{N+1}, \ldots, x_l))$$

Observation 7. *If H^N is admissible then the Galerkin projection of H^N, i.e.: $H_l^N(q_0,\dots,q_{2N},x_{N+1},\dots,x_l) := P_l H^N(\tilde{Q}_l(L_N(q_0,\dots,q_{2N}),x_{N+1},\dots,x_l))$ (for $(q_0,\dots,q_{2N},x_{N+1},\dots,x_l) \in \mathbb{R}^{2N+1}\times\mathbb{C}^{l-N}$) is a homotopy from $F^l(\tau,c+\cdot)$ to the projection of G. Analogously, the projection of H^L is a homotopy from a projection of G to a projection of G^L. There are no zeros on the boundaries for these homotopies.*

Observation 8. *If the homotopy H^N is admissible then the degree* $\deg(F_l,\tilde{P}_l(\tilde{X}_3),0)$ *is well-defined and non-zero.*

Theorem 7. *If the homotopy H^N is admissible then the equation (1.3) has a periodic orbit for some $\tau \in [\underline{\tau},\overline{\tau}]$ those Fourier coefficients x are such that $(\tau,x) \in \tilde{X}_3$.*

Proof: From Observation 8 we have that the Brouwer local degree is non-zero, thus there is a zero of F_l in $\tilde{P}_l(\tilde{X}_3)$. This happens for each $l > N$, thus from Lemma 19 we have a zero of F_l and from Theorem 1 we have a solution of the DDE.

□

4.2 Discussion of the assumption of Theorem 6

The key assumption of Theorem 6 is that:

$$\forall N : \|D_1 L_N D_2\|_\infty \leq D$$

In this section, I provide no rigorous proofs but some heuristics why I think this assumption makes sense. Let us look at the matrix:

$$M := D_2^{-1} T_N D_1^{-1}$$

of which the $D_1 L_N D_2$ is a pseudo-inverse.

The column corresponding to $\tau - \hat{\tau}$ at rows $2n-1$ and $2n$ will have the values:

$$M_{n,\tau} := \begin{bmatrix} \Re\left[(in - n^2\hat{\tau})e^{in\hat{\tau}}c_n\right]\frac{2^{1.3}}{(n+1)^2} \\ \Im\left[(in - n^2\hat{\tau})e^{in\hat{\tau}}c_n\right]\frac{2^{1.3}}{(n+1)^2} \end{bmatrix}$$

The absolute value of $(in - n^2\hat{\tau})e^{in\hat{\tau}}$ is $O(n^2)$. The absolute value of c_n decreases faster then $O(n^{-k})$ for any k (see Lemma 7). Thus, the values in the cell $M_{n,\tau}$ decreases faster then $O(n^{-k})$ for any k.

If we take the intersection of the columns $2n$ and $2n+1$ (corresponding to the x_n variable) and rows $2m-1$ and m, for $m \neq n$ we get a 2×2 cell:

$$M_{m,n} := \begin{bmatrix} (\Re\gamma_{1,n-m} + \Re\gamma_{1,m+k})\frac{(m+1)^{1.3}}{(n+1)^2} & (-\Im\gamma_{1,n-m} + \Im\gamma_{1,m+k})\frac{(m+1)^{1.3}}{(n+1)^2} \\ (\Im\gamma_{1,n-m} + \Im\gamma_{1,m+k})\frac{(m+1)^{1.3}}{(n+1)^2} & (\Re\gamma_{1,n-m} - \Re\gamma_{1,m+k})\frac{(m+1)^{1.3}}{(n+1)^2} \end{bmatrix} \tag{4.7}$$

The $\gamma_{1,n-m}$ can be proved to be $O(|n-m|^{-k})$, for any k (this can be shown like we have shown that $\gamma_{1,n} = O(n^{-2})$ in the proof of Lemma 27, but using the fact that $|c_n| \le \frac{C_k}{(n+1)^k}$). We have $n+k > n-k$, thus $\gamma_{k,n+k}$ is also $O(|n-m|^{-k})$. The $\frac{(m+1)^{1.3}}{(n+1)^2}$ can be proved to be $O(|n-m|^{1.3})$. Thus, the values in the cell $M_{m,n}$ are $O(|n-m|^{-k})$ for any k.

If $m = n$, the cell is a sum of the cell (4.7) and:

$$D_{n,n} := \begin{bmatrix} -n\hat{\tau}\cos(n\hat{\tau})(n+1)^{-0.7} & -n\hat{\tau}\sin(n\hat{\tau})(n+1)^{-0.7} \\ -n\hat{\tau}\sin(n\hat{\tau})(n+1)^{-0.7} & n\hat{\tau}\cos(n\hat{\tau})(n+1)^{-0.7} \end{bmatrix}$$

This cell corresponds to a complex number $in\hat{\tau}e^{in\hat{\tau}}(n+1)^{-0.7}$, those absolute value is greater than $\frac{1}{2}\hat{\tau}n^{0.3}$. As $ine^{in\hat{\tau}}(n+1)^{-0.7} = in^{0.3}e^{in\hat{\tau}} + O(n^{-1.3})$, one can expect that for large n, the term $ie^{in\hat{\tau}}n^{0.3}$ will dominate all the other terms in this cell and in the row and column – they are bounded and polynomially decreasing as we go further away from the diagonal $m = n$. Let us take some large N and the matrix $D_2^{-1}T_M D_1^{-1}$ for $M > N$. If we would substitute zeros for the elements other than $ie^{in\hat{\tau}}n^{0.3}$, we would get a matrix that in the complex representation looks like that:

$$\begin{bmatrix} D_2^{-1}T_N D_1^{-1} & & & & \\ & ie^{in\hat{\tau}}(N+1)^{0.3} & & 0 & \\ & & ie^{in\hat{\tau}}(N+2)^{0.3} & & \\ & 0 & & \ddots & \\ & & & & ie^{in\hat{\tau}}M^{0.3} \end{bmatrix}$$

If there exist a pseudo-inverse L_N of T_N, then the pseudo-inverse of this matrix has the form:

$$\begin{bmatrix} D_1 L_N D_2 & & & 0 & \\ & \frac{1}{ie^{in\hat{\tau}}(N+1)^{0.3}} & & & \\ & & \frac{1}{ie^{in\hat{\tau}}(N+2)^{0.3}} & & \\ & 0 & & \ddots & \\ & & & & \frac{1}{ie^{in\hat{\tau}}M^{0.3}} \end{bmatrix}$$

The supremum norm of this matrix is $\max\left\{\|L_N\|, \frac{1}{(N+1)^{0.3}}\right\}$. Thus, the norm of all the matrices for $M > N$ is uniformly bounded and, if we could replace the high terms with zeros, we would be able to choose a constant for the assertion to be fulfilled.

Of course, we cannot substitute zeros and this is not a proof. However, for the orbit in Section 3.4, the assertion seems to be satisfied. I have used Mathematica to compute some (numerical approximation of the) pseudo-inverses L_N. In the table below the norms of $\|D_1 L_N D_2\|_\infty$ are provided. The supremum norm of a matrix is the maximum of the supremum norms of rows, so I have also included the norms of some of the rows (I have chosen the rows with a big norm) to have a better insight.

N	$\|L_N\|_\infty$	row 1	row 2	row 5	row 6	row 9	row 10
N = 6	1187.68	1187.68	31.0139	79.2494	176.211	51.190	31.863
N = 12	1250.96	1250.96	53.8642	83.7091	186.011	54.275	33.888
N = 25	1267.47	1267.47	57.5289	84.7936	188.455	54.973	34.331
N = 35	1267.53	1267.53	57.5438	84.7976	188.464	54.976	34.332
N = 50	1267.53	1267.53	57.5440	84.7977	188.464	54.976	34.332

As we see, the norms of the rows seems to be convergent, the norm of the matrix is attained on the first row and higher rows usually converges to smaller values (it may not be clear from such a small number of rows, but if one computes norms for a larger number of rows, this is more visible). Thus one can expect that the norm of the matrix converges to the norm of the first row, and thus is bounded from above.

On the other hand, the assumption is not true for all T_N. If, in all dimensions high enough, there is a two-dimensional manifold of solution passing through the point (as we know, a 1-dimensional manifold is always present due to $x(\cdot + \phi)$ being also a solution), the L_N do not exist, as then we have $\dim \operatorname{Im} T_N = 2N+2-2 = 2N$ and, thus, $\dim \operatorname{Im} T_N \circ L_N = \dim \operatorname{Im} I = 2N+1$ cannot hold.

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