Numer Algor (2012) 60:545–553
DOI 10.1007/s11075-012-9578-6

ORIGINAL PAPER

Nearly optimal meshes in subanalytic sets

Wiesław Pleśniak

Received: 12 December 2011 / Accepted: 12 April 2012 /
Published online: 26 April 2012

Abstract We prove that any fat, subanalytic compact subset of $\mathbb{R}^N$ possesses a nearly optimal (polynomial) admissible mesh. It is related to particular results that have recently appeared in the literature for very special (globally semianalytic) sets like N-dimensional polynomial or analytic graph domains or polynomial and analytic polyhedrons. (Here a good source of references is the recent paper (Piazzon and Vianello, East J Approx 16(4):389–398, 2010).) We also show that an infinitely differentiable map f from a compact set Q in $\mathbb{R}^N$ onto a Markov compact set K in $\mathbb{C}^l$ ($l \leq N$) transforms a (weakly) admissible mesh in Q onto a (weakly) admissible mesh in K, which extends a result of Piazzon and Vianello (East J Approx 16(4):389–398, 2010) for analytic maps in case Q is a subset of $\mathbb{R}^N$. Versions for $\mathcal{C}^k$ maps with sufficiently large k are also given.

Keywords Admissible polynomial meshes · Optimal meshes · Subanalytic geometry · Hironaka rectilinearization theorem · Bernstein-Walsh-Siciak theorem · Jackson theorem

AMS 2000 Subject Classifications Primary 41A10; Secondary 32B20 · 32U35 · 41A17 · 41A63 · 65D05

Let K be a compact subset of the N-dimensional complex space $\mathbb{C}^N$. Let $\mathbb{P}_d = \mathbb{P}_d(\mathbb{C}^N)$ be the set of all polynomials on $\mathbb{C}^N$ of degree at most d and let $\mathbb{P} =$

W. Pleśniak (✉)
Wydział Matematyki i Informatyki, Uniwersytet Jagielloński, ul. Stanisława Łojasiewicza 6, 30-348 Kraków, Poland
e-mail: Wieslaw.Plesniak@im.uj.edu.pl

$\bigcup_{d=1}^{\infty}\mathbb{P}_d$. A family $(A(d))_{d=1}^{\infty}$ of finite subsets $A(d)$ of K is said to be a *weakly admissible mesh* if the cardinality of $A(d)$ grows polynomially when $d\to\infty$, i.e. $\#A(d)=O(d^{\alpha})$, for some $\alpha>0$, and there exists a polynomially growing sequence $\{C(d)\}$ of positive constants such that for each $d\in\mathbb{N}$ and for all $P\in\mathbb{P}_d$ one has

$$\|P\|_K \le C(A(d))\|P\|_{A(d)}. \tag{1}$$

Here $\|h\|_S$ stands for the uniform norm $\sup|h|(S)$. If moreover $\sup_d C(A(d))<\infty$, then $(A(d))$ is said to be an *admissible mesh*. Suppose that K is $\mathbb{P}$-*determining*, i.e. for each $P\in\mathbb{P}$, $P=0$ on K forces $P(z)\equiv 0$. Then by the multivariate Langrange interpolation formula (see e.g. [13, 15]) there is a weakly admissible mesh $(A(d))$ on K, where $A(d)$ is a set $\{t_1,\ldots,t_{m_d}\}$ of Fekete-Leja type extremal points of K of order $m_d:=\dim\mathbb{P}_d=\binom{N+d}{N}=\mathrm{O}(d^N)$. If K is a Markov compact subset of $\mathbb{C}^N$, i.e. a compact set that admits a Markov inequality

$$\|\nabla P\|_K \le Md^r\|P\|_K \quad \text{for all } P\in\mathbb{P}_d \tag{2}$$

with positive constants M and r depending only on K, then following [3] one can construct an admissible mesh $(A(d))$ on K with $\#A(d)=O(d^{2rN})$ (and with $O(d^{rN})$ cardinality, if $K\subset\mathbb{R}^N\cong\mathbb{R}^N+i0\subset\mathbb{C}^N$). Observe that $r\ge 1$ if $K\subset\mathbb{C}^N$ and $r\ge 2$ for any compact set $K\subset\mathbb{R}^N$ (cf also Example 7) and for computational reasons one would like to construct meshes with more modest cardinalities. On the other hand, for any $d\in\mathbb{N}$, $A(d)$ must be $\mathbb{P}_d$ - determining, whence $\#A(d)\ge m_d$. This leads to the notion of *optimal* polynomial meshes: an admissible mesh $(A(d))$ is said to be *optimal*, if $\#A(d)=O(d^N)$ as $d\to\infty$. If $\#A(d)=O((d\ln d)^N)$, it is called *nearly optimal*. The main purpose of this note is to show that nearly optimal meshes can be constructed on fat, compact subanalytic subsets of $\mathbb{R}^N$ that are known to admit Markov inequality (2) (see [8]). Let us first recall some basic notions of subanalytic geometry that was developed mainly by Łojasiewicz, Gabrielov and Hironaka.

A subset E of $\mathbb{R}^N$ is said to be semianalytic if for each point $x\in\mathbb{R}^N$ one can find a neighbourhood U of x and a finite number of real analytic functions f_{ij} and g_{ij} defined in U, such that

$$E\cap U=\bigcup_i\bigcap_j\{f_{ij}>0,\, g_{ij}=0\}.$$

The projection of a semianalytic set need not be semianalytic (cf [2, 7]). The class of sets obtained by enlarging that of semianalytic sets to include images under the projections has been called the class of subanalytic sets. More precisely, a subset E of $\mathbb{R}^N$ is said to be subanalytic if for each point $x\in\mathbb{R}^N$ there exists an open neighbourhood U of x such that $E\cap U$ is the projection of a bounded semianalytic set A in $\mathbb{R}^{N+M}$, where $M\ge 0$. If $N\ge 3$, the class of subanalytic sets is essentially larger than that of semianalytic sets, the classes being identical if $N\le 2$. The union of a locally finite family and the intersection of a finite family of semianalytic (resp. subanalytic) sets is semianalytic (resp. subanalytic). The closure, interior, boundary and complement of a semianalytic (resp. subanalytic) set is still semianalytic (resp. subanalytic), the last property in the case of subanalytic sets being a (non-trivial) theorem of Gabrielov. For an excellent survey on subanalytic geometry, the reader is referred to [2]. In particular, one can find there an elegant proof of a crucial for this theory Hironaka Rectilinearization Theorem which (in a scalar space version) reads as follows.

Theorem 1 *Let E be a subanalytic subset of $\mathbb{R}^N$. Let K be a compact subset of $\mathbb{R}^N$. Then there are finitely many real analytic mappings $\varphi_j : \mathbb{R}^N \mapsto \mathbb{R}^N$ such that:*

(1) *There is a compact subset K_j of $\mathbb{R}^N$, for each j, such that $\bigcup_j \varphi_j(K_j)$ is a neighbourhood of K in $\mathbb{R}^N$.*
(2) *$\varphi_j^{-1}(E)$ is a union of quadrants in $\mathbb{R}^N$.*

With the aid of the above theorem one can prove (see [8]) the following

Theorem 2 *Let E be a bounded, subanalytic subset of $\mathbb{R}^N$ of pure dimension N. Then there are finitely many real analytic maps $f_j : \mathbb{R}^N \mapsto \mathbb{R}^N$ such that for each j,*

$$f_j(J^N) \subset E \quad \textit{and} \quad \bigcup_j f_j(I^N) = \overline{E},$$

where $J^N := \{x \in \mathbb{R}^N : \ |x_i| < 1, \ \ i = 1, \ldots, N\}$, *and* $I^N := \{x \in \mathbb{R}^N : \ |x_i| \leq 1, \ i = 1, \ldots, N\}$.

Subanalytic geometry methods have appeared very useful in polynomial approximation, since they provide tools for investigating regularity of the pluricomplex Green's function (see e.g. [8, 9, 12, 13]). As an example, we refer the reader to an important application of Hironaka's theorem (in version of Theorem 2) which is the following

Corollary 3 [8] *If K is a fat (i.e. $K \subset \overline{\operatorname{int} K}$) compact subanalytic subset of $\mathbb{R}^N$, then it admits Markov's inequality* (2).

Actually, in [8], it has been shown essentially more, namely that the set K of the above corollary is UPC, i.e. it is *uniformly polynomially cuspidal* and consequently, its pluricomplex Green function is Hölder continuous in $\mathbb{C}^N$.

We shall need a multidimensional version of the well-known Bernstein-Walsh theorem which is due to Siciak [15].

Theorem 4 *Let K be a compact subset of the space $\mathbb{C}^N$. Assume that K is polynomially convex, i.e. $K = \hat{K} := \{z \in \mathbb{C}^N : |p(z)| \leq \|p\|_K$ for all $p \in \mathbb{P}\}$. If f is a holomorphic function in an open neighbourhood of K then*

$$\limsup_{n\to\infty} \sqrt[n]{\operatorname{dist}_K(f, \mathbb{P}_n)} < 1.$$

One can also easily prove the following

Lemma 5 (cf [13]) *If K is a Markov compact set in $\mathbb{C}^N$ then for every polynomial $P \in \mathbb{P}_d$ $(d = 1, 2 \dots)$,*

$$|P(z)| \leq e^N \|P\|_K \quad \text{if } \operatorname{dist}(z, K) \leq \frac{1}{Md^r}, \tag{3}$$

where M and r are the constants of inequality (2).

Now we can state the main result of this paper.

Theorem 6 *Let K be a fat, compact subanalytic subset of $\mathbb{R}^N$. Then one can construct an admissible mesh $(A(d))$ on K such that $\#A(d) = O((d \ln d)^N)$ as $d \to \infty$.*

Proof Let

$$f_j = (f_{j,1}, \dots, f_{j,N}) : \mathbb{R}^N \mapsto \mathbb{R}^N \ (j = 1, \dots, m)$$

be real analytic functions of Theorem 2 for $\overline{E} = K$. Let $P \in \mathbb{P}_d$. Choose a point $w \in K$ such that $|P(w)| = \|P\|_K$. Then there is $j \in \{1, \dots, m\}$ such that $w \in f_j(I)$. Now choose $x \in I$ such that $w = f_j(x)$. Since any compact set in $\mathbb{R}^N$ is polynomially convex, by Theorem 4 there exist polynomials $P_{n,k} \in \mathbb{P}_n$, $n = 1, 2, \dots$, and constants $L > 0$ and $a \in (0, 1)$ independent of n such that

$$\|f_{j,k} - P_{n,k}\| \leq La^n =: \varepsilon_n \tag{4}$$

for $k = 1, \dots, N$. Set $P_n = (P_{n,1}, \dots, P_{n,N})$. Let $w^n = P_n(x)$. Then $\|w - w^n\| = \|f_j(x) - P_n(x)\| \leq \sqrt{N}\varepsilon_n$. Let $(A(d))_{d=1}^{\infty}$ be an optimal admissible mesh in the cube I. (It is well-known that such meshes exist; e.g. one can take the Cartesian product of a one dimensional mesh $Y(d)$ on $[-1,1]$ with $\#Y(d) = O(d)$, constructed in [4], chap. 3, sec.7, Lemma 3.) By the mean value theorem, Lemma 5 and Markov's inequality (2), we have

$$|P(w) - P(w^n)| \leq \|\nabla P\|_{[w,w^n]} \|w - w^n\| \leq Ne^N Md^r \|P\|_K \varepsilon_n,$$

provided $\sqrt{N}\varepsilon_n \leq \frac{1}{Md^r}$. Hence, setting $\varphi(d, n) := Ne^N Md^r \varepsilon_n$ gives

$$\begin{aligned} \|P\|_K = |P(w)| &\leq |P(w) - P(w^n)| + |P(w^n)| \\ &\leq \varphi(d, n)\|P\|_K + C\|P\|_{P_n(A(dn))} \end{aligned} \tag{5}$$

with $C = C(A(d)) \geq 1$, as $\sqrt{N}\varepsilon_n \leq 1/Md^r$. By a similar way, we shall now estimate $\|P\|_{P_n(A(dn))}$. Let $z \in P_n(A(dn))$ be such that $|P(z)| = \|P\|_{P_n(A(dn))}$. Choose $y \in A(dn)$ so that $P_n(y) = z$. We have

$$|P(z)| \leq |P(P_n(y)) - P(f_j(y))| + |P(f_j(y))| \\ \leq \varphi(d,n)\|P\|_K + C\|P\|_{f_j(A(dn))}.$$

Hence by (5),

$$\|P\|_K \leq \varphi(d,n)\|P\|_K + C\varphi(d,n)\|P\|_K + C^2\|P\|_{A'(dn)},$$

where $A'(dn) := \bigcup_{j=1}^{m} f_j(A(dn))$, provided $\sqrt{N}\varepsilon_n \leq 1/Md^r$. Now, it is easily seen that there is a sequence $n(d) = O(\ln d)$ of positive integers such that $\varphi(d, n(d)) \leq \frac{C}{4}$ and $\sqrt{N}\varepsilon_n \leq 1/Md^r$. Then

$$\|P\|_K \leq 2C^2\|P\|_{A'(dn(d))}.$$

One also verifies that $\#A'(dn(d)) = O((d \ln d)^N)$. □

In general, Theorem 6 gives better estimates of the cardinality of accessible meshes in subanalytic sets than those yielded by [3, Theorem 5]. This is seen by the following

Example 7 Consider the set

$$K = \left\{x = (x_1, x_2) \in \mathbb{R}^2 : \ 0 \leq x_1 \leq 1, 0 \leq x_2 \leq g(x_1)\right\},$$

where g is an analytic function in an open neighbourhood of [0,1] such that $0 < g(x_1) \leq x_1^p$ for some $p \in \mathbb{N}$. Then K is a semianalytic set, whence by Corollary 3 it is Markov. Its Markov exponent r has to be greater than $M\frac{p}{\ln p}$ for p sufficiently large, which can be easily seen by considering the polynomials $P(x_1, x_2) = x_2(1 - x_1)^p$. (Actually, if $g(x_1) = x_1^p$, then by Goetgheluck [5] $r = 2p$.) Thus Markov's exponent of K could be as large as we want. By Theorem 6 one can construct an admissible mesh $(A(d))$ in K with $\#A(d) = O((d \ln d)^2)$, as $d \to \infty$, while by [3, Theorem 5] we know only that there exists an admissible mesh $(A(d))$ in K with $\#A(d) = O(d^{2r})$.

The idea of applying Markov's inequality and the mean value theorem to constructing admissible meshes goes back to Cheney and it has been described in his monograph [4] in the case of univariate polynomial approximation. In the proof of the above theorem we also exploit the possibility of rapid (geometric) approximation of analytic maps by polynomials. Such a method has also been used by the authors of the recent interesting paper [10], where they prove the following

Theorem 8 *Let K be a Markov compact subset of $\mathbb{C}^N$ and let Q be a $\mathbb{P}$-determining compact set in $\mathbb{C}^N$ such that $K = f(Q)$, where f is an analytic map in an open neighbourhood of the polynomial hull $\hat{Q}$ of Q. Let $(A(d))$ be a* *(weakly) admissible mesh for Q. Then there exists a sequence* $j(d) = O(\ln d)$ *of natural numbers such that* $(A'(d)) := ((f(A(dj(d))))$ *is a (weakly) admissible mesh for K with* $C(A'(d)) \asymp C(A(dj(d)))$ *and* $\#A'(d) \le \#A(dj(d))$.

Observe that in the above theorem we are able to let f have values in the space $\mathbb{C}^l$ with $l \le N$. Let us also note that we cannot directly apply Theorem 8 in the proof of Theorem 6, since we do not know whether the sets $f_j(I)$ are Markov. We only know, by [1], that this is the case if $\det[f_j'(x)] \ne 0$ at every point $x \in I$.

Remark 9 In a recent paper [6], Kroó constructs admissible meshes in graph domains in $\mathbb{R}^N$ that are sets of the type

$$K_g := \{(x_1, \dots, x_N) \in \mathbb{R}^N : f_k(x_1, \dots, x_{k-1}) \le x_k \le g_k(x_1, \dots, x_{k-1}),$$
$$(x_1, \dots, x_{k-1}) \in I^{k-1},\ 1 \le k \le N\},$$

where $I^k = [0, 1]^k$, $1 \le k \le N$, $f_1 \equiv 0$, $g_1 \equiv 1$ and $0 \le f_k(x) \le g_k(x) \le 1, x \in I^{k-1}, 2 \le k \le N$. (Such domains are also called "normal domains" in textbooks on multiple integrals.) He shows (Proposition 1) that in case the functions f_k and g_k are algebraic polynomials the domain K_g possesses an optimal polynomial mesh. Actually, it immediately follows from the fact that any graph set K_g is simply the image of the cube $[0, 1]^N$ by the map

$$F(t_1, \dots, t_N) := (t_1, (1 - t_2) f_2(t_1) + t_2 g_2(t_1), \dots,$$
$$(1 - t_N) f_N(t_1, \dots, t_{N-1}) + t_N g_N(t_1, \dots, t_{N-1})).$$

Indeed, if $(A(d))$ is an optimal mesh in I^N and $F = (F_1, \dots, F_N) : \mathbb{R}^N \mapsto \mathbb{R}^N$ is a polynomial map of degree $s = \max_{1 \le k \le N} \deg F_k$, then for any polynomial P in $\mathbb{R}^N$ of degree d one has

$$\|P\|_{F(I^N)} = \|P \circ F\|_{I^N} \le C\|P \circ F\|_{A(sd)} \le C\|P\|_{F(A(sd))}$$

with $\#F(A(sd)) \le \#A(sd) \le M s^N d^N$. The same holds true if K is a finite union of the images $F^j(I^N)$ of the unit cube I^N by polynomial maps $F^j : \mathbb{R}^N \mapsto \mathbb{R}^N$, in particular if K is a polytope.

If the functions f_k and g_k are traces on I^{k-1} of real analytic functions then the corresponding graph domain K_g is clearly a (global) semianalytic set. Then by Theorem 6 one can construct in K_g an admissible mesh $(A(d))$ with $\#A(d) = O((d \ln d)^N)$ which is better than the estimate $\#A(d) = O(d^N \ln^{N(N-1)} d)$ yielded in such a case by [6, Theorem 1]. Let us add that in the analytic case the cardinality result $\#A'(d) = O((d \ln d)^N)$ for K_g also follows from Corollary 3 and Theorem 7.

Other typical sets fulfilling the assumptions of Theorem 6 are analytic polyhedrons, i.e. compact subsets K of a domain Ω in $\mathbb{R}^N$ of the type

$$K := \{x \in \Omega : |h_j(x)| \le 1,\ j = 1, \dots, m\},$$

where h_j are real analytic functions in Ω.

Now we are going to show that in case Q is a subset of $\mathbb{R}^N$ Theorem 8 is also valid for $\mathcal{C}^\infty$ maps and even for $\mathcal{C}^k$ maps with sufficiently large k depending on Markov's exponent r of inequality (2) and the growth of the sequence $\{C(A(d))\}$.

Theorem 10 *Let Q be a compact set in $\mathbb{R}^N$ and let $f = (f_1, \ldots, f_l)$ be a map defined on Q, with values in $\mathbb{C}^l$ ($l \leq N$), whose components f_j are traces of $\mathcal{C}^\infty$-functions on $\mathbb{R}^N$. Suppose that the set $K = f(Q)$ is Markov. Let $(A(d))$ be a (weakly) admissible mesh in Q. Then there is a positive integer m such that $(f(A(md^2)))$ is a (weakly) admissible mesh in K.*

Proof By the multivariate Jackson theorem (applied to a cube $I \supset Q$ in $\mathbb{R}^N$), one can find polynomials $P_{j,n} \in \mathbb{P}_n$ such that the sequence $\varepsilon_{j,n} := \| f_j - P_{j,n}\|_Q$ is rapidly decreasing, i.e. for each $k > 0$, $n^k \varepsilon_{j,n} \to 0$ as $n \to \infty$ for $j = 1, \ldots, l$ (see [13, 16]). Let $P_n = (P_{1,n}, \ldots, P_{l,n})$ and $\varepsilon_n = \max_j \varepsilon_{j,n}$. We have $\| f - P_n\|_Q \leq \sqrt{l}\varepsilon_n$. Take a polynomial $W \in \mathbb{P}_d(\mathbb{C}^l)$ and choose $w \in K = f(Q)$ so that $|W(w)| = \|W\|_K$. Then, by a similar argument to that of the proof of Theorem 6 (cf also the proof of Theorem 7 in [10]) we arrive at the estimate

$$\|W\|_K \leq \psi(d, n)\|W\|_K + C(A(dn))\psi(d, n)\|W\|_K + C(A(dn))\|W\|_{f(A(dn))}$$

with $\psi(d, n) := Mle^l d^r \varepsilon_n$, provided $\sqrt{l}\varepsilon_n \leq 1/Md^r$. Observe that for each $k > 0$ we have

$$\psi(d, n) = Const.n^k \varepsilon_n \frac{d^r}{n^k} \leq Const. \sup_n (n^k \varepsilon_n) \frac{d^r}{n^k} = C(k)\frac{d^r}{n^k}.$$

Consider now two cases.

1° $C := \sup_d C(A(d)) < \infty$, that is the mesh $(A(d))$ is admissible. We may assume that $C \geq 1$. Then, setting $k = [r] + 1$, where $[r]$ denotes the entire part of r, one can find a positive integer m such that $C\psi(d, md) \leq \frac{1}{4}$ and $\varepsilon_{md} \leq 1/Md^r$. Consequently,

$$\|W\|_K \leq 2C\|W\|_{f(A(md^2)},$$

and if $\#A(d) = O(d^\alpha)$ for some $\alpha > 0$, we get $\# f(A(md^2) = O(d^{2\alpha})$. Thus $(f(A(md^2)))$ is an admissible mesh in K.

2° Suppose $C(A(d)) = O(d^\beta)$ for some $\beta > 0$. Then again, setting $k = [\beta + r] + 1$, we can find a positive integer m' such that $C(A(m'd^2))\psi(d, m'd) \leq \frac{1}{4}$ and $\varepsilon_{md} \leq 1/Md^r$. This yields the inequality

$$\|W\|_K \leq 2C(A(m'd^2))\|W\|_{f(a(m'd^2))}.$$

Moreover, if $\#A(d) = O(d^\gamma)$ for some $\gamma > 0$, then $\#f(A(m'd^2)) = O(d^{2\gamma})$. This means that the mesh $(f(A(m'd^2)))$ is weakly admissible. □

Remark 11 By a version of the multivariate Jackson theorem in [16], if a map $f = (f_1, \ldots, f_l)$ defined on Q extends to a $\mathcal{C}^{k+1}$ map from $\mathbb{R}^N$ to $\mathbb{C}^l$, then for each $j \in \{1, \ldots, l\}$,

$$\sup_n n^k \varepsilon_n \leq C(k) \sum_{|\alpha| \leq k+1} \| D^\alpha f_j \|_I \leq D(k, f),$$

where I is a compact cube in $\mathbb{R}^N$ containing the set Q. Then, if the mesh $(A(d))$ is admissible, Theorem 10 holds if f is a $\mathcal{C}^{[r]+2}$ map, and if $C(A(d)) = O(d^\beta)$ $(\beta > 0)$, then Theorem 10 is valid for any $\mathcal{C}^{[\beta+r]+2}$ map f.

Remark 12 By a non-trivial result of [11], bounded, fat and definable sets in some polynomially bounded o-minimal structures generated by special classes of C^∞ functions in $\mathbb{R}^N$ are uniformly polynomially cuspidal, whence by [8] they are Markov. This is e.g. the case of the Rolin-Speissegger-Wilkie structure (cf [14]) generated by the Denjoy-Carleman classes of quasianalytic functions with partial derivatives tempered by a strongly logarithmically convex sequence $\{M_p\}$. In [11], Pierzchała has proved a version of Theorem 2 for such a structure. Thus it should be possible to extend Theorem 6 to the case of definable sets in the Rolin-Speissegger-Wilkie o-minimal structure.

References

1. Baran, M., Pleśniak, W.: Markov's exponent of compact sets in $\mathbb{C}^n$. Proc. Am. Math. Soc. **123**, 2785–2791 (1995)
2. Bierstone, E., Milman, P.D.: Semianalytic and subanalytic sets. Publ. Math. Inst. Hautes. Études Sci. **67**, 5–42 (1988)
3. Calvi, J.-P., Levenberg, N.: Uniform approximation by discrete least squares polynomials. J. Approx. Theory **152**, 82–100 (2008)
4. Cheney, E.W.: Introduction to Approximation Theory. AMS Chelsea Publishing, Providence, Rhode Island (1982)
5. Goetgheluck, P.: Inégalité de Markov dans les ensembles effilés. J. Approx. Theory **30**, 149–154 (1980)
6. Kroó, A.: On optimal polynomial meshes. J. Approx. Theory **163**, 1107–1124 (2011)
7. Łojasiewicz, S.: Ensembles semi-analytiques. Inst. Hautes Études Sci., Bures-sur-Yvette (1964)
8. Pawłucki, W., Pleśniak, W.: Markov's inequality and $\mathcal{C}^\infty$ functions on sets with polynomial cusps. Math. Ann. **275**, 467–480 (1986)
9. Pawłucki, W., Pleśniak, W.: Extension of C^∞ functions from sets with polynomial cusps. Studia Math. **88**(3), 279–287 (1988)
10. Piazzon, F., Vianello, M.: Analytic transformations of admissible meshes. East J. Approx. **16**(4), 389–398 (2010)

11. Pierzchała, R.: UPC condition in polynomially bounded o-minimal structures. J. Approx. Theory **132**, 25–33 (2005)
12. Pleśniak, W.: L-regularity of subanalytic sets in $\mathbb{R}^n$. Bull. Acad. Polon. Sci. Sér. Sci. Math. **32**, 647–651 (1984)
13. Pleśniak, W.: Markov's inequality and the existence of an extension operator for $\mathcal{C}^\infty$ functions. J. Approx. Theory **61**, 106–117 (1990)
14. Rolin, J.-P., Speissegger, P., Wilkie, A.J.: Quasianalutic Denjoy-Carleman classes and o-minimality. J. Am. Math. Soc. **16**(4), 751–777 (2003)
15. Siciak, J.: On some extremal functions and their applications in the theory of analytic functions of several complex variables. Trans. Am. Math. Soc. **105**, 322–357 (1962)
16. Timan, A.F.: Theory of Approximation of Functions of a Real Variable. Oxford Pergamon Press (1963)