Uniwersytet Jagielloński
Wydział Matematyki i Informatyki
Instytut Informatyki

Anna Kulig

Nonlinear Evolution Inclusions and Hemivariational Inequalities for Nonsmooth Problems in Contact Mechanics

PhD Thesis

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prof. dr hab. Stanisław Migórski

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"La Meccanica è il paradiso delle scienze matematiche,
perché con quella si viene al frutto matematico."
– Milano, 1483.

⌜ *"Mechanics is the paradise of the mathematical sciences,*
because by means of it one comes to the fruits of mathematics." ⌟

Leonardo da Vinci (1452–1519)

Abstract. The dissertation deals with second order nonlinear evolution inclusions, hyperbolic hemivariational inequalities and their applications. First, we study a class of the evolution inclusions involving a Volterra integral operator and considered within the framework of an evolution triple of spaces. Combining a surjectivity result for multivalued pseudomonotone operators and the Banach Contraction Principle, we deliver a result on the unique solvability of the Cauchy problem for the inclusion. We also provide a theorem on the continuous dependence of the solution to the inclusion with respect to the operators involved in the problem. Next, we consider a class of hyperbolic hemivariational inequalities and embed these problems into a class of evolution inclusions with the multivalued term generated by the generalized Clarke subdifferential for nonconvex and nonsmooth superpotentials. Finally, we study a dynamic frictional contact problem of viscoelasticity with a general constitutive law with long memory, nonlinear viscosity and elasticity operators and the subdifferential boundary conditions. We deal with various aspects of the modeling of these contact problems and provide several examples of nonmonotone subdifferential boundary conditions which illustrate the applicability of our findings.

Keywords. Hemivariational inequality, contact, friction, nonmonotone, hyperbolic, viscoelasticity, dynamic, evolution inclusion, nonsmooth, nonconvex, multivalued, pseudomonotone operator, hemicontinuous, subdifferential, existence, uniqueness, modeling.

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Contents

1 Introduction

An important number of problems arising in Mechanics, Physics and Engineering Science lead to mathematical models expressed in terms of nonlinear inclusions. For this reason the mathematical literature dedicated to this field is extensive and the progress made in the last decades is impressive. It concerns both results on the existence, uniqueness, regularity and behavior of solutions for various classes of nonlinear inclusions as well as results on numerical approach of the solution for the corresponding problems.

The framework of evolution inclusion allows to describe dynamical systems with multivalued discontinuities and therefore this framework is more general than evolution equations. However, the great advantage of this framework over other approaches is, that physical interaction laws, such as contact and friction in mechanics can be formulated as set-valued force laws and can be incorporated in the formulation. We will therefore use the framework of nonlinear evolution inclusions in this thesis to study existence properties of nonsmooth systems which naturally arise in mechanics with inequality constraints. The abstract problem under investigation is the following second order evolution inclusion

$$\begin{cases} u''(t) + A(t, u'(t)) + B(t, u(t)) + \displaystyle\int_0^t C(t-s)u(s)\,ds + \\ \qquad\qquad + F(t, u(t), u'(t)) \ni f(t) \quad \text{a.e. } t \in (0,T), \\ u(0) = u_0, \quad u'(0) = u_1, \end{cases} \tag{$*$}$$

where A, $B\colon (0,T) \times V \to V^*$ are nonlinear operators, $C(t)$ is a bounded linear operator from V to its dual, for $t \in (0,T)$, $F\colon (0,T) \times V \times V \to 2^{Z^*}$ denotes a multivalued mapping, $f \in L^2(0,T;V^*)$, $u_0 \in V$, $u_1 \in H$, V and Z are reflexive Banach spaces with $V \subset Z$ compactly, H is a Hilbert space such that $Z \subset H$ and $0 < T < \infty$.

More precisely, we focus on the existence and uniqueness results for the Cauchy problem $(*)$. The latter is defined in the framework of an evolution triple of spaces. We approach the problem by reducing its simplified version to the first order nonlinear evolution inclusion through the introduction of the integral operator and by applying a result on the surjectivity of multivalued operators. Later we use the Banach Contraction Principle to a suitable operator and show a result on unique solvability of the evolution inclusion $(*)$. We remark that in order to prove uniqueness of solutions we need, on one hand, some restrictive hypotheses on the multivalued term, and on the other hand these hypotheses should be quite general to cope with the multifunctions which appear in the contact problems.

The Mathematical Theory of Contact Mechanics has made recently impressive progress due to the development in the field of Inequality Problems. In the latter we can distinguish two main directions: variational inequalities connected with convex energy functions and hemivariational inequalities connected with nonconvex energies. The variational inequalities have a precise physical meaning and they express the principles of virtual work and power introduced by Fourier in 1823. The prototypes of boundary value problems leading to variational inequalities are the Signorini-Fichera problem and the friction problem of elasticity. For variational inequalities the reader is referred to monographs of Duvat and Lions [27], Hlaváček et al. [36], Kikuchi and Oden [44], Kinderlehrer [45] and Panagiotopoulos [77], among others. The notion of hemivariational inequality is based on the generalized gradient of Clarke-Rockafellar [21] and has been introduced in the early 1980s by Panagiotopoulos [77, 78] to describe several important mechanical and engineering problems with nonmonotone phenomena in solid mechanics. Such inequalities appear in the modeling of the constitutive law and/or the boundary conditions. The nonsmooth and nonconvex nature of energy potentials and the resulting multivalued character of mechanical laws challenge the extension of the existing results for smooth and convex potential systems to evolution inclusions with multifunctions which are of the Clarke subdifferential form. For convex potentials the hemivariational inequalities reduce to the variational inequalities.

The evolution hemivariational inequalities have been studied for parabolic problems by Miettinen [55] who employed the regularization method with the Galerkin technique, by Carl [15, 14] (who adapted the Rauch method of [87]) and Papageorgiou [82] who both combined the method of lower and upper solutions with truncation and penalization techniques. Moreover, Liu [53] obtained existence result for parabolic hemivariational inequalities with an evolution operator of class $(S)_+$ and Miettinen and Panagiotopoulos [56] and Migórski and Ochal [63] have treated the problem using a regularized approximating model. The existence and convergence results for first order evolution hemivariational inequalities can be found in Migórski [59].

The hyperbolic hemivariational inequalities arising in nonlinear boundary value problems have been studied by Panagiotopoulos [78, 79], Panagiotopoulos and Pop [80] who used the Galerkin method as well as Gasiński [30] and Ochal [75] who employed a surjectivity result for multivalued operators. The existence results for second order nonlinear evolution inclusions can be found in Ahmed and Kerbal [2], Bian [12], Migórski [57, 58], Papageorgiou [81], and Papageorgiou and Yannakakis [83, 84], while the existence of solutions to the dynamic hemivariational inequalites of second order has been studied by Guo [33], Kulig [48], Liu and Li [51], Migórski [60, 61, 62], Migórski and Ochal [65], Park and Ha [85] and Xiao and Huang [100]. A general method for the study of dynamic viscoelastic contact problems involving subdifferential boundary conditions was presented in Migórski and Ochal [66]. Within the framework of evolutionary hemivariational inequalities, this method represents a new approach which unifies several other methods used in the study of viscoelastic contact and allows to obtain new existence and uniqueness results. Recent books and monographs on mathematical theory of hemivariational inequalities include Carl and Motreanu [16], Goeleven et al. [31], Haslinger et al. [35], Migórski et al. [70], Motreanu and Panagiotopoulos [71], Naniewicz and Panagiotopoulos [73], Panagiotopoulos [77, 78], and we refer the reader there for a wealth of additional information about these and related topics. The results on Mathematical Theory of Contact Mechanics can be found in several monographs, e.g. Eck et al. [28], Han and Sofonea [34], Shillor et al. [93], Sofonea et al. [95] and Sofonea and Matei [96].

In the thesis the hemivariational inequalities under investigation represent a particular case of nonlinear inclusions associated to the Clarke subdifferential operator. Specyfing the spaces V, Z and H as suitable Sobolev and L^2 spaces defined on an open bounded subset Ω of $\mathbb{R}^d$, considering the potential contact surface Γ_C as a measurable part of the boundary of Ω and introducing an appropriate multivalued mapping F, it can be seen that every solution to the evolution inclusion $(*)$ satisfies the hyperbolic hemivariational inequality of the form

$$\begin{cases} \langle u''(t) + A(t, u'(t)) + B(t, u(t)) + \displaystyle\int_0^t C(t-s)u(s)\,ds - f(t), v\rangle + \\ \qquad + \displaystyle\int_{\Gamma_C} g^0(x, t, \gamma u(t), \gamma u'(t), \gamma u(t), \gamma u'(t); \gamma v, \gamma v)\,d\Gamma \geq 0 \\ \qquad\qquad\qquad\qquad \text{for all } v \in V, \text{ a.e. } t \in (0,T), \\ u(0) = u_0, \ u'(0) = u_1, \end{cases} \qquad (**)$$

where g^0 denotes the generalized directional derivative of a (possibly) nonconvex function g in the sense of Clarke, γ is a trace map and $\langle \cdot, \cdot \rangle$ stands for the duality pairing between V^* and V. For the definitions of the function g and the multivalued mapping F which give a passage from $(*)$ to the hemivariational inequality $(**)$, we refer to Section 5.4.

The goals and the results of the thesis are following. First, we establish a result on unique solvability of the Cauchy problem for the second order evolution inclusion $(*)$. The inclusion $(*)$ without the Volterra memory term and with time independent operator B has been studied in Denkowski et al. [24] (with $F\colon (0,T) \times H \times H \to 2^H$), Migórski and Ochal [67] in a case B is linear, continuous, symmetric and coercive operator, and in Migórski [60], and Park and Ha [85] in a case B is linear, continuous, symmetric and nonnegative. Now, we treat the problem $(*)$ with a nonlinear Lipschitz operator $B(t, \cdot)$, and with a linear and continuous kernel operator $C(t)$ in the memory term. We underline that none of the results on nonlinear evolution inclusions in [2, 12, 57, 58, 81, 83, 84, 98] can be applied in our study because of their restrictive hypotheses on the multivalued term which was supposed to have values in H. For the hemivariational inequalities and the contact problems, the associated multivalued mapping has values in the space dual to Z which is larger than H. Moreover, we have employed a method which is different than those of [24, 60, 67, 85] and which combines a surjectivity result for pseudomonotone operators with the Banach Contraction Principle. We obtain results on local and global (under stronger hypotheses on the multifunction) unique solvability of the evolution inclusion $(*)$.

Next, we provide a result on the continuous dependence of the solution to $(*)$ with respect to the operator A, B and C. It is shown that the sequence of the unique solutions to $(*)$ corresponding to perturbed operators A_ε, B_ε and C_ε converges in a suitable sense to the unique solution corresponding to unperturbed operators A, B and C. This result is of importance from the mechanical point of view, since for vanishing relaxation operator, it indicates that the nonlinear viscoelasticity for short memory materials may be considered as a limit case of nonlinear viscoelasticity with constitutive law with long memory. This convergence result holds for the whole spectrum of nonmonotone contact conditions which we describe in this work.

Subsequently, we consider the class of evolution hemivariational inequalities of second order of the form $(**)$. Our study includes the modeling of a mechanical problem and its variational analysis. We derive the hemivariational inequality $(**)$ for the displacement field from nonconvex superpotentials through the generalized Clarke subdifferential. The mechanical properties are described by a general constitutive law which include the Kelvin-Voigt law and a viscoelastic constitutive law with long memory. The novelty of the model is to deal with nonlinear elasticity and viscosity operators and to consider the coupling between two kinds of nonmonotone possibly multivalued boundary conditions which depend on the normal (respectively, tangential) components of both the displacement and velocity. The new results concern the existence, uniqueness and regularity of the weak solution to the hemivariational inequality $(**)$ which are obtained by embedding the problem into a class of evolution inclusions of the form $(*)$ and by applying the results obtained for $(*)$. To the author's best knowledge the results obtained for hemivariational inequalities seem to be new even for the case when all/some of the potentials involved in the boundary conditions are convex functions. We also remark that the question on uniqueness of solutions to a general form of hemivariational inequality $(**)$ remains open.

Finally, in order to illustrate the cross fertilization between rigorous mathematical description and Nonlinear Analysis on one hand, and modeling and applications on the other hand, we provide examples of constitutive laws with long memory as well as several examples of contact and friction subdifferential boundary conditions. We mention that our formulation of multivalued boundary conditions covers, as particular cases, the following conditions used recently in the literature: frictionless contact, the nonmonotone normal compliance condition, the simplified Coulomb friction law, the nonmonotone normal damped response condition, the viscous contact with Tresca's friction law, the viscous contact with power-law friction boundary condition, the version of dry friction condition, the nonmonotone friction conditions depending on slip and slip rate, and the sawtooth laws generated by nonconvex superpotentials. We will also show how a suitable choice of the multivalued term in the evolution inclusion leads to different types of boundary conditions.

The thesis is organized as follows. In Section 2 we recall some preliminary material which is needed in the work. In Section 3 we study a class of second order nonlinear evolution inclusions involving a Volterra integral operator in the framework of evolution triple of spaces. For this class we give a result on the existence and uniqueness of solutions to the Cauchy problem for the inclusion under investigation. Section 4 is devoted to the study of the dependence of the solution to the abstract nonlinear evolution inclusion on the operators involved in the problem. In Section 5 we establish the link between a nonlinear evolution inclusion and the hemivariational inequality (HVI), and we apply results of Section 3 to the viscoelastic contact problem with a memory term. The review of several examples of contact and friction subdifferential boundary conditions which illustrates the applicability of our results is provided in Section 6. Section 7 contains a few results from functional analysis that are often used in the text.

A portion of the thesis concerning a mathematical model which describes dynamic viscoelastic contact problems with nonmonotone normal compliance condition and the slip displacement dependent friction has been by published by the author in [48].

2 Preliminaries

In this section we provide the background material which will be needed in the sequel. We summarize some results from the theory of vector-valued function spaces, briefly recall notions for classes of operators of monotone type, and present basic facts from the theory of the Clarke generalized differentiation of locally Lipschitz functions.

2.1 Lebesgue-Bochner and Sobolev spaces

In this part we recall some results from the theory of vector-valued function spaces which will be used in the sequel. For the details we refer to basic monographs of Adams [1], Brézis [13], Denkowski et al. [23, 24], Droniou [25], Evans [29], Grisvard [32], Hu and Papageorgiou [37], Lions [52], Showalter [94] and Zeidler [99].

Let X be a Banach space with a norm $\|\cdot\|_X$, let X^* be its dual, and let $\langle\cdot,\cdot\rangle_{X^*\times X}$ denote the duality pairing between X^* and X. Let $0 < T < \infty$ and $1 \le p \le \infty$. We denote by $L^p(0,T;X)$ the space (equivalent classes) of measurable X-valued functions $v\colon (0,T) \to X$ such that $\|v(\cdot)\|$ belongs to $L^p(0,T;\mathbb{R})$ with

$$\|v\|_{L^p(0,T;X)} = \begin{cases} \left(\displaystyle\int_0^T \|v(t)\|_X^p \, dt \right)^{1/p} & \text{if } \ 1 \le p < \infty, \\ \operatorname*{ess\,sup}_{0\le t\le T} \|u(t)\|_X & \text{if } \ p = \infty. \end{cases} \tag{1}$$

The space $C(0,T;X)$ comprises of all continuous X-valued functions $v\colon [0,T] \to X$ with

$$\|v\|_{C(0,T;X)} = \max\{ \, \|v(t)\|_X \mid t \in [0,T] \, \}.$$

Basic properties of the Lebesgue space $L^p(0,T;X)$ of Banach space valued functions are formulated below.

Proposition 1 *Let X and Y be Banach spaces. We have the following results*

(i) *The space $L^p(0,T;X)$ is a Banach space with respect to the norm (1) for $p \in [1,\infty]$.*

(ii) *If X is a Hilbert space with scalar product $\langle\cdot,\cdot\rangle_X$, then $L^2(0,T;X)$ is also a Hilbert space equipped with the scalar product*

$$\langle\langle u, v\rangle\rangle_{L^2(0,T;X)} = \int_0^T \langle u(t), v(t)\rangle_X \, dt.$$

(iii) *If X is a reflexive, separable Banach space and $p \in (1,\infty)$, then $L^p(0,T;X)$ is reflexive, separable and $(L^p(0,T;X))^* \simeq L^q(0,T;X^*)$, where $\frac{1}{p} + \frac{1}{q} = 1$, and $L^1(0,T;X)$ is separable with $(L^1(0,T;X))^* \simeq L^\infty(0,T;X^*)$.*

(iv) *Let* $1 \le r \le p < \infty$. *If the embedding* $X \subset Y$ *is continuous, then the embedding* $L^p(0,T;X) \subset L^r(0,T;Y)$ *is also continuous. For the embedding* $L^p(0,T;X) \subset L^r(0,T;X)$, *we have*

$$\|v\|_{L^r(0,T;X)} \le T^{\frac{p-r}{pr}} \|v\|_{L^p(0,T;X)} \quad \text{for all } v \in L^p(0,T;X).$$

(v) *If* $1 \le p \le \infty$ *and* $\{v_n, v\} \subset L^p(0,T;X)$, $v_n \to v$ *in* $L^p(0,T;X)$, *then there exists a subsequence* $\{v_{n_k}\} \subset \{v_n\}$ *such that* $v_{n_k}(t) \to v(t)$ *in* X *for a.e.* $t \in (0,T)$ *and* $\|v_{n_k}(t)\|_X \le h(t)$ *with* $h \in L^p(0,T)$.

(vi) *If* $1 \le p \le \infty$ *and* X *is a reflexive, separable Banach space, then for any bounded sequence* $\{v_n\}$ *in* $L^p(0,T;X)$, *there exists* $v \in L^p(0,T;X)$ *and a subsequence* $\{v_{n_k}\} \subset \{v_n\}$ *weakly convergent in* $L^p(0,T;X)$ *to* v, *i.e.*

$$\int_0^T \langle v_{n_k}(t), w(t)\rangle_{X^* \times X}\, dt \to \int_0^T \langle v(t), w(t)\rangle_{X^* \times X}\, dt \quad \text{for all} \quad w \in L^q(0,T;X^*),$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

(vii) *If* X *is a reflexive, separable Banach space, then for any bounded sequence* $\{v_n\}$ *in* $L^\infty(0,T;X)$, *there exists* $v \in L^\infty(0,T;X)$ *and a subsequence* $\{v_{n_k}\} \subset \{v_n\}$ *weakly-∗ convergent in* $L^\infty(0,T;X)$ *to* v, *i.e.*

$$\int_0^T \langle v_{n_k}(t), w(t)\rangle_{X^* \times X}\, dt \to \int_0^T \langle v(t), w(t)\rangle_{X^* \times X}\, dt \quad \text{for all} \quad w \in L^1(0,T;X^*).$$

(viii) *If* $0 \le s \le t \le T$ *and* $v \in L^1(0,T;X)$, *then*

$$\left\| \int_s^t v(\tau)\, d\tau \right\|_X \le \int_s^t \|v(\tau)\|_X\, d\tau.$$

Recall now the definition of the Bochner-Sobolev spaces. Let $1 \le p \le \infty$. By $W^{1,p}(0,T;X)$ we denote the subspace of $L^p(0,T;X)$ of functions whose first order weak derivative with respect to time belongs to $L^p(0,T;X)$, i.e.

$$W^{1,p}(0,T;X) = \{\, u \in L^p(0,T;X) \mid u' \in L^p(0,T;X)\,\}.$$

It is well known (cf. e.g. Chapter 3.4 of [24], Chapter 2 in [25]) that this space endowed with a norm $\|u\|_{W^{1,p}(0,T;X)} = \|u\|_{L^p(0,T;X)} + \|u'\|_{L^p(0,T;X)}$ becomes a Banach space and the embedding $W^{1,p}(0,T;X) \subset C(0,T;X)$ is continuous. For the definition and properties of the Bochner-Sobolev spaces $W^{k,p}(0,T;X)$ for $k \ge 1$, we refer to e.g. Chapter 23 of [99] and Chapter 3.4 of [24].

Next, we recall facts we need for the understanding of the concept of evolution triple. The space of all linear and continuous operators from a normed space X to a normed space Y will be denoted by $\mathcal{L}(X,Y)$.

Proposition 2 *Let* X *and* Y *be Banach spaces, and let* $A \in \mathcal{L}(X,Y)$. *Then the dual operator* $A^*\colon Y^* \to X^*$ *is also linear and continuous, and we have* $\|A\|_{\mathcal{L}(X,Y)} = \|A^*\|_{\mathcal{L}(Y^*,X^*)}$. *Moreover, if the linear operator* $A\colon X \to Y$ *is compact, then so is the dual operator* A^*.

Proposition 3 *Let X and Y be Banach spaces with $X \subset Y$ such that X is dense in Y, and the embedding $i\colon X \to Y$ is continuous. Then*
(i) *the embedding $Y^* \subset X^*$ is continuous and the embedding operator $\widehat{i}\colon Y^* \to X^*$ coincides with the dual operator of i, i.e. $\widehat{i} = i^*$;*
(ii) *if X is, in addition, reflexive, then Y^* is dense in X^*;*
(iii) *if the embedding $X \subset Y$ is compact, then so is the embedding $Y^* \subset X^*$;*
(iv) *for all $v \in L^2(0,T;X)$, we have $\|v\|_{L^2(0,T;Y)} \le \|i\|_{\mathcal{L}(X,Y)}\|v\|_{L^2(0,T;X)}$.*

The following notion of evolution triple, or sometimes called the Gelfand triple (cf. Chapter 23 of [99], Chapter 3.4 of [24]), is basic in the study of evolution problems.

Definition 4 *A triple of spaces (V,H,V^*) is called an evolution triple if the following properties hold*
(a) *V is a separable and reflexive Banach space, and H is separable Hilbert space endowed with the scalar product $\langle \cdot,\cdot\rangle$;*
(b) *the embedding $V \subset H$ is continuous, and V is dense in H;*
(c) *identifying H with its dual H^* by the Riesz map, we then have $H \subset V^*$ with the equality $\langle h,v\rangle_{V^*\times V} = \langle h,v\rangle$ for $h \in H \subset V^*$, $v \in V$.*

Since V is reflexive and V is dense in H, the space H^* is dense in V^*, and hence, H is dense in V^*.

Example 5 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary and let V be a closed subspace of $W^{1,p}(\Omega;\mathbb{R}^d)$ with $2 \le p < \infty$ such that $W_0^{1,p}(\Omega;\mathbb{R}^d) \subset V \subset W^{1,p}(\Omega;\mathbb{R}^d)$. Then (V,H,V^*) with $H = L^2(\Omega;\mathbb{R}^d)$ is an evolution triple with all embeddings being, in addition, compact.*

Finally, we introduce the Bochner-Sobolev space related to the Gelfand triple. Let (V,H,V^*) be an evolution triple, $1 < p < \infty$ and $\frac{1}{p}+\frac{1}{q}=1$. We set

$$W^{1,p}(0,T;V,H) = \{\, u \in L^p(0,T;V) \mid u' \in L^q(0,T;V^*)\,\},$$

where the time derivative involved in the definition is understood in the sense of vector valued distributions. We equip this space with the following norm

$$\|u\|_{W^{1,p}(0,T;V,H)} = \|u\|_{L^p(0,T;V)} + \|u'\|_{L^q(0,T;V^*)}.$$

It is well known (cf. Proposition 23.23 of [99], Theorem 3.4.13 and Proposition 3.4.14 of [24]) that the embedding $W^{1,p}(0,T;V,H) \subset C(0,T;H)$ is continuous (precisely, for each $u \in W^{1,p}(0,T;V,H)$ there exists a uniquely determined continuous function $u_1\colon [0,T] \to H$ such that $u(t) = u_1(t)$ a.e. $t \in [0,T]$) and the embedding $W^{1,p}(0,T;V,H) \subset L^p(0,T;H)$ is compact.

In the subsequent sections we will use the following notation for an evolution triple (V,H,V^*) and $p=q=2$:

$$\mathcal{V} = L^2(0,T;V), \qquad \widehat{\mathcal{H}} = L^2(0,T;H), \qquad \mathcal{V}^* = L^2(0,T;V^*),$$

$$\mathcal{W} = W^{1,2}(0,T;V,H) = \{\, v \in \mathcal{V} \mid v' \in \mathcal{V}^* \,\}.$$

With the norm introduced above, the space $\mathcal{W}$ becomes a separable reflexive Banach space and the following embeddings $\mathcal{W} \subset \mathcal{V} \subset \widehat{\mathcal{H}} \subset \mathcal{V}^*$, $\mathcal{W} \subset C(0,T;H)$ and $\{w \in \mathcal{V} \mid w' \in \mathcal{W}\} \subset C(0,T;V)$ are continuous. By Theorem 5.1 in Chapter 1 of Lions [52] the embedding $\mathcal{W} \subset \widehat{\mathcal{H}}$ is also compact. The continuity of the embedding $\mathcal{W} \subset C(0,T;H)$ entails the following result (cf. Lemma 4(b) of [69]) which will be useful in our study.

Corollary 6 *If u_n, $u \in \mathcal{W}$ and $u_n \to u$ weakly in $\mathcal{W}$, then $u_n(t) \to u(t)$ weakly in H for all $t \in [0,T]$.*

Furthermore, given a Banach space Y, we will use the following notation

$$\begin{aligned}
\mathcal{P}_{f(c)}(Y) &= \{\, A \subseteq Y \mid A \text{ is nonempty, closed, (convex)} \,\}; \\
\mathcal{P}_{(w)k(c)}(Y) &= \{\, A \subseteq Y \mid A \text{ is nonempty, (weakly) compact, (convex)} \,\}.
\end{aligned}$$

2.2 Single-valued and multivalued operators

Let X be a reflexive Banach space with the norm $\|\cdot\|$, X^* be its dual and let $\langle \cdot, \cdot \rangle$ denote the duality pairing of X^* and X. First we recall some definitions related to the single-valued and multivalued operators (cf. Denkowski et al. [23, 24], Hu and Papageorgiou [37], Naniewicz and Panagiotopoulos [73], Showalter [94] and Zeidler [99]).

Definition 7 *A mapping T from X to X^* is said to be*

(i) **bounded** *if it takes bounded sets of X into bounded sets of X^*;*

(ii) **weakly (strongly) continuous** *if for every $x_n \to x$ weakly (strongly) in X, we have $Tx_n \to Tx$ weakly (strongly) in X^*;*

(iii) **hemicontinuous** *if the real-valued function $t \to \langle T(u+tv), w\rangle$ is continuous on $[0,1]$ for all u, v, $w \in X$;*

(iv) **demicontinuous** *if for every $x_n \to x$ in X, we have $Tx_n \to Tx$ weakly in X^*;*

(v) **monotone** *if $\langle Tx - Ty, x - y\rangle \geq 0$ for all x, $y \in X$;*

(vi) **maximal monotone** *if T is monotone and for any x, $y \in X$, $w \in X^*$ such that $\langle Tx - w, x - y\rangle \geq 0$, we have $w = Ty$;*

(vii) **strongly monotone** *if there exists $c > 0$ and $p > 1$ such that for any x, $y \in X$, we have $\langle Tx - Ty, x - y\rangle \geq c\,\|x - y\|^p$;*

(viii) **pseudomonotone** *if $x_n \to x$ weakly in X and $\limsup \langle Tx_n, x_n - x\rangle \leq 0$ implies $\langle Tx, x - v\rangle \leq \liminf \langle Tx_n, x_n - v\rangle$ for all $v \in X$.*

Remark 8 *It can be shown (cf. [11]) that a mapping $T\colon X \to X^*$ is pseudomonotone according to* (viii) *of Definition 7 if and only if $x_n \to x$ weakly in X and $\limsup \langle Tx_n, x_n - x\rangle \leq 0$ implies $\lim \langle Tx_n, x_n - x\rangle = 0$ and $Tx_n \to Tx$ weakly in X^*.*

Definition 9 *A mapping T from X to 2^{X^*} is said to be*

(i) **bounded** *if set $T(C)$ is bounded in X^* for any bounded subset $C \subset X$;*

(ii) **upper semicontinuous** *if set $T^-(C) = \{x \in X \mid Tx \cap C \neq \emptyset\}$ is closed in X for any closed subset $C \subset X^*$ (cf. also Definition 77 and Remark 78);*

(iii) **monotone** *if for all x, $y \in X$, $x^* \in Tx$, $y^* \in Ty$, we have $\langle x^* - y^*, x - y\rangle \geq 0$;*

(iv) **maximal monotone** *if T is monotone and for any $x \in X$, $x^* \in X^*$ such that $\langle x^* - y^*, x - y\rangle \geq 0$ for all $y \in Y^*$, $y^* \in Ty$, we have $x^* \in Tx$;*

(vii) **strongly monotone** *if there exists $c > 0$ and $p > 1$ such that for any x, $y \in X$, $x^* \in Tx$, $y^* \in Ty$, we have $\langle x^* - y^*, x - y\rangle \geq c\,\|x - y\|^p$;*

(v) **pseudomonotone** *if it satisfies*

  (a) *for every $x \in X$, Tx is a nonempty, convex, and weakly compact set in X^*;*

  (b) *T is upper semicontinuous from every finite dimensional subspace of X into X^* endowed with the weak topology;*

  (c) *if $x_n \to x$ weakly in X, $x_n^* \in Tx_n$, and $\limsup \langle x_n^*, x_n - x\rangle \leq 0$, then for each $y \in X$ there exists $x^*(y) \in Tx$ such that $\langle x^*(y), x - y\rangle \leq \liminf \langle x_n^*, x_n - x\rangle$.*

(vi) **coercive** *if there exists a function $c\colon \mathbb{R}_+ \to \mathbb{R}$ with $\lim\limits_{r\to+\infty} c(r) = +\infty$ such that for all $x \in X$ and $x^* \in Tx$, we have $\langle x^*, x\rangle \geq c\,(\|x\|)\|x\|$;*

Let $L\colon D(L) \subset X \to X^*$ be a linear densely defined maximal monotone operator. A mapping $T\colon X \to 2^{X^*}$ is said to be

(vii) **L-pseudomonotone (pseudomonotone with respect to $D(L)$)** if and only if (v)(a), (b) and the following hold:

(d) if $\{x_n\} \subset D(L)$ is such that $x_n \to x$ weakly in X, $x \in D(L)$, $Lx_n \to Lx$ weakly in X^*, $x_n^* \in Tx_n$, $x_n^* \to x^*$ weakly in X^*, and $\limsup \langle x_n^*, x_n - x\rangle \leq 0$, then $x^* \in Tx$ and $\langle x_n^*, x_n\rangle \to \langle x^*, x\rangle$.

The following surjectivity result for L-pseudomonotone operators can be found in Theorem 1.3.73 of Denkowski et al. [24] and for the convenience of the reader we include it here.

Theorem 10 *If X is a reflexive, strictly convex Banach space, $L\colon D(L) \subset X \to X^*$ is a linear densely defined maximal monotone operator, and $T\colon X \to 2^{X^*} \setminus \{\emptyset\}$ is bounded, coercive and pseudomonotone with respect to $D(L)$, then $L+T$ is surjective.*

Finally, we recall a result which show that certain properties of the operator A are transferred to its **Nemitsky (superposition) operator** $\widehat{A}$.

Lemma 11 *Let V be a reflexive Banach space with the norm $\|\cdot\|$, the dual V^* and let $\langle\cdot,\cdot\rangle$ denote the duality pairing of V^* and V. Let $2 \leq p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and let $A\colon (0,T) \times V \to V^*$ be an operator such that*

(i) $A(\cdot, v)$ *is measurable on (0, T), for all* $v \in V$*;*
(ii) $A(t, \cdot)$ *is demicontinuous, for a.e.* $t \in (0,T)$*;*
(iii) *there exists a nonnegative function* $\overline{a}_1 \in L^q(0,T)$ *and a constant* $\overline{b}_1 > 0$ *such that*

$$\|A(t,v)\|_{V^*} \leq \overline{a}_1(t) + \overline{b}_1 \|v\|^{p-1} \text{ for all } v \in V, \ \text{a.e. } t \in (0,T);$$

(iv) *there exist constants* $\overline{b}_2 > 0$, $\overline{b}_3 \geq 0$ *and a function* $\overline{a}_2 \in L^1(0,T)$ *such that*

$$\langle A(t,v), v\rangle \geq \overline{b}_2 \|v\|^p - \overline{b}_3 \|v\|^r - \overline{a}_2(t)$$

for all $v \in V$ *and a.e.* $t \in (0,T)$ *with* $p > r$*.*

Then the Nemitsky operator $\widehat{A}\colon L^p(0,T;V) \to L^q(0,T;V^*)$ *defined by*

$$(\widehat{A}v)(t) = A(t, v(t)) \ \text{ for } v \in L^p(0,T;V)$$

has the following properties:

(i) $\widehat{A}$ *is well defined, i.e.* $\widehat{A}v \in L^q(0,T;V^*)$ *for all* $v \in L^p(0,T;V)$*;*
(ii) $\widehat{A}$ *is demicontinuous;*
(iii) *there exist constants* $\widehat{a}_1 \geq 0$ *and* $\widehat{b}_1 > 0$ *such that*

$$\|\widehat{A}v\|_{L^q(0,T;V^*)} \leq \widehat{a}_1 + \widehat{b}_1 \|v\|^{p-1}_{L^p(0,T;V)} \ \text{ for all } v \in L^p(0,T;V);$$

(iv) *there exist constants* $\overline{a}_2 > 0$ *and* $\overline{b}_2 \geq 0$ *such that*

$$\langle \widehat{A}v, v\rangle_{L^q(0,T;V^*)\times L^p(0,T;V)} \geq \overline{b}_2 \|v\|^p_{L^p(0,T;V)} - \widehat{b}_2 \|v\|^r_{L^p(0,T;V)} - \widehat{a}_2$$

for all $v \in L^p(0,T;V)$*.*

For the proof of the above lemma, we refer to Berkovits and Mustonen [11], and Ochal [75].

2.3 Clarke's generalized subdifferential

The purpose of this section is to present the basic facts of the theory of generalized differentiation for a locally Lipschitz function (cf. Clarke [21], Clarke et al. [22], Denkowski et al. [23] and Hu and Papageorgiou [37]). We also elaborate on the classes of functions which are regular in the sense of Clarke and prove a few results needed in what follows. Throughtout this section X is a Banach space, X^* is its dual and $\langle \cdot, \cdot \rangle_{X^* \times X}$ denotes the duality pairing between X^* and X.

Definition 12 **(Locally Lipschitz function)** *A function* $\varphi\colon U \to \mathbb{R}$ *defined on an open subset* U *of* X *is said to be locally Lipschitz on* U*, if for each* $x_0 \in U$ *there exists* $K > 0$ *and* $\varepsilon > 0$ *such that*

$$|\varphi(y) - \varphi(z)| \leq K\|y - z\| \ \text{ for all } \ y, z \in B(x_0, \varepsilon).$$

A function $\varphi \colon U \subseteq X \to \mathbb{R}$, which is Lipschitz continuous on bounded subsets of U is locally Lipschitz. The converse assertion is not generally true, cf. Chapter 2.5 of Carl et al. [16].

Definition 13 **(Generalized directional derivative)** *The generalized directional derivative (in the sense of Clarke) of the locally Lipschitz function $\varphi \colon U \to \mathbb{R}$ at the point $x \in U$ in the direction $v \in X$, denoted by $\varphi^0(x; v)$, is defined by*

$$\varphi^0(x; v) = \limsup_{y \to x, \ \lambda \downarrow 0} \frac{\varphi(y + \lambda v) - \varphi(y)}{\lambda}.$$

We observe that in contrast to the usual directional derivative, the generalized directional derivative φ^0 is always defined.

Definition 14 **(Generalized gradient)** *Let $\varphi \colon U \to \mathbb{R}$ be a locally Lipschitz function on an open set U of X. The generalized gradient (in the sense of Clarke) of φ at $x \in U$, denoted by $\partial\varphi(x)$, is a subset of a dual space X^* defined as follows*

$$\partial\varphi(x) = \{ \zeta \in X^* \mid \varphi^0(x; v) \geq \langle \zeta, v \rangle_{X^* \times X} \ \text{ for all } \ v \in X \}.$$

The next proposition provides basic properties of the generalized directional derivative and the generalized gradient.

Proposition 15 *If $\varphi \colon U \to \mathbb{R}$ is a locally Lipschitz function on an open set U of X, then*

(i) *for every $x \in U$ the function $X \ni v \to \varphi^0(x; v) \in \mathbb{R}$ is sublinear, finite, positively homogeneous, subadditive, Lipschitz continuous and $\varphi^0(x; -v) = (-\varphi)^0(x; v)$ for all $v \in X$;*

(ii) *the function $U \times X \ni (x, v) \to \varphi^0(x; v) \in \mathbb{R}$ is upper semicontinuous, i.e. for all $x \in U$, $v \in X$, $\{x_n\} \subset U$, $\{v_n\} \subset X$, $x_n \to x$ in U and $v_n \to v$ in X, we have $\limsup \varphi^0(x_n; v_n) \leq \varphi^0(x; v)$;*

(iii) *for every $v \in X$ we have $\varphi^0(x; v) = \max\{ \langle z, v \rangle \mid z \in \partial\varphi(x) \}$;*

(iv) *for every $x \in U$ the gradient $\partial\varphi(x)$ is nonempty, convex, and weakly-$*$ compact subset of X^* which is bounded by the Lipschitz constant $K > 0$ of φ near x;*

(v) *the graph of the generalized gradient $\partial\varphi$ is closed in $U \times$ (w-$*$-X^*)-topology, i.e. if $\{x_n\} \subset U$ and $\{\zeta_n\} \subset X^*$ are sequences such that $\zeta_n \in \partial\varphi(x_n)$ and $x_n \to x$ in X, $\zeta_n \to \zeta$ weakly-$*$ in X^*, then $\zeta \in \partial\varphi(x)$, where (w-$*$-$X^*$) denotes the space X^* equipped with weak-$*$ topology;*

(vi) *the multifunction $U \ni x \to \partial\varphi(x) \subseteq X^*$ is upper semicontinuous from U into w-$*$-X^*.*

Proof. The properties (i)-(v) can be found in Propositions 2.1.1, 2.1.2 and 2.1.5 of Clarke [21]. For the proof of (vi), we observe that from (iii), the multifunction $\partial\varphi$ is locally relatively compact (i.e. for every $x \in X$, there exists a neighborhood U_x of

x such that $\overline{\partial\varphi(U_x)}$ is a weakly-$*$ compact subset of X^*). Thus, due to Proposition 4.1.16 of [23], since the graph of $\partial\varphi$ is closed in $X\times$ (w-$*$-X^*)-topology, we obtain the upper semicontinuity of $x \mapsto \partial\varphi(x)$. □

In order to state the relations between the generalized directional derivative and classical notions of differentiability, we need the following.

Definition 16 **(Classical (one-sided) directional derivative)** *Let $\varphi\colon U \to \mathbb{R}$ be defined on an open subset U of X. The directional derivative of φ at $x \in U$ in the direction $v \in X$ is defined by*

$$\varphi'(x;v) = \lim_{\lambda\downarrow 0} \frac{\varphi(x+\lambda v) - \varphi(x)}{\lambda}, \tag{2}$$

when the limit exists.

We recall the definition of a regular function which is needed in the sequel.

Definition 17 **(Regular function)** *A function $\varphi\colon U \to \mathbb{R}$ on an open set U of X is said to be regular (in the sense of Clarke) at $x \in U$, if*

(i) *for all $v \in X$ the directional derivative $\varphi'(x;v)$ exists, and*

(ii) *for all $v \in X$, $\varphi'(x;v) = \varphi^0(x;v)$.*

The function φ is regular (in the sense of Clarke) on U if it is regular at every point $x \in U$.

Remark 18 *Directly from Definitions 13 and 16, it is clear that $\varphi'(x;v) \leq \varphi^0(x;v)$ for all $x \in U$ and all $v \in X$ when $\varphi'(x;v)$ exists.*

Definition 19 **(Gâteaux derivative)** *Let $\varphi\colon U \to \mathbb{R}$ be defined on an open subset U of X. We say that φ is Gâteaux differentiable at $x \in U$ provided that the limit in (2) exists for all $v \in X$ and there exists a (necessarily unique) element $\varphi'_G(x) \in X^*$ (called the Gâteaux derivative) that satisfies*

$$\varphi'(x;v) = \langle \varphi'_G(x), v\rangle_{X^*\times X} \quad \text{for all} \quad v \in X. \tag{3}$$

Definition 20 **(Fréchet derivative)** *Let $\varphi\colon U \to \mathbb{R}$ be defined on an open subset U of X. We say that φ is Fréchet differentiable at $x \in U$ provided that (3) holds at the point x and in addition that the convergence in (2) is uniform with respect to v in bounded subsets of X. In this case, we write $\varphi'(x)$ (the Fréchet derivative) in place of $\varphi'_G(x)$.*

The two notions of differentiability are not equivalent, even in finite dimensions. The following relations between Gâteaux and Fréchet derivative hold. If φ is Fréchet differentiable at $x \in U$, then φ is Gâteaux differentiable at x. If φ is Gâteaux differentiable in a neighborhood of x_0 and φ'_G is continuous at x_0, then φ is Fréchet differentiable at x_0 and $\varphi'(x_0) = \varphi'_G(x_0)$.

Remark 21 *If $\varphi\colon U \subset X \to \mathbb{R}$ is Fréchet differentiable in U and $\varphi'(\cdot)\colon U \to X^*$ is continuous, then we say that φ is continuously differentiable and write $\varphi \in C^1(U)$.*

The following notion of strict differentiability is intermediate between Gâteaux and continuous differentiability. It is known that the Clarke subdifferential $\partial\varphi(x)$ reduces to a singleton precisely when φ is strictly differentiable.

Definition 22 **(Strict differentiability)** *A function $\varphi\colon U \to \mathbb{R}$ be defined on an open subset U of X is strictly (Hadamard) differentiable at $x \in U$, if there exists an element $D_s\varphi(x) \in X^*$ such that*

$$\lim_{y\to x,\,\lambda\downarrow 0} \frac{\varphi(y+\lambda v) - \varphi(y)}{\lambda} = \langle D_s\varphi(x), v\rangle_{X^*\times X} \quad \text{for all} \quad v \in X$$

and provided the convergence is uniform for v in compact sets.

The following notion of subgradient of convex function generalizes the classical concept of a derivative.

Definition 23 **(Convex subdifferential)** *Let U be a convex subset of X and $\varphi\colon U \to \mathbb{R}$ be a convex function. An element $x^* \in X^*$ is called a subgradient of φ at $x \in X$ if and only if the following inequality holds*

$$\varphi(v) \geq \varphi(x) + \langle x^*, v - x\rangle_{X^*\times X} \quad \text{for all} \quad v \in X. \tag{4}$$

The set of all $x^* \in X^*$ satisfying (4) is called the subdifferential of φ at x, and is denoted by $\partial\varphi(x)$.

The following two propositions follow from Chapters 2.2 and 2.3 of [21].

Proposition 24 *Let $\varphi\colon U \to \mathbb{R}$ be defined on an open subset U of X. Then*

(i) *the function φ is strictly differentiable at $x \in U$ if and only if φ is locally Lipschitz near x and $\partial\varphi(x)$ is a singleton (which is necessarily the strict derivative of φ at x). In particular, if φ is continuously differentiable at $x \in U$, then $\varphi^0(x, v) = \varphi'(x; v) = \langle\varphi'(x), v\rangle_{X^*\times X}$ for all $v \in X$ and $\partial\varphi(x) = \{\varphi'(x)\}$;*

(ii) *if φ is regular at $x \in U$ and $\varphi'(x)$ exists, then φ is strictly differentiable at x;*

(iii) *if φ is regular at $x \in U$, $\varphi'(x)$ exists and g is locally Lipschitz near x, then $\partial(\varphi + g)(x) = \{\varphi'(x)\} + \partial g(x)$;*

(iv) *if φ is Gâteaux differentiable at $x \in U$, then $\varphi'_G(x) \in \partial\varphi(x)$;*

(v) *if U is a convex set and $\varphi\colon U \to \mathbb{R}$ is convex, then the Clarke subdifferential $\partial\varphi(x)$ at any $x \in U$ coincides with the subdifferential of φ at x in the sense of convex analysis.*

(vi) *if U is a convex set and $\varphi\colon U \to \mathbb{R}$ is convex, then the Clarke subdifferential $\partial\varphi\colon U \to 2^{X^*}$ is a monotone operator.*

The following result collects the properties of regular functions.

Proposition 25

(i) *If* $\varphi\colon U \to \mathbb{R}$ *defined on an open subset* U *of* X *is strictly differentiable at* $x \in U$*, then* φ *is regular at* x*;*

(ii) *If the open set* U *is convex and* $\varphi\colon U \to \mathbb{R}$ *is a convex function, then* φ *is locally Lipschitz and regular on* U*;*

(iii) *Any finite nonnegative linear combination of regular functions at* x*, is regular at* x*;*

(iv) *If* $\varphi\colon U \to \mathbb{R}$ *defined on an open subset* U *of* X *is regular at* $x \in U$ *and there exists the Gâteaux derivative* $\varphi'_G(x)$ *of* φ *at* x*, then* $\partial\varphi(x) = \{\varphi'_G(x)\}$*.*

In the case X is of finite dimension, we have the following characterization of the Clarke subdifferential (cf. Theorem 2.5.1 of [21]). Recall that if a function $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ is Lipschitz on an open set $U \subset \mathbb{R}^n$, then by the celebrated theorem of Rademacher (cf. e.g. Corollary 4.19 in [22]), φ is Fréchet differentiable almost everywhere on U.

Proposition 26 *Let* $\varphi\colon U \subset \mathbb{R}^n \to \mathbb{R}$ *be a locally Lipschitz near* $x \in U$*,* N *be any Lebesgue-null set in* $\mathbb{R}^n$ *and* N_φ *be the Lebesgue-null set outside of which* φ *is Fréchet differentiable. Then*

$$\partial\varphi(x) = \operatorname{co}\{\, \lim \nabla\varphi(x_i) \mid x_i \to x,\ x_i \notin N,\ x_i \notin N_\varphi \,\}.$$

Now we recall the basic calculus rules for the generalized directional derivative and the generalized gradient which are needed in the sequel.

Proposition 27 (i) *For a locally Lipschitz function* $\varphi\colon U \to \mathbb{R}$ *defined on an open subset* U *of* X *and for all* $\lambda \in \mathbb{R}$*, we have* $\partial(\lambda\varphi)(x) = \lambda\partial\varphi(x)$ *for all* $x \in U$*;*

(ii) **(The sum rules)** *For locally Lipschitz functions* φ_1*,* $\varphi_2\colon U \to \mathbb{R}$ *defined on an open subset* U *of* X*, we have*

$$\partial(\varphi_1 + \varphi_2)(x) \subset \partial\varphi_1(x) + \partial\varphi_2(x) \quad \text{for all} \quad x \in U \tag{5}$$

or equivalently

$$(\varphi_1 + \varphi_2)^0(x; v) \le \varphi_1^0(x; v) + \varphi_2^0(x; v) \quad \text{for all } v \ \in X; \tag{6}$$

(iii) *If one of* φ_1*,* φ_2 *is strictly differentiable at* $x \in U$*, then in (5) and (6) equalities hold.*

(iv) *In addition, if* φ_1*,* φ_2 *are regular at* $x \in U$*, then* $\varphi_1 + \varphi_2$ *is regular and we also have equalities in (5) and (6). The extension of (5) and (6) to finite nonnegative linear combinations is immediate.*

Proposition 28 *Let* X *and* Y *be Banach spaces,* $A \in \mathcal{L}(Y, X)$ *and let* $\varphi\colon X \to \mathbb{R}$ *be a locally Lipschitz function. Then*

(a) $\quad (\varphi \circ A)^0(x; v) \le \varphi^0(Ax; Av) \quad \text{for all} \quad x, v \in Y,$

(b) $\quad \partial(\varphi \circ A)(x) \subseteq A^* \partial\varphi(Ax)$ for all $x \in Y$,

where $A^* \in \mathcal{L}(X^*, Y^*)$ *denotes the adjoint operator to* A*. If in addition either* φ *or* $-\varphi$ *is regular at* Ax*, then either* $\varphi \circ A \colon Y \to \mathbb{R}$ *or* $(-\varphi) \circ A \colon Y \to \mathbb{R}$ *is regular and* (a) *and* (b) *hold with equalities. The equalities in* (a) *and* (b) *are also true if, instead of the regularity condition, we assume that* A *is surjective.*

Proposition 29 *Let* X_1 *and* X_2 *be Banach spaces. If* $\varphi \colon X_1 \times X_2 \to \mathbb{R}$ *is locally Lipschitz and regular at* $x = (x_1, x_2) \in X_1 \times X_2$*, then*

$$\partial\varphi(x_1, x_2) \subset \partial_1\varphi(x_1, x_2) \times \partial_2\varphi(x_1, x_2), \tag{7}$$

where by $\partial_1\varphi(x_1, x_2)$ *(respectively* $\partial_2\varphi(x_1, x_2)$*) we denote the partial generalized subdifferential of* $\varphi(\cdot, x_2)$ *(respectively* $\varphi(x_1, \cdot)$*), or equivalently*

$$\varphi^0(x_1, x_2; v_1, v_2) \le \varphi_1^0(x_1, x_2; v_1) + \varphi_2^0(x_1, x_2; v_2) \quad \text{for all} \quad (v_1, v_2) \in X_1 \times X_2,$$

where $\varphi_1^0(x_1, x_2; v_1)$ *(respectively* $\varphi_2^0(x_1, x_2; v_2)$*) denotes the partial generalized directional derivative of* $\varphi(\cdot, x_2)$ *(respectively* $\varphi(x_1, \cdot)$*) at the point* x_1 *(respectively* x_2*) in the direction* v_1 *(respectively* v_2*).*

In general in Proposition 29, without the regularity hypothesis, there is no relation between the two sets in (7), cf. Example 2.5.2 in [21].

Lemma 30 *Let* X_1 *and* X_2 *be Banach spaces and let* $\varphi \colon X_1 \times X_2 \to \mathbb{R}$ *be locally Lipschitz function at* $(x_1, x_2) \in X_1 \times X_2$*.*

(1) *If* $g \colon X_1 \to \mathbb{R}$ *is locally Lipschitz at* x_1 *and* $\varphi(y_1, y_2) = g(y_1)$ *for all* $(y_1, y_2) \in X_1 \times X_2$*, then*

(i) $\quad \varphi^0(x_1, x_2; v_1, v_2) = g^0(x_1; v_1)$ for all $(v_1, v_2) \in X_1 \times X_2$*;*

(ii) $\quad \partial\varphi(x_1, x_2) = \partial g(x_1) \times \{0\}$.

(2) *If* $h \colon X_2 \to \mathbb{R}$ *is locally Lipschitz at* x_2 *and* $\varphi(y_1, y_2) = h(y_2)$ *for all* $(y_1, y_2) \in X_1 \times X_2$*, then*

(i) $\quad \varphi^0(x_1, x_2; v_1, v_2) = h^0(x_2; v_2)$ for all $(v_1, v_2) \in X_1 \times X_2$*;*

(ii) $\quad \partial\varphi(x_1, x_2) = \{0\} \times \partial h(x_2)$.

Proof. We prove (1) since the proof of (2) is analogous. The first relation follows from the direct calculation

$$\begin{aligned}
\varphi^0(x_1, x_2; v_1, v_2) &= \limsup_{(y_1, y_2) \to (x_1, x_2),\ \lambda \downarrow 0} \frac{\varphi((y_1, y_2) + \lambda(v_1, v_2)) - \varphi(y_1, y_2)}{\lambda} = \\
&= \limsup_{(y_1, y_2) \to (x_1, x_2),\ \lambda \downarrow 0} \frac{g(y_1 + \lambda v_1) - g(y_1)}{\lambda} = \\
&= \limsup_{y_1 \to x_1,\ \lambda \downarrow 0} \frac{g(y_1 + \lambda v_1) - g(y_1)}{\lambda} = g^0(x_1; v_1)
\end{aligned}$$

for all $(v_1, v_2) \in X_1 \times X_2$. For the proof of (ii), let $(x_1^*, x_2^*) \in \partial\varphi(x_1, x_2)$. By the definition, we have

$$\langle x_1^*, v_1 \rangle_{X_1^* \times X_1} + \langle x_2^*, v_2 \rangle_{X_2^* \times X_2} \le \varphi^0(x_1, x_2; v_1, v_2)$$

for every $(v_1, v_2) \in X_1 \times X_2$. Choosing $(v_1, v_2) = (v_1, 0)$, we obtain $\langle x_1^*, v_1 \rangle_{X_1^* \times X_1} \le \varphi^0(x_1, x_2; v_1, 0) = g^0(x_1; v_1)$ for every $v_1 \in X_1$ which means that $x_1^* \in \partial g(x_1)$. Taking $(v_1, v_2) = (0, v_2)$, we get $\langle x_2^*, v_2 \rangle_{X_2^* \times X_2} \le g^0(x_1; 0) = 0$ for $v_2 \in X_2$. Since $v_2 \in X_2$ is arbitrary, we have $\langle x_2^*, v_2 \rangle_{X_2^* \times X_2} = 0$ and then $x_2^* = 0$.

Conversely, let $(x_1^*, x_2^*) \in \partial g(x_1) \times \{0\}$. For all $(v_1, v_2) \in X_1 \times X_2$, we have

$$\langle x_1^*, v_1 \rangle_{X_1^* \times X_1} + \langle x_2^*, v_2 \rangle_{X_2^* \times X_2} = \langle x_1^*, v_1 \rangle_{X_1^* \times X_1} \le g^0(x_1; v_1) = \varphi^0(x_1, x_2; v_1, v_2)$$

which implies that $(x_1^*, x_2^*) \in \partial\varphi(x_1, x_2)$. The proof is complete. □

Next, we elaborate on locally Lipschitz functions which are regular in the sense of Clarke. We consider the classes of max (min) type and d.c type (difference of convex functions). The proof of the first result can be found in Proposition 2.3.12 of [21] and Proposition 5.6.29 of [23].

Proposition 31 *Let φ_1, $\varphi_2 \colon U \to \mathbb{R}$ be locally Lipschitz functions near $x \in U$, U be an open subset of X and $\varphi = \max\{\varphi_1, \varphi_2\}$. Then φ is locally Lipschitz near x and*

$$\partial\varphi(x) \subset \mathrm{co}\,\{\partial\varphi_k(x) \mid k \in I(x)\}, \tag{8}$$

where $I(x) = \{k \in \{1, 2\} \mid \varphi(x) = \varphi_k(x)\}$ is the active index set at x. If in addition, φ_1 and φ_2 are regular at x, then φ is regular at x and (8) holds with equality.

Corollary 32 *Let φ_1, $\varphi_2 \colon U \to \mathbb{R}$ be strictly differentiable functions at $x \in U$, U be an open subset of X and $\varphi = \min\{\varphi_1, \varphi_2\}$. Then $-\varphi$ is locally Lipschitz near x, regular at x and $\partial\varphi(x) = \mathrm{co}\,\{\partial\varphi_k(x) \mid k \in I(x)\}$, where $I(x)$ is the active index set at x.*

Proof. Since φ_1 and φ_2 are strictly differentiable at $x \in U$, the functions $-\varphi_1$ and $-\varphi_2$ also have the same property. From Proposition 25(i), it follows that $-\varphi_1$ and $-\varphi_2$ are locally Lipschitz near x and regular at x. Let $g_1 = -\varphi_1$, $g_2 = -\varphi_2$ and $g = \max\{g_1, g_2\}$. It follows from Proposition 31 that g is locally Lipschitz near x, regular at x and $\partial g(x) = \mathrm{co}\,\{\partial g_k(x) \mid k \in I(x)\}$. On the other hand, we have

$$g = \max\{g_1, g_2\} = \max\{-\varphi_1, -\varphi_2\} = -\min\{\varphi_1, \varphi_2\} = -\varphi$$

and

$$\begin{aligned} -\partial\varphi(x) = \partial(-\varphi)(x) = \partial g(x) = \mathrm{co}\,\{\partial(-\varphi_k)(x) \mid k \in I(x)\} = \\ = \mathrm{co}\,\{-\partial\varphi_k(x) \mid k \in I(x)\} = -\mathrm{co}\,\{\partial\varphi_k(x) \mid k \in I(x)\}. \end{aligned}$$

Hence the conclusion of the corollary follows. □

The next proposition generalizes Lemma 14 of [68].

Proposition 33 *Let $\varphi_1, \varphi_2 \colon U \to \mathbb{R}$ be convex functions, U be an open convex subset of X, $\varphi = \varphi_1 - \varphi_2$ and $x \in U$. Assume that*

$$\partial\varphi_1(x) \text{ is singleton (or } \partial\varphi_2(x) \text{ is singleton).}$$

Then

$$-\varphi \text{ is regular at } x \text{ (or } \varphi \text{ is regular at } x \text{ respectively)}$$

and

$$\partial\varphi(x) = \partial\varphi_1(x) - \partial\varphi_2(x), \tag{9}$$

where $\partial\varphi_k$, $k = 1, 2$ are the subdifferentials in the sense of convex analysis.

Proof. From Proposition 25(ii) we know that φ_k, $k = 1, 2$ are locally Lipschitz and regular on U. Suppose $\partial\varphi_1(x)$ is a singleton. By Proposition 24(i), the function φ_1 is strictly differentiable at x. Thus $-\varphi_1$ is also strictly differentiable at x and again, by Proposition 25(ii), it follows that $-\varphi_1$ is regular at x. Hence $-\varphi = -\varphi_1 + \varphi_2$ is regular at x as the sum of two regular functions. Moreover, from Propositions 25(iii) and 27, we have

$$\begin{aligned} -\partial\varphi(x) = \partial(-\varphi)(x) &= \partial(-\varphi_1 + \varphi_2)(x) = \\ &= \partial(-\varphi_1)(x) + \partial\varphi_2(x) = -\partial\varphi_1(x) + \partial\varphi_2(x) \end{aligned}$$

which implies (9).

If $\partial\varphi_2(x)$ is a singleton, then as before by using Proposition 24(i), (ii), we deduce φ_2 is strictly differentiable at x which in turn implies that $-\varphi_2$ is strictly differentiable and regular at x. So $\varphi = \varphi_1 + (-\varphi_2)$ is regular at x being the sum of two regular functions and by Propositions 25(iii) and 27, we obtain

$$\partial\varphi(x) = \partial(\varphi_1 + (-\varphi_2))(x) = \partial\varphi_1(x) + \partial(-\varphi_2)(x) = \partial\varphi_1(x) - \partial\varphi_2(x)$$

which gives the equality (9). In view of convexity of φ_k, $k = 1, 2$ their Clarke subdifferentials coincide with the subdifferentials in the sense of convex analysis. The proof is completed. □

Lemma 34 *Let X and Y be Banach spaces and $\varphi \colon X \times Y \to \mathbb{R}$ be such that*

(i) *$\varphi(\cdot, y)$ is continuous for all $y \in Y$;*

(ii) *$\varphi(x, \cdot)$ is locally Lipschitz on Y for all $x \in X$;*

(iii) *there is a constant $c > 0$ such that for all $\eta \in \partial\varphi(x, y)$, we have*

$$\|\eta\|_{Y^*} \le c\,(1 + \|x\|_X + \|y\|_Y) \quad \text{for all } x \in X,\ y \in Y,$$

where $\partial\varphi$ denotes the generalized gradient of $\varphi(x, \cdot)$.

Then φ is continuous on $X \times Y$.

Proof. Let $x \in X$ and $y_1,\ y_2 \in Y$. By the Lebourg mean value theorem (cf. e.g. Theorem 5.6.25 of [23]), we can find y^* in the interval $[y_1, y_2]$ and $u^* \in \partial\varphi(x, y^*)$ such that $\varphi(x, y_1) - \varphi(x, y_2) = \langle u^*, y_1 - y_2 \rangle_{Y^* \times Y}$. Hence

$$\begin{aligned} |\varphi(x, y_1) - \varphi(x, y_2)| &\leq \|u^*\|_{Y^*} \|y_1 - y_2\|_Y \leq \\ &\leq c\,(1 + \|x\|_X + \|y^*\|_Y)\, \|y_1 - y_2\|_Y \leq \\ &\leq c_1\,(1 + \|x\|_X + \|y_1\|_Y + \|y_2\|_Y)\, \|y_1 - y_2\|_Y \end{aligned}$$

for some $c_1 > 0$. Let $\{x_n\} \subset X$ and $\{y_n\} \subset Y$ be such that $x_n \to x_0$ in X and $y_n \to y_0$ in Y. We have

$$\begin{aligned} |\varphi(x_n, y_n) - \varphi(x_0, y_0)| &\leq |\varphi(x_n, y_n) - \varphi(x_n, y_0)| + |\varphi(x_n, y_0) - \varphi(x_0, y_0)| \leq \\ &\leq c_1\,(1 + \|x_n\|_X + \|y_n\|_Y + \|y_0\|_Y)\, \|y_n - y_0\|_Y + \\ &+ |\varphi(x_n, y_0) - \varphi(x_0, y_0)|. \end{aligned}$$

Since $\|x_n\|_X,\ \|y_n\|_Y \leq c_2$ with a constant $c_2 > 0$ and $\varphi(\cdot, y_0)$ is continuous, we deduce that $\varphi(x_n, y_n) \to \varphi(x_0, y_0)$, which completes the proof. $\square$

We conclude this section with a result on measurability of the multifunction of the subdifferential type.

PROPOSITION 35 *Let X be a separable reflexive Banach space, $0 < T < \infty$ and $\varphi\colon (0,T) \times X \to \mathbb{R}$ be a function such that $\varphi(\cdot, x)$ is measurable for all $x \in X$ and $\varphi(t, \cdot)$ is locally Lipschitz for all $t \in (0,T)$. Then the multifunction $(0,T) \times X \ni (t,x) \mapsto \partial\varphi(t,x) \subset X^*$ is measurable, where $\partial\varphi$ denotes the Clarke generalized gradient of $\varphi(t,\cdot)$.*

Proof. Let $(t,x) \in (0,T) \times X$. First note that by Definition 13, we may express the generalized directional derivative of $\varphi(t,\cdot)$ as the upper limit of the quotient $\frac{1}{\lambda}(\varphi(t, y + \lambda v) - \varphi(t,y))$, $y \in X$, where $\lambda \downarrow 0$ taking rational values and $y \to x$ taking values in a countable dense subset of X (recall that X is separable):

$$\begin{aligned} \varphi^0(t,x;v) &= \limsup_{y \to x,\ \lambda \downarrow 0} \frac{\varphi(t, y+\lambda v) - \varphi(t,y)}{\lambda} = \inf_{r>0} \sup_{\substack{\|y - x\| \leq r \\ 0 < \lambda < r}} \frac{\varphi(t, y+\lambda v) - \varphi(t,y)}{\lambda} \\ &= \inf_{r>0} \sup_{\substack{\|y - x\| \leq r,\ 0 < \lambda < r \\ y \in D,\ \lambda \in \mathbb{Q}}} \frac{\varphi(t, y+\lambda v) - \varphi(t,y)}{\lambda} \end{aligned}$$

for all $v \in X$, where $D \subset X$ is a countable dense set. From this it follows that the function $(t,x,v) \mapsto \varphi^0(t,x;v)$ is Borel measurable as "the countable" limsup of measurable functions of (t,x,v) (note that by hypotheses, the function $(t,x) \mapsto \varphi(t,x)$ being Carathéodory, it is jointly measurable). From Lemma 69, it follows that $(t,x) \mapsto \varphi^0(t,x;v)$ is measurable for every $v \in X$.

Next, let $\Omega = (0,T) \times X$, $Y = X^*$ and $F \colon \Omega \to 2^Y$ be defined by $F(t,x) = \partial\varphi(t,x)$ for $(t,x) \in \Omega$. We already know from Proposition 15(iv) that for every $(t,x) \in \Omega$, the set $\partial\varphi(t,x)$ is nonempty, convex and weakly-$*$ compact in X^*. From Corollary 3.6.16 of [23], it follows that if X is a reflexive Banach space, then X is separable if and only if X^* is separable. Hence Y is a separable Banach space. Since the weak and weak-$*$ topologies on the dual space of a reflexive Banach space coincide (cf. e.g. [46, p.7]), the multifunction F is $\mathcal{P}_{wkc}(Y)$-valued. Using the definition of the support function (cf. Definition 73), from Proposition 15(iii), we have

$$
\begin{aligned}
\sigma(v, F(t,x)) &= \sup\{\, \langle v, a \rangle \mid a \in F(t,x) \,\} = \\
&= \max\{\, \langle v, a \rangle \mid a \in F(t,x) \,\} = \varphi^0(t,x;v)
\end{aligned}
$$

for all $v \in X$. Since $(t,x) \mapsto \varphi^0(t,x;v)$ for every $v \in X$ is a measurable function, we get that for every $v \in X$ the function $(t,x) \mapsto \sigma(v, F(t,x))$ is measurable, i.e. F is scalarly measurable. Hence by the result of Proposition 76, it follows that F is measurable. The proof is complete. □

3 Second order nonlinear evolution inclusions

The goal of this section is to study a class of second order nonlinear evolution inclusions involving a Volterra integral operator. For this class we give a result on the existence and uniqueness of solutions to the Cauchy problem for the inclusion under investigation. The proof consists of two main parts. First we consider the Cauchy problem for a nonlinear inclusion without the Volterra integral term and without a Lipschitz time dependent elasticity operator. We prove the unique solvability of this problem using the surjectivity result for pseudomonotone multivalued operators. In the second part of the proof, we apply the Banach Contraction Principle to show that a suitable contraction operator has a unique fixed point which will be the solution of the problem under consideration.

3.1 Problem statement

We begin with the notation needed for the statement of the problem. Let V and Z be separable and reflexive Banach spaces with the duals V^* and Z^*, respectively. Let H denote a separable Hilbert space and we identify H with its dual. We suppose that $V \subset H \subset V^*$ and $Z \subset H \subset Z^*$ are Gelfand triples of spaces where all embedings are continuous, dense and compact (see e.g. Chapter 23.4 of [99], Chapter 3.4 of [24]). We also assume that V is compactly embedded in Z. Let $\|\cdot\|$ and $|\cdot|$ denote the norms in V and H, respectively, and let $\langle \cdot, \cdot \rangle$ be the duality pairing between V^* and V. We also introduce the following spaces $\mathcal{V} = L^2(0,T;V)$, $\mathcal{Z} = L^2(0,T;Z)$, $\hat{\mathcal{H}} = L^2(0,T;H)$, $\mathcal{Z}^* = L^2(0,T;Z^*)$, $\mathcal{V}^* = L^2(0,T;V^*)$ and $\mathcal{W} = \{v \in \mathcal{V} \mid v' \in \mathcal{V}^*\}$. The duality pairing between $\mathcal{V}^*$ and $\mathcal{V}$ is denoted by

$$\langle\langle z, w \rangle\rangle = \int_0^T \langle z(t), w(t) \rangle \, dt \quad \text{for} \quad z \in \mathcal{V}^*, \; w \in \mathcal{V}.$$

The nonlinear evolution inclusion under consideration is as follows.

Problem $\mathcal{P}$: find $u \in \mathcal{V}$ such that $u' \in \mathcal{W}$ and

$$\begin{cases} u''(t) + A(t,u'(t)) + B(t,u(t)) + \displaystyle\int_0^t C(t-s)u(s)\,ds + \\ \qquad\qquad + F(t,u(t),u'(t)) \ni f(t) \quad \text{a.e. } t \in (0,T), \\ u(0) = u_0, \quad u'(0) = u_1, \end{cases}$$

where A, $B\colon (0,T) \times V \to V^*$ are nonlinear operators, $C(t)$ is a bounded linear operator for a.e. $t \in (0,T)$ and $F\colon (0,T) \times V \times V \to 2^{Z^*}$ is a multivalued mapping.

Let us notice that the initial conditions in Problem $\mathcal{P}$ have sense in V and H since the embeddings $\{ v \in \mathcal{V} \mid v' \in \mathcal{W} \} \subset C(0,T;V)$ and $\mathcal{W} \subset C(0,T;H)$ are continuous (cf. Section 2.1).

A solution to Problem $\mathcal{P}$ is understood as follows.

Definition 36 *A function $u \in \mathcal{V}$ is a solution of Problem $\mathcal{P}$ if and only if $u' \in \mathcal{W}$ and there exists $z \in \mathcal{Z}^*$ such that*

$$\begin{cases} u''(t) + A(t,u'(t)) + B(t,u(t)) + \displaystyle\int_0^t C(t-s)u(s)\,ds + z(t) = f(t) \quad \text{a.e. } t \in (0,T), \\ z(t) \in F(t,u(t),u'(t)) \quad \text{a.e. } t \in (0,T), \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases}$$

We will need the following hypotheses on the data.

$\underline{H(A)}$: The operator $A\colon (0,T) \times V \to V^*$ is such that

(i) $A(\cdot, v)$ is measurable on $(0,T)$ for all $v \in V$;
(ii) $A(t,\cdot)$ is pseudomonotone for a.e. $t \in (0,T)$;
(iii) $\|A(t,v)\|_{V^*} \le a_0(t) + a_1\|v\|$ for all $v \in V$, a.e. $t \in (0,T)$ with $a_0 \in L^2(0,T)$, $a_0 \ge 0$ and $a_1 > 0$;
(iv) $\langle A(t,v), v\rangle \ge \alpha\|v\|^2$ for all $v \in V$, a.e. $t \in (0,T)$ with $\alpha > 0$.

$\underline{H(A)_1}$: The operator $A\colon (0,T) \times V \to V^*$ satisfies $H(A)(i)$, (iii), (iv) and

(v) $A(t,\cdot)$ is hemicontinuous for a.e. $t \in (0,T)$;
(vi) $A(t,\cdot)$ is strongly monotone for a.e. $t \in (0,T)$, i.e. there exists $m_1 > 0$ such that $\langle A(t,v) - A(t,u), v-u\rangle \ge m_1\|v-u\|^2$ for all u, $v \in V$, a.e. $t \in (0,T)$.

Remark 37 *The hypothesis $H(A)_1$ implies $H(A)$. Indeed, strong monotonicity clearly implies monotonicity which with hemicontinuity entails (cf. Proposition 27.6(a), p.586, of Zeidler [99]) pseudomonotonicity. We also recall (cf. Remark 1.1.13 of [24]) that for monotone operators, demicontinuity and hemicontinuity are equivalent notions.*

$\underline{H(B)}$: The operator $B\colon (0,T) \times V \to V^*$ is such that

(i) $B(\cdot, v)$ is measurable on $(0,T)$ for all $v \in V$;

(ii) $B(t,\cdot)$ is Lipschitz continuous for a.e. $t \in (0,T)$, i.e. $\|B(t,u) - B(t,v)\|_{V^*} \leq L_B\|u - v\|$ for all u, $v \in V$, a.e. $t \in (0,T)$ with $L_B > 0$;

(iii) $\|B(t,v)\|_{V^*} \leq b_0(t) + b_1\|v\|$ for all $v \in V$, a.e. $t \in (0,T)$ with $b_0 \in L^2(0,T)$ and b_0, $b_1 \geq 0$.

Remark 38 *1) If the condition* $H(B)$(ii) *holds and* $B(\cdot,0) \in L^2(0,T;V^*)$*, then*

$$\|B(t,v)\|_{V^*} \leq b(t) + L_B\|v\| \text{ for all } v \in V, \text{ a.e. } t \in (0,T),$$

where $b(t) = \|B(t,0)\|_{V^*}$*,* $b \in L^2(0,T)$*,* $b \geq 0$*.*
2) If $B \in L^\infty(0,T;\mathcal{L}(V,V^*))$*, the assumption* $H(B)$(ii) *holds.*

$\underline{H(C)}$: The operator C satisfies $C \in L^2(0,T;\mathcal{L}(V,V^*))$.

$\underline{H(F)}$: The multifunction $F\colon (0,T) \times V \times V \to \mathcal{P}_{fc}(Z^*)$ is such that

(i) $F(\cdot,u,v)$ is measurable on $(0,T)$ for all u, $v \in V$;

(ii) $F(t,\cdot,\cdot)$ is upper semicontinuous from $V \times V$ into w-Z^* for a.e. $t \in (0,T)$, where $V \times V$ is endowed with $(Z \times Z)$-topology;

(iii) $\|F(t,u,v)\|_{Z^*} \leq d_0(t) + d_1\|u\| + d_2\|v\|$ for all u, $v \in V$, a.e. $t \in (0,T)$ with $d_0 \in L^2(0,T)$ and d_0, d_1, $d_2 \geq 0$.

$\underline{H(F)_1}$: The multifunction $F\colon (0,T) \times V \times V \to \mathcal{P}_{fc}(Z^*)$ satisfies $H(F)$ and

(iv) $\langle F(t,u_1,v_1) - F(t,u_2,v_2), v_1 - v_2\rangle_{Z^*\times Z} \geq -m_2\|v_1 - v_2\|^2 - m_3\|v_1 - v_2\|\|u_1 - u_2\|$ for all u_i, $v_i \in V$, $i = 1$, 2, a.e. $t \in (0,T)$ with m_2, $m_3 \geq 0$.

$\underline{(H_0)}$: $f \in \mathcal{V}^*$, $u_0 \in V$, $u_1 \in H$.

$\underline{(H_1)}$: $\alpha > 2\sqrt{3}c_e(d_1T + d_2)$, where $c_e > 0$ is the embedding constant of V into Z, i.e. $\|\cdot\|_Z \leq c_e\|\cdot\|$.

$\underline{(H_2)}$: $m_1 > m_2 + \dfrac{1}{\sqrt{2}}m_3T$.

Remark 39 *The conditions* (H_1) *and* (H_2) *give a restriction on the length of time interval* T *unless* $d_1 = m_3 = 0$*. This means that under* (H_1) *and* (H_2)*, the existence and uniqueness results of Theorems 41 and 48 below are local and hold for a sufficiently small time interval. On the other hand, if the data satisfy* (H_1) *and* (H_2) *with* $d_1 = m_3 = 0$*, then these results are global in time. For example, we observe that if the multifunction* $F(t,u,\cdot)$ *is monotone for* $u \in V$*, a.e.* $t \in (0,T)$*, i.e.* $\langle F(t,u,v_1) - F(t,u,v_2), v_1 - v_2\rangle_{Z^*\times Z} \geq 0$ *for all* u*,* $v_i \in V$*,* $i = 1$*,* 2*, a.e.* $t \in (0,T)$*, then the hypothesis* (H_2) *clearly holds with* $m_2 = m_3 = 0$ *and every* $m_1 > 0$*.*

We conclude this section with an observation concerning the existence of $\mathcal{Z}^*$ selections of the multifunction F which appears in Problem $\mathcal{P}$ (cf. Definition 36). It is known that a multifunction $\mathcal{F}\colon \Omega \times X \to 2^Y \setminus \{\emptyset\}$ which is measurable in $\omega \in \Omega$ and upper semicontinuous in $x \in X$ is not necessarily jointly measurable (see Example 7.2, Chapter 2 of [37]). In a consequence, the theorems on the existence of measurable selections of measurable multifunctions (cf. e.g. Chapter 4 of [23]) are not directly applicable in this case. Therefore it is not immediately clear that under the hypothesis $H(F)$ the multifunction $t \mapsto F(t, u(t), u'(t))$ has a measurable selection. The following lemma deals with this issue. We define a multifunction $G\colon W^{1,2}(0,T;V) \to 2^{\mathcal{Z}^*}$ by $G(u) = \{z \in \mathcal{Z}^* \mid z(t) \in F(t, u(t), u'(t)) \text{ a.e. on } (0,T)\}$.

Lemma 40 *If $F\colon (0,T) \times V \times V \to \mathcal{P}_{fc}(Z^*)$ satisfies $H(F)$, then G is $\mathcal{P}_{wkc}(\mathcal{Z}^*)$-valued.*

Proof. It is easy to see that G has convex and weakly compact values. We show that the values are nonempty. Let $u \in W^{1,2}(0,T;V)$. Then there are sequences $\{s_n\}$, $\{r_n\} \subset L^2(0,T;V) = \mathcal{V}$ of step functions such that

$$s_n(t) \to u(t), \;\; r_n(t) \to u'(t) \;\text{ in } V, \;\text{ a.e. } t \in (0,T). \tag{10}$$

From hypothesis $H(F)$(i), the multifunction $t \mapsto F(t, s_n(t), r_n(t))$ is measurable from $(0,T)$ into $\mathcal{P}_{fc}(Z^*)$. Applying the Yankov-von Neumann-Aumann selection theorem (cf. Theorem 4.3.7 of [23]), for every $n \geq 1$, there exists $z_n\colon (0,T) \to Z^*$ a measurable function such that $z_n(t) \in F(t, s_n(t), r_n(t))$ a.e. $t \in (0,T)$. Next, from $H(F)$(iii), we have

$$\|z_n\|_{\mathcal{Z}^*} \leq \sqrt{3}\, \left(\|d_0\|_{L^2(0,T)} + d_1 \|s_n\|_{\mathcal{V}} + d_2 \|r_n\|_{\mathcal{V}} \right).$$

Hence $\{z_n\}$ remains in a bounded subset of $\mathcal{Z}^*$. Thus, by passing to a subsequence, if necessary, we may suppose that $z_n \to z$ weakly in $\mathcal{Z}^*$ with $z \in \mathcal{Z}^*$. From Proposition 4.7.44 of [23], it follows that

$$z(t) \in \overline{conv}\, (w\text{-}Z^*)\text{-}\limsup \{z_n(t)\}_{n \geq 1} \;\;\text{ a.e. } t \in (0,T). \tag{11}$$

Recalling that the graph of an upper semicontinuous multifunction with closed values is closed (cf. e.g. Proposition 4.1.9 of [23]), from $H(F)$(ii), we get for a.e. $t \in (0,T)$: if $\zeta_n \in F(t, \xi_n, \eta_n)$, $\zeta_n \in Z^*$, $\zeta_n \to \zeta$ weakly in Z^*, ξ_n, $\eta_n \in V$, $\xi_n \to \xi$, $\eta_n \to \eta$ in Z, then $\zeta \in F(t, \xi, \eta)$. Hence and by (10), we have

$$(w\text{-}Z^*)\text{-}\limsup F(t, s_n(t), r_n(t)) \subset F(t, u(t), u'(t)) \;\;\text{ a.e. } t \in (0,T), \tag{12}$$

where the Kuratowski limit superior is given by

$$\begin{aligned} &(w\text{-}Z^*)\text{-}\limsup F(t, s_n(t), r_n(t)) = \\ &= \;\; \{z^* \in Z^* \mid z^* = (w\text{-}Z^*)\text{-}\lim z^*_{n_k}, z^*_{n_k} \in F(t, s_{n_k}(t), r_{n_k}(t)), n_1 < n_2 < \ldots n_k < \ldots\} \end{aligned}$$

(cf. Chapter 4.7 of [23]). So, from (11) and (12), we deduce

$$\begin{aligned} z(t) \;\; &\in \;\; \overline{conv}\, (w\text{-}Z^*)\text{-}\limsup \{z_n(t)\}_{n \geq 1} \subset \\ &\subset \;\; \overline{conv}\, (w\text{-}Z^*)\text{-}\limsup F(t, s_n(t), r_n(t)) \subset \\ &\subset \;\; F(t, u(t), u'(t)) \;\;\text{ a.e. } t \in (0,T). \end{aligned}$$

Since $z \in \mathcal{Z}^*$ and $z(t) \in F(t, u(t), u'(t))$ a.e. $t \in (0,T)$, it is clear that $z \in G(u)$. This proves that G has nonemoty values and completes the proof of the lemma. □

3.2 Evolution inclusion of Problem $\mathcal{Q}$

In this section we prove a theorem on the unique solvability of the Cauchy problem for the evolution inclusion without the Volterra integral term and without an elasticity operator. This result will play a crucial role in the proof of the solvability of Problem $\mathcal{P}$. Consider the following problem.

Problem $\mathcal{Q}$: find $u \in \mathcal{V}$ such that $u' \in \mathcal{W}$ and

$$\begin{cases} u''(t) + A(t, u'(t)) + F(t, u(t), u'(t)) \ni f(t) \ \text{ a.e. } \ t \in (0, T), \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases}$$

Theorem 41 *Under the hypotheses $H(A)$, $H(F)$, (H_0) and (H_1), Problem $\mathcal{Q}$ admits a solution. If $H(A)_1$, $H(F)_1$, (H_0), (H_1) and (H_2) hold, then the solution of Problem $\mathcal{Q}$ is unique.*

The proof of Theorem 41 will be given in several steps.

3.2.1 A priori estimate for Problem $\mathcal{Q}$

First we need the lemma on a priori estimate of a solution.

Lemma 42 *Under the hypotheses $H(A)$, $H(F)$, (H_0) and (H_1), if u is a solution to Problem $\mathcal{Q}$, then the following estimate holds*

$$\|u\|_{C(0,T;V)} + \|u'\|_{\mathcal{W}} \leq C\Big(1 + \|u_0\| + |u_1| + \|f\|_{\mathcal{V}^*}\Big) \tag{13}$$

with a constant $C > 0$.

Proof. Let u be a solution to Problem $\mathcal{Q}$, i.e. $u \in \mathcal{V}$, $u' \in \mathcal{W}$ and there is $z \in \mathcal{Z}^*$ such that

$$u''(t) + A(t, u'(t)) + z(t) = f(t) \ \text{ a.e. } \ t \in (0, T) \tag{14}$$

with $z(t) \in F(t, u(t), u'(t))$ a.e. $t \in (0, T)$, $u(0) = u_0$ and $u'(0) = u_1$. Let $t \in [0, T]$. From (14), we have

$$\int_0^t \langle u''(s), u'(s)\rangle \, ds + \int_0^t \langle A(s, u'(s)), u'(s)\rangle \, ds + \int_0^t \langle z(s), u'(s)\rangle \, ds = \int_0^t \langle f(s), u'(s)\rangle \, ds$$

for all $t \in [0, T]$. Using the integration by parts formula (cf. Proposition 3.4.14 of [24]) and the coercivity of $A(t, \cdot)$ (cf. $H(A)$(iv)), we have

$$\frac{1}{2}|u'(t)|^2 - \frac{1}{2}|u'(0)|^2 + \alpha \int_0^t \|u'(s)^2\| \, ds \leq \int_0^t \Big(\|f(s)\|_{V^*} + \|z(s)\|_{V^*}\Big)\|u'(s)\| \, ds \tag{15}$$

for all $t \in [0, T]$. From the Young inequality (cf. Lemma 85 in Section 7), we get

$$\begin{aligned} \int_0^t \Big(\|f(s)\|_{V^*} + \|z(s)\|_{V^*}\Big)\|u'(s)\| \, ds \ &\leq \ \frac{1}{2\alpha} \int_0^t \Big(\|f(s)\|_{V^*} + \|z(s)\|_{V^*}\Big)^2 ds + \\ &+ \ \frac{\alpha}{2} \int_0^t \|u'(s)\|^2 \, ds. \end{aligned}$$

From (15), it follows

$$\frac{1}{2}|u'(t)|^2 - \frac{1}{2}|u_1|^2 + \alpha \int_0^t \|u'(s)^2\|\, ds \;\le\; \frac{1}{2\alpha}\int_0^t \Big(\|f(s)\|_{V^*}^2 + \|z(s)\|_{V^*}\Big)^2 ds \le$$
$$\le\; \frac{1}{\alpha}\|f\|^2_{L^2(0,t;V)} + \frac{1}{\alpha}\|z\|^2_{L^2(0,t;V^*)}.$$

Recalling (cf. Propositions 2 and 3(iv)) that $c_e > 0$ is the embedding constant of V into Z as well as of $\mathcal{Z}^*$ into $\mathcal{V}^*$, we infer

$$|u'(t)|^2 + \alpha\, \|u'\|^2_{L^2(0,t;V)} \le |u_1|^2 + \frac{2}{\alpha}\|f\|^2_{\mathcal{V}^*} + \frac{2c_e^2}{\alpha}\|z\|^2_{L^2(0,t;Z^*)} \tag{16}$$

for all $t \in [0,T]$. On the other hand, since $u \in W^{1,2}(0,T;V)$ and V is reflexive, by Theorem 3.4.11 and Remark 3.4.9 of [23], we know that u may be identified with an absolutely continuous function with values in V and

$$u(t) = u(0) + \int_0^t u'(s)\, ds \;\; \text{for all } t \in [0,T]. \tag{17}$$

Combining the above with the Jensen inequality (cf. Lemma 81 in Section 7), we have

$$\|u(s)\|^2 \le 2\|u_0\|^2 + 2\left(\int_0^s \|u'(\tau)\|\, d\tau\right)^2 \le 2\|u_0\|^2 + 2T\int_0^s \|u'(\tau)\|^2\, d\tau$$

for all $s \in (0,t)$. Hence and from $H(F)$(iii), and Lemma 87(i), we obtain

$$\begin{aligned}
\|z\|^2_{L^2(0,t;Z^*)} &= \int_0^t \|z(s)\|^2_{Z^*}\, ds \le \int_0^t \left(d_0(s) + d_1\|u(s)\| + d_2\|u'(s)\|\right)^2 ds \le \\
&\le 3\,\|d_0\|^2_{L^2(0,t)} + 3\, d_1^2 \int_0^t \|u(s)\|^2\, ds + 3\, d_2^2\, \|u'\|^2_{L^2(0,t;V)} \le \\
&\le 3\,\|d_0\|^2_{L^2(0,T)} + 3\, d_1^2 \int_0^t \left(2\|u_0\|^2 + 2T\int_0^s \|u'(\tau)\|^2 d\,\tau\right) ds + \\
&\qquad + 3\, d_2^2\, \|u'\|^2_{L^2(0,t;V)} \le \\
&\le 3\,\|d_0\|^2_{L^2(0,T)} + 6\, d_1^2\, T\, \|u_0\|^2 + 6\, d_1^2\, T^2\, \|u'\|^2_{L^2(0,t;V)} + 3\, d_2^2\, \|u'\|^2_{L^2(0,t;V)} = \\
&= 3\,\|d_0\|^2_{L^2(0,T)} + 6\, d_1^2\, T\, \|u_0\|^2 + 3\,(2\, d_1^2\, T^2 + d_2^2)\, \|u'\|^2_{L^2(0,t;V)}
\end{aligned} \tag{18}$$

for all $t \in [0,T]$. Inserting (18) into (16), we have

$$|u'(t)|^2 + \left(\alpha - \frac{6c_e^2}{\alpha}\left(2\, d_1^2\, T^2 + d_2^2\right)\right) \|u'\|^2_{L^2(0,t;V)} \le$$
$$\le\; |u_1|^2 + \frac{2}{\alpha}\|f\|^2_{\mathcal{V}^*} + \frac{2c_e^2}{\alpha}\left(3\,\|d_0\|^2_{L^2(0,T)} + 6\, d_1^2\, T\, \|u_0\|^2\right)$$

for all $t \in [0,T]$. Since the hypothesis (H_1) implies $\alpha^2 > 6\, c_e^2\,(2d_1^2\, T^2 + d_2^2)$, we deduce that there exists a constant $C_1 > 0$ such that

$$\|u'\|_{\mathcal{V}} \le C_1\big(1 + \|u_0\| + |u_1| + \|f\|_{\mathcal{V}^*}\big). \tag{19}$$

Next, from (17), we have

$$\|u(t)\| \le \|u_0\| + \int_0^t \|u'(s)\|\, ds \le \|u_0\| + \sqrt{T}\|u'\|_{\mathcal{V}},$$

which together with (19) gives

$$\|u\|_{C(0,T;V)} = \max_{0\le t\le T} \|u(t)\| \le \|u_0\| + C_1\sqrt{T}\,(1 + \|u_0\| + |u_1| + \|f\|_{\mathcal{V}^*})\,. \tag{20}$$

From (14) and the hypothesis $H(A)$(iii), we have

$$\begin{aligned} \|u''(t)\|_{V^*} &\le \|f(t)\|_{V^*} + \|A(t,u'(t))\|_{V^*} + \|z(t)\|_{V^*} \le \\ &\le \|f(t)\|_{V^*} + a_0(t) + a_1\|u'(t)\| + \|z(t)\|_{V^*} \end{aligned}$$

for a.e. $t \in (0,T)$. Hence

$$\|u''\|^2_{\mathcal{V}^*} \le C_2 \left(\|f\|^2_{\mathcal{V}^*} + \|a_0\|^2_{L^2(0,T)} + a_1^2\,\|u'\|^2_{\mathcal{V}} + \|z\|^2_{\mathcal{V}^*} \right)$$

with a constant $C_2 > 0$. Combining this inequality with (18) and (19), we have

$$\|u''\|_{\mathcal{V}^*} \le C_3\,(1 + \|u_0\| + |u_1| + \|f\|_{\mathcal{V}^*}) \tag{21}$$

with a constant $C_3 > 0$. Now the estimate (13) is a consequence of (19), (20) and (21). The proof of the lemma is complete. □

Remark 43 *Since the embedding $\mathcal{W} \subset C(0,T;H)$ is continuous, if u is a solution to Problem $\mathcal{Q}$, then the estimate (13) implies*

$$\|u'\|_{C(0,T;H)} \le C_4\Big(1 + \|u_0\| + |u_1| + \|f\|_{\mathcal{V}^*}\Big)$$

with a constant $C_4 > 0$.

3.2.2 Existence of solutions to Problem $\mathcal{Q}$

Let us assume the hypotheses $H(A)$, $H(F)$, (H_0) and (H_1). First we define the operator $K\colon \mathcal{V} \to C(0,T;V)$ by

$$Kv(t) = \int_0^t v(s)\,ds + u_0 \quad \text{for} \quad v \in \mathcal{V}. \tag{22}$$

Problem $\mathcal{Q}$ can be now formulated as follows: find $z \in \mathcal{W}$ such that

$$\begin{cases} z'(t) + A(t,z(t)) + F(t,Kz(t),z(t)) \ni f(t) & \text{for a.e. } t \in (0,T), \\ z(0) = u_1. \end{cases} \tag{23}$$

It is obvious that $z \in \mathcal{W}$ is a solution to (23) if and only if $u = Kz$ is a solution to Problem $\mathcal{Q}$. In order to show the existence of solutions to (23), we proceed in two steps: first we assume that $u_1 \in V$ and next we pass to more general case when $u_1 \in H$.

Step 1. We suppose temporarily that $u_1 \in V$. In what follows we will need the operators $\widehat{A}_1 \colon \mathcal{V} \to \mathcal{V}^*$ and $F_1 \colon \mathcal{V} \to 2^{\mathcal{Z}^*}$ defined by

$$(\widehat{A}_1 v)(t) = A(t, v(t) + u_1), \tag{24}$$

$$F_1 v = \{ z \in \mathcal{Z}^* \mid z(t) \in F(t, K(v(t) + u_1), v(t) + u_1) \text{ a.e. } t \in (0,T) \} \tag{25}$$

for $v \in \mathcal{V}$, respectively. We remark that $\widehat{A}_1 v = \widehat{A}(v + u_1)$, where $\widehat{A} \colon \mathcal{V} \to \mathcal{V}^*$ is the Nemitsky operator corresponding to A, i.e.

$$(\widehat{A} v)(t) = A(t, v(t)) \quad \text{for } v \in \mathcal{V}. \tag{26}$$

Using these operators, from (23), we get

$$\begin{cases} z' + \widehat{A}_1 z + F_1 z \ni f, \\ z(0) = 0 \end{cases} \tag{27}$$

and note that $z \in \mathcal{W}$ is a solution to (23) if and only if $z - u_1 \in \mathcal{W}$ is a solution to (27).

Next, we recall that the generalized derivative $Lu = u'$ restricted to the subset $D(L) = \{ v \in \mathcal{W} \mid v(0) = 0 \}$ defines a linear operator $L \colon D(L) \to \mathcal{V}^*$ given by

$$\langle\langle u, v \rangle\rangle = \int_0^T \langle u'(t), v(t) \rangle \, dt \quad \text{for all } v \in \mathcal{V}.$$

From Proposition 32.10 of [99], it is well known that L is a linear, densely defined, and maximal monotone operator. The problem (27) can be now rewritten as

$$\text{find } z \in D(L) \text{ such that } (L + \mathcal{F}) z \ni f.$$

where $\mathcal{F} \colon \mathcal{V} \to 2^{\mathcal{V}^*}$ is given by $\mathcal{F} v = (\widetilde{A}_1 + F_1) v$ for $v \in \mathcal{V}$. In order to prove the existence of solutions to (27) we will show that operator $\mathcal{F}$ is bounded, coercive and pseudomonotone with respect to $D(L)$. Next, we will apply Theorem 10. In what follows we need two auxiliary results.

Lemma 44 *If $H(A)$ holds and $u_1 \in V$, then the operator $\widehat{A}_1$ defined by (24) satisfies the following:*

(1) $\|\widehat{A}_1 v\|_{\mathcal{V}^*} \le \widehat{a}_0 + \widehat{a}_1 \|v\|_{\mathcal{V}}$ *for all* $v \in \mathcal{V}$ *with* $\widehat{a}_0 \ge 0$ *and* $\widehat{a}_1 > 0$;

(2) $\langle\langle \widehat{A}_1 v, v \rangle\rangle \ge \dfrac{\alpha}{2} \|v\|^2_{\mathcal{V}} - \widehat{\alpha}_1 \|v\|_{\mathcal{V}} - \widehat{\alpha}_2$ *for all* $v \in \mathcal{V}$ *with* $\widehat{\alpha}_1, \widehat{\alpha}_2 \ge 0$;

(3) $\widehat{A}$ *is demicontinuous;*

(4) $\widehat{A}_1$ *is L-pseudomonotone.*

If $H(A)$ holds, then the Nemitsky operator $\widehat{A}$ defined by (26) satisfies the following:

(5) *For each* $\{v_n\} \subset \mathcal{W}$ *such that* $v_n \to v$ *weakly in* $\mathcal{W}$ *and* $\limsup \langle\langle \widehat{A}v_n, v_n - v\rangle\rangle \le 0$, *it follows that* $\widehat{A}v_n \to \widehat{A}v$ *weakly in* $\mathcal{V}^*$ *and* $\langle\langle \widehat{A}v_n, v_n\rangle\rangle \to \langle\langle \widehat{A}v, v\rangle\rangle$.

The proof of Lemma 44 can be found in Lemma 11 of [60].

LEMMA 45 *If* $H(F)$ *holds and* $u_1 \in V$, *then the operator* F_1 *defined by (25) satisfies the following:*

(1) $\|z\|_{\mathcal{Z}^*} \le \widehat{d}_0 + \widehat{d}_1 \|v\|_{\mathcal{V}}$ *for every* $z \in F_1 v$ *and* $v \in \mathcal{V}$ *with* $\widehat{d}_0 \ge 0$ *and* $\widehat{d}_1 > 0$;

(2) *for every* $v \in \mathcal{V}$, $F_1 v$ *is a nonempty, convex, and weakly compact subset of* $\mathcal{Z}^*$;

(3) $\langle\langle z, v\rangle\rangle \ge -\sqrt{3}\, c_e (d_1 T + d_2) \|v\|^2_{\mathcal{V}} - \widehat{d}_2 \, \|v\|_{\mathcal{V}}$ *for all* $z \in F_1 v$, $v \in \mathcal{V}$ *with* $\widehat{d}_2 \ge 0$;

(4) *for every sequence* $v_n, v \in \mathcal{V}$ *with* $v_n \to v$ *in* $\mathcal{Z}$ *and every* $z_n, z \in \mathcal{Z}^*$ *with* $z_n \to z$ *weakly in* $\mathcal{Z}^*$, *if* $z_n \in F_1 v_n$, *then* $z \in F_1 v$.

Proof. First we prove the property (1). Let $v \in \mathcal{V}$ and $z \in F_1 v$. Thus $z(t) \in F(t, K(v(t) + u_1), v(t) + u_1)$ a.e. $t \in (0, T)$. We observe that the integral operator K given by (22) is bounded from $\mathcal{V}$ into $C(0, T; H)$, i.e.

$$
\begin{aligned}
\|Kv\|_{C(0,T;V)} &= \max_{t\in[0,T]} \| \int_0^t v(s)\, ds + u_0 \| \le \\
&\le \max_{t\in[0,T]} \int_0^t \|v(s)\|\, ds + \|u_0\| \le \sqrt{T}\|v\|_{\mathcal{V}} + \|u_0\|
\end{aligned}
$$

for $v \in \mathcal{V}$. Hence, from $H(F)$(iii) and the fact that $\|u_1\|_{\mathcal{V}} = \sqrt{T}\|u_1\|$, we deduce

$$
\|z(t)\|_{Z^*} \le d_0(t) + d_1\sqrt{T}\|v\|_{\mathcal{V}} + d_1\sqrt{T}\|u_1\|_{\mathcal{V}} + d_1\|u_0\| + d_2\|v(t)\| + d_2\|u_1\|.
$$

Subsequently, using Lemma 87(i), we have

$$
\|z(t)\|^2_{Z^*} \le 3\Big(d_1^2\, T\, \|v\|^2_{\mathcal{V}} + d_2^2\, \|v(t)\|^2 + (d_0(t) + d_1\, T\, \|u_1\| + d_1\|u_0\| + d_2\|u_1\|)^2\Big)
$$

and

$$
\begin{aligned}
\|z\|^2_{\mathcal{Z}^*} = \int_0^T \|z(t)\|^2_{Z^*} &\le 3\Big(d_1^2\, T^2\, \|v\|^2_{\mathcal{V}} + d_2^2 \int_0^T \|v(t)\|^2\, dt + \\
&+ \int_0^T (d_0(t) + d_1\, T\, \|u_1\| + d_1\|u_0\| + d_2\|u_1\|)^2\, dt\Big) \le \\
&\le 3\,(d_1^2\, T^2 + d_2^2)\|v\|^2_{\mathcal{V}} + d
\end{aligned}
$$

with $d = 3\displaystyle\int_0^T \Big(d_0(t) + (d_1\, T + d_2)\, \|u_1\| + d_1\|u_0\|\Big)^2\, dt \ge 0$. Thus, by Lemma 87(ii), we have

$$
\|z\|_{\mathcal{Z}^*} \le \sqrt{3\,(d_1^2\, T^2 + d_2^2)}\, \|v\|_{\mathcal{V}} + \sqrt{d} \le \sqrt{3}\,(d_1\, T + d_2)\, \|v\|_{\mathcal{V}} + \sqrt{d}
$$

which implies that the property (1) is satisfied with $\widehat{d}_0 = \sqrt{d}$ and $\widehat{d}_1 = \sqrt{3}(d_1 T + d_2)$.

Next, from $H(F)$, by the same reasoning as in the proof of Lemma 40, we obtain that for every $v \in \mathcal{V}$ the set $F_1 v$ is nonempty in $\mathcal{Z}^*$. The fact that it is convex is clear. In order to show that $F_1 v$ is weakly compact in $\mathcal{Z}^*$, we prove that it is closed in $\mathcal{Z}^*$. Let $v \in \mathcal{V}$, $\{z_n\} \subset F_1 v$, $z_n \to z$ in $\mathcal{Z}^*$. Passing to a subsequence, if necessary, we have $z_n(t) \to z(t)$ in Z^* for a.e. $t \in (0,T)$. From the relation $z_n(t) \in F(t, K(v(t)+u_1), v(t)+u_1)$ a.e. $t \in (0,T)$, since the set is closed in Z^*, we get $z(t) \in F(t, K(v(t)+u_1), v(t)+u_1)$ a.e. $t \in (0,T)$. Hence $z \in F_1 v$ and thus $F_1 v$ is closed in $\mathcal{Z}^*$ and convex, so it is also weakly closed in space Z^*. Since $F_1 v$ is a bounded set in a reflexive Banach space Z^*, we obtain that $F_1 v$ is weakly compact in $\mathcal{Z}^*$. This implies the condition (2).

Subsequently, we provide the proof of (3). Let $v \in \mathcal{V}$ and $z \in F_1 v$. Using the property (1) and recalling that $c_e > 0$ is the embedding constant of $\mathcal{V}$ into $\mathcal{Z}$, we have

$$\begin{aligned} |\langle\langle z, v\rangle\rangle| &= |\langle\langle z, v\rangle\rangle_{\mathcal{Z}^*\times\mathcal{Z}}| \le c_e \|z\|_{\mathcal{Z}^*}\|v\|_{\mathcal{V}} \le \\ &\le \sqrt{3}\, c_e\,(d_1\,T + d_2)\|v\|^2_{\mathcal{V}} + c_e\,\sqrt{d}\,\|v\|_{\mathcal{V}}. \end{aligned}$$

Hence $\langle\langle z, v\rangle\rangle \ge -\sqrt{3}\, c_e\,(d_1\,T + d_2)\|v\|^2_{\mathcal{V}} - c_e\,\sqrt{d}\,\|v\|_{\mathcal{V}}$ and the condition (3) follows.

Finally, we prove (4). Let v_n, $v \in \mathcal{V}$, z_n, $z \in \mathcal{Z}^*$, $z_n \in F_1 v_n$ with $v_n \to v$ in $\mathcal{Z}$ and $z_n \to z$ weakly in $\mathcal{Z}^*$. Hence

$$z_n(t) \in F(t, K(v_n(t)+u_1), v_n(t)+u_1) \ \text{ a.e. } t \in (0,T). \tag{28}$$

and we may suppose (cf. Proposition 1(v)), by passing to a subsequence, if necessary that

$$v_n(t) \to v(t) \ \text{ in } Z \text{ for a.e. } t \in (0,T). \tag{29}$$

From the inequality

$$\begin{aligned} &\|K(v_n+u_1) - K(v+u_1)\|^2_{\mathcal{Z}} = \\ &= \int_0^T \left\| \int_0^t v_n(s)\,ds + u_1 t + u_0 - \int_0^t v(s)\,ds - u_1 t - u_0 \right\|^2_Z dt \le \\ &\le T\,\|v_n - v\|^2_{\mathcal{Z}}, \end{aligned}$$

we have $K(v_n+u_1) \to K(v+u_1)$ in $\mathcal{Z}$ and by passing to a further subsequence if necessary, we may assume that

$$K(v_n(t)+u_1) \to K(v(t)+u_1) \ \text{ in } Z, \text{ a.e. } t \in (0,T). \tag{30}$$

By $H(F)$(ii), (28), (29) and (30), applying the Covergence Theorem of Aubin and Cellina (cf. Proposition 83 in Section 7), we have $z(t) \in F(t, K(v(t)+u_1), v(t)+u_1)$ for a.e. $t \in (0,T)$. This implies that $z \in F_1 v$ and finishes the proof of (4). The proof of the lemma is complete. □

Now, let us continue the existence proof of the theorem.

Claim 1. The operator $\mathcal{F}$ is bounded.

From Lemmas 44(1), 45(1), and the continuity of the embeding $\mathcal{Z}^* \subset \mathcal{V}^*$, it follows easily that operator $\mathcal{F}$ maps bounded subsets of $\mathcal{V}$ into bounded subsets of $\mathcal{V}^*$, i.e. $\mathcal{F}$ is a bounded operator.

Claim 2. The operator $\mathcal{F}$ is coercive.

Let $v \in \mathcal{V}$ and $\eta \in \mathcal{F}v$, that is, $\eta = \widehat{A}_1 v + z$ with $z \in F_1 v$. From Lemmas 44(2) and 45(2), we have

$$\langle\langle \eta, v\rangle\rangle = \langle\langle \widehat{A}_1 v, v\rangle\rangle + \langle\langle z, v\rangle\rangle \geq \Big(\frac{\alpha}{2} - \sqrt{3}\, c_e\, (d_1\, T + d_2)\Big)\, \|v\|_{\mathcal{V}}^2 - \widehat{\alpha}_1 \|v\|_{\mathcal{V}} - \widehat{\alpha}_2 - \widehat{d}_2 \|v\|_{\mathcal{V}}$$

which by (H_1) immediately yields the coercivity of $\mathcal{F}$.

Claim 3. The operator $\mathcal{F}$ is pseudomonotone with respect to $D(L)$.

The fact that, for every $v \in \mathcal{V}$, $\mathcal{F}v$ is a nonempty, convex and compact subset of $\mathcal{V}^*$ follows from Lemma 45(2). Next, we prove that $\mathcal{F}$ is upper semicontinuous from $\mathcal{V}$ into $\mathcal{V}^*$ endowed with the weak topology. To this end, it is enough to show (cf. Definition 9(ii)) that if a set K is weakly closed in $\mathcal{V}^*$, then the set

$$\mathcal{F}^-(K) = \{\, v \in \mathcal{V} \mid \mathcal{F}v \cap K \neq \emptyset \,\} \ \text{ is closed in } \mathcal{V}.$$

Let $\{v_n\} \subset \mathcal{F}^-(K)$ and suppose that $v_n \to v$ in $\mathcal{V}$. For every $n \in \mathbb{N}$ we can find $\eta_n \in \mathcal{F}v_n \cap K$ which by the definition means

$$\eta_n = \widehat{A}_1 v_n + z_n \ \text{ with } z_n \in F_1 v_n. \tag{31}$$

We observe that $\{v_n\}$ is bounded in $\mathcal{V}$ and since $\mathcal{F}$ is a bounded operator, the sequence $\{\eta_n\}$ is bounded in $\mathcal{V}^*$. Hence, by passing to a subsequence if necessary, we suppose that

$$\eta_n \to \eta \ \text{ weakly in } \mathcal{V}^*, \tag{32}$$

where $\eta \in K$ by the fact that K is weakly closed in $\mathcal{V}^*$. On the other hand, by Lemma 45(1), the sequence $\{z_n\}$ is bounded in $\mathcal{Z}^*$ and again, at least for a subsequence, we may assume that

$$z_n \to z \ \text{ weakly in } \mathcal{Z}^* \text{ with } z \in \mathcal{Z}^*. \tag{33}$$

Since the embedding $\mathcal{V} \subset \mathcal{Z}$ is continuous, we know that $v_n \to v$ in $\mathcal{Z}$. Hence and from Lemma 45(4), we obtain $z \in F_1 v$. Next, from the demicontinuity of $\widehat{A}_1$ (cf. Lemma 44(3)), we have

$$\widehat{A}_1 v_n \to \widehat{A}_1 v \ \text{ weakly in } \mathcal{V}^*.$$

From this convergence, (32) and (33), by passing to the limit in (31), we obtain

$$\eta = \widehat{A}_1 v + z \ \text{ with } z \in F_1 v,$$

which means that $\eta \in \mathcal{F}v \cap K$, so $v \in \mathcal{F}^-(K)$. This proves that $\mathcal{F}^-(K)$ is closed in $\mathcal{V}$, hence $\mathcal{F}$ is upper semicountinuous from $\mathcal{V}$ into $\mathcal{V}^*$ endowed with the weak topology.

To finish the proof of the L-pseudomonotonicity of $\mathcal{F}$, it is enough to show the condition (vii)(d) in Definition 9 (see Section 2.2). Let $\{v_n\} \subset D(L)$, $v_n \to v$ weakly in $\mathcal{W}$, $\eta_n \in \mathcal{F}v_n$, $\eta_n \to \eta$ weakly in $\mathcal{V}^*$ and assume that

$$\limsup \langle\langle \eta_n, v_n - v\rangle\rangle \leq 0. \tag{34}$$

Thus, $\eta_n = \widehat{A}_1 v_n + z_n$, where $z_n \in F_1 v_n$ for all $n \in \mathbb{N}$. By the fact that F_1 is a bounded map (cf. Lemma 45(1)) and $\{v_n\}$ is bounded in $\mathcal{V}$, we infer that $\{z_n\}$ remains in a bounded subset of $\mathcal{Z}^*$. By passing to a subsequence if necessary, we may suppose

$$z_n \to z \ \text{ weakly in } \mathcal{Z}^*. \tag{35}$$

Since the embedding $V \subset Z$ is compact, from Theorem 5.1 in Chapter 1 of Lions [52], we have that $\mathcal{W} \subset \mathcal{Z}$ compactly. Therefore, we may assume that

$$v_n \to v \ \text{ in } \mathcal{Z}. \tag{36}$$

From (35), (36) and Lemma 45(4), we infer that $z \in F_1 v$. From Lemma 45(1) and (36), we obtain

$$|\langle\langle z_n, v_n - v\rangle\rangle_{\mathcal{Z}^* \times Z}| \leq \|z_n\|_{\mathcal{Z}^*} \|v_n - v\|_{\mathcal{Z}} \leq (\widehat{d}_0 + \widehat{d}_1 \, \|v_n\|_{\mathcal{V}}) \, \|v_n - v\|_{\mathcal{Z}} \to 0. \tag{37}$$

Combining (37) with (34), we infer

$$\limsup \langle\langle \widehat{A}_1 v_n, v_n - v\rangle\rangle \leq \limsup \langle\langle \eta_n, v_n - v\rangle\rangle + \limsup \langle\langle z_n, v - v_n\rangle\rangle_{\mathcal{Z}^* \times \mathcal{Z}} \leq 0.$$

From the fact that $\widehat{A}_1$ is pseudomonotone with respect to $D(L)$ (cf. Lemma 44(4)), we have

$$\widehat{A}_1 v_n \to \widehat{A}_1 v \ \text{ weakly in } \mathcal{V}^* \tag{38}$$

and

$$\langle\langle \widehat{A}_1 v_n, v_n\rangle\rangle \to \langle\langle \widehat{A}_1 v, v\rangle\rangle. \tag{39}$$

Also from (38), we conclude

$$\eta_n = \widehat{A}_1 v_n + z_n \to \widehat{A}_1 v + z =: \eta \ \text{ weakly in } \mathcal{V}^*.$$

Hence and by the fact that $z \in F_1 v$, we infer $\eta \in \mathcal{F}v$. Passing to the limit in the equation

$$\langle\langle \zeta_n, v_n\rangle\rangle = \langle\langle \widehat{A}_1 v_n, v_n\rangle\rangle + \langle\langle z_n, v_n\rangle\rangle,$$

from (37) and (39), we get $\lim \langle\langle \eta_n, v_n\rangle\rangle = \langle\langle \eta, v\rangle\rangle$ with $\eta \in \mathcal{F}v$. This proves the pseudomonotonicity of $\mathcal{F}$ with respect to $D(L)$.

It is well known (cf. Theorem of Troyanski in [99, p. 256]) that in every reflexive Banach space there exists an equivalent norm such that the space is strictly convex. Hence, we deduce that $\mathcal{V}$ is strictly convex. Thus, from Claims 1, 2, 3 and Theorem 10, we deduce that the problem (27) has a solution $z \in D(L)$, so $z + u_1$ solves (23) and $u = K(z + u_1)$ is a solution of Problem $\mathcal{Q}$ in case when $u_1 \in V$.

Step 2. Recall that we have assumed that $u_1 \in V$. Now we will remove this restriction. We assume that $u_1 \in H$. Since V is dense in H, we can find a sequence $\{u_{1n}\} \subset V$ such that $u_{1n} \to u_1$ in H as $n \to \infty$. We consider a solution u_n of Problem $\mathcal{Q}$ where u_1 is replaced with u_{1n}, i.e. a solution to the following problem

$$\begin{cases} \text{find } u_n \in \mathcal{V} \text{ such that } u_n' \in \mathcal{W} \text{ and} \\ u_n''(t) + A(t, u_n'(t)) + F(t, u_n(t), u_n'(t)) \ni f(t) \text{ a.e. } t \in (0,T), \\ u_n(0) = u_0, \quad u_n'(0) = u_{1n}. \end{cases}$$

From the first step of the proof, it follows that u_n exists for every $n \in \mathbb{N}$. We have

$$u_n''(t) + A(t, u_n'(t)) + z_n(t) = f(t) \text{ for a.e. } t \in (0,T) \tag{40}$$

with

$$z_n(t) \in F(t, u_n(t), u_n'(t)) \text{ for a.e. } t \in (0,T) \tag{41}$$

and the initial conditions $u_n(0) = u_0$, $u_n'(0) = u_{1n}$. From the estimate (13), we have

$$\|u_n\|_{C(0,T;V)} + \|u_n'\|_{\mathcal{W}} \le C\left(1 + \|u_0\| + |u_{1n}| + \|f\|_{\mathcal{V}^*}\right), \quad \text{where } C > 0.$$

Hence, as $\{u_{1n}\}$ is bounded in H, we know that $\{u_n\}$ is bounded in $\mathcal{V}$ and $\{u_n'\}$ is bounded in $\mathcal{W}$ uniformly with respect to n. So by passing to a subsequence if necessary, we may assume

$$\begin{aligned} &u_n \to u \quad \text{weakly in } \mathcal{V}, \\ &u_n' \to u' \quad \text{weakly in } \mathcal{V} \text{ and also weakly in } \mathcal{V}^*, \\ &u_n'' \to u'' \quad \text{weakly in } \mathcal{V}^* \end{aligned}$$

which implies

$$u_n \to u, \quad u_n' \to u' \quad \text{both weakly in } \mathcal{W}. \tag{42}$$

We will show that u is a solution to Problem $\mathcal{Q}$. From the above and Corollary 6, it follows that $u_n(t) \to u(t)$ and $u_n'(t) \to u'(t)$ both weakly in H for all $t \in [0,T]$. Hence $u_0 = u_n(0) \to u(0)$ weakly in H which gives $u(0) = u_0$. By a similar reason from $u_n'(0) = u_{1n}$, we obtain $u'(0) = u_1$. Using the compactness of the embedding $\mathcal{W} \subset \mathcal{Z}$, from (42), we have $u_n \to u$ and $u_n' \to u'$ both in $\mathcal{Z}$ and again for a subsequence if necessary, we may suppose

$$u_n(t) \to u(t) \text{ and } u_n'(t) \to u'(t) \text{ both in } Z \text{ for a.e. } t \in (0,T). \tag{43}$$

Subsequently, by an argument analogous to that of (18), from $H(F)$(iii), (41) and (42), we get

$$z_n \to z \quad \text{weakly in } \mathcal{Z}^*. \tag{44}$$

Using (41), (43), (44), by the convergence theorem (cf. Proposition 83), we have

$$z(t) \in F(t, u(t), u'(t)) \quad \text{a.e. } t \in (0,T). \tag{45}$$

Next, we will show that

$$\widehat{A}u_n' \to \widehat{A}u' \text{ weakly in } \mathcal{V}^*, \tag{46}$$

where $\widehat{A}$ is the Nemitsky operator defined in (26). Since $z_n \to z$ weakly in $\mathcal{Z}^*$ and $u_n' \to u'$ weakly in $\mathcal{V}$ and in $\mathcal{Z}$, from (40), we have

$$\begin{aligned} \limsup \langle\langle \widehat{A}u_n', u_n' - u'\rangle\rangle &= \lim \langle\langle f, u_n' - u'\rangle\rangle - \lim \langle\langle z_n, u_n' - u'\rangle\rangle_{\mathcal{Z}^* \times \mathcal{Z}} + \\ &+ \limsup \langle\langle u_n'', u' - u_n'\rangle\rangle = \limsup \langle\langle u_n'', u' - u_n'\rangle\rangle. \end{aligned} \tag{47}$$

Due to the integration by parts formula (Proposition 3.4.14 of [24]), we obtain

$$\begin{aligned} \langle\langle u_n'' - u'', u_n' - u'\rangle\rangle &= \frac{1}{2}\int_0^T \frac{d}{dt}|u_n'(t) - u'(t)|^2\, dt = \\ &= \frac{1}{2}\,|u_n'(T) - u'(T)|^2 - \frac{1}{2}\,|u_n'(0) - u'(0)|^2 \end{aligned}$$

which implies

$$\begin{aligned} &\limsup \langle\langle u_n'', u' - u_n'\rangle\rangle = -\liminf \langle\langle u_n'', u_n' - u'\rangle\rangle = \\ &\quad = -\liminf\, (\langle\langle u_n'' - u'', u_n' - u'\rangle\rangle + \langle\langle u'', u_n' - u'\rangle\rangle) = \\ &\quad = -\liminf\, \left(\frac{1}{2}\,|u_n'(T) - u'(T)|^2 - \frac{1}{2}\,|u_{1n} - u_1|^2\right) + \lim \langle\langle u'', u_n' - u'\rangle\rangle = \\ &\quad = -\frac{1}{2}\,\liminf\, |u_n'(T) - u'(T)|^2 \le 0. \end{aligned}$$

From (47) and the above, we deduce $\limsup \langle\langle \widehat{A}u_n', u_n' - u'\rangle\rangle \le 0$. Since $u_n' \to u'$ weakly in $\mathcal{W}$, after applying Lemma 44(5), we deduce (46). Finally, the convergences (44) and (46) allow to pass to the limit in the equation $u_n'' + \widehat{A}\,u_n' + z_n = f$ in $\mathcal{V}^*$ and we obtain $u'' + \widehat{A}\,u' + z = f$ in $\mathcal{V}^*$ which together with (45), the initial conditions $u(0) = u_0$ and $u'(0) = u_1$ implies that u is a solution to Problem $\mathcal{Q}$. The proof of the existence of solutions to Problem $\mathcal{Q}$ is complete.

3.2.3 Uniqueness of solutions to Problem $\mathcal{Q}$

Let us assume the hypotheses $H(A)_1$, $H(F)_1$, (H_0), (H_1) and (H_2). From Section 3.2.2 and Remark 37, it follows that under these hypotheses Problem $\mathcal{Q}$ admits a solution. For the proof of uniqueness, let u_1, $u_2 \in \mathcal{V}$ be two solutions to Problem $\mathcal{Q}$ such that u_1', $u_2' \in \mathcal{W}$. We have

$$u_1''(t) + A(t, u_1'(t)) + z_1(t) = f(t) \quad \text{a.e. } t \in (0,T), \tag{48}$$

$$u_2''(t) + A(t, u_2'(t)) + z_2(t) = f(t) \quad \text{a.e. } t \in (0,T), \tag{49}$$

$$z_1(t) \in F(t, u_1(t), u_1'(t)), \quad z_2(t) \in F(t, u_2(t), u_2'(t)) \quad \text{a.e. } t \in (0,T),$$

$$u_1(0) = u_2(0) = u_0, \quad u_1'(0) = u_2'(0) = u_1.$$

After substracting (49) from (48), multiplying the result by $u_1'(t) - u_2'(t)$ and using the integration by parts formula, we get

$$\begin{aligned} &\frac{1}{2}|u_1'(t) - u_2'(t)|^2 + \int_0^t \langle A(s, u_1'(s)) - A(s, u_2'(s)), u_1'(s) - u_2'(s)\rangle \, ds + \\ &\quad + \int_0^t \langle z_1(s) - z_2(s), u_1'(s) - u_2'(s)\rangle_{Z^*\times Z} \, ds = 0 \ \text{ for all } \ t \in [0, T]. \end{aligned} \tag{50}$$

Similarly as in the proof of Lemma 42 (cf. (17)), we identify u_1 and u_2 with absolutely continuous functions with values in V and

$$u_1(t) = u_1(0) + \int_0^t u_1'(s) \, ds, \quad u_2(t) = u_2(0) + \int_0^t u_2'(s) \, ds \ \text{ for all } t \in [0, T].$$

This implies

$$\|u_1(t) - u_2(t)\| \le \int_0^t \|u_1'(s) - u_2'(s)\| \, ds.$$

Hence, by the Jensen inequality (cf. Lemma 81 in Section 7), we obtain

$$\begin{aligned} \int_0^t \|u_1(s) - u_2(s)\|^2 \, ds &\le \int_0^t \left(\int_0^s \|u_1'(\tau) - u_2'(\tau)\| \, d\tau \right)^2 ds \le \\ &\le \int_0^t s \left(\int_0^s \|u_1'(\tau) - u_2'(\tau)\|^2 \, d\tau \right) ds \le \\ &\le \int_0^t s \, \|u_1' - u_2'\|^2_{L^2(0,T;V)} \, ds \le \frac{T^2}{2} \, \|u_1' - u_2'\|^2_{L^2(0,T;V)} \end{aligned}$$

for all $t \in [0, T]$. Therefore, exploiting $H(F)_1$(iv) and the Hölder inequality, we have

$$\begin{aligned} &\int_0^t \langle z_1(s) - z_2(s), u_1'(s) - u_2'(s)\rangle_{Z^*\times Z} \, ds \ge \\ &\ge - m_2 \int_0^t \|u_1'(s) - u_2'(s)\|^2 \, ds - m_3 \int_0^t \|u_1'(s) - u_2'(s)\| \, \|u_1(s) - u_2(s)\| \, ds \ge \\ &\ge - m_2 \|u_1' - u_2'\|^2_{L^2(0,t;V)} - m_3 \|u_1' - u_2'\|_{L^2(0,t;V)} \left(\int_0^t \|u_1(s) - u_2(s)\|^2 \, ds \right)^{1/2} \ge \\ &\ge - m_2 \|u_1' - u_2'\|^2_{L^2(0,t;V)} - m_3 \|u_1' - u_2'\|_{L^2(0,t;V)} \, \frac{T}{\sqrt{2}} \, \|u_1' - u_2'\|_{L^2(0,t;V)} = \\ &= - \left(m_2 + \frac{m_3 T}{\sqrt{2}} \right) \|u_1' - u_2'\|^2_{L^2(0,t;V)}. \end{aligned} \tag{51}$$

Hence, using (50), (51) and $H(A)_1$(vi), we obtain

$$\frac{1}{2}|u_1'(t) - u_2'(t)|^2 + \left(m_1 - m_2 - \frac{m_3 T}{\sqrt{2}} \right) \|u_1' - u_2'\|^2_{L^2(0,t;V)} \le 0$$

for all $t \in [0, T]$ which, together with (H_2), proves the uniqueness of the solution to Problem $\mathcal{Q}$. The proof of the theorem is complete. □

3.3 Main result for nonlinear evolution inclusion

The aim of this section is to present the proof of existence and uniqueness result for Problem $\mathcal{P}$. We begin with the following two lemmas.

LEMMA 46 *If (Y, d) is a complete metric space and $\Lambda\colon Y \to Y$ is such that the composition $\Lambda^k = \underbrace{\Lambda \circ \Lambda \circ \ldots \circ \Lambda}_{k}$ for some $k \geq 1$ is a contraction, then Λ has a unique fixed point.*

Proof. From the Banach Contraction Principle (cf. Lemma 84), there exists $y_0 \in Y$ the unique fixed point of Λ^k, i.e. $\Lambda^k y_0 = y_0$. Hence $\Lambda y_0 = \Lambda(\Lambda^k y_0) = \Lambda^k(\Lambda y_0)$ which implies that Λy_0 is also a fixed point of Λ^k. From the uniqueness of the fixed point of Λ^k, we have $\Lambda y_0 = y_0$, as claimed. □

LEMMA 47 *Let X be a Banach space with a norm $\|\cdot\|_X$ and $T > 0$. Let $\Lambda\colon L^2(0,T;X) \to L^2(0,T;X)$ be an operator satisfying*

$$\|(\Lambda\eta_1)(t) - (\Lambda\eta_2)(t)\|_X^2 \leq c \int_0^t \|\eta_1(s) - \eta_2(s)\|_X^2 \, ds \tag{52}$$

for every η_1, $\eta_2 \in L^2(0,T;X)$, a.e. $t \in (0,T)$ with a constant $c > 0$. Then Λ has a unique fixed point in $L^2(0,T;X)$, i.e. there exists a unique $\eta^* \in L^2(0,T;X)$ such that $\Lambda\eta^* = \eta^*$.

Proof. Let $t \in (0,T)$ and η_1, $\eta_2 \in L^2(0,T;X)$. By (52), we have

$$\begin{aligned}
&\|(\Lambda^2\eta_1)(t) - (\Lambda^2\eta_2)(t)\|_X^2 = \|(\Lambda(\Lambda\eta_1))(t) - (\Lambda(\Lambda\eta_2))(t)\|_X^2 \leq \\
&\leq c \int_0^t \|(\Lambda\eta_1)(s) - (\Lambda\eta_2)(s)\|_X^2 ds \leq c \int_0^t \left(c \int_0^s \|\eta_1(r) - \eta_2(r)\|_X^2 \, dr \right) ds \leq \\
&\leq c^2 \left(\int_0^t \|\eta_1(r) - \eta_2(r)\|_X^2 \, dr \right) \left(\int_0^t ds \right) = c^2 t \int_0^t \|\eta_1(r) - \eta_2(r)\|_X^2 \, dr
\end{aligned}$$

and

$$\begin{aligned}
&\|(\Lambda^3\eta_1)(t) - (\Lambda^3\eta_2)(t)\|_X^2 = \|(\Lambda(\Lambda^2\eta_1))(t) - (\Lambda(\Lambda^2\eta_2))(t)\|_X^2 \leq \\
&\leq c \int_0^t \|(\Lambda^2\eta_1)(s) - (\Lambda^2\eta_2)(s)\|_X^2 \, ds \leq c \int_0^t \left(c^2 s \int_0^s \|\eta_1(r) - \eta_2(r)\|_X^2 \, dr \right) ds \leq \\
&\leq c^3 \left(\int_0^t \|\eta_1(r) - \eta_2(r)\|_X^2 \, dr \right) \left(\int_0^t s \, ds \right) = \frac{c^3\, t^2}{2} \int_0^t \|\eta_1(r) - \eta_2(r)\|_X^2 \, dr,
\end{aligned}$$

and also

$$\begin{aligned}
&\|(\Lambda^4\eta_1)(t) - (\Lambda^4\eta_2)(t)\|_X^2 = \|(\Lambda(\Lambda^3\eta_1))(t) - (\Lambda(\Lambda^3\eta_2))(t)\|_X^2 \leq \\
&\leq c \int_0^t \|(\Lambda^3\eta_1)(s) - (\Lambda^3\eta_2)(s)\|_X^2 \, ds \leq c \int_0^t \left(\frac{c^3\, s^2}{2} \int_0^s \|\eta_1(r) - \eta_2(r)\|_X^2 \, dr \right) ds \leq \\
&\leq \frac{c^4}{2} \left(\int_0^t \|\eta_1(r) - \eta_2(r)\|_X^2 \, dr \right) \left(\int_0^t s^2 \, ds \right) = \frac{c^4\, t^3}{6} \int_0^t \|\eta_1(r) - \eta_2(r)\|_X^2 \, dr.
\end{aligned}$$

Reiterating the inequality k times, we have

$$\|(\Lambda^k \eta_1)(t) - (\Lambda^k \eta_2)(t)\|_X^2 \le \frac{c^k\, t^{k-1}}{(k-1)!} \int_0^t \|\eta_1(r) - \eta_2(r)\|_X^2 \, dr$$

which leads to

$$\|\Lambda^k \eta_1 - \Lambda^k \eta_2\|_{L^2(0,T;X)} = \left(\int_0^T \|(\Lambda^k \eta_1)(t) - (\Lambda^k \eta_2)(t)\|_X^2 \, dt \right)^{\frac{1}{2}} \le$$

$$\le \left(\int_0^T \frac{c^k\, T^{k-1}}{(k-1)!} \left(\int_0^t \|\eta_1(r) - \eta_2(r)\|_X^2 \, dr \right) dt \right)^{\frac{1}{2}} = \left(\frac{c^k\, T^k}{(k-1)!} \right)^{\frac{1}{2}} \|\eta_1 - \eta_2\|_{L^2(0,T;X)}.$$

Hence, we deduce that for k sufficiently large, Λ^k is a contraction on $L^2(0,T;X)$. Since $L^2(0,T;X)$ is a Banach space (cf. Proposition 1(i)) by Lemma 46, there exists a unique fixed point $\eta^* \in L^2(0,T;X)$ of Λ. This ends the proof of the lemma. □

We now demonstrate the main result of this section.

THEOREM 48 *Under the hypotheses $H(A)_1$, $H(B)$, $H(C)$, $H(F)_1$, (H_0), (H_1) and (H_2), Problem $\mathcal{P}$ admits a unique solution.*

Proof. Let $\eta \in \mathcal{V}^*$. We consider the following problem: find $u \in \mathcal{V}$ such that $u' \in \mathcal{W}$ and

$$\begin{cases} u''(t) + A(t, u'(t)) + F(t, u(t), u'(t)) \ni f(t) - \eta(t) \quad \text{a.e. } t \in (0,T), \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases} \tag{53}$$

From Theorem 41, we know that for every $\eta \in \mathcal{V}^*$, the problem (53) has a unique solution $u_\eta \in \mathcal{V}$ such that $u'_\eta \in \mathcal{W}$. Furthermore, by Lemma 42, we have

$$\|u_\eta\|_{C(0,T;V)} + \|u'_\eta\|_{\mathcal{W}} \le C\,(1 + \|u_0\| + |u_1| + \|f\|_{\mathcal{V}^*} + \|\eta\|_{\mathcal{V}^*}) \tag{54}$$

with a constant $C > 0$. We consider the operator $\Lambda\colon \mathcal{V}^* \to \mathcal{V}^*$ defined by

$$(\Lambda \eta)(t) = B(t, u_\eta(t)) + \int_0^t C(t-s) u_\eta(s)\, ds \quad \text{for } \eta \in \mathcal{V}^*, \text{ a.e. } t \in (0,T), \tag{55}$$

where $u_\eta \in \mathcal{V}$ is a unique solution to (53). We observe that the operator Λ is well defined. To this end it is enough to check that the integral in (55) is well defined. For $\eta \in \mathcal{V}^*$, by using the hypothesis $H(C)$, we have

$$\left\| \int_0^t C(t-s) u_\eta(s) \right\|_{V^*} \le \int_0^t \|C(t-s)\|_{\mathcal{L}(V,V^*)} \|u_\eta(s)\|\, ds \le$$

$$\le \left(\int_0^t \|C(\tau)\|^2_{\mathcal{L}(V,V^*)}\, d\tau \right)^{\frac{1}{2}} \left(\int_0^t \|u_\eta(\tau)\|\, d\tau \right)^{\frac{1}{2}} \le$$

$$\le \|C\|\, \|u_\eta\|_{L^2(0,t;V)}$$

for a.e. $t \in (0,T)$, where $\|C\| = \|C\|_{L^2(0,t;\mathcal{L}(V,V^*))}$. This implies

$$\begin{aligned}
\|\Lambda\eta\|^2_{\mathcal{V}^*} = & \int_0^T \|(\Lambda\eta)(t)\|^2_{V^*}\,dt \leq \\
\leq \;& 2\int_0^T \left(\|B(t,u_\eta(t))\|^2_{V^*} + \|\int_0^t C(t-s)u_\eta(s)\,ds\|^2_{V^*} \right) dt \leq \\
\leq \;& 2\int_0^T \left(2\left(b_0^2(t) + b_1^2\,\|u_\eta(t)\|^2_{V^*}\right) + \|C\|^2\,\|u_\eta\|^2_{L^2(0,t;V)} \right) dt \leq \\
\leq \;& 4\,\|b_0\|^2_{L^2(0,T)} + \left(4\,b_1^2 + 2\,T\,\|C\|\right)\,\|u_\eta\|^2_{L^2(0,T;V)}.
\end{aligned}$$

Hence, by (54), we obtain that the operator Λ takes values in $\mathcal{V}^*$.

Now, our goal is to show that the operator Λ has a unique fixed point. We show that Λ satisfies the hypotheses of Lemma 47. First we recall that V^* is a Banach space. Next, let η_1, $\eta_2 \in \mathcal{V}^*$. We denote by $u_1 = u_{\eta_1}$, $u_2 = u_{\eta_2}$ the unique solutions to (53) corresponding to η_1 and η_2, respectively. We have

$$\begin{aligned}
& u_1''(t) + A(t,u_1'(t)) + z_1(t) = f(t) - \eta_1(t) && \text{a.e. } t\in(0,T), && (56)\\
& u_2''(t) + A(t,u_2'(t)) + z_2(t) = f(t) - \eta_2(t) && \text{a.e. } t\in(0,T), && (57)\\
& z_1(t) \in F(t,u_1(t),u_1'(t)), \quad z_2(t) \in F(t,u_2(t),u_2'(t)) && \text{a.e. } t\in(0,T), &&\\
& u_1(0) = u_2(0) = u_0, \quad u_1'(0) = u_2'(0) = u_1. && &&
\end{aligned}$$

We substract (57) from (56) and multiply the result by $u_1'(t) - u_2'(t)$. Using the integration by parts formula, we have

$$\begin{aligned}
&\frac{1}{2}|u_1'(t) - u_2'(t)|^2 + \int_0^t \langle A(s,u_1'(s)) - A(s,u_2'(s)), u_1'(s) - u_2'(s)\rangle\,ds + \qquad (58)\\
&+ \int_0^t \langle z_1(s) - z_2(s), u_1'(s) - u_2'(s)\rangle\,ds = \int_0^t \langle \eta_1(s) - \eta_2(s), u_1'(s) - u_2'(s)\rangle\,ds
\end{aligned}$$

for every $t \in [0,T]$. Using the same reasoning as in Section 3.2.2 (cf. (51)), we arrive at the following inequalities

$$\|u_1(t) - u_2(t)\| \leq \int_0^t \|u_1'(s) - u_2'(s)\|\,ds \leq \sqrt{T}\|u_1' - u_2'\|_{L^2(0,t;V)}, \qquad (59)$$

$$\int_0^t \langle z_1(s) - z_2(s), u_1'(s) - u_2'(s)\rangle_{Z^*\times Z}\,ds \geq -\left(m_2 + \frac{m_3\,T}{\sqrt{2}}\right)\,\|u_1' - u_2'\|^2_{L^2(0,t;V)}$$

for all $t \in [0,T]$. Using the above inequalities in (58), applying $H(A)_1$(vi) and the Hölder inequality, we get

$$\begin{aligned}
\frac{1}{2}|u_1'(t) - u_2'(t)|^2 + \left(m_1 - m_2 - \frac{m_3\,T}{\sqrt{2}}\right)\,\|u_1' - u_2'\|^2_{L^2(0,t;V)} \leq \\
\leq \|\eta_1 - \eta_2\|_{L^2(0,t;V^*)}\,\|u_1' - u_2'\|_{L^2(0,t;V)}
\end{aligned}$$

for all $t \in [0,T]$. This, together with (H_2), implies

$$\|u_1' - u_2'\|_{L^2(0,t;V)} \leq \frac{1}{c} \|\eta_1 - \eta_2\|_{L^2(0,t;V^*)}, \tag{60}$$

where $c = m_1 - m_2 - \frac{m_3 T}{\sqrt{2}} > 0$. Using (59) and (60), we have

$$\|u_1(t) - u_2(t)\| \leq \frac{\sqrt{t}}{c} \|\eta_1 - \eta_2\|_{L^2(0,t;V^*)}. \tag{61}$$

By the Lipschitz continuity of the operator B (cf. $H(B)$(ii)) and $H(C)$, we infer

$$\begin{aligned}
&\|(\Lambda\eta_1)(t) - (\Lambda\eta_2)(t)\|_{V^*} \leq \\
&\qquad \leq \|B(t,u_1(t)) - B(t,u_2(t))\|_{V^*} + \int_0^t \|C(t-s)(u_1(s) - u_2(s))\|_{V^*}\, ds \leq \\
&\qquad \leq L_B \|u_1(t) - u_2(t)\| + \|C\| \, \|u_1 - u_2\|_{L^2(0,t;V)}
\end{aligned}$$

for all $t \in [0,T]$. Hence and from the inequality (61), we obtain

$$\begin{aligned}
\|(\Lambda\eta_1)(t) - (\Lambda\eta_2)(t)\|^2_{V^*} &\leq \left(L_B \|u_1(t) - u_2(t)\| + \|C\| \, \|u_1 - u_2\|_{L^2(0,t;V)}\right)^2 \leq \\
&\leq \; 2\, L_B^2 \|u_1(t) - u_2(t)\|^2 + 2\, \|C\|^2 \, \|u_1 - u_2\|^2_{L^2(0,t;V)} \leq \\
&\leq \; \frac{2L_B^2 t}{c^2} \|\eta_1 - \eta_2\|_{L^2(0,t;V^*)} + \frac{2T\|C\|^2}{c^2} t \, \|\eta_1 - \eta_2\|^2_{L^2(0,t;V^*)} \leq \\
&\leq \; \frac{2T}{c^2} \left(L_B^2 + T\|C\|^2\right) \|\eta_1 - \eta_2\|^2_{L^2(0,t;V^*)}
\end{aligned}$$

for all $t \in [0,T]$. This implies that the assumptions of the Lemma 47 hold and therefore there exists $\eta^* \in \mathcal{V}^*$ that is a unique fixed point of Λ.

We have now all the ingredients to conclude the proof of the theorem.

Existence. Let $\eta^* \in \mathcal{V}^*$ be the fixed point of the operator Λ. We denote by u the solution of the problem (53) for $\eta = \eta^*$, i.e. $u = u_{\eta^*}$. The regularity of u follows from Theorem 41. Furthermore, since $\eta^* = \Lambda\eta^*$, we have

$$\eta^*(t) = B(t, u_{\eta^*}(t)) + \int_0^t C(t-s) u_{\eta^*}(s)\, ds \quad \text{for a.e. } t \in (0,T).$$

Hence, we conclude that u is a solution of Problem $\mathcal{P}$.

Uniqueness. The uniqueness of solutions of Problem $\mathcal{P}$ is a consequence of Theorem 41 and the uniqueness of the fixed point of Λ. This concludes the proof of the theorem. $\square$

The existence result of Theorem 48 generalizes Theorem 4 of [67] where the existence of solutions for Problem $\mathcal{P}$ was obtained in a case when $C = 0$ and B is time independent, linear, bounded, symmetric and coercive operator. Theorem 48 is also a generalization of an existence result of [60] (cf. Theorem 10) and a uniqueness result of [60] (cf. Proposition 15) where Problem $\mathcal{P}$ was treated under the stronger hypotheses, i.e. $C = 0$, F is of a particular form and independent of u and B is time independent, linear, bounded, symmetric and nonnegative. The evolution equation with a single-valued mapping $F\colon (0,T) \times H \times H \to H$, $C = 0$ and B time independent, linear, bounded, symmetric and nonnegative was considered in Theorem 8.6.3 of [24].

4 A convergence result for evolution inclusions

In this section we study the dependence of the solution to Problem $\mathcal{P}$ with respect to perturbations of the operators A, B and C. To this end, for every $\varepsilon > 0$, let A_ε, B_ε and C_ε be perturbations of A, B and C, respectively, which satisfy the following hypotheses.

$\underline{H(A)_\varepsilon}$: The operators $A, A_\varepsilon\colon (0,T) \times V \to V^*$ satisfy $H(A)_1$ uniformly in ε and

$$A_\varepsilon(\cdot, w(\cdot)) \to A(\cdot, w(\cdot)) \ \text{ in } \mathcal{V}^* \ \text{ for all } w \in \mathcal{W} \text{ as } \varepsilon \to 0;$$

$\underline{H(B)_\varepsilon}$: The operators $B, B_\varepsilon\colon (0,T) \times V \to V^*$ satisfy $H(B)$ uniformly in ε and

$$B_\varepsilon(\cdot, v(\cdot)) \to B(\cdot, v(\cdot)) \ \text{ in } \mathcal{V}^* \ \text{ for all } v \in \mathcal{V} \text{ as } \varepsilon \to 0;$$

$\underline{H(C)_\varepsilon}$: $C, C_\varepsilon \in L^2(0,T;\mathcal{L}(V,V^*))$ and $C_\varepsilon \to C$ in $L^2(0,T;\mathcal{L}(V,V^*))$ as $\varepsilon \to 0$.

We consider the following sequence of the Cauchy problems. Let $\varepsilon > 0$.

Problem P_ε: find $u_\varepsilon \in \mathcal{V}$ such that $u'_\varepsilon \in \mathcal{W}$ and

$$\begin{cases} u''_\varepsilon(t) + A_\varepsilon(t, u'_\varepsilon(t)) + B_\varepsilon(t, u_\varepsilon(t)) + \displaystyle\int_0^t C_\varepsilon(t-s)\, u_\varepsilon(s)\, ds + \\ \qquad\qquad + F(t, u_\varepsilon(t), u'_\varepsilon(t)) \ni f(t) \ \text{ a.e. } \ t \in (0,T), \\ u_\varepsilon(0) = u_0, \quad u'_\varepsilon(0) = u_1. \end{cases}$$

Theorem 49 *Assume that $H(A)_\varepsilon$, $H(B)_\varepsilon$, $H(C)_\varepsilon$, $H(F)_1$, (H_0), (H_1) and (H_2) hold. Then, the sequence $\{u_\varepsilon\}$ of unique solutions of Problems $\mathcal{P}_\varepsilon$ converges to the unique solution u of Problem $\mathcal{P}$, i.e.*

$$\lim_{\varepsilon \to 0} \left(\|u_\varepsilon - u\|_{C(0,T;V)} + \|u'_\varepsilon - u'\|_{C(0,T;H)} + \|u'_\varepsilon - u'\|_{\mathcal{V}} \right) = 0.$$

Proof. Let $\varepsilon > 0$. From Theorem 48, we deduce that Problems $\mathcal{P}$ and $\mathcal{P}_\varepsilon$, for every $\varepsilon > 0$, admit unique solutions u and u_ε, respectively. Everywhere in the proof, we denote by c a positive generic constant which may depend on A, B, C, u and T but is independent of ε, and whose value may change from place to place. We have u, $u_\varepsilon \in \mathcal{V}$ with u', $u'_\varepsilon \in \mathcal{W}$ and

$$u''_\varepsilon(t) + A_\varepsilon(t, u'_\varepsilon(t)) + \eta_\varepsilon(t) + z_\varepsilon(t) = f(t) \qquad \text{a.e. } t \in (0,T), \tag{62}$$

$$u''(t) + A(t, u'(t)) + \eta(t) + z(t) = f(t) \qquad \text{a.e. } t \in (0,T), \tag{63}$$

where

$$\eta_\varepsilon(t) = B_\varepsilon(t, u_\varepsilon(t)) + \int_0^t C_\varepsilon(t-s) u_\varepsilon(s)\, ds \quad \text{a.e. } t \in (0,T),$$
$$\eta(t) = B(t, u(t)) + \int_0^t C(t-s) u(s)\, ds \qquad \text{a.e. } t \in (0,T)$$

and

$$z_\varepsilon(t) \in F(t, u_\varepsilon(t), u_\varepsilon'(t)), \quad z(t) \in F(t, u(t), u'(t)) \quad \text{a.e. } t \in (0,T).$$

From (62) and (63), we get

$$\int_0^t \langle u_\varepsilon''(s) - u''(s), u_\varepsilon'(s) - u'(s)\rangle\, ds + \int_0^t \langle A_\varepsilon(s, u_\varepsilon'(s)) - A(s, u'(s)), u_\varepsilon'(s) - u'(s)\rangle\, ds +$$
$$+ \int_0^t \langle \eta_\varepsilon(s) - \eta(s), u_\varepsilon'(s) - u'(s)\rangle\, ds + \int_0^t \langle z_\varepsilon(s) - z(s), u_\varepsilon'(s) - u'(s)\rangle_{Z^* \times Z}\, ds = 0$$

for all $t \in [0,T]$. Similarly as in Section 3.2.2 (cf. (51)), by $H(F)_1$(iv) and the Hölder inequality, we obtain

$$\int_0^t \langle z_\varepsilon(s) - z(s), u_\varepsilon'(s) - u'(s)\rangle_{Z^* \times Z}\, ds \geq - \left(m_2 + \frac{m_3 T}{\sqrt{2}} \right) \|u_\varepsilon' - u'\|^2_{L^2(0,t;V)}.$$

Hence and from the integration by parts formula, we have

$$\frac{1}{2}|u_\varepsilon'(t) - u'(t)|^2 + \int_0^t \langle A_\varepsilon(s, u_\varepsilon'(s)) - A_\varepsilon(s, u'(s)), u_\varepsilon'(s) - u'(s)\rangle\, ds +$$
$$+ \int_0^t \langle A_\varepsilon(s, u'(s)) - A(s, u'(s)), u_\varepsilon'(s) - u'(s)\rangle\, ds - \left(m_2 + \frac{m_3 T}{\sqrt{2}} \right) \|u_\varepsilon' - u'\|^2_{L^2(0,t;V)} \leq$$
$$\leq - \int_0^t \langle \eta_\varepsilon(s) - \eta(s), u_\varepsilon'(s) - u'(s)\rangle\, ds \quad \text{for all } t \in [0,T].$$

Since $A_\varepsilon(t, \cdot)$ is strongly monotone, uniformly in ε, we deduce

$$\frac{1}{2}|u_\varepsilon'(t) - u'(t)|^2 + \left(m_1 - m_2 - \frac{m_3 T}{\sqrt{2}} \right) \|u_\varepsilon' - u'\|^2_{L^2(0,t;V)} \leq$$
$$\leq \left(\|A_\varepsilon(\cdot, u'(\cdot)) - A(\cdot, u'(\cdot))\|_{L^2(0,t;V^*)} + \|\eta_\varepsilon - \eta\|_{L^2(0,t;V^*)} \right) \|u_\varepsilon' - u'\|_{L^2(0,t;V)} \qquad (64)$$

for all $t \in [0,T]$. On the other hand, using the fact that $B_\varepsilon(t, \cdot)$ is uniformly Lipschitz continuous, we have

$$\|\eta_\varepsilon(s) - \eta(s)\|_{V^*} \leq \|B_\varepsilon(s, u_\varepsilon(s)) - B_\varepsilon(s, u(s))\|_{V^*} + \|B_\varepsilon(s, u(s)) - B(s, u(s))\|_{V^*} +$$

$$+ \|\int_0^s C_\varepsilon(s-\tau)(u_\varepsilon(\tau) - u(\tau))\, d\tau\|_{V^*} + \|\int_0^s (C_\varepsilon(s-\tau) - C(s-\tau)) u(\tau)\, d\tau\|_{V^*} \leq$$

$$\leq L_B \|u_\varepsilon(s) - u(s)\| + \|B_\varepsilon(s, u(s)) - B(s, u(s))\|_{V^*} +$$

$$+ \|C_\varepsilon\|_{L^2(0,t;\mathcal{L}(V,V^*))} \|u_\varepsilon - u\|_{L^2(0,t;V)} + \|C_\varepsilon - C\|_{L^2(0,t;\mathcal{L}(V,V^*))} \|u\|_{L^2(0,t;V)}$$

for a.e. $s \in (0,t)$. Hence, we obtain

$$\|\eta_\varepsilon - \eta\|^2_{L^2(0,t;V^*)} \le c \Bigg(\|u_\varepsilon - u\|^2_{L^2(0,t;V)} + \|B_\varepsilon(\cdot, u(\cdot)) - B(\cdot, u(\cdot))\|^2_{L^2(0,t;V^*)} +$$

$$+ \|C_\varepsilon\|^2_{L^2(0,t;\mathcal{L}(V,V^*))} \|u_\varepsilon - u\|^2_{L^2(0,t;V)} + \|C_\varepsilon - C\|^2_{L^2(0,t;\mathcal{L}(V,V^*))} \|u\|^2_{L^2(0,t;V)} \Bigg) \le$$

$$\le c \left(\|u_\varepsilon - u\|^2_{L^2(0,t;V)} + \|B_\varepsilon(\cdot, u(\cdot)) - B(\cdot, u(\cdot))\|^2_{L^2(0,t;V^*)} + \|C_\varepsilon - C\|^2_{L^2(0,t;\mathcal{L}(V,V^*))} \right)$$

which implies

$$\|\eta_\varepsilon - \eta\|_{L^2(0,t;V^*)} \le c \Big(\|u_\varepsilon - u\|_{L^2(0,t;V)} + \|B_\varepsilon(\cdot, u(\cdot)) - B(\cdot, u(\cdot))\|_{L^2(0,t;V^*)} + \\ + \|C_\varepsilon - C\|_{L^2(0,t;\mathcal{L}(V,V^*))} \Big)$$

for all $t \in [0,T]$. Substituting this inequality in (64), it follows

$$\begin{aligned} &\frac{1}{2}|u'_\varepsilon(t) - u'(t)|^2 + \left(m_1 - m_2 - \frac{m_3\, T}{\sqrt{2}} \right) \|u'_\varepsilon - u'\|^2_{L^2(0,t;V)} \le \\ &\le c \Big(\|A_\varepsilon(\cdot, u'_\varepsilon(\cdot)) - A(\cdot, u'(\cdot))\|_{L^2(0,t;V^*)} + \|u_\varepsilon - u\|_{L^2(0,t;V)} + \\ &\le \|B_\varepsilon(\cdot, u(\cdot)) - B(\cdot, u(\cdot))\|_{L^2(0,t;V^*)} + \|C_\varepsilon - C\|_{L^2(0,t;\mathcal{L}(V,V^*))} \Big) \|u'_\varepsilon - u'\|_{L^2(0,t;V)} \end{aligned} \tag{65}$$

for all $t \in [0,T]$. Omitting the first term on the left hand side, by (H_2), we deduce

$$\|u'_\varepsilon - u'\|_{L^2(0,t;V)} \le c \left(\|u_\varepsilon - u\|_{L^2(0,t;V)} + r_\varepsilon \right) \tag{66}$$

where

$$r_\varepsilon = \|A_\varepsilon(\cdot, u'_\varepsilon(\cdot)) - A(\cdot, u'(\cdot))\|_{\mathcal{V}^*} + \|B_\varepsilon(\cdot, u(\cdot)) - B(\cdot, u(\cdot))\|_{\mathcal{V}^*} + \|C_\varepsilon - C\|_{L^2(0,T;\mathcal{L}(V,V^*))}.$$

Similarly as in Lemma 42 (cf. (17)), we may identify u_ε and u with absolutely continuous functions with values in V and

$$u_\varepsilon(t) = u_\varepsilon(0) + \int_0^t u'_\varepsilon(s)\, ds, \quad u(t) = u(0) + \int_0^t u'(s)\, ds \quad \text{for all } t \in [0,T],$$

and thus

$$\|u_\varepsilon(t) - u(t)\| \le \int_0^t \|u'_\varepsilon(s) - u'(s)\|\, ds \le \sqrt{T} \|u'_\varepsilon - u'\|_{L^2(0,t;V)}.$$

The latter together with (66) implies

$$\|u_\varepsilon(t) - u(t)\| \le c \left(\|u_\varepsilon - u\|_{L^2(0,t;V)} + r_\varepsilon \right) \quad \text{for all } t \in [0,T]$$

and

$$\|u_\varepsilon(t) - u(t)\|^2 \le c \left(\int_0^t \|u_\varepsilon(s) - u(s)\|^2\, ds + r_\varepsilon^2 \right) \quad \text{for all } t \in [0,T].$$

Applying now the Gronwall inequality (cf. Lemma 86), we have $\|u_\varepsilon(t) - u(t)\| \le c\, r_\varepsilon^2$ which, by hypotheses, entails

$$\lim_{\varepsilon\to 0} \|u_\varepsilon - u\|_{C(0,T;V)} = 0.$$

Next, from (66), we have $\|u_\varepsilon' - u'\|_{L^2(0,t;V)} \le c\left(\|u_\varepsilon - u\|_{C(0,T;V)} + r_\varepsilon\right)$ which implies

$$\lim_{\varepsilon\to 0} \|u_\varepsilon' - u'\|_{\mathcal{V}} = 0.$$

Finally, from (65), after omitting the second term on the left hand side, we obtain

$$\frac{1}{2}|u_\varepsilon'(t) - u'(t)|^2 \le c\left(\|u_\varepsilon - u\|_{C(0,T;V)} + r_\varepsilon\right)\, \|u_\varepsilon' - u'\|_{\mathcal{V}}.$$

Hence, we deduce

$$\lim_{\varepsilon\to 0} \|u_\varepsilon' - u'\|_{C(0,T;H)} = 0.$$

This completes the proof of the theorem. □

As a corollary we deduce a convergence result for vanishing relaxation operator.

Theorem 50 *Assume the hypotheses of Theorem 48 and let $u_\varepsilon \in \mathcal{V}$ with $u_\varepsilon' \in \mathcal{W}$ be the unique solution of the problem*

$$\begin{cases} u_\varepsilon''(t) + A(t, u_\varepsilon'(t)) + B(t, u_\varepsilon(t)) + \varepsilon \displaystyle\int_0^t C(t-s)\, u_\varepsilon(s)\, ds + \\ \qquad\qquad + F(t, u_\varepsilon(t), u_\varepsilon'(t)) \ni f(t) \quad a.e. \;\; t \in (0,T), \\ u_\varepsilon(0) = u_0, \quad u_\varepsilon'(0) = u_1 \end{cases}$$

for $\varepsilon > 0$. Then, u_ε converges to u in the following sense

$$\lim_{\varepsilon\to 0} \left(\|u_\varepsilon - u\|_{C(0,T;V)} + \|u_\varepsilon' - u'\|_{C(0,T;H)} + \|u_\varepsilon' - u'\|_{\mathcal{V}}\right) = 0,$$

where $u \in \mathcal{V}$ with $u' \in \mathcal{W}$ is the unique solution of the problem

$$\begin{cases} u''(t) + A(t, u'(t)) + B(t, u(t)) + F(t, u(t), u'(t)) \ni f(t) \quad a.e. \;\; t \in (0,T), \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases}$$

Proof. It is a consequence of Theorem 49 applied to $C_\varepsilon = \varepsilon C$. □

5 Evolution hemivariational inequalities

In this part of the thesis we apply the results of Section 3 to dynamic viscoelastic continuum systems with unilateral constraints. We investigate deformed bodies in contact with a foundation. We present a short description of the modeled process, give its weak formulation which is a hyperbolic hemivariational inequality and obtain results on existence and uniqueness of weak solutions. We concentrate on a clear presentation of the general structure of the mathematical problem and provide the reader a method which can be applied to other problems of mechanics.

5.1 Function spaces for contact problems

In this section we recall some notation for the mathematical formulations of mechanical contact problems, cf. Duvault and Lions [27], Eck et al. [28], Han and Sofonea [34], Ionescu and Sofonea [43], Nečas and Hlaváček [74], Panagiotopoulos [77, 78] and Shillor et al. [93, 95].

We denote by $\mathbb{S}^d$ the linear space of second order symmetric tensors on $\mathbb{R}^d$ ($d = 2, 3$), or equivalently, the space $\mathbb{R}_s^{d\times d}$ of symmetric matrices of order d. We define the inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ by

$$\xi \cdot \eta = \xi_i \, \eta_i, \qquad \|\xi\|_{\mathbb{R}^d} = (\xi \cdot \xi)^{1/2} \qquad \text{for all } u, v \in \mathbb{R}^d,$$

$$\sigma : \tau = \sigma_{ij} \, \tau_{ij}, \qquad \|\tau\|_{\mathbb{S}^d} = (\tau : \tau)^{1/2} \qquad \text{for all } \sigma, \tau \in \mathbb{S}^d.$$

We adopt the summation convention over repeated indices. If no confusion is possible the norm in $\mathbb{R}^d$ is simply denoted by $\|\cdot\|$.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary Γ and let ν denote the outward unit normal vector to Γ. The assumption that Γ is Lipschitz ensures that ν is defined a.e. on Γ. We use the following spaces

$$H = L^2(\Omega; \mathbb{R}^d), \qquad \mathcal{H} = \big\{ \tau = \{\tau_{ij}\} \mid \tau_{ij} = \tau_{ji} \in L^2(\Omega) \big\} = L^2(\Omega; \mathbb{S}^d),$$

$$H_1 = H^1(\Omega; \mathbb{R}^d), \qquad \mathcal{H}_1 = \{ \tau \in \mathcal{H} \mid \operatorname{Div} \tau \in H \},$$

where the deformation and the divergence operators are, respectively, given by

$$\varepsilon(u) = \{ \varepsilon_{ij}(u) \}, \quad \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \operatorname{Div} \sigma = \{ \sigma_{ij,j} \},$$

and the index following a comma indicates a partial derivative. The spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are Hilbert spaces equipped with the inner products

$$\langle u, v \rangle_H = \int_\Omega u_i \, v_i \, dx, \qquad \langle \sigma, \tau \rangle_{\mathcal{H}} = \int_\Omega \sigma : \tau \, dx,$$

$$\langle u, v \rangle_{H_1} = \langle u, v \rangle_H + \langle \varepsilon(u), \varepsilon(v) \rangle_{\mathcal{H}}, \qquad \langle \sigma, \tau \rangle_{\mathcal{H}_1} = \langle \sigma, \tau \rangle_{\mathcal{H}} + \langle \operatorname{Div} \sigma, \operatorname{Div} \tau \rangle_H.$$

Let $H_\Gamma = H^{1/2}(\Gamma; \mathbb{R}^d)$, let $H_\Gamma^* = H^{-1/2}(\Gamma; \mathbb{R}^d)$ be its dual and let $\langle \cdot, \cdot \rangle_{H_\Gamma^* \times H_\Gamma}$ denote the duality pairing between H_Γ^* and H_Γ. For every $v \in H_1$ we denote by v its trace $\overline{\gamma} v$ on Γ, where $\overline{\gamma} \colon H_1 \to H_\Gamma \subset L^2(\Gamma; \mathbb{R}^d)$ is the trace map. Given $v \in H_\Gamma$ we denote by v_ν and v_τ the usual normal and the tangential components of v on the boundary Γ, i.e.

$$v_\nu = v \cdot \nu \;\; \text{and} \;\; v_\tau = v - v_\nu \nu.$$

Similarly, for sufficiently regular (say C^1) tensor field $\sigma \colon \Omega \to \mathbb{S}^d$, we define its normal and tangential components by

$$\sigma_\nu = (\sigma \nu) \cdot \nu \;\; \text{and} \;\; \sigma_\tau = \sigma \nu - \sigma_\nu \nu.$$

We also recall the following Green formula. If $\sigma \in \mathcal{H}_1$, then there exists an element $\gamma_\nu \sigma \in H_\Gamma^*$ such that

$$\langle \gamma_\nu \sigma, \overline{\gamma} v \rangle_{H_\Gamma^* \times H_\Gamma} = \langle \sigma, \varepsilon(v) \rangle_{\mathcal{H}} + \langle \mathrm{Div}\, \sigma, v \rangle_H \quad \text{for} \quad v \in H_1.$$

Moreover, if σ is sufficiently regular (say C^1) tensor field, then

$$\langle \gamma_\nu \sigma, \overline{\gamma} v \rangle_{H_\Gamma^* \times H_\Gamma} = \int_\Gamma \sigma\nu \cdot v \, d\Gamma \quad \text{for} \quad v \in H_1.$$

For other mathematical results concerning the function spaces used in modeling of contact problems, we refer to the aformentioned textbooks.

5.2 Physical setting of the problem

The physical setting and the process are as follows. The set Ω is occupied by a viscoelastic body in $\mathbb{R}^d$ ($d = 2$, 3 in applications) which is referred to as the reference configuration. We assume that Ω is a bounded domain with Lipschitz boundary Γ which is divided into three mutually disjoint measurable parts Γ_D, Γ_N and Γ_C with $m(\Gamma_D) > 0$.

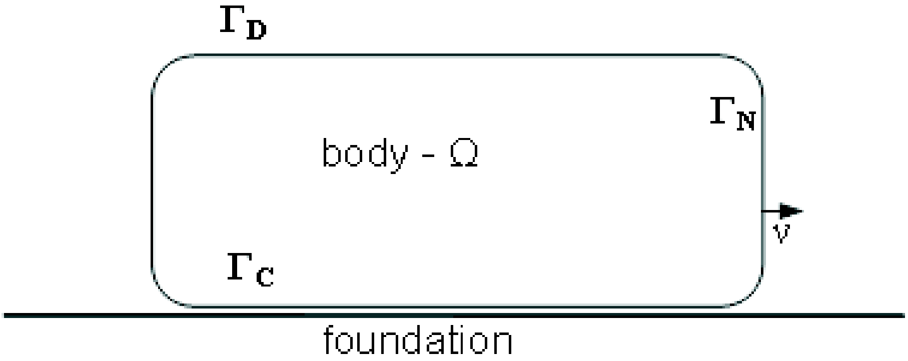

Figure 1: Physical setting; Γ_C is the potential contact surface

We study the process of evolution of the mechanical state in time interval $[0, T]$, $0 < T < \infty$. The system evolves in time as a result of applied volume forces and surface tractions. The description of this evolution is done by introducing a vector function $u = u(x,t) = (u_1(x,t), \ldots, u_d(x,t))$ which describes the displacement at time t of a particle that has the position $x = (x_1, \ldots, x_d)$ in the reference configuration. We denote by $\sigma = \sigma(x,t) = (\sigma_{ij}(x,t))$ the stress tensor and by $\varepsilon(u) = (\varepsilon_{ij}(u))$ the linearized (small) strain tensor whose components are given by (a compatibility condition)

$$\varepsilon_{ij} = \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}),$$

where i, $j = 1, \ldots, d$ and where a comma separates the components from partial derivatives, i.e. $u_{i,j} = \partial u_i / \partial u_j$. In cases where an index appears twice, we use the summation convention. We also put $Q = \Omega \times (0, T)$.

Since the process is dynamic, we deal with the dynamic equation of motion representing momentum conservation (cf. [34, 77]) and governing the evolution of the state of the body

$$u''(t) - \operatorname{Div} \sigma(t) = f_0(t) \quad \text{in} \quad Q,$$

where Div denotes the divergence operator for tensor valued functions and f_0 is the density of applied volume forces such as gravity. We assume that the mass density is constant and set equal to one. We remark that when the system configuration and the external forces and tractions vary in time in such a way that the accelerations of the system are rather small, then the inertial terms $u'' = \partial^2 u / \partial t^2$ can be neglected. In this case we obtain the quasistatic approximation for the equation of motion considered in e.g. [27, 34, 89, 93, 95, 97]. This situation is not studied in the present work, we deal with a full dynamic equation describing the motion.

In the model the material is assumed to be viscoelastic and for its description we suppose a general constitutive law (the relationship between strain and stress) of the form

$$\sigma(t) = \mathcal{A}(t, \varepsilon(u'(t))) + \mathcal{B}(t, \varepsilon(u(t))) + \int_0^t \mathcal{C}(t-s)\, \varepsilon(u(s))\, ds \quad \text{in} \quad Q. \tag{67}$$

Here $\mathcal{A}$ is a nonlinear operator describing the purely viscous properties of the material while $\mathcal{B}$ and $\mathcal{C}$ are the nonlinear elasticity and the linear relaxation operators, respectively. Note that the operators $\mathcal{A}$ and $\mathcal{B}$ may depend explicitely on the time variable and this is the case when the viscosity and elasticity properties of the material depend on the temperature field which plays the role of a parameter and whose evolution in time is prescribed. When $\mathcal{C} = 0$ the constitutive law (67) reduces to a viscoelastic constitutive law (the so called Kelvin-Voigt law) with short memory

$$\sigma(t) = \mathcal{A}(t, \varepsilon(u'(t))) + \mathcal{B}(t, \varepsilon(u(t))) \quad \text{in} \quad Q,$$

and in the case when $\mathcal{A} = 0$, it reduces to an elastic constitutive law with long memory

$$\sigma(t) = \mathcal{B}(t, \varepsilon(u(t))) + \int_0^t \mathcal{C}(t-s)\, \varepsilon(u(s))\, ds \quad \text{in} \quad Q.$$

In linear viscoelasticity the Kelvin-Voigt law takes the form

$$\sigma_{ij}(t) = a_{ijkl}\, \varepsilon_{kl}(u'(t)) + b_{ijkl}\, \varepsilon_{kl}(u(t)) \quad \text{in} \quad Q,$$

where $\mathcal{A} = \{a_{ijkl}\}$ and $\mathcal{B} = \{b_{ijkl}\}$, i, j, k, $l = 1, \ldots, d$ are the viscosity and elasticity tensors respectively, which may be functions of position. Two simple one-dimensional constitutive laws of the form (67) will be given in Section 6.1. For more details on the construction of rheological models which lead to the law (67), see [27] and [34].

Next we describe the boundary conditions. The body is supposed to be held fixed on the part Γ_D of the surface, so the displacement $u = 0$ on $\Gamma_D \times (0, T)$. On the part

Γ_N a prescribed surface force (traction) $f_1 = f_1(x,t)$ is applied, thus we have the condition $\sigma(t)\,\nu = f_1$ on $\Gamma_N \times (0,T)$. Here $\nu \in \mathbb{R}^d$ denotes the outward unit normal to Γ and $\sigma(t)\,\nu$ represents the boundary stress vector. The body may come in contact over the part Γ_C of its surface. As it is met in the literature (cf. [27, 34, 93, 95]) the conditions on the contact surface are naturally divided to conditions in the normal direction and those in the tangential direction, cf. Section 5.4 of [34] for the normal approach and the tangential process. In the model under consideration, the frictional contact on the part Γ_C is described by the subdifferential boundary conditions of the form

$$-\sigma_\nu(t) \in \partial j_1(x,t,u(t),u'(t),u_\nu(t)) + \partial j_2(x,t,u(t),u'(t),u'_\nu(t))$$

and

$$-\sigma_\tau(t) \in \partial j_3(x,t,u(t),u'(t),u_\tau(t)) + \partial j_4(x,t,u(t),u'(t),u'_\tau(t))$$

on $\Gamma_C \times (0,T)$, where σ_ν and σ_τ, u_ν and u_τ, u'_ν and u'_τ denote the normal and the tangential components of the stress tensor, the displacement and the velocity, respectively. The functions j_k, $k = 1, \ldots, 4$ are prescibed and locally Lipschitz in their last variables. The component σ_τ represents the friction force on the contact surface and ∂j_k, $k = 1, \ldots, 4$ denote the Clarke subdifferentials of the superpotentials j_k, $k = 1, \ldots, 4$ with respect to their last variables. Since the superpotentials depend on the spatial variable the multivalued boundary conditions can be different at distinct points. The explicit dependence of superpotentials on the time variable allows (as it is for the viscosity and elasticity operators) to model situations when the frictional contact conditions depend on the prescribed evolution of the temperature. Concrete examples of contact models which lead to subdifferential boundary conditions of the form (72) and (73) will be provided in Section 6.2.

Finally, we prescribe the initial conditions for the displacement and the velocity, i.e.

$$u(0) = u_0 \quad \text{and} \quad u'(0) = u_1 \quad \text{in} \quad \Omega,$$

where u_0 and u_1 denote the initial displacement and the initial velocity, respectively. In what follows we skip occasionally the dependence of various functions on the spatial variable $x \in \Omega \cup \Gamma$.

Collecting the equations and conditions described above, we obtain the following formulation of the mechanical problem: find a displacement field $u\colon Q \to \mathbb{R}^d$ and a stress field $\sigma\colon Q \to \mathcal{S}_d$ such that

$$u''(t) - \operatorname{Div} \sigma(t) = f_0(t) \quad \text{in } Q, \tag{68}$$

$$\sigma(t) = \mathcal{A}(t, \varepsilon(u'(t))) + \mathcal{B}(t, \varepsilon(u(t))) + \int_0^t \mathcal{C}(t-s)\, \varepsilon(u(s))\, ds \quad \text{in } Q, \tag{69}$$

$$u(t) = 0 \quad \text{on } \Gamma_D \times (0,T), \tag{70}$$

$$\sigma(t)\, \nu = f_1 \quad \text{on } \Gamma_N \times (0,T), \tag{71}$$

$$-\sigma_\nu(t) \in \partial j_1(t, u(t), u'(t), u_\nu(t)) + \partial j_2(t, u(t), u'(t), u'_\nu(t)) \quad \text{on } \Gamma_C \times (0,T), \tag{72}$$

$$-\sigma_\tau(t) \in \partial j_3(t, u(t), u'(t), u_\tau(t)) + \partial j_4(t, u(t), u'(t), u'_\tau(t)) \quad \text{on } \Gamma_C \times (0,T), \tag{73}$$

$$u(0) = u_0, \quad u'(0) = u_1 \quad \text{in } \Omega. \tag{74}$$

The above problem represents the classical formulation of the viscoelastic frictional contact problem. The conditions (72) and (73) introduce one of the main difficulties to the problem since the superpotentials are nonconvex and nonsmooth in general. This is the reason why the problem (68)-(74) has no classical solutions, i.e. solutions which posses all necessary classical derivatives and satisfy the relations in the usual sense at each point and at each time instant. In the following we formulate the above problem in a weak sense.

5.3 Weak formulation of the problem

In this section we give a weak formulation of the classical viscoelastic frictional contact problem (68)–(74). Due to the Clarke subdifferential boundary conditions (72) and (73) this formulation will be a hyperbolic hemivariational inequality. We introduce

$$V = \{\, v \in H_1 \mid v = 0 \ \text{ on } \Gamma_D \,\}.$$

This is the closed subspace of H_1 and so it is a Hilbert space with the inner product and the corresponding norm given by

$$(u, v)_V = (\varepsilon(u), \varepsilon(v))_{\mathcal{H}}, \quad \|v\| = \|\varepsilon(v)\|_{\mathcal{H}} \ \text{ for } u, v \in V.$$

By the Korn inequality $\|v\|_{H_1} \le c\, \|\varepsilon(v)\|_{\mathcal{H}}$ for $v \in V$ with $c > 0$ (cf. Section 6.3 of [74]), it follows that $\|\cdot\|_{H_1}$ and $\|\cdot\|$ are the equivalent norms on V. Identifying $H = L^2(\Omega; \mathbb{R}^d)$ with its dual, we have an evolution triple of spaces (V, H, V^*) (see Definition 4 and Example 5) with dense, continuous and compact embeddings. For this evolution triple, analogously as in Section 3.1, we define the spaces $\mathcal{V} = L^2(0,T;V)$, $\widehat{\mathcal{H}} = L^2(0,T;H)$, $\mathcal{V}^* = L^2(0,T;V^*)$ and $\mathcal{W} = \{v \in \mathcal{V} \mid v' \in \mathcal{V}^*\}$. The duality pairing between V^* and V, and between $\mathcal{V}^*$ and $\mathcal{V}$ are, respectively, denoted by $\langle \cdot, \cdot \rangle$ and $\langle\langle \cdot, \cdot \rangle\rangle$.

We admit the following hypotheses on the data of the problem (68)-(74).

$\underline{H(\mathcal{A})}$: The viscosity operator $\mathcal{A}\colon Q \times \mathbb{S}^d \to \mathbb{S}^d$ is such that

(i) $\mathcal{A}(\cdot, \cdot, \varepsilon)$ is measurable on Q for all $\varepsilon \in \mathbb{S}^d$;

(ii) $\mathcal{A}(x, t, \cdot)$ is continuous on $\mathbb{S}^d$ for a.e. $(x, t) \in Q$;

(iii) $\|\mathcal{A}(x, t, \varepsilon)\|_{\mathbb{S}^d} \leq \widetilde{a_1}(x, t) + \widetilde{a_2}\, \|\varepsilon\|_{\mathbb{S}^d}$ for all $\varepsilon \in \mathbb{S}^d$, a.e. $(x, t) \in Q$ with $\widetilde{a_1} \in L^2(Q)$, $\widetilde{a_1}$, $\widetilde{a_2} \geq 0$;

(iv) $(\mathcal{A}(x, t, \varepsilon_1) - \mathcal{A}(x, t, \varepsilon_2)) : (\varepsilon_1 - \varepsilon_2) \geq 0$ for all ε_1, $\varepsilon_2 \in \mathbb{S}^d$, a.e. $(x, t) \in Q$;

(v) $\mathcal{A}(x, t, \varepsilon) : \varepsilon \geq \widetilde{a_3}\, \|\varepsilon\|^2_{\mathbb{S}^d}$ for all $\varepsilon \in \mathbb{S}^d$, a.e. $(x, t) \in Q$ with $\widetilde{a_3} > 0$.

<u>$H(\mathcal{A})_1$</u> : The viscosity operator $\mathcal{A}\colon Q \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies $H(\mathcal{A})$(i), (ii), (iii), (v) and

(vi) $(\mathcal{A}(x, t, \varepsilon_1) - \mathcal{A}(x, t, \varepsilon_2)) : (\varepsilon_1 - \varepsilon_2) \geq \widetilde{a_4}\, \|\varepsilon_1 - \varepsilon_2\|^2_{\mathbb{S}^d}$ for all ε_1, $\varepsilon_2 \in \mathbb{S}^d$, a.e. $(x, t) \in Q$ with $\widetilde{a_4} > 0$.

REMARK 51 *It should be remarked that the hypothesis $H(\mathcal{A})$ is more general than the ones considered in the literature, cf. e.g. conditions (6.34) in Chapter 6.3 of [34] and assumption (6.4.4) in Chapter 6.4 of [93]. The growth condition $H(\mathcal{A})$(iii) is a substantial assumption, it excludes terms with power greater than one, but is satisfied within linearized viscoelasticity, and is satisfied by truncated operators, cf. [34, 93]. The condition $H(\mathcal{A})$(iv) means that the viscosity operator is monotone. This assumption together with the coercivity condition $H(\mathcal{A})$(v) is quite natural. It is clear that if $\mathcal{A}(x, t, \cdot)$ is Lipschitz continuous, i.e. $\|\mathcal{A}(x, t, \varepsilon_1) - \mathcal{A}(x, t, \varepsilon_2)\|_{\mathbb{S}^d} \leq L_{\mathcal{A}} \|\varepsilon_1 - \varepsilon_2\|_{\mathbb{S}^d}$ for all ε_1, $\varepsilon_2 \in \mathbb{S}^d$, a.e. $(x, t) \in Q$ with $L_{\mathcal{A}} > 0$ and $\mathcal{A}(\cdot, \cdot, 0) \in L^2(Q; \mathbb{S}^d)$, then $H(\mathcal{A})$(iii) holds with $\widetilde{a_1}(x, t) = \|\mathcal{A}(x, t, 0)\|_{\mathbb{S}^d}$, $\widetilde{a_1} \in L^2(Q)$ and $\widetilde{a_2} = L_{\mathcal{A}}$.*

<u>$H(\mathcal{B})$</u> : The elasticity operator $\mathcal{B}\colon Q \times \mathbb{S}^d \to \mathbb{S}^d$ is such that

(i) $\mathcal{B}(\cdot, \cdot, \varepsilon)$ is measurable on Q for all $\varepsilon \in S_d$;

(ii) $\|\mathcal{B}(x, t, \varepsilon)\|_{\mathbb{S}^d} \leq \widetilde{b_1}(x, t) + \widetilde{b_2}\, \|\varepsilon\|_{\mathbb{S}^d}$ for all $\varepsilon \in \mathbb{S}^d$, a.e. $(x, t) \in Q$ with $\widetilde{b_1} \in L^2(Q)$, $\widetilde{b_1}$, $\widetilde{b_2} \geq 0$;

(iii) $\|\mathcal{B}(x, t, \varepsilon_1) - \mathcal{B}(x, t, \varepsilon_2)\|_{\mathbb{S}^d} \leq L_{\mathcal{B}} \|\varepsilon_1 - \varepsilon_2\|_{\mathbb{S}^d}$ for all ε_1, $\varepsilon_2 \in \mathbb{S}^d$, a.e. $(x, t) \in Q$ with $L_{\mathcal{B}} > 0$.

REMARK 52 *1) If the condition $H(\mathcal{B})$(iii) holds and $\mathcal{B}(\cdot, \cdot, 0) \in L^2(Q; \mathbb{S}^d)$, then*

$$\|\mathcal{B}(x, t, \varepsilon)\|_{\mathbb{S}^d} \leq \widetilde{b}(x, t) + L_{\mathcal{B}} \|\varepsilon\|_{\mathbb{S}^d} \text{ for all } \varepsilon \in \mathbb{S}^d, \text{ a.e. } (x, t) \in Q,$$

where $\widetilde{b}(x, t) = \|\mathcal{B}(x, t, 0)\|_{\mathbb{S}^d}$, $\widetilde{b} \in L^2(Q)$, $\widetilde{b} \geq 0$.
2) If $\mathcal{B}(x, t, \cdot) \in \mathcal{L}(\mathbb{S}^d, \mathbb{S}^d)$ for a.e. $(x, t) \in Q$, the conditions $H(\mathcal{B})$(ii) and (iii) hold. Thus the hypothesis $H(\mathcal{B})$ is more general than the ones considered in [60, 61, 62, 65, 66, 75, 85] where the elasticity operator is assumed to be linear (which corresponds to the Hooke law).

<u>$H(\mathcal{C})$</u> : The relaxation operator $\mathcal{C}\colon Q \times \mathbb{S}^d \to \mathbb{S}^d$ is of the form $\mathcal{C}(x, t, \varepsilon) = c(x, t)\, \varepsilon$ and $c(x, t) = \{c_{ijkl}(x, t)\}$ with $c_{ijkl} = c_{jikl} = c_{lkij} \in L^\infty(Q)$.

<u>$H(f)$</u> : $f_0 \in L^2(0, T; H)$, $f_1 \in L^2(0, T; L^2(\Gamma_N; \mathbb{R}^d))$, $u_0 \in V$, $u_1 \in H$.

The functions j_k for $k = 1$, 2 satisfy the following

<u>$H(j_k)_1$</u> : The function $j_k\colon \Gamma_C \times (0, T) \times (\mathbb{R}^d)^2 \times \mathbb{R} \to \mathbb{R}$ is such that

(i) $j_k(\cdot,\cdot,\zeta,\rho,r)$ is measurable for all $\zeta,\ \rho\in\mathbb{R}^d$, $r\in\mathbb{R}$,
$j_k(\cdot,\cdot,v(\cdot),w(\cdot),0)\in L^1(\Gamma_C\times(0,T))$ for all $v,\ w\in L^2(\Gamma_C;\mathbb{R}^d)$;

(ii) $j_k(x,t,\cdot,\cdot,r)$ is continuous for all $r\in\mathbb{R}$, a.e. $(x,t)\in\Gamma_C\times(0,T)$,
$j_k(x,t,\zeta,\rho,\cdot)$ is locally Lipschitz for all $\zeta,\ \rho\in\mathbb{R}^d$, a.e. $(x,t)\in\Gamma_C\times(0,T)$;

(iii) $|\partial j_k(x,t,\zeta,\rho,r)|\le c_{k0}+c_{k1}\|\zeta\|+c_{k2}\|\rho\|+c_{k3}|r|$ for all $\zeta,\ \rho\in\mathbb{R}^d$, $r\in\mathbb{R}$, a.e. $(x,t)\in\Gamma_C\times(0,T)$ with $c_{kj}\ge 0$, $j=0,\ 1,\ 2,\ 3$, where ∂j_k denotes the Clarke subdifferential of $j_k(x,t,\zeta,\rho,\cdot)$.

The functions j_k for $k=3,\ 4$ satisfy the following

$\underline{H(j_k)_1}:$ The function $j_k\colon\Gamma_C\times(0,T)\times(\mathbb{R}^d)^3\to\mathbb{R}$ is such that

(i) $j_k(\cdot,\cdot,\zeta,\rho,\theta)$ is measurable for all $\zeta,\ \rho,\ \theta\in\mathbb{R}^d$,
$j_k(\cdot,\cdot,v(\cdot),w(\cdot),0)\in L^1(\Gamma_C\times(0,T))$ for all $v,\ w\in L^2(\Gamma_C;\mathbb{R}^d)$;

(ii) $j_k(x,t,\cdot,\cdot,\theta)$ is continuous for all $\theta\in\mathbb{R}^d$, a.e. $(x,t)\in\Gamma_C\times(0,T)$,
$j_k(x,t,\zeta,\rho,\cdot)$ is locally Lipschitz for all $\zeta,\ \rho\in\mathbb{R}^d$, a.e. $(x,t)\in\Gamma_C\times(0,T)$;

(iii) $\|\partial j_k(x,t,\zeta,\rho,\theta)\|\le c_{k0}+c_{k1}\|\zeta\|+c_{k2}\|\rho\|+c_{k3}\|\theta\|$ for all $\zeta,\ \rho,\ \theta\in\mathbb{R}^d$, a.e. $(x,t)\in\Gamma_C\times(0,T)$ with $c_{kj}\ge 0$, $j=0,\ 1,\ 2,\ 3$, where ∂j_k denotes the Clarke subdifferential of $j_k(x,t,\zeta,\rho,\cdot)$.

REMARK 53 *The results of this thesis remain valid if the hypotheses $H(j_k)_1$(iii) for $k=1,\ldots,4$ are replaced, respectively, by the following conditions*

(iii)' $|\partial j_k(x,t,\zeta,\rho,r)|\le a(x,t)+c_{k1}\|\zeta\|+c_{k2}\|\rho\|+c_{k3}|r|$ *for all* $\zeta,\ \rho\in\mathbb{R}^d$, $r\in\mathbb{R}$, *a.e.* $(x,t)\in\Gamma_C\times(0,T)$ *with* $c_{kj}\ge 0$, $j=1,\ 2,\ 3$ *and* $a\in L^2(\Gamma_C\times(0,T))$ *for* $k=1,\ 2$,

and

(iii)' $\|\partial j_k(x,t,\zeta,\rho,\theta)\|\le\overline{a}(x,t)+c_{k1}\|\zeta\|+c_{k2}\|\rho\|+c_{k3}\|\theta\|$ *for all* $\zeta,\ \rho,\ \theta\in\mathbb{R}^d$, *a.e.* $(x,t)\in\Gamma_C\times(0,T)$ *with* $c_{kj}\ge 0$, $j=1,\ 2,\ 3$ *and* $\overline{a}\in L^2(\Gamma_C\times(0,T))$ *for* $k=3,\ 4$.

For simplicity of further notation, we restrict ourselves to the conditions given in the hypotheses $H_1(j_k)$ for $k=1,\ldots,4$.

Moreover, we need the following hypotheses.

$\underline{H(j)_{reg}}:$ The functions $j_k\colon\Gamma_C\times(0,T)\times(\mathbb{R}^d)^2\times\mathbb{R}\to\mathbb{R}$ for $k=1,2$ and functions $j_k\colon\Gamma_C\times(0,T)\times(\mathbb{R}^d)^3\to\mathbb{R}$ for $k=3,4$ are such that for all $\zeta,\ \rho\in\mathbb{R}^d$, a.e. $(x,t)\in\Gamma_C\times(0,T)$, either all $j_k(x,t,\zeta,\rho,\cdot)$ are regular or all $-j_k(x,t,\zeta,\rho,\cdot)$ are regular for $k=1,\ldots,4$.

For $k=1,\ 2$, we introduce

$\underline{H(j_k)_2}:$ The function $j_k\colon\Gamma_C\times(0,T)\times(\mathbb{R}^d)^2\times\mathbb{R}\to\mathbb{R}$ satisfies $H(j_k)_1$ and

(iv) $j_k^0(x,t,\cdot,\cdot,\cdot;s)$ is upper semicontinuous on $(\mathbb{R}^d)^2\times\mathbb{R}$ for all $s\in\mathbb{R}$, a.e. $(x,t)\in\Gamma_C\times(0,T)$, where j_k^0 denotes the generalized directional derivative of Clarke of $j_k(x,t,\zeta,\rho,\cdot)$ in the direction s.

For $k = 3, 4$, we introduce

$\underline{H(j_k)_2}$: The function $j_k \colon \Gamma_C \times (0,T) \times (\mathbb{R}^d)^3 \to \mathbb{R}$ satisfies $H(j_k)_1$ and

(iv) $j_k^0(x,t,\cdot,\cdot,\cdot;\sigma)$ is upper semicontinuous on $(\mathbb{R}^d)^3$ for all $\sigma \in \mathbb{R}^d$, a.e. $(x,t) \in \Gamma_C \times (0,T)$, where j_k^0 denotes the generalized directional derivative of Clarke of $j_k(x,t,\zeta,\rho,\cdot)$ in the direction σ.

The above hypotheses are realistic with respect to the physical data and the process modeling. We will see this in the specific examples of contact laws which are given in Section 6.2.

Next, let $v \in V$. We define $f \in \mathcal{V}^*$ by

$$\langle f(t), v\rangle_{V^*\times V} = \langle f_0(t), v\rangle_H + \langle f_1(t), v\rangle_{L^2(\Gamma_N;\mathbb{R}^d)}$$

for a.e. $t \in (0,T)$. Assuming that the functions in the problem (68)–(74) are sufficiently regular, using the equation of motion (68) and the Green formula, we obtain

$$\langle u''(t), v\rangle + \langle \sigma(t), \varepsilon(v)\rangle_{\mathcal{H}} - \int_\Gamma \sigma(t)\nu \cdot v(x)\, d\Gamma = \langle f_0(t), v\rangle_H$$

for a.e. $t \in (0,T)$. From the boundary conditions (70) and (71), we have

$$\int_\Gamma \sigma(t)\nu \cdot v\, d\Gamma = \int_{\Gamma_N} f_1(t)\cdot v\, d\Gamma + \int_{\Gamma_C} \left(\sigma_\tau(t)\cdot v_\tau + \sigma_\nu(t) v_\nu\right) d\Gamma.$$

On the other hand, the subdifferential boundary conditions (72) and (73) imply

$$-\sigma_\nu(t)\, r \le j_1^0(x,t,u(t),u'(t),u_\nu(t);r) + j_2^0(x,t,u(t),u'(t),u'_\nu(t);r) \;\text{ for all } r \in \mathbb{R},$$

$$-\sigma_\tau(t)\cdot\xi \le j_3^0(x,t,u(t),u'(t),u_\tau(t);\xi) + j_4^0(x,t,u(t),u'(t),u'_\tau(t);\xi) \;\text{ for all } \xi \in \mathbb{R}^d.$$

Hence

$$\begin{aligned}
-\int_{\Gamma_C} \sigma_\nu(t)\, v_\nu\, d\Gamma \;&\le\; \int_{\Gamma_C} \Big(j_1^0(x,t,u(t),u'(t),u_\nu(t);v_\nu) + \\
&\qquad\qquad + j_2^0(x,t,u(t),u'(t),u'_\nu(t);v_\nu)\Big)\, d\Gamma, \\
-\int_{\Gamma_C} \sigma_\tau(t)\cdot v_\tau\, d\Gamma \;&\le\; \int_{\Gamma_C} \Big(j_3^0(x,t,u(t),u'(t),u_\tau(t);v_\tau) + \\
&\qquad\qquad + j_4^0(x,t,u(t),u'(t),u'_\tau(t);v_\tau)\Big)\, d\Gamma
\end{aligned}$$

for $t \in (0,T)$. Using the constitutive law (69) and the above relations, we obtain the following weak formulation of the problem (68)–(74).

Problem (HVI): find $u \colon (0,T) \to V$ such that $u \in \mathcal{V}$, $u' \in \mathcal{W}$ and

$$\begin{cases}
\langle u''(t), v\rangle + \langle \mathcal{A}(t,\varepsilon(u'(t))) + \mathcal{B}(t,\varepsilon(u(t))) + \displaystyle\int_0^t \mathcal{C}(t-s)\,\varepsilon(u(s))\,ds, \varepsilon(v)\rangle_{\mathcal{H}} + \\
\quad + \displaystyle\int_{\Gamma_C} \Big(j_1^0(x,t,u(t),u'(t),u_\nu(t);v_\nu) + j_2^0(x,t,u(t),u'(t),u'_\nu(t);v_\nu) + \\
\quad + \; j_3^0(x,t,u(t),u'(t),u_\tau(t);v_\tau) + j_4^0(x,t,u(t),u'(t),u'_\tau(t);v_\tau)\Big)\, d\Gamma \ge \\
\quad \ge \langle f(t), v\rangle \;\text{ for all } v \in V, \text{ a.e. } t \in (0,T), \\
u(0) = u_0,\; u'(0) = u_1.
\end{cases}$$

This problem is the hyperbolic hemivariational inequality (HVI). In the next section it will be associated with the nonlinear evolution inclusion of second order.

5.4 Evolution inclusion for hemivariational inequality

The aim of this section is to formulate the hemivariational inequality in Problem (HVI) as an evolution inclusion which has the form of Problem $\mathcal{P}$ of Section 3.1. This formulation needs a series of auxiliary results. We begin with the definitions and properties of operators associated to the viscosity, elasticity and relaxation operators.

Let us define the following operators A, B, $C\colon (0,T)\times V\to V^*$ by

$$\langle A(t,u),v\rangle = (\mathcal{A}(x,t,\varepsilon(u)),\varepsilon(v))_{\mathcal{H}}, \tag{75}$$

$$\langle B(t,u),v\rangle = (\mathcal{B}(x,t,\varepsilon(u)),\varepsilon(v))_{\mathcal{H}} \tag{76}$$

and

$$\langle C(t,u),v\rangle = (\mathcal{C}(t)\varepsilon(u)),\varepsilon(v))_{\mathcal{H}} \tag{77}$$

for u, $v\in V$, a.e. $t\in(0,T)$.

Lemma 54 *Under the hypothesis $H(\mathcal{A})$, the operator $A\colon (0,T)\times V\to V^*$ defined by (75) satisfies $H(A)$ with $a_0(t)=\sqrt{2}\,\|\widetilde{a_1}(t)\|_{L^2(\Omega)}$, $a_1=\sqrt{2}\,\widetilde{a_2}$ and $\alpha=\widetilde{a_3}$, and $H(A)_1(v)$. Under the hypothesis $H(\mathcal{A})_1$, the operator A satisfies $H(A)_1(vi)$ with $m_1=\widetilde{a_4}$.*

Proof. Let us suppose $H(\mathcal{A})$. By $H(\mathcal{A})$(iii) and Hölder's inequality, we have

$$\begin{aligned}|\langle A(t,v),w\rangle| &\le \int_\Omega \|\mathcal{A}(x,t,\varepsilon(v))\|_{\mathbb{S}^d}\|\varepsilon(w)\|_{\mathbb{S}^d}\,dx\le\\ &\le \int_\Omega (\widetilde{a_1}(x,t)+\widetilde{a_2}\,\|\varepsilon(v)\|_{\mathbb{S}^d})\|\varepsilon(w)\|_{\mathbb{S}^d}\,dx\le\\ &\le \sqrt{2}\left(\|\widetilde{a_1}(t)\|_{L^2(\Omega)}+\widetilde{a_2}\,\|v\|\right)\|w\| \end{aligned}\tag{78}$$

for all v, $w\in V$, a.e. $t\in(0,T)$. Hence the function $(x,t)\mapsto\mathcal{A}(x,t,\varepsilon(v))\colon\varepsilon(w)$ is integrable for all v, $w\in V$. By Fubini's theorem (cf. Lemma 79), we have that $t\mapsto\int_\Omega\mathcal{A}(x,t,\varepsilon(v))\colon\varepsilon(w)\,dx=\langle A(t,v),w\rangle$ is measurable for all v, $w\in V$. Hence, for all $v\in V$, the function $t\mapsto A(t,v)$ is weakly measurable from $(0,T)$ into V^*. Since the latter is separable, from the Pettis measurability theorem, it follows that $t\mapsto A(t,v)$ is measurable for all $v\in V$, i.e. $H(A)$(i) holds. Also from (78) we obtain that $H(A)$(iii) is satisfied with $a_0(t)=\sqrt{2}\,\|\widetilde{a_1}(t)\|_{L^2(\Omega)}$ and $a_1=\sqrt{2}\,\widetilde{a_2}$.

From the hypothesis $H(\mathcal{A})$(v), it follows

$$\langle A(t,v),v\rangle=\int_\Omega\mathcal{A}(x,t,\varepsilon(v)):\varepsilon(v)\,dx\ge\widetilde{a_3}\int_\Omega\|\varepsilon(v)\|^2_{\mathbb{S}^d}\,dx=\widetilde{a_3}\,\|v\|^2$$

for all $v\in V$, a.e. $t\in(0,T)$. Hence $H(A)$(iv) holds with $\alpha=\widetilde{a_3}$. Similarly $H(\mathcal{A})$(iv) implies that $A(t,\cdot)$ is monotone for a.e. $t\in(0,T)$. From Proposition 26.12 of Zeidler [99], we know that the operator $A(t,\cdot)$ is continuous for a.e. $t\in(0,T)$. Hence, in particular, it is hemicontinuous and monotone, thus by Proposition 27.6(a) of [99], it is also pseudomonotone. This proves that $H(A)$ and $H(A)_1$(v) are satisfied.

Assume now $H(\mathcal{A})_1$. From $H(\mathcal{A})_1$(vi), it follows

$$\begin{aligned}\langle A(t,u)-A(t,v),u-v\rangle &= (\mathcal{A}(x,t,\varepsilon(u))-\mathcal{A}(x,t,\varepsilon(v)),\varepsilon(u)-\varepsilon(v))_{\mathcal{H}} = \\ &= \int_\Omega (\mathcal{A}(x,t,\varepsilon(u))-\mathcal{A}(x,t,\varepsilon(v)))\,:\,(\varepsilon(u)-\varepsilon(v))\,dx \geq \\ &\geq \widetilde{a_4}\int_\Omega \|\varepsilon(u-v)\|^2_{\mathbb{S}^d}\,dx = \widetilde{a_4}\,\|u-v\|^2\end{aligned}$$

for all u, $v \in V$, a.e. $t\in(0,T)$. This shows $H(A)_1$(vi) and ends the proof of the lemma. □

Lemma 55 *Under the hypothesis $H(\mathcal{B})$, the operator $B\colon (0,T)\times V\to V^*$ defined by (76) satisfies $H(B)$ with $L_B = L_{\mathcal{B}}$, $b_0(t)=\sqrt{2}\,\|\widetilde{b_1}(t)\|_{L^2(\Omega)}$ and $b_1=\sqrt{2}\,\widetilde{b_2}$.*

Proof. The measurability of $B(\cdot,v)$ for all $v\in V$ is shown analogously as in the proof of Lemma 54. Indeed, using $H(\mathcal{B})$(ii) and Hölder's inequality, we have

$$|\langle B(t,v),w\rangle| \leq \sqrt{2}\left(\|\widetilde{b_1}(t)\|_{L^2(\Omega)}+\widetilde{b_2}\,\|v\|\right)\|w\| \tag{79}$$

for all v, $w\in V$, a.e. $t\in(0,T)$. From Fubini's theorem, we know that $t\mapsto\langle B(t,v),w\rangle$ is measurable for all v, $w\in V$. Clearly $t\mapsto B(t,v)$ is weakly measurable from $(0,T)$ into V^* for all $v\in V$ and since V^* is separable, by the Pettis measurability theorem, we deduce that $t\mapsto B(t,v)$ is measurable for all $v\in V$. This proves $H(B)$(i).

Using (79), we easily obtain that $H(B)$(iii) holds with $b_0(t)=\sqrt{2}\,\|\widetilde{b_1}(t)\|_{L^2(\Omega)}$ and $b_1=\sqrt{2}\,\widetilde{b_2}$. Next, from $H(\mathcal{B})$(iii) and Hölder's inequality, we get

$$\begin{aligned}|\langle B(t,u)-B(t,v),w\rangle| &= |\int_\Omega(\mathcal{B}(x,t,\varepsilon(u))-\mathcal{B}(x,t,\varepsilon(v)))\,:\,\varepsilon(w)\,dx| \leq \\ &\leq L_{\mathcal{B}}\int_\Omega\|\varepsilon(u)-\varepsilon(v)\|_{\mathbb{S}^d}\|\varepsilon(w)\|_{\mathbb{S}^d}\,dx \leq L_{\mathcal{B}}\|u-v\|\|w\|\end{aligned}$$

for all u, v, $w\in V$, a.e. $t\in(0,T)$. Hence, $H(B)$(ii) follows. The proof of the lemma is thus complete. □

Lemma 56 *Under the hypothesis $H(\mathcal{C})$, the operator C defined by (77) satisfies $H(C)$.*

Proof. From the hypothesis $H(\mathcal{C})$, we have

$$\langle C(t,u),v\rangle = \int_\Omega c(x,t)\,\varepsilon(u):\varepsilon(v)\,dx \quad \text{for } u,v\in V, \text{ a.e. } t\in(0,T).$$

Since $c(x,t)=\{c_{ijkl}(x,t)\}$ and $c_{ijkl}\in L^\infty(Q)$, using the Hölder inequality we readily obtain that $C\in L^2(0,T;\mathcal{L}(V,V^*))$. □

We also observe that if $H(f)$ holds then (H_0) is satisfied as well. Now, in order to formulate the hemivariational inequality (HVI) in the form of evolution inclusion, we extend the pointwise relations (72) and (73) to relations involving multifunctions. This needs some work and is carried out below.

We consider the function $g\colon \Gamma_C \times (0,T) \times (\mathbb{R}^d)^4 \to \mathbb{R}$ defined by

$$
\begin{aligned}
g(x,t,\zeta,\rho,\xi,\eta) &= j_1(x,t,\zeta,\rho,\xi_\nu) + j_2(x,t,\zeta,\rho,\eta_\nu) + \\
&+ j_3(x,t,\zeta,\rho,\xi_\tau) + j_4(x,t,\zeta,\rho,\eta_\tau)
\end{aligned}
\tag{80}
$$

for $\zeta,\ \rho,\ \xi,\ \eta \in \mathbb{R}^d$ and a.e. $(x,t) \in \Gamma_C \times (0,T)$.

In what follows, we will need the following hypothesis.

<u>$H(g)_1$:</u> The function $g\colon \Gamma_C \times (0,T) \times (\mathbb{R}^d)^4 \to \mathbb{R}$ satisfies the following

(i) $g(\cdot,\cdot,\zeta,\rho,\xi,\eta)$ is measurable for all $\zeta,\ \rho,\ \xi,\ \eta \in \mathbb{R}^d$,
$g(\cdot,\cdot,v(\cdot),w(\cdot),0,0) \in L^1(\Gamma_C \times (0,T))$ for all $v,\ w \in L^2(\Gamma_C;\mathbb{R}^d)$;

(ii) $g(x,t,\cdot,\cdot,\xi,\eta)$ is continuous for all $\xi,\ \eta \in \mathbb{R}^d$, a.e. $(x,t) \in \Gamma_C \times (0,T)$,
$g(x,t,\zeta,\rho,\cdot,\cdot)$ is locally Lipschitz for all $\zeta,\ \rho \in \mathbb{R}^d$, a.e. $(x,t) \in \Gamma_C \times (0,T)$;

(iii) $\|\partial g(x,t,\zeta,\rho,\xi,\eta)\|_{(\mathbb{R}^d)^2} \le c_{g0} + c_{g1}(\|\zeta\| + \|\xi\|) + c_{g2}(\|\rho\| + \|\eta\|)$ for all $\zeta,\ \rho,\ \xi,\ \eta \in \mathbb{R}^d$, a.e. $(x,t) \in \Gamma_C \times (0,T)$ with $c_{g0},\ c_{g1},\ c_{g2} \ge 0$, where ∂g denotes the Clarke subdifferential of $g(x,t,\zeta,\rho,\cdot,\cdot)$.

<u>$H(g)_{reg}$:</u> The function $g\colon \Gamma_C \times (0,T) \times (\mathbb{R}^d)^4 \to \mathbb{R}$ is such that either $g(x,t,\zeta,\rho,\cdot,\cdot)$ or $-g(x,t,\zeta,\rho,\cdot,\cdot)$ is regular for all $\zeta,\rho \in \mathbb{R}^d$, a.e. $(x,t) \in \Gamma_C \times (0,T)$.

<u>$H(g)_2$:</u> The function $g\colon \Gamma_C \times (0,T) \times (\mathbb{R}^d)^4 \to \mathbb{R}$ satisfies $H(g)_1$ and

(iv) $g^0(x,t,\cdot,\cdot,\cdot,\cdot;\chi,\sigma)$ is upper semicontinuous on $(\mathbb{R}^d)^4$ for a.e. $(x,t) \in \Gamma_C \times (0,T)$ and all $\chi,\ \sigma \in \mathbb{R}^d$ where g^0 denotes the generalized directional derivative of Clarke of $g(x,t,\zeta,\rho,\cdot,\cdot)$ in the direction (χ,σ).

Lemma 57 *1) Assume that $H(j_k)_1$ for $k = 1,\ldots,4$ hold. Then the function g defined by (80) satisfies $H(g)_1$ with*

$$
c_{g0} = \max_{1\le k\le 4} c_{k0}, \quad c_{g1} = \max\{\max_{1\le k\le 4} c_{k1}, c_{13}, c_{33}\}, \quad c_{g2} = \max\{\max_{1\le k\le 4} c_{k2}, c_{23}, c_{43}\}
$$

and

$$
\begin{aligned}
g^0(x,t,\zeta,\rho,\xi,\eta;\chi,\sigma) &\le j_1^0(x,t,\zeta,\rho,\xi_\nu;\chi_\nu) + j_2^0(x,t,\zeta,\rho,\eta_\nu;\sigma_\nu) + \\
&+ j_3^0(x,t,\zeta,\rho,\xi_\tau;\chi_\tau) + j_4^0(x,t,\zeta,\rho,\eta_\tau;\sigma_\tau)
\end{aligned}
\tag{81}
$$

for $\zeta,\ \rho,\ \xi,\ \eta,\ \chi,\ \sigma \in \mathbb{R}^d$ and a.e. $(x,t) \in \Gamma_C \times (0,T)$ where j_k^0 denotes the generalized directional derivative of $j_k(x,t,\zeta,\rho,\cdot)$ for $k = 1,\ldots,4$. If in addition $H(j)_{reg}$ is satisfied then $H(g)_{reg}$ is satisfied as well and (81) holds with equality.

2) Under the hypotheses $H(j_k)_2$ for $k = 1,\ldots,4$ and $H(j)_{reg}$, the function g defined by (80) satisfies $H(g)_2$.

Proof. First under the hypotheses $H(j_k)_1$ for $k = 1, \ldots, 4$ we establish $H(g)_1$. The conditions $H(g)_1$(i) and (ii) follow directly from the hypotheses on j_k for $k = 1, \ldots, 4$. For the proof of (81), let ζ, ρ, ξ, $\eta \in \mathbb{R}^d$ and $(x,t) \in \Gamma_C \times (0,T)$. By the definition of g, we can write

$$g(x,t,\zeta,\rho,\xi,\eta) = \sum_{k=1}^{4} \widehat{j}_k(x,t,\zeta,\rho,\xi,\eta),$$

where the functions $\widehat{j}_k \colon \Gamma_C \times (0,T) \times (\mathbb{R}^d)^4 \to \mathbb{R}$ are defined by

$$\begin{aligned}
\widehat{j}_1(x,t,\zeta,\rho,\xi,\eta) &= j_1(x,t,\zeta,\rho,\xi_\nu),\\
\widehat{j}_2(x,t,\zeta,\rho,\xi,\eta) &= j_2(x,t,\zeta,\rho,\eta_\nu),\\
\widehat{j}_3(x,t,\zeta,\rho,\xi,\eta) &= j_3(x,t,\zeta,\rho,\xi_\tau),\\
\widehat{j}_4(x,t,\zeta,\rho,\xi,\eta) &= j_4(x,t,\zeta,\rho,\eta_\tau).
\end{aligned}$$

By Proposition 27(ii), we have

$$g^0(x,t,\zeta,\rho,\xi,\eta;\chi,\sigma) \le \sum_{k=1}^{4} (\widehat{j}_k)^0(x,t,\zeta,\rho,\xi,\eta;\chi,\sigma), \tag{82}$$

for every direction $(\chi,\sigma) \in (\mathbb{R}^d)^2$. Consider now the operators $N_1 \in \mathcal{L}(\mathbb{R}^d,\mathbb{R})$ and $N_2 \in \mathcal{L}(\mathbb{R}^d,\mathbb{R}^d)$ given by $N_1\xi = \xi_\nu$ and $N_2\xi = \xi_\tau$ for $\xi \in \mathbb{R}^d$. From Proposition 28(a) applied to the functions j_k, $k = 1, \ldots, k$ and the operators N_1 and N_2, respectively, we get[1]

$$\begin{aligned}
(\widehat{j}_1)^0(x,t,\zeta,\rho,\xi,\eta;\chi,\sigma) &\le j_1^0(x,t,\zeta,\rho,\xi_\nu;\chi_\nu),\\
(\widehat{j}_2)^0(x,t,\zeta,\rho,\xi,\eta;\chi,\sigma) &\le j_2^0(x,t,\zeta,\rho,\eta_\nu;\sigma_\nu),\\
(\widehat{j}_3)^0(x,t,\zeta,\rho,\xi,\eta;\chi,\sigma) &\le j_3^0(x,t,\zeta,\rho,\xi_\tau;\chi_\tau),\\
(\widehat{j}_4)^0(x,t,\zeta,\rho,\xi,\eta;\chi,\sigma) &\le j_4^0(x,t,\zeta,\rho,\eta_\tau;\sigma_\tau).
\end{aligned}$$

The latter four inequalities together with (82) imply (81).

[1] Alternatively, we can do a direct calculation, for example for j_1, as follows

$$\begin{aligned}
(\widehat{j}_1)^0(x,t,\zeta,\rho,\xi,\eta;\chi,\sigma) &= \limsup_{(\xi',\eta')\to(\xi,\eta),\ \lambda\downarrow 0} \frac{1}{\lambda}\Big(\widehat{j}_1(x,t,\zeta,\rho,(\xi',\eta')+\lambda(\chi,\sigma)) - \widehat{j}_1(x,t,\zeta,\rho,\xi',\eta')\Big) =\\
&= \limsup_{(\xi',\eta')\to(\xi,\eta),\ \lambda\downarrow 0} \frac{1}{\lambda}\Big(\widehat{j}_1(x,t,\zeta,\rho,\xi'_\nu+\lambda\chi_\nu) - \widehat{j}_1(x,t,\zeta,\rho,\xi'_\nu)\Big) \le\\
&\le \limsup_{\xi'_\nu\to\xi_\nu,\ \lambda\downarrow 0} \frac{1}{\lambda}\Big(\widehat{j}_1(x,t,\zeta,\rho,\xi'_\nu+\lambda\chi_\nu) - \widehat{j}_1(x,t,\zeta,\rho,\xi'_\nu)\Big) =\\
&= j_1^0(x,t,\zeta,\rho,\xi_\nu;\chi_\nu),
\end{aligned}$$

where j_1^0 denotes the generalized directional derivative of $j_1(x,t,\zeta,\rho,\cdot)$ in the direction χ_ν.

In order to show $H(g)_1$(iii), let us take $(\overline{\chi}, \overline{\sigma}) \in \partial g(x, t, \zeta, \rho, \xi, \eta)$, where ζ, ρ, ξ, η, $\overline{\chi}$, $\overline{\sigma} \in \mathbb{R}^d$ and $(x, t) \in \Gamma_C \times (0, T)$. By the definition of the subdifferential and (81), for all χ, $\sigma \in \mathbb{R}^d$, we have

$$\begin{aligned} \langle(\overline{\chi}, \overline{\sigma}), (\chi, \sigma)\rangle_{(\mathbb{R}^d)^2} &\leq g^0(x, t, \zeta, \rho, \xi, \eta; \chi, \sigma) \leq \\ &\leq j_1^0(x, t, \zeta, \rho, \xi_\nu; \chi_\nu) + j_2^0(x, t, \zeta, \rho, \eta_\nu; \sigma_\nu) + \\ &+ j_3^0(x, t, \zeta, \rho, \xi_\tau; \chi_\tau) + j_4^0(x, t, \zeta, \rho, \eta_\tau; \sigma_\tau). \end{aligned}$$

Using Proposition 15(iii) and $H(j_1)_1$(iii), we deduce

$$\begin{aligned} j_1^0(x, t, \zeta, \rho, \xi_\nu; \chi_\nu) &= \max\{r\,\chi_\nu \mid r \in \partial j_1(x, t, \zeta, \rho, \xi_\nu)\} \leq \\ &\leq |\chi_\nu| \max\{|r| \mid r \in \partial j_1(x, t, \zeta, \rho, \xi_\nu)\} \leq \\ &\leq |\chi_\nu|\,(c_{10} + c_{11}\|\zeta\| + c_{12}\|\rho\| + c_{13}|\xi_\nu|). \end{aligned}$$

Analogously, by $H(j_k)_1$(iii) for $k = 2$, 3, 4, we obtain

$$\begin{aligned} j_2^0(x, t, \zeta, \rho, \eta_\nu; \sigma_\nu) &\leq |\sigma_\nu|(c_{20} + c_{21}\|\zeta\| + c_{22}\|\rho\| + c_{23}|\eta_\nu|), \\ j_3^0(x, t, \zeta, \rho, \xi_\tau; \chi_\tau) &\leq \|\chi_\tau\|(c_{30} + c_{31}\|\zeta\| + c_{32}\|\rho\| + c_{33}\|\xi_\tau\|), \\ j_4^0(x, t, \zeta, \rho, \eta_\tau; \sigma_\tau) &\leq \|\sigma_\tau\|(c_{40} + c_{41}\|\zeta\| + c_{42}\|\rho\| + c_{43}\|\eta_\tau\|). \end{aligned}$$

Recalling that $|\xi_\nu| \leq \|\xi\|$ and $\|\xi_\tau\| \leq \|\xi\|$ for all $\xi \in \mathbb{R}^d$, from the above, we have

$$\begin{aligned} \langle(\overline{\chi}, \overline{\sigma}), (\chi, \sigma)\rangle_{(\mathbb{R}^d)^2} &\leq \Big(\max_{1\leq k\leq 4} c_{k0} + \|\zeta\| \max_{1\leq k\leq 4} c_{k1} + \|\rho\| \max_{1\leq k\leq 4} c_{k2} + \\ &+ \|\xi\| \max\{c_{13}, c_{33}\} + \|\eta\| \max\{c_{23}, c_{43}\}\Big)(\|\chi\| + \|\sigma\|) \leq \\ &\leq \Big(c_{g0} + c_{g1}(\|\zeta\| + \|\xi\|) + c_{g2}(\|\rho\| + \|\eta\|)\Big)\|\,(\chi, \sigma)\|_{(\mathbb{R}^d)^2}, \end{aligned}$$

where $c_{g0} = \max c_{k0}$, $c_{g1} = \max\{\max c_{k1}, c_{13}, c_{33}\}$, $c_{g2} = \max\{\max c_{k2}, c_{23}, c_{43}\}$. Hence $H(g)_1$(iii) holds.

Now, let ζ, $\rho \in \mathbb{R}^d$ and $(x, t) \in \Gamma_C \times (0, T)$, and suppose that $j_k(x, t, \zeta, \rho, \cdot)$ for $k = 1, \ldots, 4$ are regular in the sense of Clarke. This means, by definition, that for all $r \in \mathbb{R}$ and $\theta \in \mathbb{R}^d$ the usual directional derivatives $j_k'(x, t, \zeta, \rho, r; s)$ for $k = 1$, 2 and $j_k'(x, t, \zeta, \rho, \theta; \sigma)$ for $k = 3$, 4 exist and

$$\begin{cases} j_k'(x, t, \zeta, \rho, r; s) = j_k^0(x, t, \zeta, \rho, r; s) & \text{for } k = 1, 2 \\ j_k'(x, t, \zeta, \rho, \theta; \sigma) = j_k^0(x, t, \zeta, \rho, \theta; \sigma) & \text{for } k = 3, 4 \end{cases} \tag{83}$$

for all directions $s \in \mathbb{R}$ and $\sigma \in \mathbb{R}^d$. Hence we deduce that the directional derivative $g'_{\xi\eta}$ of the function $g(x, t, \zeta, \rho, \cdot, \cdot)$ also exists at every point $(\xi, \eta) \in \mathbb{R}^d \times \mathbb{R}^d$ and in any direction $(\chi, \sigma) \in (\mathbb{R}^d)^2$. Indeed, we have

$$\begin{aligned}
g'_{\xi\eta}(x,t,\zeta,\rho,\xi,\eta;\chi,\sigma) &= \lim_{\lambda\downarrow 0} \frac{1}{\lambda}\Big(g(x,t,\zeta,\rho,(\xi,\eta)+\lambda(\chi,\sigma)) - g(x,t,\zeta,\rho,\xi,\eta)\Big) = \\
&= \lim_{\lambda\downarrow 0} \frac{1}{\lambda}\Big(j_1(x,t,\zeta,\rho,\xi_N+\lambda\chi_\nu) + j_2(x,t,\zeta,\rho,\eta_\nu+\lambda\sigma_\nu) + \\
&\qquad + j_3(x,t,\zeta,\rho,\xi_\tau+\lambda\chi_\tau) + j_4(x,t,\zeta,\rho,\eta_\tau+\lambda\sigma_\tau) - \\
&\qquad - j_1(x,t,\zeta,\rho,\xi_\nu) - j_2(x,t,\zeta,\rho,\eta_\nu) - \\
&\qquad - j_3(x,t,\zeta,\rho,\xi_\tau) - j_4(x,t,\zeta,\rho,\eta_\tau)\Big) = \\
&= \lim_{\lambda\downarrow 0} \frac{1}{\lambda}\Big(j_1(x,t,\zeta,\rho,\xi_\nu+\lambda\chi_\nu) - j_1(x,t,\zeta,\rho,\xi_\nu)\Big) + \\
&+ \lim_{\lambda\downarrow 0} \frac{1}{\lambda}\Big(j_2(x,t,\zeta,\rho,\eta_\nu+\lambda\sigma_\nu) - j_2(x,t,\zeta,\rho,\eta_\nu)\Big) + \\
&+ \lim_{\lambda\downarrow 0} \frac{1}{\lambda}\Big(j_3(x,t,\zeta,\rho,\xi_\tau+\lambda\chi_\tau) - j_3(x,t,\zeta,\rho,\xi_\tau)\Big) + \\
&+ \lim_{\lambda\downarrow 0} \frac{1}{\lambda}\Big(j_4(x,t,\zeta,\rho,\eta_\tau+\lambda\sigma_\tau) - j_4(x,t,\zeta,\rho,\eta_\tau)\Big) = \\
&= j_1'(x,t,\zeta,\rho,\xi_\nu;\chi_\nu) + j_2'(x,t,\zeta,\rho,\eta_\nu;\sigma_\nu) + \\
&+ j_3'(x,t,\zeta,\rho,\xi_\tau;\chi_\tau) + j_4'(x,t,\zeta,\rho,\eta_\tau;\sigma_\tau).
\end{aligned}$$

Furthermore, by (81) and (83), the latter implies

$$\begin{aligned}
g'_{\xi\eta}(x,t,\zeta,\rho,\xi,\eta;\chi,\sigma) &= j_1^0(x,t,\zeta,\rho,\xi_\nu;\chi_\nu) + j_2^0(x,t,\zeta,\rho,\eta_\nu;\sigma_\nu) + \\
&+ j_3^0(x,t,\zeta,\rho,\xi_\tau;\chi_\tau) + j_4^0(x,t,\zeta,\rho,\eta_\tau;\sigma_\tau) \geq \\
&\geq g^0(x,t,\zeta,\rho,\xi,\eta;\xi,\sigma)
\end{aligned}$$

for all ξ, η, χ, $\sigma \in \mathbb{R}^d$. The opposite inequality $g'_{\xi\eta} \leq g^0$ is always true (cf. Remark 18), so we deduce that

$$g'_{\xi\eta}(x,t,\zeta,\rho,\xi,\eta;\chi,\sigma) = g^0(x,t,\zeta,\rho,\xi,\eta;\chi,\sigma)$$

for ζ, ρ, ξ, η, χ, $\sigma \in \mathbb{R}^d$, a.e. $(x,t) \in \Gamma_C \times (0,T)$, which means that $g(x,t,\zeta,\rho,\cdot,\cdot)$ is regular in the sense of Clarke. Thus (81) holds with equality.

If for $k = 1,\ldots,k$ the functions $-j_k$ are regular in their last variables, then we proceed in the same way as above and deduce the reguarity of $-g(x,t,\zeta,\rho,\cdot,\cdot)$ for $(x,t) \in \Gamma_C \times (0,T)$ and ζ, $\rho \in \mathbb{R}^d$. We use the property $(-g)^0(x,t,\zeta,\rho,\xi,\eta;\chi,\sigma) = g^0(x,t,\zeta,\rho,\xi,\eta;-\chi,-\sigma)$ for all ζ, ρ, ξ, η, χ, $\sigma \in \mathbb{R}^d$, a.e. $(x,t) \in \Gamma_C \times (0,T)$ (cf. Proposition 15(i)), and again get the equality in (81).

Finally, we suppose the hypotheses $H(j_k)_2$ for $k = 1,\ldots,4$ and $H(j)_{reg}$. In order to prove $H(g)_2$, it is enough to show that $g^0(x,t,\cdot,\cdot,\cdot,\cdot;\chi,\sigma)$ is upper semicontinuous on $(\mathbb{R}^d)^4$ for all χ, $\sigma \in \mathbb{R}^d$ and a.e. $(x,t) \in \Gamma_C \times (0,T)$. Let χ, $\sigma \in \mathbb{R}^d$ and $(x,t) \in \Gamma_C \times (0,T)$, and let $\{\zeta_n\}$, $\{\rho_n\}$, $\{\xi_n\}$, $\{\eta_n\}$ be sequences in $\mathbb{R}^d$ such that $\zeta_n \to \zeta$,

$\rho_n \to \rho$, $\xi_n \to \xi$ and $\eta_n \to \eta$. By the hypothesis $H(j_k)_2$ for $k = 1, \ldots, 4$ and equality (81), we find

$$
\begin{aligned}
&\limsup g^0(x, t, \zeta_n, \rho_n, \xi_n, \eta_n; \chi, \sigma) \le \\
&\quad \le \ \limsup j_1^0(x, t, \zeta_n, \rho_n, \xi_{n\nu}; \chi_\nu) + \limsup j_2^0(x, t, \zeta_n, \rho_n, \eta_{n\nu}; \sigma_\nu) + \\
&\quad + \ \limsup j_3^0(x, t, \zeta_n, \rho_n, \xi_{n\tau}; \chi_\tau) + \limsup j_4^0(x, t, \zeta_n, \rho_n, \eta_{n\tau}; \sigma_\tau) \le \\
&\quad \le \ j_1^0(x, t, \zeta, \rho, \xi_\nu; \chi_\nu) + j_2^0(x, t, \zeta, \rho, \eta_\nu; \sigma_\nu) + \\
&\quad + \ j_3^0(x, t, \zeta, \rho, \xi_\tau; \chi_\tau) + j_4^0(x, t, \zeta, \rho, \eta_\tau; \sigma_\tau) = \\
&\quad = \ g^0(x, t, \zeta, \rho, \xi, \eta; \chi, \sigma).
\end{aligned}
$$

Hence the condition $H(g)_2$ follows. The proof of the lemma is complete. □

The next step is to study the integral functional corresponding to superpotentials which appear in the boundary conditions. Let us consider the functional $G\colon (0,T) \times L^2(\Gamma_C; \mathbb{R}^d)^4 \to \mathbb{R}$ defined by

$$
G(t, w, z, u, v) = \int_{\Gamma_C} g(x, t, w(x), z(x), u(x), v(x)) \, d\Gamma \tag{84}
$$

for w, z, u, $v \in L^2(\Gamma_C; \mathbb{R}^d)$, $t \in (0,T)$, where the integrand g is given by (80).

We introduce the following conditions.

$\underline{H(G)_1}$: The functional $G\colon (0,T) \times L^2(\Gamma_C; \mathbb{R}^d)^4 \to \mathbb{R}$ is such that

(i) $G(\cdot, w, z, u, v)$ is measurable for all w, z, u, $v \in L^2(\Gamma_C; \mathbb{R}^d)$, $G(\cdot, w, z, 0, 0) \in L^1(0,T)$ for all w, $z \in L^2(\Gamma_C; \mathbb{R}^d)$;

(ii) $G(t, w, z, \cdot, \cdot)$ is Lipschitz continuous on bounded subsets of $L^2(\Gamma_C; \mathbb{R}^d)^2$ for all w, $z \in L^2(\Gamma_C; \mathbb{R}^d)$, a.e. $t \in (0,T)$;

(iii) $\|\partial G(t, w, z, u, v)\|_{L^2(\Gamma_C;\mathbb{R}^d)^2} \le c_{G0} + c_{G1}(\|w\| + \|u\|) + c_{G2}(\|z\| + \|v\|)$ for all w, z, u, $v \in L^2(\Gamma_C; \mathbb{R}^d)$, a.e. $t \in (0,T)$ with c_{G0}, c_{G1}, $c_{G2} \ge 0$, where ∂G denotes the Clarke subdifferential of $G(t, w, z, \cdot, \cdot)$;

(iv) For all w, z, u, v, $\overline{u}$, $\overline{v} \in L^2(\Gamma_C; \mathbb{R}^d)$, a.e. $t \in (0,T)$, we have

$$
G^0(t, w, z, u, v; \overline{u}, \overline{v}) \le \int_{\Gamma_C} g^0(x, t, w(x), z(x), u(x), v(x); \overline{u}(x), \overline{v}(x)) \, d\Gamma, \tag{85}
$$

where G^0 denotes the generalized directional derivative of $G(t, w, z, \cdot, \cdot)$ at a point (u, v) in the direction $(\overline{u}, \overline{v})$.

$\underline{H(G)_{reg}}$: The functional $G\colon (0,T) \times L^2(\Gamma_C; \mathbb{R}^d)^4 \to \mathbb{R}$ is such that either $G(t, w, z, \cdot, \cdot)$ or $-G(t, w, z, \cdot, \cdot)$ is regular for all w, $z \in L^2(\Gamma_C; \mathbb{R}^d)$, a.e. $t \in (0,T)$.

$\underline{H(G)_2}$: The functional $G\colon (0,T) \times L^2(\Gamma_C; \mathbb{R}^d)^4 \to \mathbb{R}$ is such that $H(G)_1$ holds and

(v) $G^0(t, \cdot, \cdot, \cdot, \cdot; \overline{u}, \overline{v})$ is upper semicontinuous on $L^2(\Gamma_C; \mathbb{R}^d)^4$ for all $\overline{u}, \overline{v} \in L^2(\Gamma_C; \mathbb{R}^d)$, a.e. $t \in (0,T)$.

Lemma 58 *1) Under the hypotheses $H(g)_1$ the functional G defined by (84) satisfies $H(G)_1$ with $c_{G0} = c_{g0}\sqrt{5\, m(\Gamma_C)}$, $c_{G1} = c_{g1}\sqrt{5}$ and $c_{G2} = c_{g2}\sqrt{5}$. If in addition $H(g)_{reg}$ holds, then $H(G)_{reg}$ is satisfied as well and (85) holds with equality.*

2) Under the hypotheses $H(g)_2$ and $H(g)_{reg}$, the functional G satisfies $H(G)_2$.

Proof. First, from $H(g)_1$(ii) and Lemma 34, it follows that $g(x,t,\cdot,\cdot,\cdot,\cdot)$ is continuous on $(\mathbb{R}^d)^4$ which together with $H(g)_1$(i) implies that g is a Carathéodory function. Hence $(x,t) \mapsto g(x,t,w(x),z(x),u(x),v(x))$ is measurable for all w, z, u, $v \in L^2(\Gamma_C;\mathbb{R}^d)$ and subsequently the integrand of (84) is a measurable function of x.

Next, applying the Lebourg mean value theorem (cf. e.g. Theorem 5.6.25 of [23]) to a locally Lipschitz function $g(x,t,\zeta,\rho,\cdot,\cdot)$ (cf. $H(g)_1$(ii)), we deduce that there exist $(\overline{\xi},\overline{\eta})$ in the interval $[0,(\xi,\eta)] \subset (\mathbb{R}^d)^2$ and $(\xi^*,\eta^*) \in \partial g(x,t,\zeta,\rho,\overline{\xi},\overline{\eta})$ such that

$$g(x,t,\zeta,\rho,\xi,\eta) - g(x,t,\zeta,\rho,0,0) = ((\xi^*,\eta^*),(\xi,\eta))_{(\mathbb{R}^d)^2}$$

for all ζ, ρ, ξ, $\eta \in \mathbb{R}^d$, a.e. $(x,t) \in \Gamma_C \times (0,T)$. Hence, by $H(g)_1$(iii), we obtain

$$\begin{aligned} &g(x,t,w(x),z(x),u(x),v(x)) \le g(x,t,w(x),z(x),0,0) + \\ &\quad + \ c\,(\|u(x)\| + \|v(x)\|)\ (c_{g0} + c_{g1}(\|w(x)\| + \|u(x)\|) + c_{g2}(\|z(x)\| + \|v(x)\|)) \end{aligned}$$

for all w, z, u, $v \in L^2(\Gamma_C;\mathbb{R}^d)$, a.e. $(x,t) \in \Gamma_C \times (0,T)$ with a constant $c > 0$. From $H(g)_1$(i), it is easy to see that $(x,t) \mapsto g(x,t,w(x),z(x),u(x),v(x))$ is integrable and from Fubini's theorem, we infer that $G(\cdot,w,z,u,v)$ is measurable and $H(G)_1$(i) holds.

Now, let w, $z \in L^2(\Gamma_C;\mathbb{R}^d)$ and let $\widetilde{g}\colon \Gamma_C \times (0,T) \times (\mathbb{R}^d)^2 \to \mathbb{R}$ be defined by

$$\widetilde{g}(x,t,\xi,\eta) = g(x,t,w(x),z(x),\xi,\eta) \ \text{ for } \xi,\eta \in \mathbb{R}^d, \ \text{ a.e. } (x,t) \in \Gamma_C \times (0,T).$$

From (i) and (ii) of $H(g)_1$, it follows that $\widetilde{g}(\cdot,\cdot,\xi,\eta)$ is measurable for all ξ, $\eta \in \mathbb{R}^d$, $\widetilde{g}(\cdot,t,0,0) \in L^1(\Gamma_C)$ for a.e. $t \in (0,T)$ (by invoking again Fubini's theorem) and $\widetilde{g}(x,t,\cdot,\cdot)$ is locally Lipschitz for a.e. $(x,t) \in \Gamma_C \times (0,T)$. Moreover, by employing $H(g)_1$(iii), we have

$$\begin{aligned} \|\partial\widetilde{g}(x,t,\xi,\eta)\|_{(\mathbb{R}^d)^2} &= \|\partial g(x,t,w(x),z(x),\xi,\eta)\|_{(\mathbb{R}^d)^2} \le \\ &\le c_{g0} + c_{g1}(\|w(x)\| + \|\xi\|) + c_{g2}(\|z(x)\| + \|\eta\|) = \\ &= \omega(x) + \max\{c_{g1},c_{g2}\}\,(\|\xi\| + \|\eta\|) \end{aligned}$$

with $\omega \in L^2(\Gamma_C)$. At this stage we appeal to Aubin-Clarke's theorem (cf. Lemma 82) to deduce that the functional $G(t,w,z,\cdot,\cdot)$ is well defined, finite and Lipschitz continuous on bounded subsets of $L^2(\Gamma_C;\mathbb{R}^d)$ for all w, $z \in L^2(\Gamma_C;\mathbb{R}^d)$, a.e. $t \in (0,T)$. Hence $H(G)_1$(ii) is satisfied. Furthermore, for w, z, u, $v \in L^2(\Gamma_C;\mathbb{R}^d)$ and a.e. $t \in (0,T)$, we have

$$\begin{aligned} &\partial G(t,w,z,u,v) \subset \\ &\subset \{(\overline{u},\overline{v}) \in L^2(\Gamma_C;\mathbb{R}^d)^2 \ \mid \ (\overline{u}(x),\overline{v}(x)) \in \partial g(x,t,w(x),z(x),u(x),v(x)) \text{ a.e. } x \in \Gamma_C\}. \end{aligned} \tag{86}$$

Hence, by $H(g)_1$(iii), we thus obtain that for all $(\overline{u},\overline{v}) \in \partial G(t,w,z,u,v)$, $\overline{u}$, $\overline{v} \in L^2(\Gamma_C;\mathbb{R}^d)$, we have

$$\|(\overline{u}(x),\overline{v}(x))\|_{(\mathbb{R}^d)^2} \le c_{g0} + c_{g1}(\|w(x)\| + \|u(x)\|) + c_{g2}(\|z(x)\| + \|v(x)\|)$$

for a.e. $x \in \Gamma_C$. Hence

$$\|(\overline{u},\overline{v})\|_{L^2(\Gamma_C;\mathbb{R}^d)^2} \le \sqrt{5}\,\left(c_{g0}\sqrt{m(\Gamma_C)} + c_{g1}(\|w\| + \|u\|) + c_{g2}(\|z\| + \|v\|)\right)$$

which entails that the condition $H(G)_1$(iii) holds with the aformentioned constants c_{g0}, c_{g1} and c_{g2}.

Next, by the Fatou lemma (cf. Lemma 80), we have

$$\begin{aligned}
&G^0(t,w,z,u,v;\overline{u},\overline{v}) = \\
&\quad= \limsup_{(u',v')\to(u,v),\ \lambda\downarrow 0} \frac{1}{\lambda}\Big(G(t,w,z,(u',v')+\lambda(\overline{u},\overline{v})) - G(t,w,z,u',v')\Big) = \\
&\quad= \limsup_{(u',v')\to(u,v),\ \lambda\downarrow 0} \int_{\Gamma_C} \frac{1}{\lambda}\Big(g(x,t,w(x),z(x),u'(x)+\lambda\overline{u}(x),v'(x)+\lambda\overline{v}(x)) - \\
&\qquad\qquad -g(x,t,w(x),z(x),u'(x),v'(x))\Big)\,d\Gamma \le \\
&\quad\le \int_{\Gamma_C} \limsup_{(u',v')\to(u,v),\ \lambda\downarrow 0} \frac{1}{\lambda}\Big(g(x,t,w(x),z(x),u'(x)+\lambda\overline{u}(x),v'(x)+\lambda\overline{v}(x)) - \\
&\qquad\qquad -g(x,t,w(x),z(x),u'(x),v'(x))\Big)\,d\Gamma = \\
&\quad= \int_{\Gamma_C} g^0(x,t,w(x),z(x),u(x),v(x);\overline{u}(x),\overline{v}(x))\,d\Gamma
\end{aligned}$$

for all w, z, u, v, $\overline{u}$, $\overline{v} \in L^2(\Gamma_C;\mathbb{R}^d)$, a.e. $t \in (0,T)$, which implies (85).

Next, we assume in addition that $g(x,t,\zeta,\rho,\cdot,\cdot)$ is regular in the sense of Clarke. Again by exploiting the Fatou lemma and (85), we obtain

$$\begin{aligned}
&G^0(t,w,z,u,v;\overline{u},\overline{v}) \ge \liminf_{\lambda\downarrow 0} \frac{1}{\lambda}\Big(G(t,w,z,(u,v)+\lambda(\overline{u},\overline{v})) - G(t,w,z,u,v)\Big) = \\
&\quad= \liminf_{\lambda\downarrow 0} \int_{\Gamma_C} \frac{1}{\lambda}\Big(g(x,t,w(x),z(x),u(x)+\lambda\overline{u}(x),v(x)+\lambda\overline{v}(x)) - \\
&\qquad\qquad -g(x,t,w(x),z(x),u(x),v(x))\Big)\,d\Gamma \ge \\
&\quad\ge \int_{\Gamma_C} \liminf_{\lambda\downarrow 0} \frac{1}{\lambda}\Big(g(x,t,w(x),z(x),u(x)+\lambda\overline{u}(x),v(x)+\lambda\overline{v}(x)) - \\
&\qquad\qquad -g(x,t,w(x),z(x),u(x),v(x))\Big)\,d\Gamma = \\
&\quad= \int_{\Gamma_C} g'_{\xi\eta}(x,t,w(x),z(x),u(x),v(x);\overline{u}(x),\overline{v}(x))\,d\Gamma = \\
&\quad= \int_{\Gamma_C} g^0(x,t,w(x),z(x),u(x),v(x);\overline{u}(x),\overline{v}(x))\,d\Gamma \ge \\
&\quad\ge G^0(t,w,z,u,v;\overline{u},\overline{v})
\end{aligned}$$

for all w, z, u, v, $\overline{u}$, $\overline{v} \in L^2(\Gamma_C;\mathbb{R}^d)$, a.e. $t \in (0,T)$. Hence $G'_{(u,v)}(t,w,z,u,v;\overline{u},\overline{v})$ exists and

$$G'_{(u,v)}(t,w,z,u,v;\overline{u},\overline{v}) = G^0(t,w,z,u,v;\overline{u},\overline{v})$$

which means that $G(t,w,z,\cdot,\cdot)$ is regular for all w, $z \in L^2(\Gamma_C;\mathbb{R}^d)$ and a.e. $t \in (0,T)$. The above also implies that (85) holds with equality.

When $-g(x,t,\zeta,\rho,\cdot,\cdot)$ is regular in the sense of Clarke, we proceed analogously as above and deduce the regularity of $-G(t,w,z,\cdot,\cdot)$. From the property

$$(-G)^0(t,w,z,u,v;\overline{u},\overline{v}) = G^0(t,w,z,u,v;-\overline{u},-\overline{v})$$

all w, z, u, v, $\overline{u}$, $\overline{v} \in L^2(\Gamma_C;\mathbb{R}^d)$, for a.e. $t \in (0,T)$ (cf. Proposition 15(i)), we again get the equality in (85).

Finally, we suppose the hypotheses $H(g)_2$ and $H(g)_{reg}$. Let $t \in (0,T)$, w, z, u, v, $\overline{u}$, $\overline{v} \in L^2(\Gamma_C;\mathbb{R}^d)$ and $\{w_n\}$, $\{z_n\}$, $\{u_n\}$, $\{v_n\}$ be sequences in $L^2(\Gamma_C;\mathbb{R}^d)$ such that $w_n \to w$, $z_n \to z$, $u_n \to u$ and $v_n \to v$ in $L^2(\Gamma_C;\mathbb{R}^d)$. We may assume by passing to subsequences, if necessary, that $w_n(x) \to w(x)$, $z_n(x) \to z(x)$, $u_n(x) \to u(x)$ and $v_n(x) \to v(x)$ in $\mathbb{R}^d$ for a.e. $x \in \Gamma_C$, $\|w_n(x)\| \le w_0(x)$, $\|z_n(x)\| \le z_0(x)$, $\|u_n(x)\| \le u_0(x)$ and $\|v_n(x)\| \le v_0(x)$ with w_0, z_0, u_0, $v_0 \in L^2(\Gamma_C;\mathbb{R}^d)$. By the Fatou lemma and $H(g)_2$, we obtain

$$\begin{aligned}
&\limsup G^0(t,w_n,z_n,u_n,v_n,\overline{u},\overline{v}) = \\
&\quad = \limsup \int_{\Gamma_C} g^0(x,t,w_n(x),z_n(x),u_n(x),v_n(x);\overline{u}(x),\overline{v}(x))\, d\Gamma \le \\
&\quad \le \int_{\Gamma_C} \limsup g^0(x,t,w_n(x),z_n(x),u_n(x),v_n(x);\overline{u}(x),\overline{v}(x))\, d\Gamma \le \\
&\quad \le \int_{\Gamma_C} g^0(x,t,w(x),z(x),u(x),v(x);\overline{u}(x),\overline{v}(x))\, d\Gamma = G^0(t,w,z,u,v,\overline{u},\overline{v})
\end{aligned}$$

for all $\overline{u}$, $\overline{v} \in L^2(\Gamma_C;\mathbb{R}^d)$ and a.e. $t \in (0,T)$. This means that $G^0(t,\cdot,\cdot,\cdot,\cdot,\overline{u},\overline{v})$ is upper semicontinuous on $L^2(\Gamma_C;\mathbb{R}^d)^4$ for all $\overline{u}$, $\overline{v} \in L^2(\Gamma_C;\mathbb{R}^d)$ and a.e. $t \in (0,T)$. This completes the proof that the functional G satisfies $H(G)_2$. The proof of the lemma is done. □

Now we are in a position to carry out the last step of the construction of the multifunction which will appear in the evolution inclusion. To this end, let $Z = H^{1/2}(\Omega;\mathbb{R}^d)$ and $\gamma\colon Z \to L^2(\Gamma_C;\mathbb{R}^d)$ be the trace operator. Let $\gamma^*\colon L^2(\Gamma_C;\mathbb{R}^d) \to Z^*$ stand for the adjoint operator to γ. We introduce the following operators

$$\begin{array}{lll}
R\colon Z \times Z \to L^2(\Gamma_C;\mathbb{R}^d)^2 & \text{by } \; R(z_1,z_2) = (\gamma z_1, \gamma z_2) & \text{for all } z_1, z_2 \in Z, \\
R^*\colon L^2(\Gamma_C;\mathbb{R}^d)^2 \to Z^* \times Z^* & \text{by } \; R^*(u,v) = (\gamma^* u, \gamma^* v) & \text{for all } u, v \in L^2(\Gamma_C;\mathbb{R}^d), \\
S\colon Z^* \times Z^* \to Z^* & \text{by } \; S(z_1^*,z_2^*) = z_1^* + z_2^* & \text{for all } z_1^*, z_2^* \in Z^*.
\end{array}$$

We define the following multivalued mapping $F\colon (0,T) \times V \times V \to 2^{Z^*}$ by

$$F(t,u,v) = S\, R^*\, \partial G(t,R(u,v),R(u,v)) \;\text{ for } u,v \in V, \text{ a.e. } t \in (0,T), \qquad (87)$$

where ∂G denotes the Clarke subdifferential of the functional $G = G(t, w, z, u, v)$ defined by (84) with respect to (u, v).

Before we establish the properties of the multifunction F given by (87), we need the following auxiliary lemma. Recall that for a Banach space X, the symbol w-X stands for X endowed with the weak topology.

LEMMA 59 *Let (Ω, Σ) be a mesurable space, Y_1, Y_2 be separable Banach spaces, $A \in \mathcal{L}(Y_1, Y_2)$ and let $G\colon \Omega \to \mathcal{P}_{wkc}(Y_1)$ be measurable. Then the multifunction $F\colon \Omega \to \mathcal{P}_{wkc}(Y_2)$ given by $F(\omega) = AG(\omega)$ for $\omega \in \Omega$ is measurable.*

Proof. First we recall that if $A \in \mathcal{L}(Y_1, Y_2)$, then $A \in \mathcal{L}(w\text{-}Y_1, w\text{-}Y_2)$. Hence it follows that F is $\mathcal{P}_{wkc}(Y_2)$-valued. Given an open set $U \subset Y_2$, we will show that $F^-(U) = \{\omega \in \Omega \mid F(\omega) \cap U \neq \emptyset\} \in \Sigma$. From the definition of F, we have $F^-(U) = \{\omega \in \Omega \mid G(\omega) \cap A^{-1}(U) \neq \emptyset\} = G^-(U')$, where $U' = A^{-1}(U)$. Since the mapping $A\colon Y_1 \to Y_2$ is continuous, for every open set $U \subset Y_2$, the inverse image $A^{-1}(U) \subset Y_1$ is an open set. From the definition of measurability of G, we have $G^-(U') \in \Sigma$. Therefore $F^-(U) \in \Sigma$ which implies that F is measurable as claimed. $\square$

LEMMA 60 *If the hypothesis $H(G)_2$ holds, then the multifunction $F\colon (0, T) \times V \times V \to 2^{Z^*}$ defined by (87) satisfies $H(F)$ with $d_0(t) = c_{G0}\|\gamma\|$, $d_1 = 2c_e c_{G1}\|\gamma\|^2$ and $d_2 = 2c_e c_{G2}\|\gamma\|^2$.*

Proof. The fact that the mapping F has nonempty and convex values follows from the nonemptiness and convexity of values of the Clarke subdifferential of G (cf. Proposition 15(iv)). Because the values of the subdifferential $\partial G(t, w, z, \cdot, \cdot)$ are weakly closed subsets of $L^2(\Gamma_C; \mathbb{R}^d)$ (which follows from Proposition 15(v)), using $H(G)_1$, we can also easily check that the mapping F has closed values in Z^*.

To show that $F(\cdot, u, v)$ is measurable on $(0, T)$ for all u, $v \in V$, let w, z, $\overline{u}$, $\overline{v} \in L^2(\Gamma_C; \mathbb{R}^d)$. Since, by the hypothesis $H(G)_1$, $G(\cdot, w, z, \overline{u}, \overline{v})$ is measurable and $G(t, w, z, \cdot, \cdot)$ is locally Lipschitz on $L^2(\Gamma_C; \mathbb{R}^d)^2$ (being Lipschitz continuous on bounded subsets) for a.e. $t \in (0, T)$, according to Lemma 35, we know that

$$(0, T) \times L^2(\Gamma_C; \mathbb{R}^d)^2 \ni (t, \overline{u}, \overline{v}) \mapsto \partial G(t, w, z, \overline{u}, \overline{v}) \subset L^2(\Gamma_C; \mathbb{R}^d)^2$$

is measurable. Hence, by Lemma 69, we infer that also the multifunction $(0, T) \ni t \mapsto \partial G(t, w, z, \overline{u}, \overline{v})$ is measurable, and clearly it is $\mathcal{P}_{wkc}(L^2(\Gamma_C; \mathbb{R}^d)^2)$-valued. On the other hand, we can readily verify that $SR^*\colon L^2(\Gamma_C; \mathbb{R}^d)^2 \to Z^*$ is a linear continuous operator. These properties ensure the applicability of Lemma 59. So we have that $(0, T) \ni t \mapsto SR^*\partial G(t, w, z, \overline{u}, \overline{v})$ is measurable. As a consequence the multifunction $F(\cdot, u, v)$ is measurable for all u, $v \in V$.

Next we will prove the upper semicontinuity of $F(t, \cdot, \cdot)$ for a.e. $t \in (0, T)$. According to Remark 78, we show that for every weakly closed subset K of Z^*, the set

$$F^-(K) = \{(u, v) \in V \times V \mid F(t, u, v) \cap K \neq \emptyset\}$$

is closed in $Z \times Z$. Let $t \in (0,T)$, $\{(u_n, v_n)\} \subset F^-(K)$ and $(u_n, v_n) \to (u, v)$ in $Z \times Z$. We can find $\zeta_n \in F(t, u_n, v_n) \cap K$ for $n \in \mathbb{N}$. By the definition of F, we have $\zeta_n = \zeta_n^1 + \zeta_n^2$, $(\zeta_n^1, \zeta_n^2) = (\gamma^* \eta_n^1, \gamma^* \eta_n^2)$ with $(\eta_n^1, \eta_n^2) \in L^2(\Gamma_C; \mathbb{R}^d)$ and

$$(\eta_n^1, \eta_n^2) \in \partial G(t, \gamma u_n, \gamma v_n, \gamma u_n, \gamma v_n) \;\; \text{for a.e. } t \in (0,T). \tag{88}$$

Using the continuity of the trace operator (cf. e.g. Theorem 1.5.1.2 in Grisvard [32]), we have

$$\gamma u_n \to \gamma u, \;\; \gamma v_n \to \gamma v \;\; \text{in } L^2(\Gamma_C; \mathbb{R}^d).$$

Since by $H(G)_1$(iii) the operator $\partial G(t, \cdot, \cdot, \cdot, \cdot)$ is bounded (it maps bounded sets into bounded sets), from (88), it follows that the sequence $\{(\eta_n^1, \eta_n^2)\}$ remains in a bounded subset of $L^2(\Gamma_C; \mathbb{R}^d)^2$. Thus, by passing to a subsequence, if necessary, we may suppose that

$$\eta_n^1 \to \eta^1, \;\; \eta_n^2 \to \eta^2 \;\; \text{weakly in } L^2(\Gamma_C; \mathbb{R}^d)$$

for some η^1, $\eta^2 \in L^2(\Gamma_C; \mathbb{R}^d)$. Now, we will use the fact that the graph of $\partial G(t, \cdot, \cdot, \cdot, \cdot)$ is closed in $L^2(\Gamma_C; \mathbb{R}^d)^4 \times (w\text{-}L^2(\Gamma_C; \mathbb{R}^d)^2)$-topology for a.e. $t \in (0,T)$, which will be showed at the end of this proof. Hence and from (88), we obtain

$$(\eta^1, \eta^2) \in \partial G(t, \gamma u, \gamma v, \gamma u, \gamma v).$$

Furthermore, since $\{\zeta_n\}$ also remains in a bounded subset of Z^*, we may assume that $\zeta_n \to \zeta$ weakly in Z^*. Because $\zeta_n \in K$ and K is weakly closed in Z^*, it follows that $\zeta \in K$. By the continuity and linearity of the operator γ^*, we obtain

$$\gamma^* \eta_n^1 \to \gamma^* \eta^1, \;\; \gamma^* \eta_n^2 \to \gamma^* \eta^2 \;\; \text{weakly in } Z^*.$$

Hence

$$\zeta_n = \gamma^* \eta_n^1 + \gamma^* \eta_n^2 \to \gamma^* \eta^1 + \gamma^* \eta^2 = \zeta^1 + \zeta^2 \;\; \text{weakly in } Z^*$$

and $\zeta = \zeta^1 + \zeta^2$, where $(\zeta^1, \zeta^2) = (\gamma^* \eta^1, \gamma^* \eta^2)$ and $(\eta^1, \eta^2) \in \partial G(t, \gamma u, \gamma v, \gamma u, \gamma v)$. This, by the definition of F implies that $\zeta \in F(t, u, v)$. As a consequence, once $\zeta \in K$, we know that $F^-(K)$ is closed in $Z \times Z$. Hence $H(F)$(ii) follows.

Next, we show that F satisfies $H(F)$(iii). Let $t \in (0,T)$, u, $v \in V$ and $z^* \in Z^*$, $z^* \in F(t, u, v)$. The latter is equivalent to $z^* = z_1^* + z_2^*$, z_1^*, $z_2^* \in Z^*$, $(z_1^*, z_2^*) = (\gamma^* \eta_1, \gamma^* \eta_2)$ where η_1, $\eta_2 \in L^2(\Gamma_C; \mathbb{R}^d)$ and $(\eta_1, \eta_2) \in \partial G(t, \gamma u, \gamma v, \gamma u, \gamma v)$. Using the estimate $H(G)_1$(iii), we have

$$\begin{aligned} \|z^*\|_{Z^*} &= \|\gamma^*(\eta_1 + \eta_2)\|_{Z^*} \le \|\gamma^*\| \|\eta_1 + \eta_2\|_{L^2(\Gamma_C; \mathbb{R}^d)} \le \\ &\le \|\gamma^*\| \left(c_{G0} + 2\, c_{G1} \, \|\gamma u\|_{L^2(\Gamma_C; \mathbb{R}^d)} + 2\, c_{G2} \, \|\gamma v\|_{L^2(\Gamma_C; \mathbb{R}^d)} \right) \le \\ &\le \|\gamma^*\| \left(c_{G0} + 2\, c_{G1} \, \|\gamma\| \|u\|_Z + 2\, c_{G2} \, \|\gamma\| \|v\|_Z \right) \le \\ &\le c_{G0} \|\gamma\| + 2 c_e c_{G1} \|\gamma\|^2 \|u\| + 2 c_e c_{G2} \|\gamma\|^2 \|v\| \end{aligned}$$

where $\|\gamma^*\| = \|\gamma\|$ denotes the norm in $\mathcal{L}(L^2(\Gamma_C; \mathbb{R}^d), Z^*)$ and $c_e > 0$ is the embedding constant of V into Z. This implies that F satisfies $H(F)$(iii) with $d_0(t) = c_{G0} \|\gamma\|$, $d_1 = 2 c_e c_{G1} \|\gamma\|^2$ and $d_2 = 2 c_e c_{G2} \|\gamma\|^2$.

To complete the proof, it is enough to show that the graph of $\partial G(t,\cdot,\cdot,\cdot,\cdot)$ is closed in $L^2(\Gamma_C;\mathbb{R}^d)^4 \times (w\text{-}L^2(\Gamma_C;\mathbb{R}^d)^2)$-topology for a.e. $t \in (0,T)$. This is a simple consequence of $H(G)_2$. Indeed, let $t \in (0,T)$, $\{w_n\}$, $\{z_n\}$, $\{u_n\}$, $\{v_n\}$ be sequences in $L^2(\Gamma_C;\mathbb{R}^d)$ such that $w_n \to w$, $z_n \to z$, $u_n \to u$, $v_n \to v$ in $L^2(\Gamma_C;\mathbb{R}^d)$, let $\{(\eta_n^1,\eta_n^2)\} \subset L^2(\Gamma_C;\mathbb{R}^d)^2$, $(\eta_n^1,\eta_n^2) \to (\eta^1,\eta^2)$ weakly in $L^2(\Gamma_C;\mathbb{R}^d)^2$ and $(\eta_n^1,\eta_n^2) \in \partial G(t,w_n,z_n,u_n,v_n)$. The latter means that

$$\langle (\eta_n^1,\eta_n^2),(\overline{u},\overline{v})\rangle_{L^2(\Gamma_C;\mathbb{R}^d)^2} \le G^0(t,w_n,z_n,u_n,v_n;\overline{u},\overline{v}) \ \text{ for all } \overline{u},\overline{v} \in L^2(\Gamma_C;\mathbb{R}^d).$$

The hypothesis $H(G)_2$ implies

$$\langle (\eta^1,\eta^2),(\overline{u},\overline{v})\rangle_{L^2(\Gamma_C;\mathbb{R}^d)^2} \le \limsup G^0(t,w_n,z_n,u_n,v_n;\overline{u},\overline{v}) \le G^0(t,w,z,u,v;\overline{u},\overline{v})$$

for all $\overline{u}$, $\overline{v} \in L^2(\Gamma_C;\mathbb{R}^d)$ which entails $(\eta^1,\eta^2) \in \partial G(t,w,z,u,v)$. The above finishes the proof that the graph is closed. This argument completes the proof of the lemma. □

In order to prove that the multifunction F defined by (87) satisfies the hypothesis $H(F)_1$, we need additional conditions on the superpotentials j_k for $k = 1,\ldots,4$.

<u>$H(j_1)_3$</u>: $j_1\colon \Gamma_C \times (0,T) \times (\mathbb{R}^d)^2 \times \mathbb{R} \to \mathbb{R}$ is such that

$$|\partial j_1(x,t,\zeta_1,\rho_1,r_1) - \partial j_1(x,t,\zeta_2,\rho_2,r_2)| \le L_1 \left(\|\zeta_1-\zeta_2\| + \|\rho_1-\rho_2\| + |r_1-r_2|\right)$$

for all ζ_1, ζ_2, ρ_1, $\rho_2 \in \mathbb{R}^d$, r_1, $r_2 \in \mathbb{R}$, a.e. $(x,t) \in \Gamma_C \times (0,T)$ with a constant $L_1 \ge 0$.

<u>$H(j_2)_3$</u>: $j_2\colon \Gamma_C \times (0,T) \times (\mathbb{R}^d)^2 \times \mathbb{R} \to \mathbb{R}$ is such that

$$\begin{aligned}&\left(\partial j_2(x,t,\zeta_1,\rho_1,r_1) - \partial j_2(x,t,\zeta_2,\rho_2,r_2)\right)(r_1-r_2) \ge \\ &\qquad \ge \ -L_2 \left(\|\zeta_1-\zeta_2\| + \|\rho_1-\rho_2\| + |r_1-r_2|\right)|r_1-r_2|\end{aligned}$$

for all ζ_1, ζ_2, ρ_1, $\rho_2 \in \mathbb{R}^d$, r_1, $r_2 \in \mathbb{R}$, a.e. $(x,t) \in \Gamma_C \times (0,T)$ with a constant $L_2 \ge 0$.

<u>$H(j_3)_3$</u>: $j_3\colon \Gamma_C \times (0,T) \times (\mathbb{R}^d)^3 \to \mathbb{R}$ is such that

$$\|\partial j_3(x,t,\zeta_1,\rho_1,\theta_1) - \partial j_3(x,t,\zeta_2,\rho_2,\theta_2)\| \le L_3 \left(\|\zeta_1-\zeta_2\| + \|\rho_1-\rho_2\| + \|\theta_1-\theta_2\|\right)$$

for all ζ_1, ζ_2, ρ_1, ρ_2, θ_1, $\theta_2 \in \mathbb{R}^d$, a.e. $(x,t) \in \Gamma_C \times (0,T)$ with a constant $L_3 \ge 0$.

<u>$H(j_4)_3$</u>: $j_4\colon \Gamma_C \times (0,T) \times (\mathbb{R}^d)^3 \to \mathbb{R}$ is such that

$$\begin{aligned}&\left(\partial j_4(x,t,\zeta_1,\rho_1,\theta_1) - \partial j_4(x,t,\zeta_2,\rho_2,\theta_2), \theta_1-\theta_2\right) \ge \\ &\qquad \ge \ -L_4 \left(\|\zeta_1-\zeta_2\| + \|\rho_1-\rho_2\| + \|\theta_1-\theta_2\|\right)\|\theta_1-\theta_2\|\end{aligned}$$

for all ζ_1, ζ_2, ρ_1, ρ_2, θ_1, $\theta_2 \in \mathbb{R}^d$, a.e. $(x,t) \in \Gamma_C \times (0,T)$ with a constant $L_4 \ge 0$.

In the conditions $H(j_k)_3$ for $k = 1,\ldots,4$, ∂j_k denotes the subdifferential of j_k with respect to its last variable.

Remark 61 *The hypothesis $H(j_2)_3$ (and $H(j_4)_3$) has been introduced and used earlier in [60] (under the name of relaxed monotonicity condition) in the case when j_2 (and j_4) does not depend on the variables ζ and ρ, i.e. when this condition is of the form*

$$(\partial j_2(x,t,r_1) - \partial j_2(x,t,r_2))\,(r_1 - r_2) \geq -L_2|r_1 - r_2|^2$$

for all r_1, $r_2 \in \mathbb{R}$, a.e. $(x,t) \in \Gamma_C \times (0,T)$ with $L_2 \geq 0$.

Lemma 62 *Assume that the hypotheses $H(j_k)_2$ hold for $k = 1, \ldots, 4$, and that either*

$$j_k(x,t,\zeta,\rho,\cdot) \text{ are regular and } j_k \text{ satisfy } H(j_k)_3 \text{ for } k = 1,\ldots,4 \tag{89}$$

or

$$-j_k(x,t,\zeta,\rho,\cdot) \text{ are regular and } -j_k \text{ satisfy } H(j_k)_3 \text{ for } k = 1,\ldots,4. \tag{90}$$

Then the multifunction $F\colon (0,T) \times V \times V \to 2^{Z^*}$ defined by (87) with the functional G given by (84) and its integrand g defined by (80), satisfies the condition $H(F)_1$ with $m_2 = c_e k_1 \|\gamma\|^2$ and $m_3 = c_e k_2 \|\gamma\|^2$.

Proof. It is clear that under the hypotheses, the condition $H(j)_{reg}$ holds. By Lemma 57, under $H(j_k)_2$ for $k = 1, \ldots, 4$ and $H(j)_{reg}$, we know that the integrand g given by (80) satisfies $H(g)_2$ and $H(g)_{reg}$. Hence by Lemma 58, it follows that the functional G given by (84) satisfies $H(G)_2$. Using Lemma 60, under $H(G)_2$, we obtain that the multifunction F satisfies $H(F)$.

Now, it is enough to prove that the multifunction F satisfies $H(F)_1$(iv). We suppose (89), the case when (90) holds can be treated analogously. We show that the following inequality holds

$$\begin{aligned}(\partial g(x,t,\xi_1,\eta_1,\xi_1,\eta_1) - \partial g(x,t,\xi_2,\eta_2,\xi_2,\eta_2), (\eta_1-\eta_2,\eta_1-\eta_2))_{\mathbb{R}^d\times\mathbb{R}^d} \geq \\ \geq\ -k_1\|\eta_1-\eta_2\|^2 - k_2\|\eta_1-\eta_2\|\|\xi_1-\xi_2\|\end{aligned} \tag{91}$$

for all ξ_i, $\eta_i \in \mathbb{R}^d$, $i = 1$, 2, a.e. $(x,t) \in \Gamma_C \times (0,T)$ with k_1, $k_2 \geq 0$. Under (89), it follows that $g(x,t,\zeta,\rho,\cdot,\cdot)$ is regular for all ζ, $\rho \in \mathbb{R}^d$, a.e. $(x,t) \in \Gamma_C \times (0,T)$. Using this regularity, by Propositions 28(b) and 29, we have

$$\begin{aligned}
\partial g(x,t,\zeta,\rho,\xi,\eta) &\subset \partial_\xi g(x,t,\zeta,\rho,\xi,\eta) \times \partial_\eta g(x,t,\zeta,\rho,\xi,\eta) = \\
&= \partial_\xi\Big(j_1(x,t,\zeta,\rho,N_1\xi) + j_3(x,t,\zeta,\rho,N_2\xi)\Big)\times \\
&\qquad \times \partial_\eta\Big(j_2(x,t,\zeta,\rho,N_1\eta) + j_4(x,t,\zeta,\rho,N_2\eta)\Big) = \\
&= \Big(N_1^*\partial j_1(x,t,\zeta,\rho,N_1\xi) + N_2^*\partial j_3(x,t,\zeta,\rho,N_2\xi)\Big)\times \\
&\qquad \times \Big(N_1^*\partial j_2(x,t,\zeta,\rho,N_1\eta) + N_2^*\partial j_4(x,t,\zeta,\rho,N_2\eta)\Big) = \\
&= \Big(\partial j_1(x,t,\zeta,\rho,\xi_\nu)\,\nu + (\partial j_3(x,t,\zeta,\rho,\xi_\tau))_\tau\Big)\times \\
&\qquad \times \Big(\partial j_2(x,t,\zeta,\rho,\eta_\nu)\,\nu + (\partial j_4(x,t,\zeta,\rho,\eta_\tau))_\tau\Big),
\end{aligned}$$

where ∂g denotes the subdifferential of g with respect to (ξ, η), $N_1 \in \mathcal{L}(\mathbb{R}^d, \mathbb{R})$, $N_2 \in \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d)$ are operators defined by

$$N_1 \xi = \xi_\nu, \quad N_2 \xi = \xi_\tau \quad \text{for all } \xi \in \mathbb{R}^d$$

with their adjoints $N_1^* \in \mathcal{L}(\mathbb{R}, \mathbb{R}^d)$, $N_2^* \in \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d)$ given by

$$N_1^* r = r\,\nu, \quad N_2^* \xi = \xi_\tau \quad \text{for all } r \in \mathbb{R},\ \xi \in \mathbb{R}^d,$$

i.e. $N_2^* = N_2$. Let $(\overline{\chi}_i, \overline{\sigma}_i) \in \partial g(x, t, \xi_i, \eta_i, \xi_i, \eta_i)$, $(x, t) \in \Gamma_C \times (0, T)$ with ξ_i, $\eta_i \in \mathbb{R}^d$, $i = 1, 2$. For simplicity of notation we omit the dependence on (x, t). Then

$$\begin{aligned} \overline{\chi}_1 &\in \partial j_1(x, t, \xi_1, \eta_1, \xi_{1\nu})\,\nu + (\partial j_3(x, t, \xi_1, \eta_1, \xi_{1\tau}))_\tau\,, \\ \overline{\sigma}_1 &\in \partial j_2(x, t, \xi_1, \eta_1, \eta_{1\nu})\,\nu + (\partial j_4(x, t, \xi_1, \eta_1, \eta_{1\tau}))_\tau \end{aligned}$$

and

$$\begin{aligned} \overline{\chi}_2 &\in \partial j_1(x, t, \xi_2, \eta_2, \xi_{2\nu})\,\nu + (\partial j_3(x, t, \xi_2, \eta_2, \xi_{2\tau}))_\tau\,, \\ \overline{\sigma}_2 &\in \partial j_2(x, t, \xi_2, \eta_2, \eta_{2\nu})\,\nu + (\partial j_4(x, t, \xi_2, \eta_2, \eta_{2\tau}))_\tau\,, \end{aligned}$$

which means that

$$\begin{aligned} \overline{\chi}_1 &= \alpha_1\,\nu + \gamma_{1\tau}, \qquad \overline{\sigma}_1 = \beta_1\,\nu + \delta_{1\tau}, \\ \overline{\chi}_2 &= \alpha_2\,\nu + \gamma_{2\tau}, \qquad \overline{\sigma}_2 = \beta_2\,\nu + \delta_{2\tau} \end{aligned}$$

with

$$\begin{aligned} \alpha_i &\in \partial j_1(x, t, \xi_i, \eta_i, \xi_{i\nu}), \qquad \beta_i \in \partial j_2(x, t, \xi_i, \eta_i, \eta_{i\nu}), \\ \gamma_i &\in \partial j_3(x, t, \xi_i, \eta_i, \xi_{i\tau}), \qquad \delta_i \in \partial j_4(x, t, \xi_i, \eta_i, \eta_{i\tau}) \end{aligned}$$

for $i = 1, 2$. By the hypotheses $H(j_k)_3$ for $k = 1, \ldots, 4$, we have

$$\begin{aligned} &|\,(\partial j_1(x, t, \xi_1, \eta_1, \xi_{1\,\nu}) - \partial j_1(x, t, \xi_2, \eta_2, \xi_{2\,\nu}))\,(\eta_{1\,\nu} - \eta_{2\,\nu})| \le \\ &\quad \le\ L_1\,(\|\xi_1 - \xi_2\| + \|\eta_1 - \eta_2\| + |\xi_{1\,\nu} - \xi_{2\,\nu}|)\,|\eta_{1\,\nu} - \eta_{2\,\nu}| \le \\ &\quad \le\ L_1\,(2\|\xi_1 - \xi_2\| + \|\eta_1 - \eta_2\|)\,\|\eta_1 - \eta_2\|, \end{aligned}$$

$$\begin{aligned} &(\partial j_2(x, t, \xi_1, \eta_1, \eta_{1\,\nu}) - \partial j_2(x, t, \xi_2, \eta_2, \eta_{2\,\nu}))\,(\eta_{1\,\nu} - \eta_{2\,\nu}) \ge \\ &\quad \ge\ -L_2\,(\|\xi_1 - \xi_2\| + \|\eta_1 - \eta_2\| + |\eta_{1\,\nu} - \eta_{2\,\nu}|)\,|\eta_{1\,\nu} - \eta_{2\,\nu}| \ge \\ &\quad \ge\ -L_2\,(\|\xi_1 - \xi_2\| + 2\|\eta_1 - \eta_2\|)\,\|\eta_1 - \eta_2\|, \end{aligned}$$

$$\begin{aligned} &|\,(\partial j_3(x, t, \xi_1, \eta_1, \xi_{1\,\tau}) - \partial j_3(x, t, \xi_2, \eta_2, \xi_{2\,\tau}), \eta_{1\,\tau} - \eta_{2\,\tau})_{\mathbb{R}^d}\,| \le \\ &\quad \le\ L_3\,(\|\xi_1 - \xi_2\| + \|\eta_1 - \eta_2\| + |\xi_{1\,\tau} - \xi_{2\,\tau}|)\,\|\eta_{1\,\tau} - \eta_{2\,\tau}\| \le \\ &\quad \le\ L_3\,(2\|\xi_1 - \xi_2\| + \|\eta_1 - \eta_2\|)\,\|\eta_1 - \eta_2\| \end{aligned}$$

and

$$
\begin{aligned}
&(\partial j_4(x,t,\xi_1,\eta_1,\eta_{1\tau}) - \partial j_4(x,t,\xi_2,\eta_2,\eta_{2\tau}), \eta_{1\tau} - \eta_{2\tau})_{\mathbb{R}^d} \geq \\
&\quad \geq \; -L_4 \left(\|\xi_1 - \xi_2\| + \|\eta_1 - \eta_2\| + |\eta_{1\tau} - \eta_{2\tau}| \right) \|\eta_{1\tau} - \eta_{2\tau}\| \geq \\
&\quad \geq \; -L_4 \left(\|\xi_1 - \xi_2\| + 2\|\eta_1 - \eta_2\| \right) \|\eta_1 - \eta_2\|.
\end{aligned}
$$

Using the last four inequalities and the fact that $(\zeta_\tau, \rho)_{\mathbb{R}^d} = (\zeta, \rho_\tau)_{\mathbb{R}^d}$ for all ζ, $\rho \in \mathbb{R}^d$, we calculate

$$
\begin{aligned}
&(\partial g(x,t,\xi_1,\eta_1,\xi_1,\eta_1) - \partial g(x,t,\xi_2,\eta_2,\xi_2,\eta_2), (\eta_1 - \eta_2, \eta_1 - \eta_2))_{\mathbb{R}^d \times \mathbb{R}^d} = \\
&= \; ((\overline{\chi}_1, \overline{\sigma}_1) - (\overline{\chi}_2, \overline{\sigma}_2), (\eta_1 - \eta_2, \eta_1 - \eta_2))_{\mathbb{R}^d \times \mathbb{R}^d} = \\
&= \; (\overline{\chi}_1 - \overline{\chi}_2, \eta_1 - \eta_2)_{\mathbb{R}^d} + (\overline{\sigma}_1 - \overline{\sigma}_2, \eta_1 - \eta_2)_{\mathbb{R}^d} = \\
&= \; (\alpha_1 \nu + \gamma_{1\tau} - \alpha_2 \nu - \gamma_{2\tau}, \eta_1 - \eta_2)_{\mathbb{R}^d} + (\beta_1 \nu + \delta_{1\tau} - \beta_2 \nu - \delta_{2\tau}, \eta_1 - \eta_2)_{\mathbb{R}^d} = \\
&= \; (\alpha_1 - \alpha_2)\,(\nu, \eta_1 - \eta_2)_{\mathbb{R}^d} + (\gamma_{1\tau} - \gamma_{2\tau}, \eta_1 - \eta_2)_{\mathbb{R}^d} + \\
&\qquad + (\beta_1 - \beta_2)\,(\nu, \eta_1 - \eta_2)_{\mathbb{R}^d} + (\delta_{1\tau} - \delta_{2\tau}, \eta_1 - \eta_2)_{\mathbb{R}^d} = \\
&= \; (\alpha_1 - \alpha_2)(\eta_{1\nu} - \eta_{2\nu}) + (\beta_1 - \beta_2)(\eta_{1\nu} - \eta_{2\nu}) + \\
&\qquad + (\gamma_1 - \gamma_2, \eta_{1\tau} - \eta_{2\tau})_{\mathbb{R}^d} + (\delta_1 - \delta_2, \eta_{1\tau} - \eta_{2\tau})_{\mathbb{R}^d} \geq \\
&\geq \; -L_1 \left(2\|\xi_1 - \xi_2\| + \|\eta_1 - \eta_2\| \right) \|\eta_1 - \eta_2\| - \\
&\quad\; -L_2 \left(\|\xi_1 - \xi_2\| + 2\|\eta_1 - \eta_2\| \right) \|\eta_1 - \eta_2\| - \\
&\quad\; -L_3 \left(2\|\xi_1 - \xi_2\| + \|\eta_1 - \eta_2\| \right) \|\eta_1 - \eta_2\| - \\
&\quad\; -L_4 \left(\|\xi_1 - \xi_2\| + 2\|\eta_1 - \eta_2\| \right) \|\eta_1 - \eta_2\| = \\
&= \; -k_1 \|\eta_1 - \eta_2\|^2 - k_2 \|\eta_1 - \eta_2\| \, \|\xi_1 - \xi_2\|
\end{aligned}
$$

with $k_1 = \max\{L_1, 2L_2, L_3, 2L_4\}$ and $k_2 = \max\{2L_1, L_2, 2L_3, L_4\}$. Hence the proof of the property (91) is complete.

Next we will prove that the subdifferential ∂G of the functional G defined by (84) satisfies the condition

$$
\begin{aligned}
&\langle \partial G(t,w_1,z_1,w_1,z_1) - \partial G(t,w_2,z_2,w_2,z_2), (z_1 - z_2, z_1 - z_2) \rangle_{L^2(\Gamma_C;\mathbb{R}^d)^2} \geq \\
&\quad \geq \; -k_1 \|z_1 - z_2\|^2_{L^2(\Gamma_C;\mathbb{R}^d)} - k_2 \|z_1 - z_2\|_{L^2(\Gamma_C;\mathbb{R}^d)} \|w_1 - w_2\|_{L^2(\Gamma_C;\mathbb{R}^d)} \qquad (92)
\end{aligned}
$$

for all w_i, $z_i \in L^2(\Gamma_C;\mathbb{R}^d)$, a.e. $t \in (0,T)$ with k_1, $k_2 \geq 0$, where ∂G denotes the subdifferential of $G(t,w,z,\cdot,\cdot)$. Similarly as in the proof of Lemma 58, (cf. (86)) and Theorem 2.7.5 of Clarke [21], we use the property that if

$$
(\overline{u}, \overline{v}) \in \partial G(t,w,z,u,v) \;\; \text{for a.e. } t \in (0,T),
$$

then

$$
(\overline{u}(x), \overline{v}(x)) \in \partial g(x,t,w(x),z(x),u(x),v(x)) \;\; \text{for a.e. } (x,t) \in \Gamma_C \times (0,T),
$$

for every w, z, u, v, $\overline{u}$, $\overline{v} \in L^2(\Gamma_C;\mathbb{R}^d)$. For the proof of (92), let w_i, z_i, $\overline{u}_i$, $\overline{v}_i \in L^2(\Gamma_C;\mathbb{R}^d)$ with $(\overline{u}_i,\overline{v}_i) \in \partial G(t,w_i,z_i,w_i,z_i)$ for $i = 1, 2$, a.e. $t \in (0,T)$. From the aformentioned property, we known that

$$(\overline{u}_i(x),\overline{v}_i(x)) \in \partial g(x,t,w_i(x),z_i(x),w_i(x),z_i(x))$$

for a.e. $(x,t) \in \Gamma_C \times (0,T)$. Exploiting the inequality (91), we have

$$\begin{aligned} &((\overline{u}_1(x),\overline{v}_1(x)) - (\overline{u}_2(x),\overline{v}_2(x)),(z_1(x)-z_2(x),z_1(x)-z_2(x)))_{\mathbb{R}^d\times\mathbb{R}^d} \geq \\ &\geq \ -k_1\,\|z_1(x)-z_2(x)\|^2 - k_2\,\|w_1(x)-w_2(x)\|\|z_1(x)-z_2(x)\| \end{aligned}$$

for a.e. $x \in \Gamma_C$. Integrating this inequality over Γ_C and applying the Hölder inequality, we obtain

$$\begin{aligned} &\langle(\overline{u}_1,\overline{v}_1) - (\overline{u}_2,\overline{v}_2),(z_1-z_2,z_1-z_2)\rangle_{L^2(\Gamma_C;\mathbb{R}^d)^2} = \\ &= \ \langle\overline{u}_1-\overline{u}_2,z_1-z_2\rangle_{L^2(\Gamma_C;\mathbb{R}^d)} + \langle\overline{v}_1-\overline{v}_2,z_1-z_2\rangle_{L^2(\Gamma_C;\mathbb{R}^d)} = \\ &= \ \int_{\Gamma_C}\Big((\overline{u}_1(x)-\overline{u}_2(x))\cdot(z_1(x)-z_2(x)) + (\overline{v}_1(x)-\overline{v}_2(x))\cdot(z_1(x)-z_2(x))\Big)\,d\Gamma \geq \\ &\geq \ -k_1\int_{\Gamma_C}\|z_1(x)-z_2(x)\|^2\,d\Gamma - k_2\int_{\Gamma_C}\|w_1(x)-w_2(x)\|\,\|z_1(x)-z_2(x)\|\,d\Gamma \geq \\ &\geq \ -k_1\,\|z_1-z_2\|^2_{L^2(\Gamma_C;\mathbb{R}^d)} - k_2\,\|z_1-z_2\|_{L^2(\Gamma_C;\mathbb{R}^d)}\|w_1-w_2\|_{L^2(\Gamma_C;\mathbb{R}^d)} \end{aligned}$$

which means that (92) is satisfied.

Finally we show that the multifunction F defined by (87) satisfies $H(F)_1$(iv). Let u_i, $v_i \in V$, $t \in (0,T)$ and $z_i \in F(t,u_i,v_i)$ for $i = 1, 2$. By the definition of F, we have

$$z_1 = a_1 + a_2,\ \ (a_1,a_2) = R^*(\eta_1,\eta_2) = (\gamma^*\eta_1,\gamma^*\eta_2),\ \ (\eta_1,\eta_2) \in \partial G(t,\gamma u_1,\gamma v_1,\gamma u_1,\gamma v_1),$$

$$z_2 = b_1 + b_2,\ \ \ \ (b_1,b_2) = R^*(\xi_1,\xi_2) = (\gamma^*\xi_1,\gamma^*\xi_2),\ \ (\xi_1,\xi_2) \in \partial G(t,\gamma u_2,\gamma v_2,\gamma u_2,\gamma v_2)$$

with a_i, $b_i \in Z^*$ and η_i, $\xi_i \in L^2(\Gamma_C;\mathbb{R}^d)$, $i = 1, 2$. Exploiting (92) and the continuity of the trace operator, we obtain

$$\begin{aligned} &\langle z_1-z_2,v_1-v_2\rangle_{Z^*\times Z} = \langle a_1+a_2-b_1-b_2,v_1-v_2\rangle_{Z^*\times Z} = \\ &= \ \langle\gamma^*\eta_1+\gamma^*\eta_2-\gamma^*\xi_1-\gamma^*\xi_2,v_1-v_2\rangle_{Z^*\times Z} = \\ &= \ \langle(\eta_1-\xi_1)+(\eta_2-\xi_2),\gamma v_1-\gamma v_2\rangle_{L^2(\Gamma_C;\mathbb{R}^d)} = \\ &= \ \langle(\eta_1,\eta_2)-(\xi_1,\xi_2),(\gamma v_1-\gamma v_2,\gamma v_1-\gamma v_2)\rangle_{L^2(\Gamma_C;\mathbb{R}^d)^2} \geq \\ &\geq \ -k_1\,\|\gamma v_1-\gamma v_2\|^2_{L^2(\Gamma_C;\mathbb{R}^d)} - k_2\,\|\gamma v_1-\gamma v_2\|_{L^2(\Gamma_C;\mathbb{R}^d)}\|\gamma u_1-\gamma u_2\|_{L^2(\Gamma_C;\mathbb{R}^d)} \geq \\ &\geq \ -k_1\,c_e\,\|\gamma\|^2\|v_1-v_2\|^2 - k_2\,c_e\,\|\gamma\|^2\|v_1-v_2\|\,\|u_1-u_2\|, \end{aligned}$$

where $c_e > 0$ is the embedding constant of V into Z and $\|\gamma\|$ is the norm of the trace operator. Thus the condition $H(F)_1$(iv) holds with $m_2 = c_e\,k_1\,\|\gamma\|^2$ and $m_3 = c_e\,k_2\,\|\gamma\|^2$. The proof of the lemma is complete. □

5.5 Unique solvability of hemivariational inequality

In this section we provide a result on the existence of solutions to Problem (HVI). To this aim, we associate with Problem (HVI) an evolution inclusion of the form which appears in Problem $\mathcal{P}$. In order to establish existence of solution to Problem (HVI), we show that the associated evolution inclusion has a solution and that every solution of the inclusion is also a solution to the hemivariational inequality.

Consider the following nonlinear evolution inclusion of second order associated with Problem (HVI): find $u \in \mathcal{V}$ with $u' \in \mathcal{W}$ such that

$$\begin{cases} u''(t) + A(t, u'(t)) + B(t, u(t)) + \displaystyle\int_0^t C(t-s)u(s)\, ds + \\ \qquad\qquad + F(t, u(t), u'(t)) \ni f(t) \quad \text{a.e. } t \in (0, T), \\ u(0) = u_0, \quad u'(0) = u_1, \end{cases} \tag{93}$$

where the multivalued mapping is of the form (87), i.e.

$$F(t, u, v) = S\, R^* \, \partial G(t, R(u, v), R(u, v)) \quad \text{for } u, v \in V, \; t \in (0, T)$$

with $G\colon (0, T) \times L^2(\Gamma_C; \mathbb{R}^d)^4 \to \mathbb{R}$ of the form (84) and its integrand $g\colon \Gamma_C \times (0, T) \times (\mathbb{R}^d)^4 \to \mathbb{R}$ given by (80), and the operators A, B and C are defined by (75), (76) and (77), respectively.

In order to formulate and prove the results on the existence and uniqueness of solutions to the hemivariational inequality in Problem (HVI) we need the following two lemmas.

Lemma 63 *Under hypotheses $H(\mathcal{A})_1$, $H(\mathcal{B})$, $H(\mathcal{C})$, $H(f)$ and $H(j_k)_1$ for $k = 1, \ldots, 4$, every solution of the inclusion (93) is a solution to Problem (HVI).*

Proof. Let $u \in \mathcal{V}$ with $u' \in \mathcal{W}$ be a solution of the inclusion (93). Then there exists $z \in \mathcal{Z}^*$ such that

$$u''(t) + A(t, u'(t)) + B(t, u(t)) + \int_0^t C(t-s)\, u(s)\, ds + z(t) = f(t) \text{ a.e. } t, \tag{94}$$

$$z(t) \in S\, R^* \, \partial G(t, R(u(t), u'(t)), R(u(t), u'(t))) \quad \text{a.e. } t \in (0, T),$$

$$u(0) = u_0, \quad u'(0) = u_1.$$

Hence, by the definition of the multivalued term, we obtain $z(t) = z_1(t) + z_2(t)$, $(z_1(t), z_2(t)) = (\gamma^* \eta_1(t), \gamma^* \eta_2(t))$ and

$$(\eta_1(t), \eta_2(t)) \in \partial G(t, \gamma u(t), \gamma u'(t), \gamma u(t), \gamma u'(t)) \quad \text{for a.e. } t \in (0, T),$$

where $\eta_i \in L^2(0, T; L^2(\Gamma_C; \mathbb{R}^d))$, $i = 1, 2$. The last inclusion, by the definition of the subdifferential is equivalent to

$$\langle \eta_1(t), \overline{u} \rangle_{L^2(\Gamma_C; \mathbb{R}^d)} + \langle \eta_2(t), \overline{v} \rangle_{L^2(\Gamma_C; \mathbb{R}^d)} \leq G^0(t, \gamma u(t), \gamma u'(t), \gamma u(t), \gamma u'(t); \overline{u}, \overline{v}) \tag{95}$$

for all $\overline{u}$, $\overline{v} \in L^2(\Gamma_C;\mathbb{R}^d)$ and a.e. $t \in (0,T)$. On the other hand, by 1) of Lemma 57 (cf. (81)) and 1) of Lemma 58 (cf. (85)), we have

$$\begin{aligned} G^0(t, \gamma u(t), \gamma u'(t), \gamma u(t), \gamma u'(t); \overline{u}, \overline{v}) \leq \\ \leq \int_{\Gamma_C} g^0(x, t, \gamma u(t), \gamma u'(t), \gamma u(t), \gamma u'(t); \overline{u}, \overline{v})\, d\Gamma \leq \\ \leq \int_{\Gamma_C} \Big(j_1^0(x,t,u(t),u'(t),u_\nu(t);\overline{u}_\nu) + j_2^0(x,t,u(t),u'(t),u'_\nu(t);\overline{v}_\nu) + \\ + j_3^0(x,t,u(t),u'(t),u_\tau(t);\overline{u}_\tau) + j_4^0(x,t,u(t),u'(t),u'_\tau(t);\overline{v}_\tau)\Big)\, d\Gamma \end{aligned} \tag{96}$$

for all $\overline{u}$, $\overline{v} \in L^2(\Gamma_C;\mathbb{R}^d)$, a.e. $t \in (0,T)$. By (94), (95) and (96), for all $v \in V$ and a.e. $t \in (0,T)$, we deduce

$$\begin{aligned} \langle f(t) - u''(t) - A(t,u'(t)) - B(t,u(t)) - \int_0^t C(t-s)u(s)\,ds, v\rangle = \\ = \langle z(t), v\rangle_{Z^*\times Z} = \langle \gamma^*\eta_1(t), v\rangle_{Z^*\times Z} + \langle \gamma^*\eta_2(t), v\rangle_{Z^*\times Z} = \\ = \langle \eta_1(t), \gamma v\rangle_{L^2(\Gamma_C;\mathbb{R}^d)} + \langle \eta_2(t), \gamma v\rangle_{L^2(\Gamma_C;\mathbb{R}^d)} \leq \\ \leq \int_{\Gamma_C} \Big(j_1^0(x,t,u(t),u'(t),u_\nu(t);v_\nu) + j_2^0(x,t,u(t),u'(t),u'_\nu(t);v_\nu) + \\ + j_3^0(x,t,u(t),u'(t),u_\tau(t);v_\tau) + j_4^0(x,t,u(t),u'(t),u'_\tau(t);v_\tau)\Big)\, d\Gamma \end{aligned}$$

which means that u is a solution to Problem (HVI). The proof of the lemma is complete. □

LEMMA 64 *Assume the hypotheses of Lemma 63 and $H(j)_{reg}$. If either $j_1 = j_3 = 0$ or $j_2 = j_4 = 0$, then u is a solution to Problem (HVI) if and only if u is a solution to the evolution inclusion (93).*

Proof. In view of Lemma 63, it is enough to show that every solution to Problem (HVI) is a solution to the evolution inclusion (93). Let $u \in \mathcal{V}$ with $u' \in \mathcal{W}$ be a solution of Problem (HVI), i.e. $u(0) = u_0$, $u'(0) = u_1$ and

$$\begin{aligned} \langle f(t) - u''(t) - A(t,u'(t)) - B(t,u(t)) - \int_0^t C(t-s)\,u(s)\,ds, v\rangle \leq \\ + \int_{\Gamma_C} \Big(j_1^0(x,t,u(t),u'(t),u_\nu(t);v_\nu) + j_2^0(x,t,u(t),u'(t),u'_\nu(t);v_\nu) + \\ + j_3^0(x,t,u(t),u'(t),u_\tau(t);v_\tau) + j_4^0(x,t,u(t),u'(t),u'_\tau(t);v_\tau)\Big)\, d\Gamma \end{aligned} \tag{97}$$

for all $v \in V$, a.e. $t \in (0,T)$, where the operators A, B and C are defined by (75), (76) and (77), respectively. From $H(j)_{reg}$, by Lemmas 57 and 58, we know that (81) and (85) hold with equalities, which implies

$$\begin{aligned}
&\int_{\Gamma_C} \Big(j_1^0(x,t,u(t),u'(t),u_\nu(t);v_\nu) + j_2^0(x,t,u(t),u'(t),u'_\nu(t);v_\nu) \; + \\
&\quad + \quad j_3^0(x,t,u(t),u'(t),u_\tau(t);v_\tau) + j_4^0(x,t,u(t),u'(t),u'_\tau(t);v_\tau) \Big) \, d\Gamma = \\
&\quad = \int_{\Gamma_C} g^0(x,t,\gamma u(t),\gamma u'(t),\gamma u(t),\gamma u'(t);\gamma v,\gamma v) \, d\Gamma = \\
&\quad = G^0(t,\gamma u(t),\gamma u'(t),\gamma u(t),\gamma u'(t);\gamma v,\gamma v) \qquad (98)
\end{aligned}$$

for all $v \in V$, a.e. $t \in (0,T)$. Suppose now that $j_1 = j_3 = 0$. Then g is given by

$$g(x,t,\zeta,\rho,\xi,\eta) = j_2(x,t,\zeta,\rho,\eta_\nu) + j_4(x,t,\zeta,\rho,\eta_\tau)$$

for all $\zeta, \rho, \xi, \eta \in \mathbb{R}^d$, a.e. $(x,t) \in \Gamma_C \times (0,T)$ and is independent of ξ, and consequently G is given by

$$\begin{aligned}
G(t,\widehat{w},\widehat{z},\widehat{u},\widehat{v}) &= \int_{\Gamma_C} g(x,t,\widehat{w}(x),\widehat{z}(x),\widehat{u}(x),\widehat{v}(x)) \, d\Gamma = \\
&= \int_{\Gamma_C} \Big(j_2^0(x,t,\widehat{w}(x),\widehat{z}(x),\widehat{v}_\nu(x)) + j_4^0(x,t,\widehat{w}(x),\widehat{z}(x),\widehat{v}_\tau(x)) \Big) \, d\Gamma
\end{aligned}$$

for $\widehat{w}$, $\widehat{z}$, $\widehat{u}$, $\widehat{v} \in L^2(\Gamma_C;\mathbb{R}^d)$, a.e. $t \in (0,T)$ and is independent of $\widehat{u}$. We denote the latter by G_1, i.e.

$$G(t,\widehat{w},\widehat{z},\widehat{u},\widehat{v}) = G_1(t,\widehat{w},\widehat{z},\widehat{v}) \ \text{ for } \widehat{w},\widehat{z},\widehat{u},\widehat{v} \in L^2(\Gamma_C;\mathbb{R}^d), \ \text{a.e. } t \in (0,T) \qquad (99)$$

with $G_1 \colon (0,T) \times L^2(\Gamma_C;\mathbb{R}^d)^3 \to \mathbb{R}$. Applying Lemma 30, we have

$$G^0(t,\widehat{w},\widehat{z},\widehat{u},\widehat{v};\overline{u},\overline{v}) = G_1^0(t,\widehat{w},\widehat{z},\widehat{v};\overline{v}) \qquad (100)$$

for all $\widehat{w}$, $\widehat{z}$, $\widehat{u}$, $\widehat{v}$, $\overline{u}$, $\overline{v} \in L^2(\Gamma_C;\mathbb{R}^d)$, a.e. $t \in (0,T)$, where G_1^0 denotes the generalized derivative of $G_1(t,\widehat{w},\widehat{z},\cdot)$, and

$$\partial G(t,\widehat{w},\widehat{z},\widehat{u},\widehat{v}) = \{0\} \times \partial G_1(t,\widehat{w},\widehat{z},\widehat{v}) \qquad (101)$$

for all $\widehat{w}$, $\widehat{z}$, $\widehat{u}$, $\widehat{v} \in L^2(\Gamma_C;\mathbb{R}^d)$, a.e. $t \in (0,T)$, where ∂G_1 denotes the generalized gradient of $G_1(t,\widehat{w},\widehat{z},\cdot)$. From (97)–(100), we obtain

$$\begin{aligned}
&\langle f(t) - u''(t) - A(t,u'(t)) - B(t,u(t)) - \int_0^t C(t-s)\, u(s)\, ds, v \rangle \le \\
&\qquad \le G_1^0(t,\gamma u(t),\gamma u'(t),\gamma u'(t);\gamma v) \qquad (102)
\end{aligned}$$

for all $v \in V$, a.e. $t \in (0,T)$. Using the equality

$$G_1^0(t,\gamma u(t),\gamma u'(t),\gamma u'(t);\gamma v) = (G_1 \circ \gamma)^0(t,\gamma u(t),\gamma u'(t),u'(t);v)$$

(which is a consequence of Proposition 28(a) and the regularity of $G_1(t, \widehat{w}, \widehat{z}, \cdot)$), from (102), it follows that

$$\begin{aligned} f(t) - u''(t) - A(t, u'(t)) - B(t, u(t)) - \int_0^t C(t-s)\, u(s)\, ds &\in \\ \in \ \partial (G_1 \circ \gamma)(t, \gamma u(t), \gamma u'(t), u'(t)) &= \\ = \ \gamma^* \, \partial G_1 (t, \gamma u(t), \gamma u'(t), \gamma u'(t)) & \end{aligned} \tag{103}$$

for a.e. $t \in (0, T)$. The last equality follows from Proposition 28(b).

On the other hand, we observe that the multifunction F defined by (87), with G given by (99), is now of the form

$$\begin{aligned} F(t, \widetilde{u}, v) &= S\, R^* \, \partial G(t, R(\widetilde{u}, v), R(\widetilde{u}, v)) = S\, R^* \left(\{0\} \times \partial G_1(t, R(\widetilde{u}, v), \gamma v)\right) = \\ &= S\left(\{0\}, \gamma^* \, \partial G_1(t, R(\widetilde{u}, v), \gamma v)\right) = \gamma^* \, \partial G_1(t, R(\widetilde{u}, v), \gamma v) = \\ &= \gamma^* \, \partial G_1(t, \gamma \widetilde{u}, \gamma v, \gamma v) \end{aligned}$$

for all $\widetilde{u}$, $v \in V$, a.e. $t \in (0, T)$. Therefore, from (103), we have

$$f(t) - u''(t) - A(t, u'(t)) - B(t, u(t)) - \int_0^t C(t-s)\, u(s)\, ds \in F(t, u(t), u'(t))$$

for a.e. $t \in (0, T)$ which means that u is a solution to the inclusion (93).

The case when $j_2 = j_4 = 0$ can be treated in an analogous way. This completes the proof of the lemma. □

The following is the existence result for the hemivariational inequality in Problem (HVI).

THEOREM 65 *Under the hypotheses $H(\mathcal{A})_1$, $H(\mathcal{B})$, $H(\mathcal{C})$, $H(f)$, $H(j_k)_2$ for $k = 1, \ldots, 4$, either (89) or (90), and the following conditions*

$$\widetilde{a}_3 > 4\sqrt{15}\, c_e^2 \, \|\gamma\|^2 \left(T \, \max\{ \max_{1 \le k \le 4} c_{k1}, c_{13}, c_{33}\} + \max\{ \max_{1 \le k \le 4} c_{k2}, c_{23}, c_{43}\} \right)$$

and

$$\widetilde{a}_4 > c_e \|\gamma\|^2 \left(\max\{L_1, 2L_2, L_3, 2L_4\} + \frac{T}{\sqrt{2}} \max\{2L_1, L_2, 2L_3, L_4\} \right),$$

Problem (HVI) admits a solution.

Proof. It is enough to show that the evolution inclusion (93) admits a solution and then apply Lemma 63. In order to establish the existence of a solution to evolution inclusion (93), we apply Theorem 48. From Lemma 54, it follows that under $H(\mathcal{A})_1$, the operator A satisfies $H(A)_1$. It is clear from Lemmas 55 and 56 that under $H(\mathcal{B})$ and $H(\mathcal{C})$, the operators B and C satisfy $H(B)$ and $H(C)$, respectively. The condition (H_0) holds as a consequence of $H(f)$. Next, under $H(j_k)_2$ and either (89) or (90), we know, by Lemma 62, that the multifunction F satisfies $H(F)_1$. Finally, we readily check using the constants delivered in Lemmas 54, 57, 58, 60 and 62 that (H_1) and (H_2) are satisfied. Thus, we deduce that the hypotheses $H(A)_1$, $H(B)$, $H(C)$, $H(F)_1$, (H_0), (H_1) and (H_2) of Theorem 48 hold. Hence, by applying this theorem, we obtain that the evolution inclusion (93) has a unique solution, and hence also Problem (HVI) admits a solution. □

The result on the uniqueness of solutions to Problem (HVI) is a consequence of Theorem 65 and Lemma 64.

THEOREM 66 *Assume the hypotheses of Theorem 65 and $H(j)_{reg}$. In addition, if either $j_1 = j_3 = 0$ or $j_2 = j_4 = 0$, then the hemivariational inequality in Problem (HVI) admits a unique solution.*

6 Applications to viscoelastic mechanical problems

The aim of this section is to explain, by providing several examples, the origins and formulations of unilateral boundary conditions of mechanics. We consider boundary conditions resulting from convex or nonconvex and nonsmooth potentials using the concept of subdifferential. We restrict ourselves to one-dimensional examples, refering to Chapter 4.6 of [73] for two- and three-dimensional contact laws.

6.1 Examples of constitutive laws with long memory

In this part we provide one dimensional examples of the constitutive law of the form

$$\sigma(t) = \mathcal{A}(t, \varepsilon(u'(t))) + \mathcal{B}(t, \varepsilon(u(t))) + \int_0^t \mathcal{C}(t-s)\, \varepsilon(u(s))\, ds \quad \text{in} \quad Q. \tag{104}$$

First, consider a dashpot connected in parallel with a Maxwell model. In this case an additive formula holds

$$\sigma = \sigma^V + \sigma^R, \tag{105}$$

where σ, σ^V and σ^R denote the total stress, the stress in the dashpot and the stress in the Maxwell model, respectively. We have

$$\sigma^V = A\, \varepsilon(u') \tag{106}$$

and

$$(\sigma^R)' = E\, \varepsilon(u') - \frac{E}{\eta}\, \sigma^R, \tag{107}$$

where A and η are positive viscosity coefficients, $E > 0$ is the Young modulus of the Maxwell material and ε denotes the strain of the model. It is well known that, assuming the initial conditions $\sigma^R(0) = 0$ and $\varepsilon(u(0)) = 0$, the Maxwell constitutive equation (107) is equivalent to the integral equation

$$\sigma^R(t) = E\, \varepsilon(u(t)) - \frac{E^2}{\eta} \int_0^t e^{-\frac{E}{\eta}(t-s)} \varepsilon(u(s))\, ds. \tag{108}$$

We combine now (105), (106) and (108) to obtain

$$\sigma(t) = A\,\varepsilon(u'(t)) + E\,\varepsilon(u(t)) - \frac{E^2}{\eta}\int_0^t e^{-\frac{E}{\eta}(t-s)}\varepsilon(u(s))\,ds,$$

which represents a constitutive equation of the form (104).

The second example can be obtained replacing the Maxwell model above with a linear standard viscoelastic constitutive model. In this case we have

$$\frac{(\sigma^R)'}{E} + \frac{\sigma^R}{\eta} = \Big(1 + \frac{E_1}{E}\Big)\varepsilon(u') + \frac{E_1}{\eta}\,\varepsilon(u), \tag{109}$$

where E, E_1 and η are positive constants. We integrate (109) with the initial conditions $\sigma^R(0) = 0$ and $\varepsilon(u(0)) = 0$ to obtain

$$\sigma(t) = A\,\varepsilon(u'(t)) + (E + E_1)\,\varepsilon(u(t)) - \frac{E^2}{\eta}\int_0^t e^{-\frac{E}{\eta}(t-s)}\varepsilon(u(s))\,ds. \tag{110}$$

Combining now (105), (106) and (110) we find again a viscoelastic constitutive law of the form (104).

More details on the one-dimensional laws of the form (104) as well as on the construction of rheological models obtained by connecting springs and dashpots can be found in Chapter 6 of Han and Sofonea [34].

We would like to mention that all materials exhibit some viscoelastic response with their deformation depending on load, time and temperature. For example, an amorphous solid such as glass may act more like a liquid at elevated temperatures, at which its time dependent response can be measured in seconds. On the other hand, at room temperature, its stiffness is much greater, so glass may still flow, but the time dependent response is measured in years or decades. Viscoelastic behavior is similarly found in other materials such as polymers (e.g. amorphous, semicrystalline, biopolymers, thermoplastic, organic), numerous metals (e.g. aluminium, quartz) at a temperature close to their melting point, steel, concrete (e.g. fresh, reinforced, asphalt concrete), bitumen materials, cement-based materials, rock-soils, geological materials, plastics, rubber, ceramics, natural and synthetic fibers, composites (e.g. dental, reinforced composites), elastomers, several materials including brass, aluminum alloys, solid rocket propellants, etc. Materials of biological origin contain natural polymers, and therefore they can be expected to exhibit viscoelastic behavior. For example, natural viscoelastic materials include wood, human and animal bones, biological soft tissues such as brain, skin, kidney, spleen, etc. In some applications, even a small viscoelastic response can be significant. To be complete, an analysis or design involving such materials must incorporate their viscoelastic behavior. Knowledge of the viscoelastic response of a material is based on measurements.

6.2 Examples of subdifferential boundary conditions

In this section we present specific examples of contact and friction laws which can be met in mechanics and which lead to the subdifferential boundary conditions of the form

$$
\begin{aligned}
-\sigma_\nu(t) &\in \partial j_1(x,t,u(t),u'(t),u_\nu(t)) + \partial j_2(x,t,u(t),u'(t),u'_\nu(t)), && (111)\\
-\sigma_\tau(t) &\in \partial j_3(x,t,u(t),u'(t),u_\tau(t)) + \partial j_4(x,t,u(t),u'(t),u'_\tau(t)) && (112)
\end{aligned}
$$

on $\Gamma_C \times (0,T)$. In these examples the conditions on the contact surface are divided into the boundary conditions in the normal and in the tangential directions. For a detailed discussion of various contact and friction conditions, we refer to the extensive literature.

6.2.1 Frictionless contact

In the simplest case when $j_3 = j_4 = 0$, we are lead to frictionless contact. It is a situation if the reaction of the foundation towards the body is in the normal direction only. Thus, the friction force on the contact surface vanishes, i.e.

$$\sigma_\tau = 0 \quad \text{on } \Gamma_C \times (0,T).$$

This condition is used when the contact surfaces are fully lubricated and it represents a first approximation of more realistic conditions involving friction, cf. [20].

6.2.2 Prescribed normal stress and nonmonotone friction laws

Let us consider the following boundary conditions on $\Gamma_C \times (0,T)$:

$$-\sigma_\nu(t) = S(t), \tag{113}$$

$$-\sigma_\tau(t) \in \partial j_4(x,t,u(t),u'(t),u'_\tau(t)). \tag{114}$$

The equation (113) states that the normal stress is prescribed on $\Gamma_C \times (0,T)$ and is given by $S = S(x,t) \geq 0$. Such a condition makes sense when the real contact area is close to the nominal one and the surfaces are conforming. Then $S = S(x,t)$ is the contact pressure and it is given by the ratio of the total applied force to the nominal contact area. It is considered (see Chapters 2.6 and 10.1 of Shillor et al. [93]) to be a good approximation when the load is light and the contact surface is transmitted by the asperity tips only. This law is of the form (111) with $j_1(x,t,\zeta,\rho,r) = S(x,t)\,r$ and $j_2 = 0$, where $S \in L^\infty(\Gamma_C \times (0,T))$, $S \geq 0$ is a given normal stress. It is clear that $j_1(x,t,\zeta,\rho,\cdot)$ is convex (hence regular), and that $H(j_1)_1$ with $c_{10} = \|S\|_{L^\infty(\Gamma_C\times(0,T))}$, $c_{11} = c_{12} = c_{13} = 0$, $H(j_1)_2$ (by Proposition 15(ii)) and $H(j_1)_3$ hold.

6.2.2.1 Nonmonotone friction independent of slip and slip rate. We consider the nonmonotone friction laws which are independent of the slip displacement and the slip rate. This is the case when the superpotential $j_4 = j_4(x,t,\zeta,\rho,\theta)$ is independent of (ζ,ρ) and nonconvex in θ. Then the friction law (114) takes the form

$$-\sigma_\tau(t) \in \partial j_4(x,t,u'_\tau(t)) \quad \text{on } \Gamma_C \times (0,T). \tag{115}$$

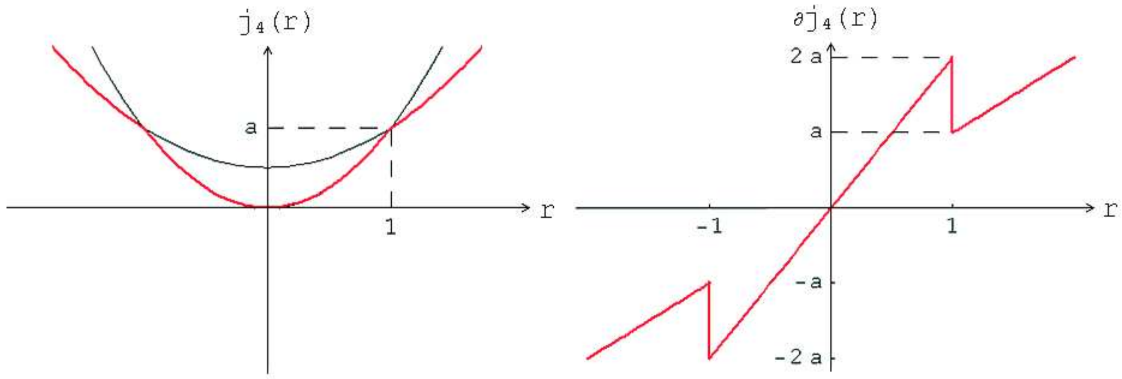

Figure 2: Zig-zag friction law

This law appears (cf. Section 7.2 of Panagiotopoulos [78]) in the tangential direction of the adhesive interface and describes the partial cracking and crushing of the adhesive bonding material. Several examples of zig-zag friction laws from Section 2.4 of Panagiotopoulos [78] can be formulated in the form (115). For instance, let $j_4 \colon \mathbb{R} \to \mathbb{R}$ be given by $j_4(r) = \min\{\varphi_1(r), \varphi_2(r)\}$, where $\varphi_1(r) = ar^2$, $\varphi_2(r) = \frac{a}{2}(r^2+1)$, $r \in \mathbb{R}$ (for simplicity we also drop the (x,t)-dependence) and $a > 0$. Its subdifferential is as follows

$$\partial j_4(r) = \begin{cases} ar & r \in (-\infty,-1) \cup (1,+\infty), \\ 2ar & r \in (-1,1), \\ [a,2a] & r = 1, \\ [-2a,-a] & r = -1 \end{cases}$$

(see Figure 2). Using Proposition 26, we know that $\partial j_4(r) \subset \mathrm{co}\{\varphi_1'(r), \varphi_2'(r)\}$. Hence the subdifferential ∂j_4 has at most linear growth and $H(j_4)_1$ holds with $c_{40} = c_{41} = c_{42} = 0$ and $c_{43} = 2a$. Since j_4 is the minimum of the strictly differentiable functions, by Corollary 32, the function $-j_4$ is regular. By Proposition 15(ii), it follows that $H(j_4)_2$ is satisfied. The above model example can also be modified to obtain nonmonotone zig-zag relations which describe the adhesive contact problems and contact laws for a granular material and a reinforced concrete, cf. Sections 2.4 and 7.2 of Panagiotopoulos [78], Section 4.6 of Naniewicz and Panagiotopoulos [73] and Section 2.8 of Goeleven et al. [31]. Furthermore, if the function $j_4 \colon \mathbb{R} \to \mathbb{R}$ in (115) is continuously differentiable, then $\partial j_4(r) = \{j_4'(r)\}$ for $r \in \mathbb{R}$ and (115) reduces to the equation

$$-\sigma_\tau(t) = j_4'(u_\tau'(t)) \quad \text{on } \Gamma_C \times (0,T).$$

For example, when $j_4(r) = \frac{\mu}{2} r^2$ ($\mu > 0$ being the constant friction coefficient) then (115) takes the form

$$-\sigma_\tau(t) = \mu\, u_\tau'(t) \quad \text{on } \Gamma_C \times (0,T),$$

which simply means that the tangential shear is proportional to the tangential velocity.

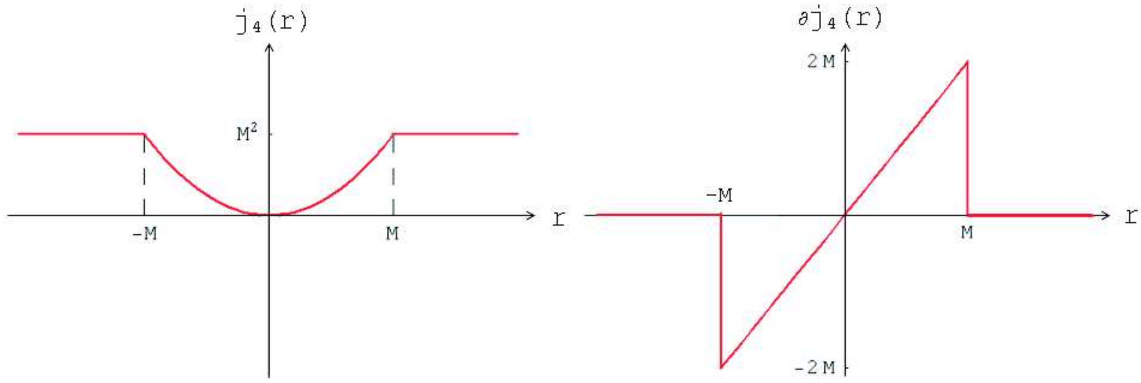

Figure 3: Nonmonotone friction law

Another example of nonmonotone friction law can be obtained from the nonconvex function $j_4\colon \mathbb{R}^d \to \mathbb{R}$ given by

$$j_4(\xi) = \begin{cases} \|\xi\|^2 & \text{if } \|\xi\| \le M, \\ M^2 & \text{if } \|\xi\| > M \end{cases}$$

for $\xi \in \mathbb{R}^d$, where M is a positive constant (see Figure 3 for $d = 1$). This function can be represented as a difference of convex functions, i.e. $j_4(\xi) = \varphi_1(\xi) - \varphi_2(\xi)$ for $\xi \in \mathbb{R}^d$, where $\varphi_1(\xi) = \|\xi\|^2$ and

$$\varphi_2(\xi) = \begin{cases} 0 & \text{if } \|\xi\| \le M, \\ \|\xi\|^2 - M^2 & \text{if } \|\xi\| > M. \end{cases}$$

Since $\partial\varphi_1(\xi)$ is a singleton for $\xi \in \mathbb{R}^d$, by Proposition 33, we deduce that $-j_4$ is regular and $\partial j_4(\xi) = \partial\varphi_1(\xi) - \partial\varphi_2(\xi)$ for $\xi \in \mathbb{R}^d$. In addition, it is easy to observe that j_4 satisfies $H(j_4)_1$ with $c_{40} = c_{41} = c_{42} = 0$, $c_{43} = 2M$, and $H(j_4)_2$ (by Proposition 15(ii)).

6.2.2.2 Nonmonotone friction depending on slip and slip rate. We consider the nonmonotone friction conditions which depend on both the slip and the slip rate. This is the case when the superpotential $j_4 = j_4(x, t, \zeta, \rho, \theta)$ depends on ζ and ρ, and it is nonconvex in θ. As an example of this function we choose

$$j_4(x, t, \zeta, \rho, \theta) = \psi(x, t, \zeta, \rho)\, h(\theta) \ \text{ for } \zeta,\ \rho,\ \theta \in \mathbb{R}^d,\ \text{ a.e. } t \in (0, T), \tag{116}$$

where $\psi\colon \Gamma_C \times (0, T) \times (\mathbb{R}^d)^2 \to \mathbb{R}$ satisfies

$$\begin{cases} \psi(\cdot, \cdot, \zeta, \rho) \text{ is measurable for all } \zeta,\ \rho \in \mathbb{R}^d; \\ \psi(x, t, \cdot, \cdot) \text{ is continuous for a.e. } (x, t) \in \Gamma_C \times (0, T); \\ 0 \le \psi(x, t, \zeta, \rho) \le \psi_0(1 + \|\zeta\| + \|\rho\|) \text{ for all } \zeta, \rho \in \mathbb{R}^d, \\ \qquad\qquad\qquad \text{a.e. } (x, t) \in \Gamma_C \times (0, T) \text{ with } \psi_0 > 0 \end{cases}$$

and $h\colon \mathbb{R}^d \to \mathbb{R}$ is a locally Lipschitz function such that $h(0) = 0$ and

$$\|\partial h(\theta)\| \le h_0 \ \text{ for } \theta \in \mathbb{R}^d \text{ with } h_0 > 0.$$

Under these hypotheses on ψ and h, the function j_4 given by (116) satisfies $H(j_4)_1$ with $c_{40} = c_{41} = c_{42} = h_0\,\psi_0$, $c_{43} = 0$. The friction law (114) takes now the form

$$-\sigma_\tau(t) \in \psi(x,t,u(t),u'(t))\,\partial h(u'_\tau(t)) \ \text{ on } \Gamma_C \times (0,T). \tag{117}$$

It is clear that $j_4(x,t,\zeta,\rho,\cdot)$ is regular if and only if h is regular. Next, let $(\zeta_n,\rho_n,\theta_n) \in (\mathbb{R}^d)^3$, $(\zeta_n,\rho_n,\theta_n) \to (\zeta,\rho,\theta)$ and $\sigma \in \mathbb{R}^d$. We have

$$\begin{aligned}
&\limsup\, j_4^0(x,t,\zeta_n,\rho_n,\theta_n;\sigma) = \limsup\, \psi(x,t,\zeta_n,\rho_n)\,h^0(\theta_n;\sigma) = \\
&\quad = \limsup\left[(\psi(x,t,\zeta_n,\rho_n) - \psi(x,t,\zeta,\rho))\,h^0(\theta_n;\sigma) + \psi(x,t,\zeta,\rho)\,h^0(\theta_n;\sigma)\right] \le \\
&\quad \le h_0\|\sigma\|\lim\left(\psi(x,t,\zeta_n,\rho_n) - \psi(x,t,\zeta,\rho)\right) + \psi(x,t,\zeta,\rho)\limsup\, h^0(\theta_n;\sigma) \le \\
&\quad \le \psi(x,t,\zeta,\rho)\,h^0(\theta;\sigma) = j_4^0(x,t,\zeta,\rho,\theta;\sigma)
\end{aligned}$$

for a.e. $(x,t) \in \Gamma_C \times (0,T)$. Hence $H(j_4)_2$ holds. Moreover, if for instance $\psi(x,t,\cdot,\cdot)$ is Lipschitz continuous for a.e. $(x,t) \in \Gamma_C \times (0,T)$ (i.e. $|\psi(x,t,\zeta_1,\rho_1) - \psi(x,t,\zeta_2,\rho_2)| \le L_\psi\,(\|\zeta_1-\zeta_2\| + \|\rho_1-\rho_2\|)$ for all ζ_1, ζ_2, ρ_1, $\rho_2 \in \mathbb{R}^d$, a.e. $(x,t) \in \Gamma_C \times (0,T)$) and h is convex, then

$$\begin{aligned}
&(\partial j_4(x,t,\zeta_1,\rho_1,\theta_1) - \partial j_4(x,t,\zeta_2,\rho_2,\theta_2), \theta_1 - \theta_2) = \\
&\quad = ((\psi(x,t,\zeta_1,\rho_1) - \psi(x,t,\zeta_2,\rho_2))\,\partial h(\theta_1), \theta_1-\theta_2) + \\
&\qquad + \psi(x,t,\zeta_2,\rho_2)\,(\partial h(\theta_1) - \partial h(\theta_2), \theta_1 - \theta_2) \ge \\
&\quad \ge -L_\psi\,h_0\,(\|\zeta_1-\zeta_2\| + \|\rho_1-\rho_2\|)\,\|\theta_1-\theta_2\|
\end{aligned}$$

for all ζ_1, ζ_2, ρ_1, $\rho_2 \in \mathbb{R}^d$, a.e. $(x,t) \in \Gamma_C \times (0,T)$ which implies that $H(j_4)_3$ is satisfied with $L_4 = L_\psi h_0$.

By choosing $h\colon \mathbb{R}^d \to \mathbb{R}$, $h(\theta) = \|\theta\|$ for $\theta \in \mathbb{R}^d$ and a suitable function ψ, we obtain a number of well-known monotone friction laws which are popular and formulated below.

Contact with simplified Coulomb's friction law

Consider the contact problem modeled by a simplified version of Coulomb's law of dry friction, that is

$$\begin{cases}
-\sigma_\nu(t) = S(t), \\
\|\sigma_\tau\| \le \mu|\sigma_\nu| \text{ with} \\
\qquad \|\sigma_\tau\| < \mu|\sigma_\nu| \implies u'_\tau = 0, \\
\qquad \|\sigma_\tau\| = \mu|\sigma_\nu| \implies \sigma_\tau = -\lambda u'_\tau = 0 \ \text{ with some } \lambda \ge 0
\end{cases}$$

on $\Gamma_C \times (0,T)$. Here, as above, $S \in L^\infty(\Gamma_C \times (0,T))$, $S \ge 0$ is a given normal stress and the coefficient of friction $\mu \in L^\infty(\Gamma_C)$ is such that $\mu \ge 0$ a.e. on Γ_C. This law has been studied e.g. in Duvaut and Lions [27], Pangiotopoulos [77], Ionescu and Sofonea [43], Awbi et al. [10], Motreanu and Sofonea [72], Migórski and Ochal [66]. In the contact between a hard rigid smooth tool and an elastic-plastic workpiece, the Coulomb condition is useful within the boundary lubrication regime and when the nominal contact pressure is relatively small as compared to the hardness of the workpiece material. In such a case contact takes place at the tips of the asperities, and there is a considerable difference between the averaged contact pressure and the maximal pointwise pressure at the tips. The simplified Coulomb friction law is of the form (117) with $\psi(x,t,\zeta,\rho) = S(x,t)\,\mu(x)$ and $h(\theta) = \|\theta\|$. Since

$$\partial\|\theta\| = \begin{cases} \overline{B}(0,1) & \text{if } \theta = 0, \\ \dfrac{\theta}{\|\theta\|} & \text{if } \theta \neq 0, \end{cases}$$

where $\overline{B}(0,1)$ denotes the closed unit ball in $\mathbb{R}^d$ (see Figure 4 for $d = 1$), this boundary condition is equivalent to

$$\begin{cases} -\sigma_\nu(t) = S(t), \\ \|\sigma_\tau(t)\| \leq S(x,t)\mu(x) \quad \text{if } u'_\tau(t) = 0, \\ -\sigma_\tau(t) = S(x,t)\mu(x)\dfrac{u'_\tau(t)}{\|u'_\tau(t)\|} \quad \text{if } u'_\tau(t) \neq 0. \end{cases}$$

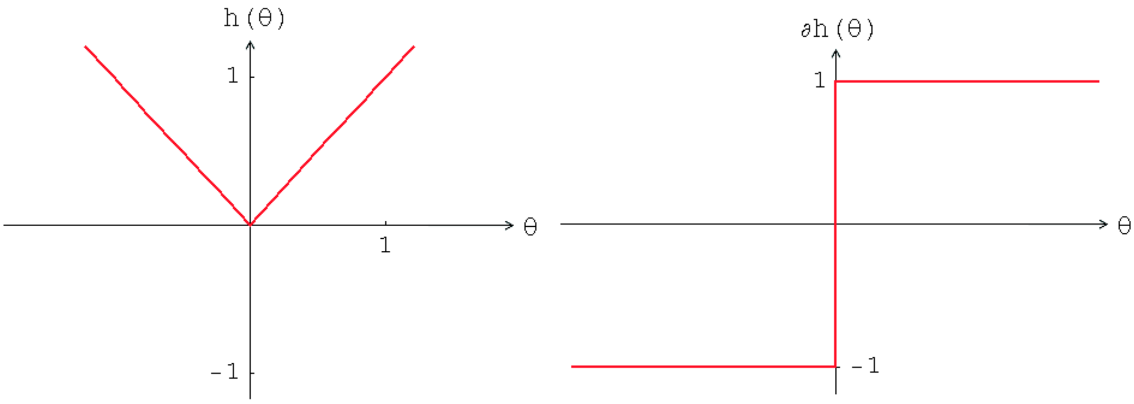

Figure 4: Monotone friction boundary condition

It is clear that simplified Coulomb's friction law corresponds to (111), (112) with $j_1(x,t,\zeta,\rho,r) = S(x,t)\,r$, $j_2 = j_3 = 0$ and $j_4(x,t,\zeta,\rho,\theta) = S(x,t)\,\mu(x)\|\theta\|$. The properties of j_1 are stated in Section 6.2.2. The function j_4 satisfies $H(j_4)_1$ with $c_{40} = c_{41} = c_{42} = 0$, $c_{43} = \|S\|_{L^\infty(\Gamma_C\times(0,T))}\|\mu\|_{L^\infty(\Gamma_C)}$ and $H(j_4)_2$ (by Proposition 15(ii)). It is convex (thus regular) in its last variable, so its subdifferential is monotone (cf. Proposition 24(vi)) and $H(j_4)_3$ holds with $L_4 = 0$.

Contact with slip dependent friction

The contact problem with slip dependent friction is modeled with a condition in which the normal stress on the contact surface is prescribed and the friction coefficient depends on the slip $\|u_\tau\|$, i.e.

$$\begin{cases} -\sigma_\nu(t) = S(t), \\ \|\sigma_\tau(t)\| \le \mu(x,t,\|u_\tau(t)\|)S(t) \ \text{ if } u'_\tau(t) = 0, \\ -\sigma_\tau(t) = \mu(x,t,\|u_\tau(t)\|)S(t)\, \dfrac{u'_\tau(t)}{\|u'_\tau(t)\|} \ \text{ if } u'_\tau(t) \neq 0 \end{cases} \tag{118}$$

on $\Gamma_C \times (0,T)$. The physical model of slip-dependent friction was introduced by Rabinowicz [86] in the geophysical context of earthquakes' modeling. This model of friction was studied by Ionescu and Paumier [41], Ionescu and Nguyen [39], Ionescu et al. [40], Shillor et al. in Chapter 10.1 of [93] and Migórski and Ochal [66]. It is clear that this law is of the form (117) with $\psi(x,t,\zeta,\rho) = S(x,t)\mu(x,t,\|\zeta_\tau\|)$ and $h(\theta) = \|\theta\|$. It can be observed that if the normal stress $S \in L^\infty(\Gamma_C \times (0,T))$, $S \ge 0$ a.e. on $\Gamma_C \times (0,T)$ and the coefficient of friction satisfies the following conditions

$\underline{H(\mu)}$: $\quad \mu\colon \Gamma_C \times (0,T) \times \mathbb{R}_+ \to \mathbb{R}_+$ is such that

$\mu(\cdot,\cdot,r)$ is measurable for all $r \in \mathbb{R}$;
$\mu(x,t,\cdot)$ is continuous for a.e. $(x,t) \in \Gamma_C \times (0,T)$;
$0 \le \mu(x,t,r) \le \mu_0(1+|r|)$ for all $r \in \mathbb{R}_+$, a.e. $(x,t) \in \Gamma_C \times (0,T)$ with $\mu_0 > 0$,

then the function $j_4(x,t,\zeta,\rho,\theta) = S(x,t)\mu(x,t,\|\zeta_\tau\|)\|\theta\|$ satisfies $H(j_4)_1$ with $c_{40} = c_{41} = \mu_0\|S\|_{L^\infty(\Gamma_C\times(0,T))}$ and $c_{42} = c_{43} = 0$ and $H(j_4)_2$; it is convex in θ (hence also regular). If, in addition, $\mu(x,t,\cdot)$ is Lipschitz continuous for a.e. $(x,t) \in \Gamma_C \times (0,T)$, then $H(j_4)_3$ holds. The relations (118) assert that the tangential stress is bounded by the normal stress multiplied by the value of the time-dependent friction coefficient $\mu(x,t,\|u_\tau(x)\|)$. If such a limit is not attained, sliding does not occur. Otherwise, the friction stress is opposed to the slip rate and its absolute value depends on the slip. The function μ depends on $x \in \Gamma_C$ to model the local roughness of the contact surface.

Contact with a version of dry friction law

The classical formulation of frictional contact with normal damped response is as follows

$$\begin{cases} -\sigma_\nu(t) = p_\nu(u'_\nu(t)), \\ \|\sigma_\tau(t)\| \le p_\tau(u'_\nu(t)) \ \text{ with} \\ \qquad \|\sigma_\tau(t)\| < p_\tau(u'_\nu(t)) \ \text{ if } u'_\tau(t) = 0, \\ \qquad \|\sigma_\tau(t)\| = p_\tau(u'_\nu(t))\, \dfrac{u'_\tau(t)}{\|u'_\tau(t)\|} \ \text{ if } u'_\tau(t) \neq 0 \end{cases} \tag{119}$$

on $\Gamma_C \times (0,T)$. There is a number of ways we may choose functions p_ν and p_τ (see e.g. Chapter 8.6 of Shillor et al. [93]). Let $p_\nu(x,r) = S(x)$, where $S \in L^\infty(\Gamma_C)$ is a given positive function (cf. (8.6.9) in [93]), that is, the normal stress is prescribed on Γ_C. This type of contact condition in which the normal stress is given arises in the study of some mechanisms and was considered e.g. in Duvaut and Lions [27] and Panagiotopoulos [77], see also the normal damped response condition of Section 6.2.4. The friction condition in (119) is of the form (117) with $\psi(x,t,\zeta,\rho) = p_\tau(x,t,\rho_\nu)$ and $h(\theta) = \|\theta\|$. If the function $p_\tau \colon \Gamma_C \times (0,T) \times \mathbb{R} \to \mathbb{R}_+$ satisfies the following conditions

$$\left\{ \begin{array}{l} p_\tau(\cdot,\cdot,r) \text{ is measurable for all } r \in \mathbb{R}; \\ p_\tau(x,t,\cdot) \text{ is continuous for a.e. } (x,t) \in \Gamma_C \times (0,T); \\ 0 \le p_\tau(x,t,r) \le p_0(1+|r|) \text{ for all } r \in \mathbb{R}, \\ \qquad\qquad\qquad\qquad \text{a.e. } (x,t) \in \Gamma_C \times (0,T) \text{ with } p_0 > 0, \end{array} \right.$$

then the function $j_4(x,t,\zeta,\rho,\theta) = p_\tau(x,t,\rho_\nu)\,\|\theta\|$ satisfies $H(j_4)_1$ with $c_{40} = c_{42} = p_0$ and $c_{41} = c_{43} = 0$ and $H(j_4)_2$; $j_4(x,t,\zeta,\rho,\cdot)$ is convex (thus regular). If, in addition, $p_\tau(x,t,\cdot)$ is Lipschitz continuous for a.e. $(x,t) \in \Gamma_C \times (0,T)$, then $H(j_4)_3$ holds.

Contact with slip rate dependent friction

Consider the friction condition (117) with $\psi(x,t,\zeta,\rho) = \omega(x,t,\zeta_\nu)\mu(x,t,\|\rho_\tau\|)$ and $h(\theta) = \|\theta\|$. We admit the following assumption

$\underline{H(\omega)}$: $\quad \omega \colon \Gamma_C \times (0,T) \times \mathbb{R} \to \mathbb{R}_+$ satisfies

$\omega(\cdot,\cdot,r)$ is measurable for all $r \in \mathbb{R}$;
$\omega(x,t,\cdot)$ is continuous for a.e. $(x,t) \in \Gamma_C \times (0,T)$;
$0 \le \omega(x,t,r) \le \omega_0$ for all $r \in \mathbb{R}$, a.e. $(x,t) \in \Gamma_C \times (0,T)$ with $\omega_0 > 0$.

We can show that if $H(\omega)$ and $H(\mu)$ (introduced in the paragraph on contact with slip dependent friction) hold, then the function

$$j_4(x,t,\zeta,\rho,\theta) = \omega(x,t,\zeta_\nu)\,\mu(x,t,\|\rho_\tau\|)\,\|\theta\|$$

satisfies $H(j_4)_1$ with $c_{40} = c_{42} = \mu_0\omega_0$, $c_{41} = c_{43} = 0$ and $H(j_4)_2$. Moreover, it can be seen that if $\mu(x,t,\cdot)$ and $\omega(x,t,\cdot)$ are nonnegative, bounded from above and Lipschitz continuous functions for a.e. $(x,t) \in \Gamma_C \times (0,T)$, then $H(j_4)_3$ also holds. The friction condition (117) takes the form

$$\left\{ \begin{array}{ll} \|\sigma_\tau(t)\| \le \omega(x,t,u_\nu(t))\mu(x,t,0) & \text{if } u'_\tau(t) = 0, \\ -\sigma_\tau(t) = \omega(x,t,u_\nu(t))\,\mu(x,t,\|u'_\tau(t)\|)\,\dfrac{u'_\tau(t)}{\|u'_\tau(t)\|} & \text{if } u'_\tau(t) \ne 0. \end{array} \right.$$

Since the friction coefficient μ is a function of u'_τ, the friction model is slip rate or velocity dependent. In most geological publications dealing with the motion of tectonic plates, the friction coefficient is assumed to be dependent on the slip rate. For more details on the interpretation of this friction law, we refer to Rabinowicz [86], Ionescu et al. [40], Ionescu and Paumier [41] and the references therein.

6.2.3 Contact with nonmonotone normal compliance

This contact condition describes reactive foundation assigning a reactive normal traction or pressure that depends on the interpenetration of the asperities on the body surface and those on the foundation. It is of the form (111) with $j_2 = 0$. We comment on it in a simple case when

$$-\sigma_\nu(t) \in \partial j_1(u_\nu(t)) \quad \text{on } \Gamma_C \times (0,T) \tag{120}$$

with $j_1 \colon \mathbb{R} \to \mathbb{R}$ defined by

$$j_1(r) = \int_0^r p(s)\, ds, \quad \text{for } r \in \mathbb{R}.$$

We admit the following hypothesis in the integrand of j_1.

$\underline{H(p)}$: $\quad p \colon \mathbb{R} \to \mathbb{R}$ is a function such that

$$p \in L^\infty_{loc}(\mathbb{R}), \quad |p(s)| \le p_1(1+|s|) \text{ for } s \in \mathbb{R} \text{ with } p_1 > 0.$$

It is well known (cf. [18, 31]) that $\partial j_1(s) = \widehat{p}(s)$ for $s \in \mathbb{R}$, where the multivalued function $\widehat{p} \colon \mathbb{R} \to 2^{\mathbb{R}}$ is given by $\widehat{p}(s) = [p^{(1)}(s), p^{(2)}(s)]$ ($[\cdot,\cdot]$ denotes an interval in $\mathbb{R}$) and

$$p^{(1)}(r) = \lim_{\varepsilon \to 0^+} \operatorname*{ess\,inf}_{|\tau - r| \le \varepsilon} p(\tau), \qquad p^{(2)}(r) = \lim_{\varepsilon \to 0^+} \operatorname*{ess\,sup}_{|\tau - r| \le \varepsilon} p(\tau).$$

In this case j_1 is a locally Lipschitz function, $|\partial j_1(r)| \le p_1(1+|r|)$ for $r \in \mathbb{R}$ and (120) takes the form

$$-\sigma_\nu(t) \in \widehat{p}(u_\nu(t)) \quad \text{on } \Gamma_C \times (0,T).$$

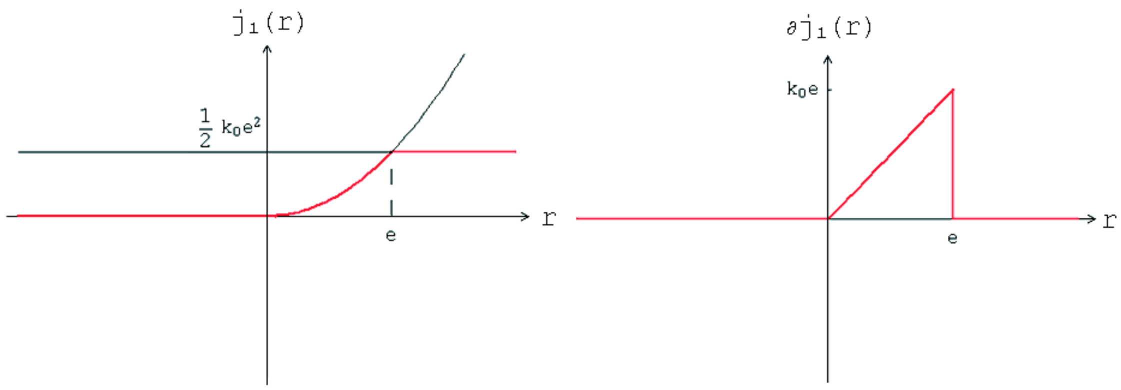

Figure 5: Nonmonotone Winkler's law

We provide a concrete example which is the nonmonotone Winkler law. This is a boundary condition between a body and a Winkler-type support which may sustain only limited values of efforts. Let $\vartheta \in L^\infty_{loc}(\mathbb{R})$ be given by

$$\vartheta(r) = \begin{cases} 0 & \text{if } r \in (-\infty, 0) \cup (e, +\infty), \\ k_0 r & \text{if } r \in [0, e], \end{cases}$$

where e is a small positive constant and $k_0 > 0$ is the Winkler coefficient. Then $|\vartheta(r)| \leq k_0\, e$ for $r \in \mathbb{R}$ and $j_1(r) = \min\{\psi_1(r), \psi_2(r)\}$, where

$$\psi_1(s) = \begin{cases} 0 & \text{if } r < 0, \\ \dfrac{k_0}{2} r^2 & \text{if } r \geq 0 \end{cases}$$

and $\psi_2(r) = \frac{k_0}{2} e^2$ for $r \in \mathbb{R}$. Assuming that the tangential forces are known $\sigma_\tau = C_\tau$, $C_\tau = C_\tau(x)$ is given on $\Gamma_C \times (0, T)$, the condition (120) can be interpreted as follows

$$\begin{cases} \sigma_\nu(t) = 0 & \text{if } u_\nu(t) < 0 \text{ and } u_\nu(t) > e, \\ -\sigma_\nu(t) = k_0\, u_\nu(t) & \text{if } 0 \leq u_\nu(t) < e, \\ -\sigma_\nu(t) \in [0, k_0 e] & \text{if } u_\nu(t) = e. \end{cases}$$

In the noncontact region $u_\nu < 0$ and we have $\sigma_\nu = 0$. For $u_\nu \in [0, e)$ the contact is idealized by the Winkler law $-\sigma_\nu = k_0 u_\nu$. If $u_\nu = e$, the condition deals with destruction of the support and we have $-\sigma_\nu \in [0, k_0 e]$. When $u_\nu > e$, then $\sigma_\nu = 0$ and it holds in a region where the support has been destructed. The support can maintain the maximal value of reactions given by $k_0 e$. For more details, cf. Section 2.8 of Goeleven et al. [31]. For the nonmonotone Winkler law, the potential $j_1(r) = \int_0^r \vartheta(s)\, ds$ for $r \in \mathbb{R}$ and its subdifferential satisfy (see Figure 5)

$$j_1(r) = \begin{cases} 0 & \text{if } r < 0, \\ \frac{1}{2} k_0 r^2 & \text{if } 0 \leq r < e, \\ \frac{1}{2} k_0 e^2 & \text{if } r \geq e, \end{cases} \qquad \partial j_1(r) = \begin{cases} 0 & \text{if } r < 0, \\ k_0 r & \text{if } 0 \leq r < e, \\ [0, k_0 e] & \text{if } r = e, \\ 0 & \text{if } r > e. \end{cases}$$

It is easy to check that j_1 satisfies $H(j_1)_1$ with $c_{10} = c_{11} = c_{12} = 0$ and $c_{13} = k_0$. Moreover, since the function j_1 is the minimum of strictly differentiable functions, by Corollary 32, $-j_1$ is regular, and by Proposition 15(ii), the condition $H(j_1)_2$ holds.

We also observe that if, in addition, $p\colon \mathbb{R} \to \mathbb{R}$ is a continuous function, then the inclusion (120) reduces to

$$-\sigma_\nu(t) = p(u_\nu(t)) \quad \text{on } \Gamma_C \times (0, T). \tag{121}$$

The latter is an expression introduced for the first time by Martins and Oden [54, 76] and used in many models, for instance, in Han and Sofonea [34], Anderson [6], Kikuchi and Oden [44], Klarbring et al. [47], Rochdi et al. [88]. A commonly used form of the function p is $p(r) = c_\nu r_+$ or $p(r) = c_\nu (r_+)^m$, where $c_\nu > 0$ is the surface stiffness coefficient, $m \geq 1$ and $r_+ = \max\{0, r\}$ denotes the positive part of r.

If $p(r) = c_\nu r_+$, then the corresponding superpotential $j_1\colon \mathbb{R} \to \mathbb{R}$ is the following

$$j_1(r) = \int_0^r p(s)\, ds = \begin{cases} 0 & \text{if } r \leq 0, \\ \dfrac{c_\nu}{2} r^2 & \text{if } r > 0. \end{cases}$$

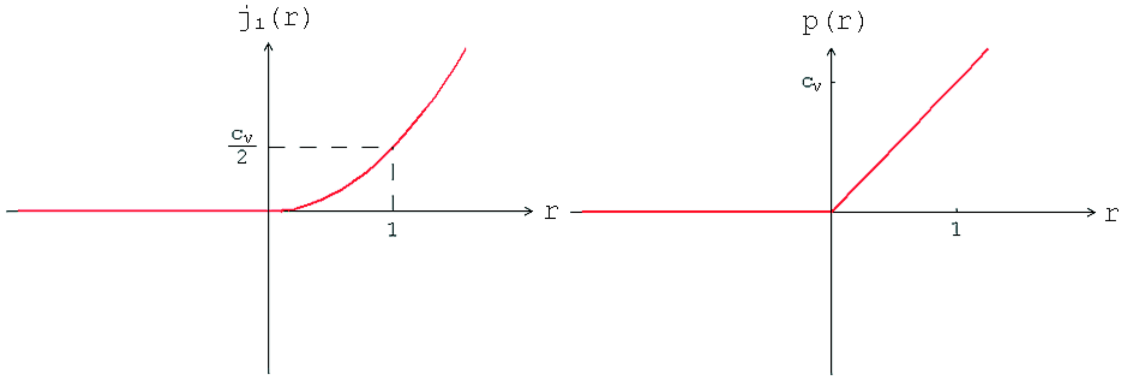

Figure 6: Normal compliance function

The function j_1 is continuously differentiable, its subdifferential $\partial j_1(r) = \{p(r)\}$ is a singleton for all $r \in \mathbb{R}$ (see Figure 6) and $|\partial j_1(r_1) - \partial j_1(r_2)| = |p(r_1) - p(r_2)| \leq c_\nu |r_1 - r_2|$ for all $r_1, r_2 \in \mathbb{R}$. Hence the function j_1 satisfies $H(j_1)_2$ and $H(j_1)_3$.

We can also consider the following truncated normal compliance function (cf. [34])

$$p(r) = \begin{cases} c_\nu r_+ & \text{if } r \leq r_0, \\ c_\nu r_0 & \text{if } r > r_0, \end{cases}$$

where $r_0 > 0$ is a constant related to the wear and the hardness of the surface of the body. In this case the equation (121) means that when the penetration is too large, i.e. when it exceeds the value r_0, the obstacle offers no additional resistance to penetration. For the truncated normal compliance function, the superpotential has the form

$$j_1(r) = \begin{cases} 0 & \text{if } r \leq 0, \\ \dfrac{c_\nu}{2} r^2 & \text{if } r \in (0, r_0), \\ c_\nu \, r_0 \, r - \dfrac{c_\nu r_0^2}{2} & \text{if } r \geq r_0, \end{cases}$$

(see Figure 7). It satisfies $H(j_1)_1$ with $c_{10} = c_{11} = c_{12} = 0$, $c_{13} = c_\nu r_0$, $H(j_1)_2$, $H(j_1)_3$ and since it is convex, it is also regular.

We remark that when the surface stiffness coefficient becomes infinite, i.e. $c_\nu \to +\infty$ (and thus the interpenetration is not allowed), the normal compliance condition leads formally to the Signorini contact condition

$$u_\nu \leq 0, \quad \sigma_\nu \leq 0, \text{ and } \sigma_\nu \, u_\nu = 0.$$

The latter is an idealization of the normal compliance and corresponds to contact of the body with a rigid support. The Signorini condition can be regarded as the limiting case of contact with deformable support whose resistance to compression increases. The result of the previous sections can not be applied to the Signorini contact condition since it does not satisfy the growth condition $H(j_1)_1$(iii).

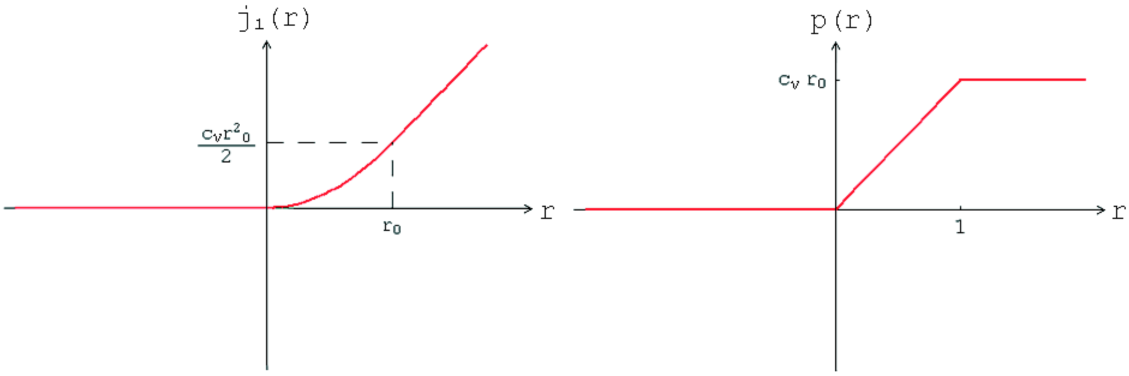

Figure 7: Truncated normal compliance function

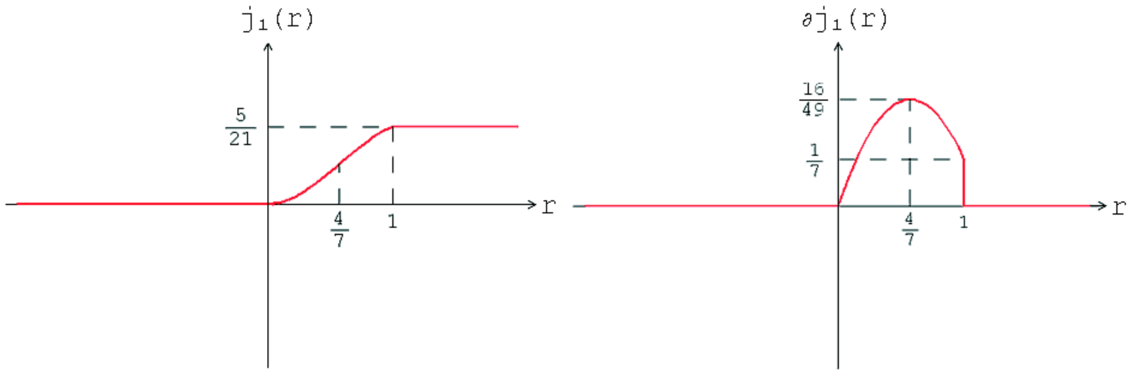

Figure 8: Nonmonotone law for a granular material

The following example of the nonmonotone normal compliance relation is the normal contact law which can be expressed in the form (120) which holds between a deformable body and a support of a granular material or concrete. It was described in Chapter 2.4 of [78] (cf. Figure 2.4.1). In this case the superpotential $j_1\colon \mathbb{R} \to \mathbb{R}$ and its subdifferential take the form (see Figure 8)

$$j_1(r) = \begin{cases} 0 & \text{if } r < 0, \\ -\frac{1}{3}r^3 + \frac{4}{7}r^2 & \text{if } 0 \le r < 1, \\ \frac{5}{21} & \text{if } r \ge 1, \end{cases} \qquad \partial j_1(r) = \begin{cases} 0 & \text{if } r < 0, \\ -r^2 + \frac{8}{7}r & \text{if } 0 \le r < 1, \\ [0, \frac{1}{7}] & \text{if } r = 1, \\ 0 & \text{if } r > 1. \end{cases}$$

It is easy to observe that the function j_1 satisfies $H(j_1)_2$ with $c_{10} = 16/49$ and $c_{11} = c_{12} = c_{13} = 0$. It can also be represented (see Figure 9) as the difference of convex functions, $j_1(r) = \varphi_1(r) - \varphi_2(r)$, $r \in \mathbb{R}$, where

$$\varphi_1(r) = \begin{cases} 0 & \text{if } r < 0, \\ -\frac{1}{3}r^3 + \frac{4}{7}r^2 & \text{if } 0 \le r < \frac{4}{7}, \\ \frac{16}{49}r - \frac{64}{1029} & \text{if } r \ge \frac{4}{7}, \end{cases} \quad \varphi_2(r) = \begin{cases} 0 & \text{if } r < \frac{4}{7}, \\ \frac{1}{3}r^3 - \frac{4}{7}r^2 + \frac{16}{49}r - \frac{64}{1029} & \text{if } \frac{4}{7} \le r < 1, \\ \frac{16}{49}r - \frac{103}{343} & \text{if } r \ge 1. \end{cases}$$

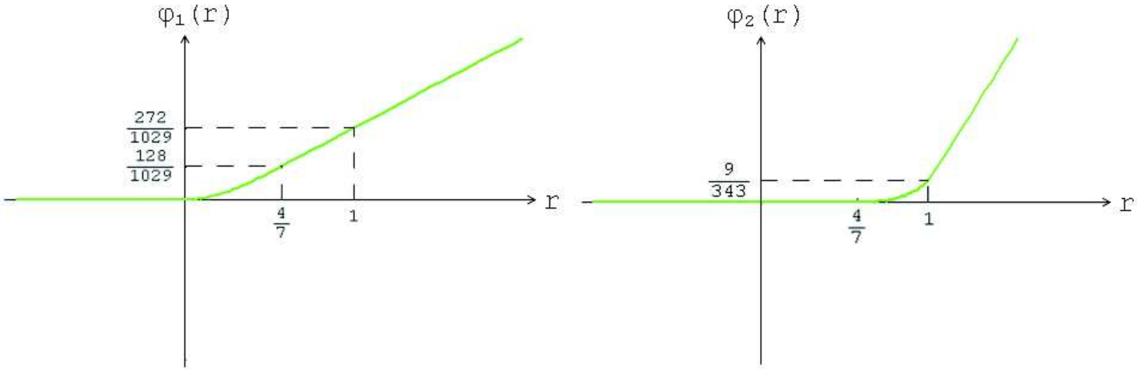

Figure 9: Two convex functions for the potential in Figure 8

Since φ_1, φ_2 are convex functions and $\partial\varphi_1$ is a singleton, by Proposition 33, we infer that the function $-j_1$ is regular.

6.2.4 Contact with nonmonotone normal damped response

This contact condition is of the form (111) with $j_1 = 0$ and it models the situations with granular or wet surfaces in which the response of the foundation depends on the normal speed of the body. For simplicity, we describe the case when

$$-\sigma_\nu(t) \in \partial j_2(u'_\nu(t)) \quad \text{on } \Gamma_C \times (0, T).$$

Analogously as in Section 6.2.3, we consider the superpotential $j_2 \colon \mathbb{R} \to \mathbb{R}$ given by $j_2(r) = \int_0^r p(s)\, ds$ for $r \in \mathbb{R}$ where the function p satisfies hypothesis $H(p)$ of Section 6.2.3. In this case, we obtain

$$-\sigma_\nu(t) \in \widehat{p}\,(u'_\nu(t)) \quad \text{on } \Gamma_C \times (0, T).$$

When, in addition, p is a continuous function, then the above reduces to $-\sigma_\nu(t) = p(u'_\nu(t))$ on $\Gamma_C \times (0, T)$ which is the relation frequently studied in the literature, cf. Awbi et al. [10] and Shillor et al. [93]. If $p(r) = k_1 r$ with $k_1 > 0$, we have $-\sigma_\nu = k_1 u'_\nu$ on $\Gamma_C \times (0, T)$ which means that the resistance of the foundation to penetration is proportional to the normal velocity. This type of boundary condition was considered by Sofonea and Shillor [92] and models the motion of a deformable body on a support of granular material. If $p(r) = k_2 r_+ + k_3$, where $k_2 > 0$ and $k_3 \geq 0$, we get the model studied by Rochdi et al. [88] in which the contact surface Γ_C was supposed to be covered with a lubricant that contains solid particles, such as one of the new smart lubricants or with worn metallic particles. The constant k_2 denotes the damping resistance whereas k_3 represents the prescribed oil pressure. This contact condition models the phenomenon that the oil layer presents damping or resistance, only when the surface moves towards the foundation. The particular form of the normal damped response condition has been studied in the dynamic case in Chau et al. [19], where $-\sigma_\nu = p(x, u'_\nu)$ is considered with $p(x, \cdot)$ continuous and monotone. The corresponding quasistatic case was treated in Awbi et al. [9, 10].

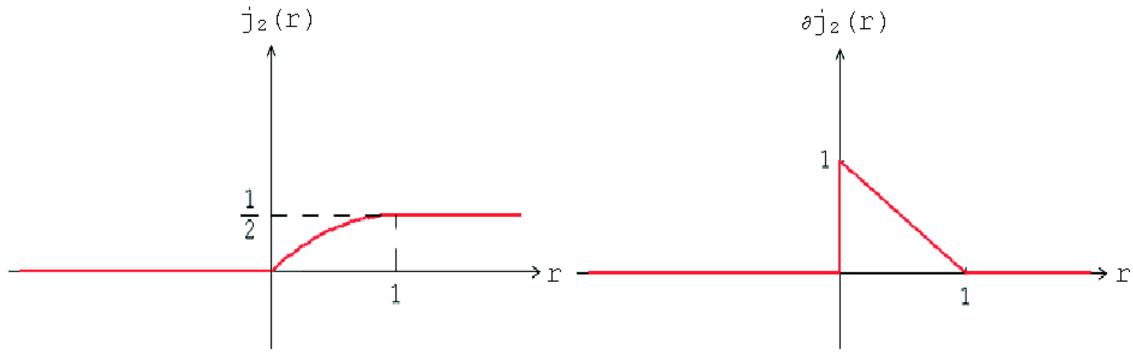

Figure 10: Nonmonotone normal damped response condition

The specific example of the nonmonotone normal damped response condition is given by the following nonconvex, regular, d.c. function which together with its subdifferential is depicted in Figure 10:

$$j_2(r) = \begin{cases} 0 & \text{if } r < 0, \\ -\frac{1}{2}r^2 + r & \text{if } 0 \le r < 1, \\ \frac{1}{2} & \text{if } r \ge 1, \end{cases} \qquad \partial j_2(r) = \begin{cases} 0 & \text{if } r < 0, \\ [0,1] & \text{if } r = 0, \\ -r+1 & \text{if } 0 < r < 1, \\ 0 & \text{if } r \ge 1. \end{cases}$$

It is clear that $|\partial j_2(r)| \le 1 + |r|$ for $r \in \mathbb{R}$, i.e. $H(j_2)_1$ holds with $c_{20} = c_{23} = 1$, $c_{21} = c_{22} = 0$. Next, we verify that $\eta_1 \le \eta_2 - (r_1 - r_2)$ for all $r_1 < r_2$ and $\eta_i \in \partial j_2(r_i)$, $i = 1, 2$ which implies relaxed monotonicity condition $(\partial j_2(r_1) - \partial j_2(r_2))(r_1 - r_2) \ge -|r_1 - r_2|^2$ (cf. Remark 61), and $H(j_2)_3$ with $L_2 = 1$. The function j_2 can be represented (see Figure 11) as the difference of convex functions, i.e. $j_2(r) = \varphi_1(r) - \varphi_2(r)$, $r \in \mathbb{R}$, where

$$\varphi_1(r) = \begin{cases} \frac{1}{2}r^2 - r + 1 & \text{if } r < 0, \\ 1 & \text{if } 0 \le r < 1, \\ \frac{1}{2}r^2 - r + \frac{3}{2} & \text{if } r \ge 1, \end{cases} \qquad \varphi_2(r) = \frac{1}{2}r^2 - r + 1.$$

Since φ_1, φ_2 are convex functions, $\partial\varphi_1$, $\partial\varphi_2$ have a sublinear growth with $\partial\varphi_2$ being a singleton, we deduce by Proposition 33 that j_2 is regular with $\partial j_2(r) = \partial\varphi_1(r) - \partial\varphi_2(r)$ for $r \in \mathbb{R}$. Moreover, by Proposition 15(ii), it is obvious that $H(j_2)_2$ holds.

6.2.5 Viscous contact with Tresca's friction law

We consider a model of damped response contact with time-dependent Tresca's friction law. In this model the contact is characterized by the following boundary conditions

$$\begin{cases} -\sigma_\nu(t) = k(x)|u'_\nu(t)|^{q-1}u'_\nu(t), \\ \|\sigma_\tau(t)\| \le \psi(t) \text{ with} \\ \qquad \|\sigma_\tau(t)\| < \psi(t) \Rightarrow u'_\tau(t) = 0, \\ \qquad \|\sigma_\tau(t)\| = \psi(t) \Rightarrow \exists\, \lambda \ge 0 \ : \ \sigma_\tau(t) = -\lambda u'_\tau(t) \end{cases}$$

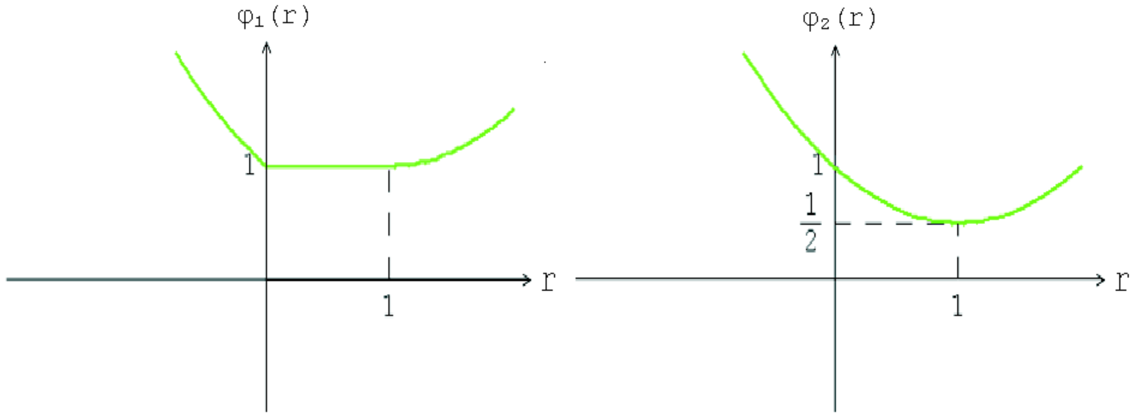

Figure 11: Two convex functions for the potential in Figure 10

on $\Gamma_C \times (0,T)$, where $k \in L^\infty(\Gamma_C)$, $k > 0$ a.e. on Γ_C, $0 < q \le 1$, $\psi \in L^\infty(\Gamma_C \times (0,T))$ and $\psi \ge 0$ a.e. on $\Gamma_C \times (0,T)$, cf. Shillor and Sofonea [92] and Chapter 13 of Han and Sofonea [34]. These boundary conditions are of the form (111) and (112) with $j_1 = j_3 = 0$, $j_2(x,t,\zeta,\rho,r) = \frac{k(x)}{q+1}|r|^{q+1}$ and $j_4(x,t,\zeta,\rho,\theta) = \psi(x,t)\|\theta\|$. Therefore

$$\partial j_2(x,t,\zeta,\rho,r) = k(x)|r|^{q-1}r,$$

$$\partial j_4(x,t,\zeta,\rho,\theta) = \psi(x,t)\,\partial\|\theta\| = \begin{cases} \psi(x,t)\,\overline{B}(0,1) & \text{if } \eta = 0, \\ \psi(x,t)\dfrac{\eta}{\|\eta\|} & \text{if } \eta \neq 0. \end{cases}$$

Thus $H(j_2)_1$ holds with $c_{20} = c_{23} = \|k_0\|_{L^\infty(\Gamma_C)}$, $c_{21} = c_{22} = 0$ while j_4 satisfies $H(j_4)_1$ with $c_{40} = \|\psi\|_{L^\infty(\Gamma_C\times(0,T))}$, $c_{41} = c_{42} = c_{43} = 0$ and $H(j_4)_2$; j_4 is also convex (so regular) in θ and $H(j_4)_3$ holds (by the argument of Section 6.2.2.2). Classically the Tresca friction law is characterized by a given constant friction bound, that is, $\psi(x,t) = \text{const.}$, cf. e.g. Amassad and Fabre [3], Amassad and Sofonea [4, 5], Duvaut and Lions [27], Han and Sofonea [34], Panagiotopoulos [77], Selmani and Sofonea [92].

6.2.6 Viscous contact with power-law friction condition

In this model, the boundary conditions are of the form (111) and (112) with $j_1 = j_3 = 0$, j_2 is as in Section 6.2.5 and $j_4(x,t,\zeta,\rho,\theta) = \frac{\mu(x)}{p+1}\|\theta\|^{p+1}$, where $\mu \in L^\infty(\Gamma_C)$, $\mu > 0$ a.e. on Γ_C and $0 < p \le 1$. This choice leads to the following contact and friction laws

$$\begin{cases} -\sigma_\nu(t) = k(x)|u'_\nu(t)|^{q-1}u'_\nu(t), \\ -\sigma_\tau(t) = \mu(x)\|u'_\tau(t)\|^{p-1}u'_\tau(t) \quad \text{on } \Gamma_C \times (0,T), \end{cases}$$

with $k \in L^\infty(\Gamma_C)$, $k > 0$ a.e. on Γ_C.

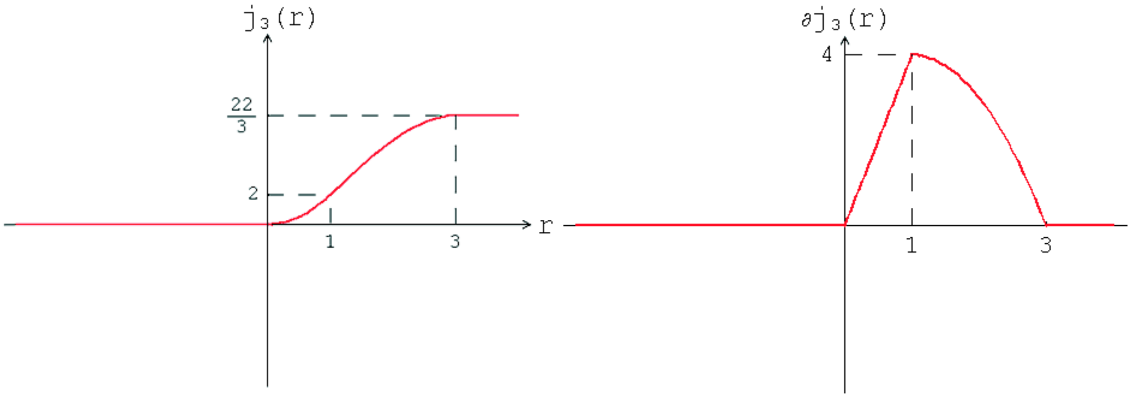

Figure 12: Nonmonotone law between reinforcement and concrete

6.2.7 Other nonmonotone friction contact laws

In this part we comment on the boundary conditions expressed in the form

$$-\sigma_\tau(t) \in \partial j_3(x, t, u(t), u'(t), u_\tau(t)). \tag{122}$$

This relation may be considered both in the framework of a small or a large deformation theory. It describes the tangential contact law between reinforcement and concrete in a concrete structure. In literature, cf. Chapter 2.4 in Panagiotopoulos [78] (diagrams of Figure 2.4.1), Chapter 1.4 in Naniewicz and Panagiotopoulos [73] (diagrams of Figure 1.4.3), one can find a couple of examples of the superpotential j_3 which describes such type of contact. We give two examples of nonconvex functions which appear in (122).

In the first example the superpotential $j_3\colon \mathbb{R} \to \mathbb{R}$ and its subdifferential are of the form (see Figure 12).

$$j_3(r) = \begin{cases} 0 & \text{if } r < 0, \\ 2r^2 & \text{if } 0 \le r < 1, \\ -\frac{1}{3}r^3 + r^2 + 3r - \frac{5}{3} & \text{if } 1 \le r < 3, \\ \frac{22}{3} & \text{if } r \ge 3, \end{cases} \quad \partial j_3(r) = \begin{cases} 0 & \text{if } r < 0, \\ 4r & \text{if } 0 \le r < 1, \\ -r^2 + 2r + 3 & \text{if } 1 \le r < 3, \\ 0 & \text{if } r \ge 3. \end{cases}$$

It is easy to check that the function j_3 satisfies $H(j_3)_1$ with $c_{30} = 4$, $c_{31} = c_{32} = c_{33} = 0$. Furthermore, j_3 can be represented (see Figure 13) as the difference of convex functions, $j_3(r) = \varphi_1(r) - \varphi_2(r)$, $r \in \mathbb{R}$ with

$$\varphi_1(r) = \begin{cases} 0 & \text{if } r < 0, \\ 2r^2 & \text{if } r \ge 0, \end{cases} \qquad \varphi_2(r) = \begin{cases} 0 & \text{if } r < 1, \\ \frac{1}{3}r^3 + r^2 - 3r + \frac{5}{3} & \text{if } 1 \le r < 3, \\ 2r^2 - \frac{22}{3} & \text{if } r \ge 3. \end{cases}$$

Since φ_1, φ_2 are convex functions and $\partial\varphi_1$ is a singleton, from Proposition 33 we deduce that the function $-j_3$ is regular.

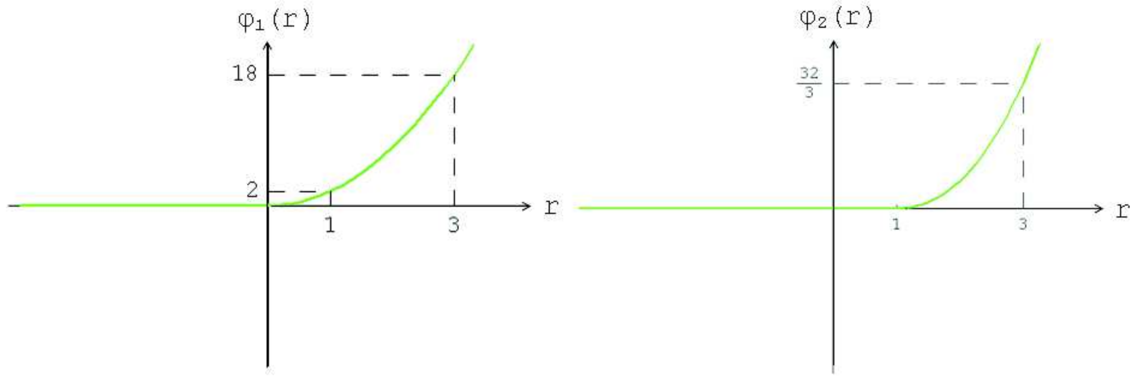

Figure 13: Two convex functions for the potential in Figure 12

In the second example, we consider (see Figure 14) the function $j_3\colon \mathbb{R} \to \mathbb{R}$ such that

$$j_3(r) = \begin{cases} 0 & \text{if } r < 0, \\ r^2 & \text{if } 0 \le r < 1, \\ \frac{1}{8}r^4 - r^3 + \frac{9}{4}r^2 - \frac{3}{8} & \text{if } 1 \le r < 3, \\ 3 & \text{if } r \ge 3, \end{cases} \quad \partial j_3(r) = \begin{cases} 0 & \text{if } r < 0, \\ 2r & \text{if } 0 \le r < 1, \\ \frac{1}{2}r^3 - 3r^2 + \frac{9}{2}r & \text{if } 1 \le r < 3, \\ 0 & \text{if } r \ge 3. \end{cases}$$

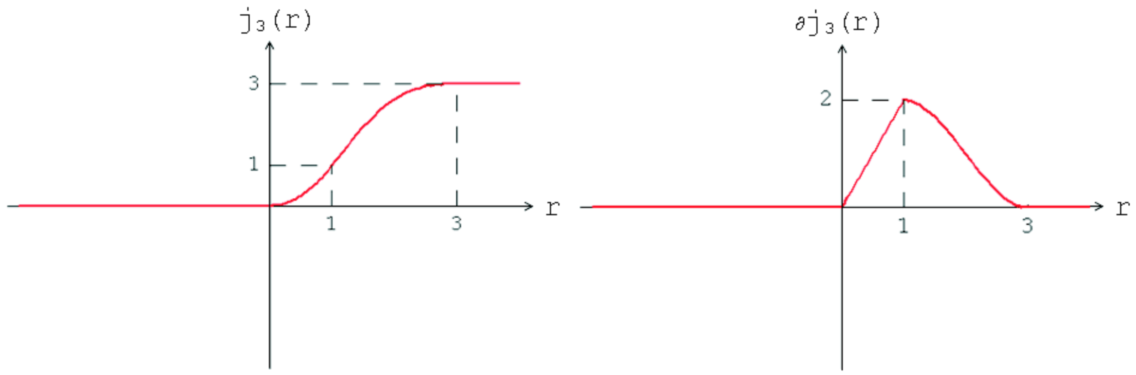

Figure 14: Tangential contact law for a concrete structure

Similarly to the previous case, j_3 satisfies $H(j_3)_1$ with $c_{30} = 2$, $c_{31} = c_{32} = c_{33} = 0$ and $H(j_3)_3$. It can be also represented (Fig. 15) as the difference of convex functions, $j_3(r) = \varphi_1(r) - \varphi_2(r)$, $r \in \mathbb{R}$, where

$$\varphi_1(r) = \begin{cases} 0 & \text{if } r < 0, \\ r^2 & \text{if } r \ge 0, \end{cases} \qquad \varphi_2(r) = \begin{cases} 0 & \text{if } r < 1, \\ -\frac{1}{8}r^4 + r^3 - \frac{5}{4}r^2 + \frac{3}{8} & \text{if } 1 \le r < 3, \\ r^2 - 3 & \text{if } r \ge 3. \end{cases}$$

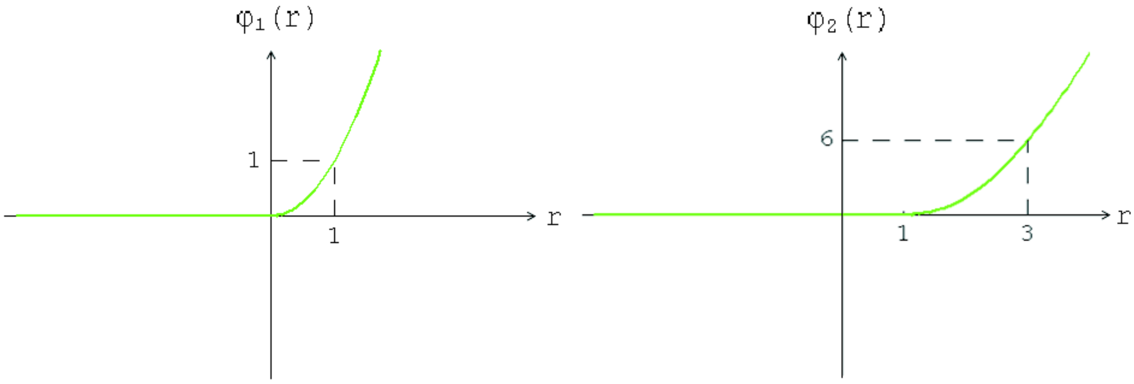

Figure 15: Two convex functions for the potential in Figure 14

Again, from the fact that φ_1 and φ_2 are convex functions and $\partial\varphi_1$ is a singleton, we conclude that $-j_3$ is regular.

We end this section with indications on specific applications of research on contact problems. It is of importance to provide various applications of the theoretical results to contact problems arising in real world. It is clear that economic profits may come from more accurate prediction and the evaluation of frictional contact processes in mechanical and civil engineering. The reduction of costs can be achieved by numerical simulations that will model the time dependent behavior of considered systems. For instance, the applications concern the following areas:

Construction and exploitation of machines. The understanding of contact problems are extremely important in various branches of engineering such as structural foundations, bearings, metal forming processes, rubber sealings, aeronautics, drilling problems, the simulation of car crashes, the car braking system, rolling contact between car tyres and the road, contact of train wheels with the rails, a shoe with the floor, tectonic plates, machine tools, bearings, motors, turbines, cooling of electronic devices, joints in mechanical devices, ski lubricants, and many more, cf. e.g. Andrews et al. [7], Chau et al. [19], Kuttler and Shillor [49, 50], Rochdi et al. [88] and Sofonea and Matei [96].

Biomechanics. The applications concerns the medical field of arthoplasty where bonding between the bone implant and the tissue is of considerable importance since debonding may lead to decrease in the persons ability to use the artificial limb or joint. Artificial implants of knee and hip prostheses (both cemented and cement-less) demonstrate that the adhesion is important at the bone-implant interface. These applications are related to contact modeling and design of biomechanal parts like human joints, implants or teeth, cf. Panagiotopoulos [78], Rojek and Telega [90], Rojek et al. [91], Shillor et al. [93] and Sofonea et al. [95].

Plate tectonics and earthquakes predictions. Results may be applicable to models with nonmonotone strain-stress laws in rock layers. Frictional contact between rocks are described by several models, cf. Dumont et al. [26], Ionescu et al. [38, 40], Ionescu and Nguyen [39], Ionescu and Paumier [41, 42] and Rabinowicz [86].

Medicine and biology. Results are applicable to nonmonotone semipermeable membranes and walls (biological and artificial), cf. Duvaut and Lions [27]. In particular, contact problems for piezoelectric materials will continue to play a decisive role in the field of ultrasonic tranducers for imaging applications, e.g. medical imaging (sonogram), nondestructive testing and high power applications (medical treatment, sonochemistry and industrial processing), cf. Shillor et al. [93], Sofonea et al. [95].

7 Appendix

In this section for the convenience of the reader, we recall some definitions and results from nonlinear analysis which are frequently used in this work. Most of the prerequisite material presented here can be found in standard textbooks such as Aubin and Cellina [8], Castaing and Valadier [17], Denkowski et al. [23, 24], Evans [29], Hu and Papageorgiou [37], Kisielewicz [46], and Zeidler [99].

Definition 67 *A measurable space is a pair (Ω, Σ) where Ω is a set and Σ is a σ-algebra of subsets of Ω. A collection Σ of subsets of Ω is called σ-algebra if*

(i) $\emptyset \in \Sigma$;

(ii) if $A \in \Sigma$ then $\Omega \setminus A \in \Sigma$;

(iii) if $A_n \in \Sigma$, $n \in \mathbb{N}$ then $\cup_{n=1}^{\infty} A_n \in \Sigma$.

The elements of Σ are called measurable sets. If Ω is a topological space, then the smallest σ-algebra containing all open sets is called the Borel σ-algebra and it is denoted by $\mathcal{B}(\Omega)$.

Definition 68 (i) *If (Ω_1, Σ_1) and (Ω_2, Σ_2) are measurable spaces, then $f\colon \Omega_1 \to \Omega_2$ is called measurable (or (Σ_1, Σ_2)-measurable) when $f^{-1}(\Sigma_2) \subseteq \Sigma_1$.*

(ii) *If Y_1, Y_2 are Hausdorff topological spaces, then $f\colon Y_1 \to Y_2$ is called Borel measurable when $f^{-1}(\mathcal{B}(Y_2)) \subseteq \mathcal{B}(Y_1)$.*

(iii) *If (Ω, Σ) is a measurable space and Y is a Hausdorff topological space, then $f\colon \Omega \to Y$ is called measurable when $f^{-1}(\mathcal{B}(Y)) \subseteq \Sigma$.*

Lemma 69 (cf. Proposition 2.4.3 of [23]) *Let (Ω_1, Σ_1) and (Ω_2, Σ_2) be measurable spaces and $f\colon \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a $\Sigma_1 \times \Sigma_2$-measurable function. Then $f(\omega_1, \cdot)$ is Σ_2-measurable for each $\omega_1 \in \Omega_1$ and $f(\cdot, \omega_2)$ is Σ_1-measurable for each $\omega_2 \in \Omega_2$.*

Definition 70 (cf. Definition 2.5.18 of [23]) *Let (Ω, Σ) be a measurable space and Y_1, Y_2 be topological spaces. A function $f\colon \Omega \times Y_1 \to Y_2$ is said to be a Carathéodory function if $f(\cdot, y)$ is $(\Sigma, \mathcal{B}(Y_2))$-measurable for every $y \in Y_1$ and $f(\omega, \cdot)$ is continuous for every $\omega \in \Omega$.*

The following is an important property of Carathéodory functions.

Lemma 71 (cf. Theorem 2.5.22 of [23]) *If (Ω, Σ) is a measurable space, Y_1 is a separable metric space, Y_2 is a metric space, $f\colon \Omega \times Y_1 \to Y_2$ is a Carathéodory function and $x\colon \Omega \to Y_1$ is Σ-measurable, then $\Omega \ni \omega \mapsto f(\omega, x(\omega)) \in Y_2$ is Σ-measurable.*

Definition 72 *Let* (Ω, Σ) *be a measurable space. A set function* $\mu\colon \Sigma \to [0,\infty]$ *is a measure on* Σ *if* $\mu(\emptyset) = 0$ *and* $\mu\left(\cup_{n=1}^{\infty} A_n\right) = \Sigma_{n=1}^{\infty}\mu(A_n)$ *for every infinite sequence* $\{A_n\}$ *of pairwise disjoint sets from* Σ*. A measure on* Σ *is said to be finite if* $\mu(\Omega) < \infty$*. A measure on* Σ *is called* σ*-finite if* $\Omega = \cup_{n=1}^{\infty}\Omega_n$*,* $\Omega_n \in \Sigma$ *and* $\mu(\Omega_n) < \infty$ *for all* $n \geq 1$*. If* (Ω, Σ) *is a measurable space and* μ *is a measure on* Σ*, then the triple* (Ω, Σ, μ) *is called a measure space.*

Definition 73 *Let* Y *be a normed space and* $A \in 2^Y \setminus \{\emptyset\}$*. The support function of the set* A *is defined by* $Y^* \ni y^* \mapsto \sigma(y^*, A) = \sup\{\, \langle y^*, a\rangle \mid a \in A \,\} \in \mathbb{R} \cup \{+\infty\}$*, where* $\langle \cdot, \cdot \rangle$ *denotes the duality pairing of* Y^* *and* Y*.*

In what follows (Ω, Σ) is assumed to be a measurable space.

Definition 74 *Let* Y *be a separable metric space. A multifunction (set-valued function)* $F\colon \Omega \to 2^Y$ *is said to be measurable if for every* $U \subset Y$ *open, the weak inverse image* $F^-(U) = \{\omega \in \Omega \mid F(\omega) \cap U \neq \emptyset\} \in \Sigma$*.*

Definition 75 *Let* Y *be a separable Banach space. A multifunction* $F\colon \Omega \to 2^Y$ *is said to be scalarly measurable if for every* $y^* \in Y^*$ *the function* $\Omega \ni \omega \mapsto \sigma(y^*, F(\omega)) \in \mathbb{R} \cup \{+\infty\}$ *is measurable.*

It is known (see Proposition 4.3.16 of [23]) that for $\mathcal{P}_{wkc}$-valued multifunctions scalar measurability is equivalent to measurability.

Lemma 76 *Let* (Ω, Σ) *be a measurable space and* Y *be a separable Banach space. If* $F\colon \Omega \to \mathcal{P}_{wkc}(Y)$*, then* F *is measurable if and only if* F *is scalarly measurable.*

Definition 77 *Let* Y_1 *and* Y_2 *be Hausdorff topological spaces and* $F\colon Y_1 \to Y_2$ *be a multifunction. We say that* F *is upper semicontinuous at* $y_0 \in Y_1$*, if for all* $V \subseteq Y_2$ *open such that* $F(y_0) \subseteq V$*, we can find a neighborhood* $U \in \mathcal{N}(y_0)$ *such that* $F(U) \subseteq V$*. We say that* F *is upper semicontinuous, if it is upper semicontinuous at every* $y_0 \in Y_1$*.*

Remark 78 *It can be shown (cf. Proposition 4.1.4 of [23]) that* $F\colon Y_1 \to Y_2$ *is upper semicontinuous if and only if for every* $C \subseteq Y_2$ *closed, the weak inverse image* $F^-(C)$ *is closed in* Y_1*.*

For an impressive list of criteria of measurability and semicontinuity of multifunctions, cf. [17], Chapter 4 of [23] and Chapter 2 of [37].

Lemma 79 (Fubini's theorem) (cf. Theorem 2.4.10 of [23]) *Let* $(\Omega_1, \Sigma_1, \mu_1)$*,* $(\Omega_2, \Sigma_2, \mu_2)$ *be* σ*-finite measure spaces, and let* $f\colon \Omega_1 \times \Omega_2 \to [-\infty, \infty]$ *be* $\mu_1 \times \mu_2$ *integrable function. Then for* μ_1*-almost all* $\omega_1 \in \Omega_1$*, we have*

$$\text{the function } f(\omega_1, \cdot) \text{ is } \mu_2\text{-integrable;}$$

$$\text{the function } \int_{\Omega_2} f(\cdot, \omega_2)\, d\mu_2(\omega_2) \text{ is } \mu_1\text{-integrable.}$$

Similarly for μ_2-almost all $\omega_2 \in \Omega_2$, we have

$$\text{the function } f(\cdot, \omega_2) \text{ is } \mu_1\text{-integrable;}$$
$$\text{the function } \int_{\Omega_1} f(\omega_1, \cdot)\, d\mu_1(\omega_1) \text{ is } \mu_2\text{-integrable.}$$

Moreover,

$$\begin{aligned} \int_{\Omega_1 \times \Omega_2} f(\omega_1, \omega_2)\, d(\mu_1 \times \mu_2) &= \int_{\Omega_1} \Big(\int_{\Omega_2} f(\omega_1, \omega_2)\, d\mu_2(\omega_2) \Big)\, d\mu_1(\omega_1) \\ &= \int_{\Omega_2} \Big(\int_{\Omega_1} f(\omega_1, \omega_2)\, d\mu_1(\omega_1) \Big)\, d\mu_2(\omega_2). \end{aligned}$$

Lemma 80 (Fatou's lemma) (cf. Theorem 2.2.17 of [23]) *Let (Ω, Σ, μ) be a measure space and $f_n \colon \Omega \to \mathbb{R}$ be a sequence of measurable functions such that there is $h \in L^1(\Omega)$ with $f_n \le h$ μ-a.e. on Ω. Then*

$$\limsup \int_\Omega f_n\, d\mu \le \int_\Omega \limsup f_n\, d\mu.$$

If there is a function $h_1 \in L^1(\Omega)$ such that $f_n \ge h_1$ μ-a.e. on Ω, then

$$\int_\Omega \liminf f_n\, d\mu \le \liminf \int_\Omega f_n\, d\mu.$$

Lemma 81 (Jensen's inequality) (cf. Theorem 2.2.51 of [23]) *Let (Ω, Σ, μ) be a finite measure space, $I \subset \mathbb{R}$ be an open interval, $\varphi \colon I \to \mathbb{R}$ be a convex function, $f \in L^1(\Omega)$ with $f(\Omega) \subseteq I$ and $\varphi \circ f \in L^1(\Omega)$. Then*

$$\varphi\Big(\frac{1}{\mu(\Omega)} \int_\Omega f\, d\mu \Big) \le \frac{1}{\mu(\Omega)} \int_\Omega (\varphi \circ f)\, d\mu.$$

Subsequently, we present a result on the differentiation of locally Lipschitz integral functionals.

Let $1 < p < \infty$, $1/p + 1/q = 1$ and D be a bounded subset of $\mathbb{R}^n$. Let $j \colon D \times \mathbb{R}^N \to \mathbb{R}$ be a function such that

(i) $j(\cdot, \xi)$ is measurable for all $\xi \in \mathbb{R}^N$ and $j(\cdot, 0)$ is (finitely) integrable;

(ii) $j(x, \cdot)$ is locally Lipschitz for a.e. $x \in D$;

(iii) there are a constant $c > 0$ and a function $a \in L^q(D)$ such that for all $\eta \in \partial j(x, \xi)$, we have
$$\|\eta\| \le a(x) + c\|\xi\|^{p-1} \quad \text{for all } \xi \in \mathbb{R}^N, \text{ a.e. } x \in D.$$

We set
$$J(v) = \int_D j(x, v(x))\, dx \quad \text{for all } v \in L^p(D; \mathbb{R}^N). \tag{123}$$

Lemma 82 (Aubin-Clarke's theorem) (cf. Theorem 5.6.39 of [23]) *Under the above hypotheses, the functional* $J\colon L^p(D;\mathbb{R}^N)\to\mathbb{R}$ *given by (123) is well defined, it is finite, Lipschitz continuous on bounded subsets of* $L^p(D;\mathbb{R}^N)$*, and*

$$\partial J(v)\subset\int_D\partial j(x,v(x))\,dx\quad\text{for all } v\in L^p(D;\mathbb{R}^N),$$

in the sense that for every $\zeta\in\partial J(v)$*, there is a function* $z\in L^q(D;\mathbb{R}^N)$ *satisfying* $z(x)\in\partial j(x,v(x))$ *a.e.* $x\in D$ *and such that*

$$\langle\zeta,y\rangle_{L^q(D;\mathbb{R}^N)\times L^p(D;\mathbb{R}^N)}=\int_D\langle z(x),y(x)\rangle_{\mathbb{R}^N}\,dx\quad\text{for all } y\in L^p(D;\mathbb{R}^N).$$

We recall also the convergence theorem of Aubin and Cellina.

Proposition 83 (Convergence theorem) *Let* F *be an upper semicontinuous map from a Hausdorff locally convex space* X *to the closed convex subsets of a Banach space* Y *endowed with the weak topology. Let* $x_k(\cdot)$ *and* $y_k(\cdot)$ *be measurable function from* $(0,T)$ *to* X *and* Y*, respectively satisfying the following condition: for almost all* $t\in(0,T)$*, for every neighborhood* $\mathcal{N}_0$ *of* 0 *in* $X\times Y$ *there exists* k_0 *such that* $(x_k(t),y_k(t))\in graph(F)+\mathcal{N}_0$ *for all* $k\geq k_0$*. If*

(i) $x_k(\cdot)$ *converges almost everywhere to a function* $x(\cdot)$ *from* $(0,T)$ *to* X*,*

(ii) $y_k(\cdot)$ *belongs to* $L^1(0,T;Y)$ *and converges weakly to* $y(\cdot)$ *in* $L^1(0,T;Y)$*,*

then $(x(t),y(t))\in graph(F)$*, i.e.* $y(t)\in F(x(t))$ *for a.e.* $t\in(0,T)$*.*

The proof of Proposition 83 can be found in Theorem 5 of Aubin and Cellina [8, p. 60], which contains a more general case of upper hemicontinuous map. Considering the fact that any upper semicontinuous map from X to Y endowed with the weak topology is upper hemicontinuous (cf. Proposition 1 of [8]), we conclude that Proposition 83 holds.

Lemma 84 (Banach Contraction Principle) (cf. Theorem 6.7.3 od [23]) *If* (X,d) *is a complete metric space and* $f\colon X\to X$ *is a* k*-contraction (i.e. for all* x*,* $y\in X$ *we have* $d(f(x),f(y))\leq k\,d(x,y)$ *with* $k<1$*), then* f *has a unique fixed point.*

Lemma 85 (Young's inequality) *Let* $1<p<\infty$*,* $1/p+1/q=1$ *and* $\varepsilon>0$*. Then*

$$a\,b\leq\frac{\varepsilon^p}{p}|a|^p+\frac{1}{\varepsilon^q q}|b|^q\quad\text{for all}\quad a,b\in\mathbb{R}.$$

Lemma 86 (Gronwall's inequality) *If* $f\colon[0,T]\to\mathbb{R}$ *is a continuous function,* h*,* $k\in L^1(0,T)$*,* $k\geq 0$ *and*

$$f(t)\leq h(t)+\int_0^t k(s)\,f(s)\,ds\quad\text{for all } t\in[0,T],$$

then

$$f(t)\leq h(t)+\int_0^t exp\Big(\int_s^t k(r)dr\Big)k(s)\,h(s)\,ds\quad\text{for all } t\in[0,T].$$

Lemma 87 *If a_i, $i = 1, \ldots, m$, are nonnegative reals, then we have*

$$\text{(i)} \qquad \sum_{i=1}^{m} |a_i|^p \leq \left|\sum_{i=1}^{m} a_i\right|^p \leq |m|^{p-1} \sum_{i=1}^{m} |a_i|^p \quad \text{for} \quad 1 \leq p < +\infty,$$

$$\text{(ii)} \qquad |m|^{p-1} \sum_{i=1}^{m} |a_i|^p \leq \left|\sum_{i=1}^{m} a_i\right|^p \leq \sum_{i=1}^{m} |a_i|^p \quad \text{for} \quad 0 < p \leq 1.$$

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