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Expansivity and Cone-fields in Metric Spaces

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Abstract Due to the results of Lewowicz and Tolosa expansivity can be characterized with the aid of Lyapunov function. In this paper we study a similar problem for uniform expansivity and show that it can be described using generalized cone-fields on metric spaces. We say that a function $f: X \to X$ is uniformly expansive on a set $\Lambda \subset X$ if there exist $\varepsilon > 0$ and $\alpha \in (0, 1)$ such that for any two orbits $\mathrm{x}: \{-N, \dots, N\} \to \Lambda$, $\mathrm{v}: \{-N, \dots, N\} \to X$ of f we have

$$\sup_{-N \le n \le N} d(\mathrm{x}_n, \mathrm{v}_n) \le \varepsilon \implies d(\mathrm{x}_0, \mathrm{v}_0) \le \alpha \sup_{-N \le n \le N} d(\mathrm{x}_n, \mathrm{v}_n).$$

It occurs that a function is uniformly expansive iff there exists a generalized cone-field on X such that f is cone-hyperbolic.

Keywords Cone-field · Hyperbolicity · Expansive map · Lyapunov function

Mathematics Subject Classification 37D20

1 Introduction

In 1892 Lyapunov [9] introduced the idea of Lyapunov functions to study stability of equilibria of differential equations. The Lyapunov approach allows to assess the stability of equilibrium points of a system without solving the differential equations that describe the system. This theory is widely used in qualitative theory of dynamical systems.

In Lewowicz [7,8]proposed to use Lyapunov functions of two variables to study structural stability and similar concepts, such as topological stability and persistence. The method has

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been applied in particular to study hyperbolic diffeomorphisms on manifolds. For the survey of the results, methods and possible generalizations see [12].

Let us quote one of the most interesting results from [12]. Let $f: M \to M$ be a homeomorphism of a compact manifold M. For $U: M \times M \to \mathbb{R}$ we define

$$\Delta_f U(x, y) := U(f(x), f(y)) - U(x, y) \text{ for } x, y \in M.$$

We say that U is a Lyapunov function for f if it is continuous, vanishes on the diagonal, and $\Delta_f U(x, y)$ is positive for (x, y) on a neighborhood of the diagonal, $x \neq y$.

The following result characterizes expansive homeomorphisms in terms of Lyapunov functions.

Theorem *[12, Theorem 3.2]. Let f be a homeomorphism of a compact manifold M. The following conditions are equivalent:*

i) f is expansive;
ii) there exists a Lyapunov function for f.

The proof of this result for diffeomorphisms f can be found in [7]; see Sect. 4 and Lemma 3.3 of that paper. Additional arguments required for the case of a homeomorphism are discussed in [6, Sect. 1]. See also [12], where Tolosa, basing on the results of Lewowicz, characterized the expansivity on metric spaces with the using Lyapunov functions.

In this paper we use a generalized notion of cone-fields on metric space to describe uniform expansivity. The notions of cone-fields and cone condition [4,10] originally appeared in the late 60's in the works of Alekseev, Anosov, Moser and Sinai. Recently, Sheldon Newhouse [10] obtained new conditions for dominated and hyperbolic splittings on compact invariant sets with the use of cone-fields. It is also worth mentioning that the notion of cone-field can be very useful in the study of hyperbolicity [1,3,4,10].

Let us briefly describe the contents of this paper. In Sect. 2 we discuss the notion of uniform expansivity. We show that if f is uniformly expansive then it is also expansive. In Sect. 3 we recall our generalization of cone-fields to metric spaces which we presented in paper [11] and show that the existence of hyperbolic cone fields guarantees uniform expansivity. In Sect. 4 we show how to construct functions c_s, c_u for a uniformly expansive f such that f is cone-hyperbolic with respect to the cone-field (c_s, c_u). The main result of the section can be summarized as follows:

Main Result [see Theorem 3]. *Let X be a metric space and let $f : X \rightharpoonup X$ be an L-bilipschitz map. Assume that $\Lambda \subset X$ is an invariant set for f such that f is uniformly expansive on Λ.*

Then there exists a cone-field on Λ such that f is cone-hyperbolic on Λ.

2 Uniform Expansivity

First we define uniform expansivity of f and show that this notion is stronger than the classical expansivity.

By a partial map from X to Y (written as $f : X \rightharpoonup Y$) we denote a function which domain is subset of X [2, Chapter 2]. By $\mathrm{dom}(f)$ we denote the domain of a partial map $f : X \rightharpoonup Y$, and by $\mathrm{im}(f)$ we denote its inverse image. For a given $f : X \rightharpoonup X$ we say that a sequence $\mathrm{x}: I \to X$ defined on a subinterval[1] I of $\mathbb{Z}$ is an *orbit of* f if

$$\mathrm{x}_n \in \mathrm{dom}(f) \text{ and } \mathrm{x}_{n+1} = f(\mathrm{x}_n) \text{ for } n \in I \text{ such that } n+1 \in I.$$

[1] We say the I is a subinterval of $\mathbb{Z}$ if $[k, l] \cap \mathbb{Z} \subset I$ for any $k, l \in I$.

We recall the classical definition of expansivity. We say that $f\colon X \rightharpoonup X$ is *expansive* on $\Lambda \subset X$ if there exists an $\varepsilon > 0$ such that for any two orbits $\mathrm{x}\colon \mathbb{Z} \to \Lambda$, $\mathrm{v}\colon \mathbb{Z} \to X$ if $\sup_{n\in\mathbb{Z}} d(\mathrm{x}_n, \mathrm{v}_n) \leq \varepsilon$ then $\mathrm{x} = \mathrm{v}$.

Definition 1 Let $N \in \mathbb{N}$, $\varepsilon > 0$ and $\alpha \in (0, 1)$ be given. We say that $f\colon X \rightharpoonup X$ is (N, ε, α)*-uniformly expansive on a set* $\Lambda \subset X$ if for any two orbits $\mathrm{x}\colon \{-N, \dots, N\} \to \Lambda$, $\mathrm{v}\colon \{-N, \dots, N\} \to X$ we have

$$d_{\sup}(\mathrm{x}, \mathrm{v}) \leq \varepsilon \implies d(\mathrm{x}_0, \mathrm{v}_0) \leq \alpha d_{\sup}(\mathrm{x}, \mathrm{v}),$$

where

$$d_{\sup}(\mathrm{x}, \mathrm{v}) := \sup_{-N\leq n\leq N} d(\mathrm{x}_n, \mathrm{v}_n).$$

This notion is more useful because it does not need an infinite trajectory.

Example 1 Consider a rotation of $f\colon S^1 \to S^1$ by an angle α. Then f is an isometry, and therefore is not expansive, and consequently not (N, ε, α)-uniformly expansive on $\Lambda = S^1$.

Example 2 Let us consider the function $f\colon \mathbb{R}_+ \ni x \mapsto x + \sqrt{x} \in \mathbb{R}_+$. One can easily check that this function is expansive because its derivative at each point is strongly greater than 1. On the other hand, f is not uniformly expansive because for sufficiently large x the derivative of the function at x can become as close to 1 as we want.

One can easily verify that uniform expansivity implies classical expansivity (this result can also be easily deduced from Theorem 1 below).

Observation 1 *[11, Observation 4.1] Let $N \in \mathbb{N}$, $\varepsilon > 0$, $\alpha \in (0, 1)$, $\Lambda \subset X$ and $f\colon X \rightharpoonup X$ be given. If f is (N, ε, α)-uniformly expansive on Λ, then it is also expansive on Λ.*

Given $L \geq 1$ and $f\colon X \rightharpoonup Y$ we call f *L-bilipschitz* if

$$L^{-1}d(x, y) \leq d(f(x), f(y)) \leq Ld(x, y) \text{ for } x, y \in \mathrm{dom}(f). \tag{2.1}$$

Note that if a function f is L-bilipschitz then it is injective.

For $\delta > 0$ and a set $A \subset X$ we define the *δ-neighbourhood of A* as

$$A_\delta := \bigcup_{x\in A} B(x, \delta).$$

Let an injective map $f\colon X \rightharpoonup X$ be given. We call $A \subset \mathrm{dom}(f)$ an *invariant set for f* if $f(x)$ and $f^{-1}(x) \in A$ for every $x \in A$.

Now we show how to change the metric so that the function f which is $(N, \cdot, \cdot)$-uniformly expansive becomes $(1, \cdot, \cdot)$-uniformly expansive.

Theorem 1 *Let $f\colon X \rightharpoonup X$ be an L-bilipschitz map for some $L > 1$ and $\alpha \in (0, 1)$. Let $\Lambda \subset X$ and $\delta > 0$ be such that $\Lambda_\delta \subset dom(f) \cap im(f)$. We assume that Λ is an invariant set for f and that f is (N, δ, α)-uniformly expansive on Λ.*

Then there exists a metric ρ on $\Lambda_{\delta L^{-N+1}}$ such that

$$d(x, v) \leq \rho(x, v) \leq L^{N-1}d(x, v) \text{ for } x, v \in \Lambda_{\delta L^{-N+1}}, \tag{2.2}$$

that f is $(1, \delta L^{-N+1}, \sqrt[N]{\alpha})$-uniformly expansive on $\Lambda_{\delta L^{-N+1}}$ and $\max\{\alpha^{-1/N}, L\}$-bilipschitz map with respect to the metric ρ.

Proof Let $\beta = \sqrt[N]{\alpha}$. We put

$$\rho(x, v) := \max_{k \in \{-N+1,\ldots,N-1\}} \beta^{|k|} d(f^k(x), f^k(v)) \text{ for } x, v \in \Lambda_{\delta L^{-N+1}}.$$

Inequalities (2.2) follow from the definition and (2.1). Note that for $k \in \{-N+1, \ldots, N-1\}$ we have

$$x, v \in \Lambda_{\delta L^{-N+1}} \Longrightarrow f^k(x), f^k(v) \in \Lambda_{\delta L^{-N+1+|k|}}.$$

This means that ρ is well defined on $\Lambda_{\delta L^{-N+1}}$.

First we show that f is $\max\{\beta^{-1}, L\}$-bilipschitz map with respect to the metric ρ. Since f is L-bilipschitz in the metric d, we know that $d(f^N(x), f^N(v)) \le L d(f^{N-1}(x), f^{N-1}(v))$ and finally we get

$$\begin{aligned}
\rho(f(x), f(v)) &= \max_{k \in \{-N+1,\ldots,N-1\}} \beta^{|k|} d(f^k(f(x)), f^k(f(v))) \\
&= \max\{\beta^{|-N+1|} d(f^{-N+2}(x), f^{-N+2}(v)), \ldots, \beta^{N-1} d(f^N(x), f^N(v))\} \\
&= \max\{\beta^{|-N+1|}\beta^{-1}\beta d(f^{-N+2}(x), f^{-N+2}(v)), \ldots, \beta^1\beta^{-1}\beta d(x, v), \\
&\qquad \beta^0\beta\beta^{-1} d(f(x), f(v)), \ldots, \beta^{N-2}\beta\beta^{-1} d(f^{N-1}(x), f^{N-1}(v)), \\
&\qquad \beta^{N-1} d(f^N(x), f^N(v))\} \\
&= \max\{\beta\beta^{|-N+2|} d(f^{-N+2}(x), f^{-N+2}(v)), \ldots, \beta\beta^0 d(x, v), \\
&\qquad \beta^{-1}\beta^1 d(f(x), f(v)), \ldots, \beta^{-1}\beta^{N-1} d(f^{N-1}(x), f^{N-1}(v)), \\
&\qquad \beta^{N-1} d(f^N(x), f^N(v))\} \\
&\le \max\{\beta\beta^{|-N+2|} d(f^{-N+2}(x), f^{-N+2}(v)), \ldots, \beta\beta^0 d(x, v), \\
&\qquad \beta^{-1}\beta^1 d(f(x), f(v)), \ldots, \beta^{-1}\beta^{N-1} d(f^{N-1}(x), f^{N-1}(v)), \\
&\qquad \beta^{N-1} L d(f^{N-1}(x), f^{N-1}(v))\} \\
&\le \max\{\beta, \beta^{-1}, L\} \cdot \rho(x, v) = \max\{\beta^{-1}, L\} \cdot \rho(x, v).
\end{aligned}$$

Similarly, as for the opposite inequality, we know that $L^{-1} d(f^{N-1}(x), f^{N-1}(v)) \le d(f^N(x), f^N(v))$ and $L^{-1} d(f^{-N+1}(x), f^{-N+1}(v)) \le d(f^{-N+2}(x), f^{-N+2}(v))$. Hence

$$\begin{aligned}
\rho(f(x), f(v)) &= \max\{\beta\beta^{|-N+2|} d(f^{-N+2}(x), f^{-N+2}(v)), \ldots, \beta\beta^0 d(x, v), \\
&\qquad \beta^{-1}\beta^1 d(f(x), f(v)), \ldots, \beta^{-1}\beta^{N-1} d(f^{N-1}(x), f^{N-1}(v)), \\
&\qquad \beta^{N-1} d(f^N(x), f^N(v))\} \\
&\ge \max\{\beta\beta^{|-N+2|} d(f^{-N+2}(x), f^{-N+2}(v)), \ldots, \beta\beta^0 d(x, v), \\
&\qquad \beta^{-1}\beta^1 d(f(x), f(v)), \ldots, \beta^{-1}\beta^{N-1} d(f^{N-1}(x), f^{N-1}(v)), \\
&\qquad \beta^{N-1} L^{-1} d(f^{N-1}(x), f^{N-1}(v))\} \\
&\ge \min\{\beta, L^{-1}\} \cdot \rho(x, v).
\end{aligned}$$

Now we show that for $x \in \Lambda$ and $v \in \Lambda_{\delta L^{-N+1}}$ such that

$$\max\left\{\rho(f^{-1}(x), f^{-1}(v)), \rho(x, v), \rho(f(x), f(v))\right\} \le \delta L^{-N+1} \tag{2.3}$$

the following inequality holds:

$$\rho(x, v) \le \beta \max(\rho(f(x), f(v)), \rho(f^{-1}(x), f^{-1}(v))).$$

We have to show that for $k = -N+1, \ldots, N-1$

$$\beta^{|k|} d(f^k(x), f^k(v)) \leq \beta \max \\ \times \left(\max_{k=-N+1,\ldots,N-1} \beta^{|k|} d(f^{k+1}(x), f^{k+1}(v)), \max_{k=-N+1,\ldots,N-1} \beta^{|k|} d(f^{k-1}(x), f^{k-1}(v)) \right).$$

For $k < 0$ or $k > 0$ it is straightforward. Consider the case $k = 0$. From (2.2) and (2.3) we get

$$\max \left\{ d(f^{-1}(x), f^{-1}(v)), d(x, v), d(f(x), f(v)) \right\} \leq \delta L^{-N+1},$$

which together with (2.1) implies that $d(f^k(x), f^k(v)) \leq \delta$ for $k = -N, \ldots, N$. By the uniform expansivity and the fact that $\beta < 1$ we get

$$d(x, v) \leq \alpha \max_{|k| \leq N} d(f^k(x), f^k(v)) \leq \beta \max_{|k| \leq N} (\beta^{N-1} d(f^k(x), f^k(v))) \\ \leq \beta \max \left(\max_{|k| \leq N-1} \beta^{|k|} d(f^{k+1}(x), f^{k+1}(v)), \max_{|k| \leq N-1} \beta^{|k|} d(f^{k-1}(x), f^{k-1}(v)) \right).$$

3 Cone-fields and Cone-hyperbolic Maps

In this section, for the convenience of the reader, we recall basic definitions concerning generalization of cone-fields to metric spaces (for more information and motivation see [5,11]).

Definition 2 [11, Definition 3.1] Let $\delta > 0$ and $\Lambda \subset X$ be nonempty. We say that a pair of functions $c_s, c_u : U \to \mathbb{R}_+$ for some $U \subset X \times X$ forms a δ-*cone-field on* Λ if

$$\{x\} \times B(x, \delta) \subset U \quad \text{for } x \in \Lambda.$$

We put $c(x, v) := \max\{c_s(x, v), c_u(x, v)\}$. If there exists $K > 0$ such that:

$$\frac{1}{K} d(x, v) \leq c(x, v) \leq K d(x, v) \text{ for } (x, v) \in U$$

then we call it (K, δ)-*cone-field on* Λ or *uniform* δ-*cone-field on* Λ.

For each point $x \in \Lambda$ we introduce *unstable* and *stable cones* by the formula

$$C_x^u(\delta) := \{v \in B(x, \delta) \,:\, c_s(x, v) \leq c_u(x, v)\},$$
$$C_x^s(\delta) := \{v \in B(x, \delta) \,:\, c_s(x, v) \geq c_u(x, v)\}.$$

We consider a partial map $f : X \rightharpoonup Y$ between metric spaces X and Y and $\Lambda \subset \text{dom}(f)$. Assume that X is equipped with a uniform δ-cone-field on Λ and Y is equipped with a uniform δ-cone-field on a subset Z of Y such that $f(\Lambda) \subset Z$.

For every $x \in \text{dom}(f)$ we put

$$B_f(x, \delta) := \{v \in B(x, \delta) \cap \text{dom}(f) \,:\, f(v) \in B(f(x), \delta)\}.$$

Now we define $u_x(f; \delta)$ and $s_x(f; \delta)$, the expansion and the contraction rates of f, respectively. These rates are a modification of the classical definition from [10], but we do not assume that the function f is invertible (for more information see [11]).

Definition 3 [11, Definition 3.2] Let $x \in \operatorname{dom}(f)$ and $\delta > 0$ be given. We define

$$u_x(f;\delta) := \sup\{R \in [0,\infty] \mid c(f(x), f(v)) \geq Rc(x,v),\, v \in B_f(x,\delta);\, v \in C^u_x(\delta)\},$$
$$s_x(f;\delta) := \inf\{R \in [0,\infty] \mid c(f(x), f(v)) \leq Rc(x,v),\, v \in B_f(x,\delta);\, f(v) \in C^s_{f(x)}(\delta)\}.$$

Let $u_\Lambda(f;\delta) := \inf_{x\in\Lambda}\{u_x(f;\delta)\}$ and $s_\Lambda(f;\delta) := \sup_{x\in\Lambda}\{s_x(f;\delta)\}$.

Definition 4 We say that f is *δ-cone-hyperbolic* on Λ if

$$s_\Lambda(f;\delta) < 1 < u_\Lambda(f;\delta).$$

The next proposition is a simple analogue of [10, Lemma 1.1].

Proposition 1 *[11, Proposition 3.1] Every δ-cone-hyperbolic is δ-cone-invariant, i.e. for $x \in \Lambda$ and $v \in B_f(x,\delta)$ we have*

$$v \in C^u_x(\delta) \implies f(v) \in C^u_{f(x)}(\delta),$$

and

$$f(v) \in C^s_{f(x)}(\delta) \implies v \in C^s_x(\delta).$$

Theorem 2 *[11, Theorem 4.1] Suppose that for $K > 0$ and $\delta > 0$ we are given a (K,δ)-cone-field on $\Lambda \subset X$. Let $f\colon \Lambda_\delta \rightharpoonup X$ be δ-cone-hyperbolic on Λ and let $\lambda > 1$ be chosen such that*

$$s_\Lambda(f;\delta) \leq \lambda^{-1},\ u_\Lambda(f;\delta) \geq \lambda.$$

Then f is $(N,\delta,K^2/\lambda^N)$-uniformly expansive on Λ for every $N \in \mathbb{N}$, $N > 2\log_\lambda K$.

Example 3 Let $f\colon T^2 \to T^2$ be defined by $f(x,y) = (2x+y, x+y)$, where $T^2 = \mathbb{R}^2/\mathbb{Z}^2$.
We know that f is expansive (see [4, Sect. 1.8]). It is easy to show that

$$s_{T^2}(f;\delta) \leq \frac{3-\sqrt{5}}{2} < 1, \quad u_{T^2}(f;\delta) \geq \frac{3+\sqrt{5}}{2} > 1.$$

From Theorem 2 we conclude that f is uniformly expansive on $\Lambda = T^2$.

4 Expansivity and Cone-fields

In this section we show that uniform expansiveness of f on an invariant set Λ lets us construct a cone-field on Λ such that f is cone-hyperbolic on Λ. In our reasoning we will need the notion of ε-quasiconvexity.

Definition 5 Let I be a subinterval of $\mathbb{Z}$, and let $\varepsilon \geq 0$ be fixed. We call a sequence $\alpha\colon I \to \mathbb{R}$ *ε-quasiconvex* if

$$\alpha_n \leq \max\{\alpha_{n-1}, \alpha_{n+1}\} - \varepsilon \ \text{ for } \ n \in I :\ n-1, n+1 \in I.$$

Now we show some properties of ε-quasiconvex sequences, which will be used later.

Observation 2 *Let $\varepsilon \geq 0$ and $\alpha\colon I \to \mathbb{R}$ be an ε-quasiconvex sequence.*
Then

i) if $m, m+2 \in I$ *and* $\alpha_{m+1} > \alpha_m - \varepsilon$ *then*

$$\alpha_{n+1} \geq \alpha_n + \varepsilon \text{ for } n \geq m+1 \text{ such that } n, n+1 \in I. \tag{4.1}$$

ii) if $m-1, m+1 \in I$ *and* $\alpha_{m+1} < \alpha_m + \varepsilon$ *then*

$$\alpha_{n+1} \leq \alpha_n - \varepsilon \text{ for } n < m \text{ such that } n, n+1 \in I. \tag{4.2}$$

Proof The above statements are similar so we show the first one. The proof proceeds on induction. Suppose that $m, m+2 \in I$ and $\alpha_{m+1} > \alpha_m - \varepsilon$. Since α is ε-quasiconvex,

$$\alpha_{m+1} \leq \max\{\alpha_m, \alpha_{m+2}\} - \varepsilon = \max\{\alpha_m - \varepsilon, \alpha_{m+2} - \varepsilon\}.$$

But $\alpha_{m+1} > \alpha_m - \varepsilon$, so we get

$$\alpha_{m+1} \leq \alpha_{m+2} - \varepsilon,$$

and hence

$$\alpha_{m+2} \geq \alpha_{m+1} + \varepsilon.$$

It implies that (4.1) is valid for $n = m+1$. Suppose now that (4.1) holds for some $n \geq m+1$, i.e. that $n; n+1 \in I$ and $\alpha_{n+1} \geq \alpha_n + \varepsilon$. Assume additionally that $n+2 \in I$. Then we get

$$\alpha_{n+1} \leq \alpha_{n+2} - \varepsilon,$$

thus

$$\alpha_{n+2} \geq \alpha_{n+1} + \varepsilon,$$

which completes the proof.

The following proposition will be a basic tool in the proof of our main result, Theorem 3.

Proposition 2 *Let* $\varepsilon > 0$, $L > 1$, $\beta \in (0,1)$ *and let* (Y, ρ) *be a metric space. Let* $\Lambda \subset Y$ *be given and* $f : Y \rightharpoonup Y$ *be an L-bilipschitz map such that* $\Lambda_\varepsilon \subset dom(f) \cap im(f)$. *Assume that* Λ *is an invariant set for* f *and that* f *is* $(1, \varepsilon, \beta)$*-uniformly expansive on* Λ.

Then

$$\begin{aligned} c_s(x,v) &:= \inf\{\rho(f^k(x), f^k(v)) \mid k \in (-\infty, 0) \cap \mathbb{Z} : f^l(v) \in B(f^l(x), \varepsilon) \\ &\quad \text{for } l \in [k, 0] \cap \mathbb{Z}\}, \\ c_u(x,v) &:= \inf\{\rho(f^k(x), f^k(v)) \mid k \in [0, \infty) \cap \mathbb{Z} : f^l(v) \in B(f^l(x), \varepsilon) \\ &\quad \text{for } l \in [0, k] \cap \mathbb{Z}\}, \end{aligned} \tag{4.3}$$

define an $(L, \varepsilon/L)$ *cone-field on* Λ. *Moreover,* f *is cone-hyperbolic on* Λ *and*

$$s_\Lambda(f; \varepsilon/L) \leq \beta < \frac{1}{\beta} \leq u_\Lambda(f; \varepsilon/L). \tag{4.4}$$

Proof First we show that $c_s(x,v)$ and $c_u(x,v)$ defined above are $(L, \varepsilon/L)$ cone-field on Λ, i.e.

$$\frac{1}{L}\rho(x,v) \leq c(x,v) \leq L\rho(x,v) \ \text{ for } \ (x,v) \in \big\{(x,v) \ : \ x \in \Lambda, v \in B(x, \varepsilon/L)\big\},$$

where $c(x,v) := \max\{c_s(x,v), c_u(x,v)\}$.

Choose an arbitrary point $x \in \Lambda$ and $v \in B(x, \varepsilon/L)$. We can assume that $x \neq v$, because the case $x = v$ is trivial ($c_s(x,v) = c_u(x,v) = 0 = \rho(x,v)$).

Let I be the biggest subinterval of $\mathbb{Z}$ containing 0 such that

$$\sup\{\rho(f^n(x), f^n(v)) : n \in I\} \leq \varepsilon. \tag{4.5}$$

Since f is L-bilipschitz, we know that $f^{-1}(v) \in B(f^{-1}(x), \varepsilon)$, and therefore $\{-1, 0\} \subset I$. This yields $c(x, v) < \infty$.

Now we define a sequence $\{a_n\}_{n \in I} \subset \mathbb{R}$ by the formula

$$a_n := \ln \rho(f^n(x), f^n(v)) \text{ for } n \in I. \tag{4.6}$$

Observe that a_n is well-defined because $\rho(f^n(x), f^n(v)) > 0$ for all $n \in I$.

Let

$$I_- := \{n \in I : n < 0\} \text{ and } I_+ := \{n \in I : n \geq 0\}.$$

We have the following relations:

$$c_s(x, v) = \exp\left(\inf_{n \in I_-} \{a_n\}\right) \text{ and } c_u(x, v) = \exp\left(\inf_{n \in I_+} \{a_n\}\right),$$

where we use the convention $\exp(-\infty) = 0$.

We show that the sequence $\{a_n\}$ is $\ln(1/\beta)$-quasiconvex. Let $n \in I$ be such that $n-1$, $n+1 \in I$. By (4.5) we observe that

$$\max\{\rho(f^{n-1}(x), f^{n-1}(v)), \rho(f^n(x), f^n(v)), \rho(f^{n+1}(x), f^{n+1}(v))\} \leq \varepsilon.$$

Consequently, by $(1, \varepsilon, \beta)$-uniform expansivity of f we get

$$\rho(f^n(x), f^n(v)) \leq \beta \max\{\rho(f^{n-1}(x), f^{n-1}(v)), \rho(f^{n+1}(x), f^{n+1}(v))\},$$

which implies that $a_n \leq \max\{a_{n-1}, a_{n+1}\} - \ln(1/\beta)$.

Now we consider two cases. If $a_{-1} \leq a_0$ then by Observation 2 i) we get

$$a_{n+1} \geq a_n + \ln \frac{1}{\beta} \text{ for } n \geq 0, n \in I,$$

which yields

$$\inf_{n \in I_-} \{a_n\} \leq a_{-1} \leq a_0 = \inf_{n \in I_+} \{a_n\},$$

Hence

$$c_s(x, v) \leq c_u(x, v) = c(x, v) = e^{a_0} = \rho(x, v).$$

On the other hand if $a_{-1} \geq a_0$ then by Observation 2 ii) we get

$$a_{n+1} \leq a_n - \ln \frac{1}{\beta} \text{ for } n < -1, n \in I.$$

Therefore

$$\inf_{n \in I_-} \{a_n\} = a_{-1} \geq a_0 \geq \inf_{n \in I_+} \{a_n\},$$

and consequently

$$c_u(x, v) \leq c_s(x, v) = c(x, v) = e^{a_{-1}} = \rho(f^{-1}(x), f^{-1}(v)).$$

Since f is L-bilipschitz, we get that c_s, c_u define an $(L, \varepsilon/L)$ cone-field on Λ.

Now we check that f is cone-hyperbolic on Λ. Let us take $x \in \Lambda$ and $v \in B_f(x, \varepsilon/L)$ such that $f(v) \in C^s_{f(x)}(\varepsilon/L)$. We define the sequence $\{a_n\}_{n\in I}$ as in (4.6).

We show that $a_0 \geq a_1$. Suppose that, on the contrary, $a_0 < a_1$. By Observation 2 i) we get

$$a_{n+1} \geq a_n \quad \text{for} \quad n \geq 1, n \in I.$$

Hence

$$\ln(c_u(f(x), f(v))) = \inf_{n\geq 1, n\in I}\{a_n\} = a_1 > a_0 \geq \inf_{n<1, n\in I}\{a_n\} = \ln(c_s(f(x), f(v))),$$

which is a contradiction with $f(v) \in C^s_{f(x)}(\varepsilon/L)$. So we have $a_1 \leq a_0$. By the Observation 2 ii) we get

$$a_{n+1} \leq a_n - \ln(1/\beta) \text{ for } n < 0 \text{ such that } n, n+1 \in I.$$

In particular,

$$a_0 \leq a_{-1} - \ln(1/\beta). \tag{4.7}$$

Consequently,

$$\begin{aligned} c_u(f(x), f(v)) &= \exp\left(\inf_{n\geq 1, n\in I}\{a_n\}\right) \leq \exp(a_1) \leq \exp(a_0) \\ &= \exp\left(\inf_{n<1, n\in I}\{a_n\}\right) = c_s(f(x), f(v)) = \underline{c(f(x), f(v))} \\ &\overset{(4.7)}{\leq} \beta\exp(a_{-1}) = \beta\exp\left(\inf_{n\in I_-}\{a_n\}\right) \leq \underline{\beta c(x, v)}. \end{aligned}$$

Therefore

$$s_\Lambda(f; \varepsilon/L) = \sup_{x\in\Lambda}\{s_x(f; \varepsilon/L)\} \leq \beta < 1.$$

Now we consider an $x \in \Lambda$ and $v \in B_f(x, \varepsilon/L)$ such that $v \in C^u_x(\varepsilon/L)$. We show that $a_0 \geq a_{-1}$. Suppose the contrary, $a_0 < a_{-1}$. By Observation 2 ii) we get

$$a_{n+1} \geq a_n \quad \text{for} \quad n < -1, n \in I.$$

Hence

$$\inf_{n\in I_-}\{a_n\} = a_{-1} > a_0 \geq \inf_{n\in I_+}\{a_n\},$$

which is contradiction with $v \in C^u_x(\varepsilon/L)$. So we have $a_0 \geq a_{-1}$. By the Observation 2 i) we get

$$a_{n+1} \geq a_n + \ln(1/\beta) \text{ for } n \geq 0 \text{ such that } n, n+1 \in I.$$

In particular,

$$a_1 \geq a_0 + \ln(1/\beta). \tag{4.8}$$

Finally

$$\begin{aligned} c_s(f(x), f(v)) &= \exp\left(\inf_{n<1, n\in I}\{a_n\}\right) \leq \exp(a_0) \leq \exp(a_1) \\ &= \exp\left(\inf_{n\geq 1, n\in I}\{a_n\}\right) = c_u(f(x), f(v)) = c(f(x), f(v)), \end{aligned}$$

which yields

$$\exp(a_1) = \underline{c(f(x), f(v))} \overset{4.8}{\geq} \frac{1}{\beta}\exp(a_0) = \frac{1}{\beta}\exp\left(\inf_{n\in I_+}\{a_n\}\right) = \underline{\frac{1}{\beta}c(x, v)}.$$

This shows that

$$u_\Lambda(f; \varepsilon/L) = \inf_{x\in\Lambda}\{u_x(f; \varepsilon/L)\} \geq \frac{1}{\beta} > 1.$$

Therefore f is cone-hyperbolic on Λ.

As a consequence of earlier results we obtain the following theorem.

Theorem 3 *Let* $\varepsilon > 0$, $L > 1$, $N \in \mathbb{N}$, $\alpha \in (0, 1)$ *be fixed. Let* (X, d) *be a metric space and* $\Lambda \subset X$ *be given. Let* $f : X \rightharpoonup X$ *be an L-bilipschitz map such that* $\Lambda_\varepsilon \subset dom(f) \cap im(f)$. *Assume that* Λ *is an invariant set for* f *and that* f *is* (N, ε, α)*-uniformly expansive on* Λ.

Then there exists an $(\max\{\alpha^{-1/N}L^{N-1}, L^N\}, \min\{\varepsilon L^{-2N+1}\sqrt[N]{\alpha}, \varepsilon L^{-2N}\})$ *cone-field on* Λ *such that* f *is cone-hyperbolic on* Λ *and*

$$s_\Lambda(f, \min\{\varepsilon L^{-2N+1}\sqrt[N]{\alpha}, \varepsilon L^{-2N}) \leq \sqrt[N]{\alpha} < \frac{1}{\sqrt[N]{\alpha}} \leq u_\Lambda(f, \min\{\varepsilon L^{-2N+1}\sqrt[N]{\alpha}, \varepsilon L^{-2N}).$$

Proof We will apply Proposition 2. By applying Theorem 1 (for $\delta = \varepsilon$) we obtain the metric ρ which is equivalent to d on $U = \{x : d(x, \Lambda) < \varepsilon L^{-N+1}\}$ and such that

i) $d(x, v) \leq \rho(x, v) \leq L^{N-1}d(x, v)$ for $x, v \in U$,
ii) f is $(1, \varepsilon L^{-N+1}, \sqrt[N]{\alpha})$-uniformly expansive on U with respect to the metric ρ,
iii) f is $\max\{\alpha^{-1/N}, L\}$-bilipschitz map on U with respect to the metric ρ.

Let $\widetilde{Y} = \{y : d(y, \Lambda) < L^{-N+1}\varepsilon\}$ and $\widetilde{L} = \max\{\alpha^{-1/N}, L\}$. We use Proposition 2 (for $\widetilde{\varepsilon} = \varepsilon L^{-N}$, $\widetilde{L}$, $\widetilde{\beta} = \sqrt[N]{\alpha}$, $\widetilde{f} = f|_{\{x : d(x,\Lambda)<\varepsilon L^{-N}\}}$) and construct functions $\widetilde{c_s}$, $\widetilde{c_u}$ which define an $(\widetilde{L}, \widetilde{\delta})$ cone-field on U such that $\widetilde{f}$ is $\widetilde{\delta}$-cone-hyperbolic with respect to the metric ρ, where $\widetilde{\delta} = \varepsilon L^{-N}/\widetilde{L}$.

Now we need to "translate" the results from the metric ρ to the original metric d. For clarity of notation we use the subscript $(.)_d$ to denote objects with respect to the metric d and $(.)_\rho$ to denote objects with respect to the metric ρ.

By the definition of $(\widetilde{L}, \widetilde{\delta})$ cone-field on U and i) we get

$$\begin{aligned}\frac{1}{\widetilde{L}L^{N-1}}d(x, v) &\leq \frac{1}{\widetilde{L}}d(x, v) \leq \frac{1}{\widetilde{L}}\rho(x, v) \leq c(x, v) \leq \widetilde{L}\rho(x, y)\\ &\leq \widetilde{L}L^{N-1}d(x, y) \text{ for } (x, v) \in \left\{x \in U, v \in B(x, \widetilde{\delta})_\rho\right\}.\end{aligned}$$

From i) we have

$$B(x, \widetilde{\delta}/L^{N-1})_d \subset B(x, \widetilde{\delta})_\rho, \quad B_f(x, \widetilde{\delta}/L^{N-1})_d \subset B_f(x, \widetilde{\delta})_\rho,$$

and

$$C^u_x(\widetilde{\delta}/L^{N-1})_d \subset C^u_x(\widetilde{\delta})_\rho, \quad C^s_x(\widetilde{\delta}/L^{N-1})_d \subset C^s_x(\widetilde{\delta})_\rho.$$

Consequently, from Definition 3 for an arbitrary $x \in U$ we get

$$u_x(f; \widetilde{\delta})_\rho \leq u_x(f; \widetilde{\delta}/L^{N-1})_d, \quad s_x(f; \widetilde{\delta})_\rho \geq s_x(f; \widetilde{\delta}/L^{N-1})_d.$$

Hence

$$u_U(f; \widetilde{\delta})_\rho \leq u_U(f; \widetilde{\delta}/L^{N-1})_d, \quad s_U(f; \widetilde{\delta})_\rho \geq s_U(f; \widetilde{\delta}/L^{N-1})_d.$$

From the above inequalities and (4.4)

$$s_U(f;\widetilde{\delta})_\rho \leq \widetilde{\beta} < 1 < \frac{1}{\widetilde{\beta}} \leq u_U(f;\widetilde{\delta})_\rho,$$

we obtain that f is $(\widetilde{\delta}/L^{N-1})$-cone-hyperbolic in metric d and

$$s_U(f;\widetilde{\delta}/L^{N-1})_d \leq \widetilde{\beta} < 1 < \frac{1}{\widetilde{\beta}} \leq u_U(f;\widetilde{\delta}/L^{N-1})_d.$$

Finally we conclude that $\widetilde{c_s}$ and $\widetilde{c_u}$ are $(\max\{\alpha^{-1/N}L^{N-1}, L^N\}, \widetilde{\delta}/L^{N-1})$-cone-field on Λ such that f is $(\widetilde{\delta}/L^{N-1})$-cone-hyperbolic on Λ with respect to metric d.

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