Wydział Matematyki i Informatyki
Uniwersytet Jagielloński

Frank Weilandt

Rigorous Numerical Computation of the Conley Index for Flows

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prof. dra hab. Mariana Mrozka

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Contents

Chapter 1

Introduction

This thesis deals with the numerical computation of the Conley index, an invariant which is useful for analyzing dynamical systems. It was developed in the 1970s for flows, i.e., continuous maps $\varphi\colon X \times \mathbb{R} \to X$ with some additional properties which are satisfied if $\varphi(x, \mathbb{R})$ for each $x \in X$ is a solution of an ordinary differential equation $\dot{x} = v(x)$, where $v\colon \mathbb{R}^d \to \mathbb{R}^d$ is a continuously differentiable vector field. A flow is a special kind of dynamical system; the second main example is the iteration of a continuous map $f\colon X \to X$.

In both cases, we are interested in the long-time behavior of the system. The concept of an invariant set is essential for this purpose. A subset of X is invariant if going forward by any time step maps the set to itself, for example $f(S) = S$. We call an invariant set S isolated if it is the largest invariant set inside some topological neighborhood of itself. For example, in the flow case we mean that there is a compact set M with $S \subset \operatorname{int} M$ such that whenever $\varphi(x, \mathbb{R}) \subset M$ for some $x \in M$, then $x \in S$.

For these sets, the Conley index is defined as the homotopy type of a certain topological space. At least for flows. In the case of a map f, an invariant with analogous properties was developed in a series of publications [RS88, Szy95, FR00]. The definitions therein are closely related and they are using basic homotopy theory. One can also define such a notion using homology groups [Mro90b]. We present these central definitions in Chapter 2.

The Conley index of S tells us something about the behavior of the system close to S, but it does not describe its behavior inside S. An interesting feature is that we can compute this index without knowing S precisely, but simply by knowing the behavior of the flow on the boundary of an arbitrary neighborhood M as above. This way, one can detect for a given set M whether it contains a non-empty invariant set. Computing Conley indices of several invariant sets, one can also detect if there are orbits connecting two of them (meaning the orbit approaches one set in backward time and another in forward time). If X is a differentiable manifold and v is the gradient of a so-called Morse function, the Conley index generalizes the index in Morse theory (where it is assumed that S is just a point).

We do not go further into motivating the Conley index. Good introductions containing example computations for several dynamical systems are [Con78] and [MM02]. As the title reveals, here we are interested in the rigorous numerical computation of the Conley index. *Rigorous numerical computations* yield mathematically strict results as opposed to approximate numerical computations. These approximate computations are quite popular because implementations usually run fast compared to the methods we apply here.

In Chapter 3, we recall existing ideas how one can compute the Conley index for a map $f : X \to X$. Using $f(x) := \varphi(x, h)$ for some $h > 0$, this also allows computing the Conley index for a flow φ. The main message is that even though defining the Conley index for flows is easier than for maps, numerical computations can be hard. An interesting feature of the Conley index is that one does not need to know the trajectories of all initial conditions precisely in order to compute the Conley index – at least in terms of homology. Our numerical methods are based on a subdivision of the phase space X into boxes and on numerically computing for each box a set of boxes covering its image under f.

In Chapter 4, we compute the Conley index for a Poincaré map P coming from a non-autonomous ordinary differential equation $\dot{x} = f(t, x)$ which is periodic, i.e., there is a number $T > 0$ such that $f(t + T, x) = f(t, x)$ for all (t, x). Finding the Conley index of P using the standard approach discussed above would require enclosing the image of boxes under P after time T. This is infeasible if the solution curves expand very quickly. Another difficulty is that P is often not defined on every point (t, x), but only on a subset.

We present a theorem to deal with this situation, which allows us to compute the Conley index of P integrating the system for a time much smaller than the return time T of the Poincaré map. This theorem is adjusted to the numerical methods presented herein. We only sketch its proof and then present an algorithm which shows that the theorem is indeed useful for numerical applications. The algorithm together with an analysis of its correctness constitutes the main content of this thesis.

In Chapter 5, we recall how the numerical methods from Chapter 3 can also be used to find a Morse decomposition. This concept, which was introduced by Charles Conley, allows us to describe the behavior of the dynamical system inside an isolated invariant set. More precisely, such a decomposition consists of Morse sets $S_i \subset S$ for the given isolated invariant set S and some information about “forbidden” trajectories, i.e., one can encode that there is no trajectory from a certain Morse set S_i to another one. Unfortunately, we cannot encode the existence of a trajectory directly. But what looks like a rather sloppy perspective at first sight, makes this description numerically computable in the rigorous sense.

Also in Chapter 5, we propose a slightly more flexible approach to the discretization strategy from Chapter 3. In the classical strategy, one would use a fixed time step $h > 0$ in the whole phase space. This idea has its problems: There is no reasonable heuristic for its choice. Quite often, this does not cause trouble because a large range of values for h can yield useful results. But it is not obvious how one could avoid the bad choices where the algorithms yield only insufficient information because subdividing the space into arbitrarily small boxes is not an option since time and memory are limited.

We present a more flexible strategy in which we let $\tau : X \to (0, \infty)$ be any continuous function and $f(x) := \varphi(x, \tau(x))$ be the map with which we work numerically. This allows us to propose a heuristic for choosing $\tau(x)$ depending on $\|v(x)\|$, the norm of the given vector field in x. This requires a careful adaptation of the theory cited in Chapter 3 to this new setting. As one would expect, the numerical results can be significantly better for a system in which the norm of v varies a lot within a region.

Even though most of the scientific contributions of this thesis are contained in Chapters 4 and 5, Chapter 2 containing prerequisites is also quite long. This has two reasons: First, we want to compute a topological invariant for dynamical systems. This requires us to recall the definition of this invariant and how we use homology to describe it. Second, there are several definitions of the Conley index for maps. We compute one of its versions for the Poincaré map. In order to make this text accessible to readers who are more familiar with other versions, we recall the relations between these definitions in Section 2.5. We also propose an additional definition that is unpublished at the moment. It seems interesting because it encodes information about the dynamics in the homotopy type of a space – just like in the original definition for flows. But we do not go into details so as not to get carried away from the main line of thought.

The results appear in logical order in this thesis. Chapter 4 uses results from Chapters 2 and 3. Chapter 5 also uses results from these two chapters, but no results from Chapter 4. A reader who would like to see the numerical examples before the theory might want to read the corresponding subsections first. All the numerical examples were computed on one core of an Intel i5 CPU. The laptop we used has 6 GB of RAM, which was useful in Example 4.5.2. The source code of the algorithms is publicly available on the author's website [web].

The author's contribution

In Chapter 2, only Subsection 2.5.5 contains original ideas from the author. Section 2.5 shows more clearly than the existing literature that Theorem 2.5.13 suffices to define all discrete time Conley indices we listed. But the ideas leading to this seminal theorem in Conley index theory for maps are already contained in [RS88].

Similarly, Chapter 3 does not contain new theoretical ideas, but a numerical simulation showing how important it is to choose a good time step h in Section 3.5.

For Chapter 4, the author developed all the algorithms which are presented in pseudocode together with proofs of their correctness in Section 4.4. A series of proofs culminates in Theorem 4.4.7. The ideas in Sections 4.3 to 4.5 are the author's, who also implemented the algorithms.

For Chapter 5, the author found the numerically hard example in Section 5.4 and a time step heuristic. The algorithmic contribution here is rather small: The author used existing algorithms and their implementations from [CAPD] and [AKK+09]. The author's main theoretical result in Chapter 5 are Lemmas 5.3.4 and 5.3.5 leading to criterion (B) in Theorem 5.3.6.

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1.1 List of symbols (with page of first appearance)

Chapter 2

Mathematical background

The Conley index is a topological invariant describing the behavior of a dynamical system around certain invariant sets (which we call isolated). In this chapter, we recall its definition and main properties for the cases of continuous and discrete dynamical systems. The notions introduced here are essential for the later chapters. Notions which are used only in one of the following chapters are introduced at the beginning of the respective chapter.

Given an isolated invariant set of a flow or of a discrete dynamical system, its Conley index is well-defined. Even though we are interested in the Conley index for flows (as described in [Con78]), we use algorithms developed for the computation of Conley indices of maps. Therefore, we need to consider both situations here.

2.1 Local and global flows

Definition 2.1.1. A *flow* on a topological space X is a continuous map $\varphi : X' \to X$, where

1. X' is an open subset of $X \times \mathbb{R}$ such that
   (i) $X \times \{0\} \subset X'$, and
   (ii) for every $x \in X$, the set $I_x := \{ t \in \mathbb{R} \mid (x,t) \in X' \}$ is open in $\mathbb{R}$ and connected.
2. For every $x \in X$,
   (i) $\varphi(x,0) = x$ and
   (ii) $\varphi(x,s+t) = \varphi(\varphi(x,s),t)$ whenever $(x,s),(x,s+t),(\varphi(x,s),t) \in X'$.

Replacing $\mathbb{R}$ in the definition above by $\mathbb{R}^+$, we call the map φ a *semiflow*. If $X' = X \times \mathbb{R}$, we say that φ is a *global flow*. If $X' \subsetneq X \times \mathbb{R}$, we say that φ is a *local flow*.

A standard way to define a flow is via an ordinary differential equation $\dot{x} = v(x)$ for a map $v \in C^1(\mathbb{R}^d, \mathbb{R}^d)$, a *vector field*. Given such a v, there is a maximal subset

$X' \subset \mathbb{R}^d \times \mathbb{R}$ and a flow $\varphi : X' \to X$ differentiable with respect to t such that

$$\frac{\partial}{\partial t}\varphi(x,t) = v(\varphi(x,t)) \text{ for all } (x,t) \in X'.$$

This is a classical result about ordinary differential equations [Tes12, Theorem 6.1]. Other basic properties of flows can be found in the same source and similar introductions to dynamical systems. We do not always cite them explicitly.

All our example flows are induced by such a C^1 vector field. For these flows, we also have:

Lemma 2.1.2. *Let $I_x^+ := I_x \cap \mathbb{R}^+$ and $I_x^- := I_x \cap \mathbb{R}^-$. If $\varphi(x, I_x^\pm) \subset K$ for a compact set $K \subset \mathbb{R}^d$, then $I_x^\pm = \mathbb{R}^\pm$.*

This means that if I_x^+ is not $\mathbb{R}^+$, then the solution curve starting in x leaves any compact set in forward time. Analogously for I_x^-.

Remark 2.1.3. All the invariants we introduce later only consider a given flow inside a compact region K of the phase space X. Using the lemma, it does not matter for these invariants if a flow is only local or global. We sometimes assume that φ is a global flow to make the presentation simpler, tacitly having this in mind.

2.2 Basic notions of homotopy theory

Whenever we write (X,A) for topological spaces X and A, we mean that $A \subset X$ with A having the subspace topology and call it a *pair of spaces*. A *compact pair* is a pair of spaces (X,A) such that X is a compact topological space and A is closed in X (therefore A is also compact). A *pointed space* is a pair of spaces $(X, \{x_0\})$. In this situation, we usually call X pointed and denote the base point x_0 by $*$.

If (X,A) and (Y,B) are pairs of spaces and $f : X \to Y$ such that $f(A) \subset B$, then we write $f : (X,A) \to (Y,B)$.

We call two maps $f, g : X \to Y$ are *homotopic relative to A* (written $f \simeq g \operatorname{rel} A$) if there is a continuous map $H : X \times [0,1]$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for $x \in X$, and additionally $H(x,t) = f(x) = g(x)$ for all $x \in A$, $t \in [0,1]$. If $A = \varnothing$, we call f and g *homotopic* and write $f \simeq g$. Two maps of pairs $(X,A) \to (Y,B)$ are homotopic if they are homotopic relative to A.

We also need to consider pointed maps on certain quotient spaces. For a topological space X and a subspace $Y \subset X$, we let the underlying set of the space X/Y be the disjoint union of the sets $(X \setminus Y)$ and $\{*\}$, where $\{*\}$ is the one-point space. We let $q : X \to X/Y$ be defined as $q(x) = [x]$ for $x \in X \setminus Y$ and $q(x) = *$ for $x \in Y$. As usual, we endow X/Y with the finest topology making q continuous, i.e., we let $Z \subset X/Y$ be open if $q^{-1}(Z)$ is open in X. We consider the quotient X/Y a pointed space with base point $*$. If $Y = \varnothing$, we get $X/\varnothing = X \coprod \{*\}$, the disjoint union (sum, coproduct) of X and the one-point space.

If X is a pointed space and Y is an unpointed space, then

$$X \rtimes Y := \frac{X \times Y}{\{*\} \times Y} \tag{2.1}$$

is a pointed space. We call pointed maps $f, g\colon X \to Y$ *homotopic* if there is a continuous map $H\colon X \rtimes [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. This is equivalent to saying that f and g are homotopic relative to the base point.

The following definitions apply to pointed and unpointed continuous maps: Two spaces X and Y are *homotopy equivalent* if there are continuous maps $r\colon X \to Y$ and $s\colon Y \to X$ such that $sr \simeq \mathrm{id}_X$ and $rs \simeq \mathrm{id}_Y$. This is written as $X \simeq Y$. The equivalence class of all spaces homotopy equivalent to X is denoted by $[X]$ and is called the *homotopy type* of X.

2.3 Singular homology

Throughout this thesis, we let $\mathbb{F}$ be a field. A *graded vector space* is a sequence of vector spaces $V = V_* = \{V_n \mid n \in \mathbb{N}\}$ over the same field $\mathbb{F}$. A *linear map* $\alpha\colon V \to W$ between graded vector spaces is a sequence of linear maps $\alpha_n\colon V_n \to W_n$.

Given a pair of topological spaces (X,A), the graded vector space $\mathrm{H}_*(X,A)$ is defined as the relative singular homology of (X,A). We do not recall the complete construction of singular homology here. The most common construction – simplicial singular homology – is presented in standard textbooks on algebraic topology, e.g. [Spa66, Hat02]. An equivalent alternative – cubical singular homology – is sketched in Section 3.2.

A continuous map $f : (X,A) \to (Y,B)$ induces a linear map $f_* = \mathrm{H}_*(f)\colon \mathrm{H}_*(X,A) \to \mathrm{H}_*(Y,B)$. Singular homology satisfies the following **axioms of homology**:

1. It is a covariant functor, i.e.,
   (a) $\mathrm{H}_*(\mathrm{id}_{(X,A)}) = \mathrm{id}_{\mathrm{H}_*(X,A)}$;
   (b) if the composition gf exists for maps f and g, then $\mathrm{H}_*(gf) = \mathrm{H}_*(g)\mathrm{H}_*(f)$.
2. (Homotopy) If $f \simeq g\colon (X,A) \to (Y,B)$, then $f_* = g_*\colon \mathrm{H}_*(X,A) \to \mathrm{H}_*(Y,B)$.
3. (Long exact sequence) There are linear maps $\partial_n\colon \mathrm{H}_n(X,A) \to \mathrm{H}_{n-1}(A)$ for all $n \geq 1$ such that:
   (a) For the inclusions $i\colon A \to X$ and $j\colon X = (X,\varnothing) \to (X,A)$, the sequence of vector spaces and linear maps

   $$\cdots \longrightarrow \mathrm{H}_n(A) \xrightarrow{i_n} \mathrm{H}_n(X) \xrightarrow{j_n} \mathrm{H}_n(X,A) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(A) \xrightarrow{i_{n-1}} \cdots$$

   is exact.

   (b) A map $f: (X,A) \to (Y,B)$ induces a morphism $f_* = \mathrm{H}_*(f)$ between the long exact sequences of these pairs with $\partial_n f_n = f_{n-1}\partial_n$ for $n \geq 1$.

4. (Excision) If $U \subset X$ is an open set with $\operatorname{cl} U \subset \operatorname{int} A$, then the inclusion

$$k\colon (X \setminus U, A \setminus U) \to (X,A)$$

induces an isomorphism $k_*\colon \mathrm{H}_*(X \setminus U, A \setminus U) \to \mathrm{H}_*(X,A)$.

5. For the one-point space $\{*\}$, $\mathrm{H}_n(\{*\}) = \begin{cases} \mathbb{F} & \text{if } n = 0, \\ 0 & \text{if } n > 0. \end{cases}$

6. (Additivity) For a disjoint union $X = \coprod X_k$ of spaces X_k, the inclusions $i_k : X_k \to X$ induce an isomorphism $\bigoplus_k \mathrm{H}_n(X_k) \to \mathrm{H}_n(X)$.

If a space X is pointed with base point $* \in X$, we let its *reduced homology* be $\widetilde{\mathrm{H}}_*(X) = \mathrm{H}_*(X, \{*\})$. For any choice of this base point $*$ in X, these vector spaces $\widetilde{\mathrm{H}}_*(X)$ are isomorphic. Given an unpointed space X, we can construct the pointed space $X_+ := X \coprod \{*\}$. Then we always have $\widetilde{\mathrm{H}}_*(X_+) = \mathrm{H}_*(X)$. We often write simply $*$ for the topological space $\{*\}$. In order to relate $\mathrm{H}_*(X,A)$ and $\widetilde{\mathrm{H}}_*(X/A)$, we use the following condition on a pair of spaces.

Definition 2.3.1 ([Hat02, Section 2.1]). A pair (X,A) of spaces is called a *good pair* if

(i) X is Hausdorff,

(ii) A is a non-empty closed subspace of X, and

(iii) A is a deformation retract of a neighborhood in X.

For good pairs, the singular homology of the quotient and the relative singular homology are naturally isomorphic in the following sense.

Proposition 2.3.2 ([Hat02, Proposition 2.22]). *For good pairs (X,A), the quotient map $q\colon (X,A) \to (X/A, A/A)$ induces an isomorphism*

$$q_*\colon \mathrm{H}_*(X,A) \to \mathrm{H}_*(X/A, A/A) = \mathrm{H}_*(X/A, *) = \widetilde{\mathrm{H}}_*(X/A).$$

The following version of excision is useful for defining a homological version of the Conley index in Section 2.5.

Proposition 2.3.3 ([Spa66, Section 4.8, Theorem 9]). *Given a compact pair (X,A) and an open subset $U \subset X$ such that $(X \setminus U, A \setminus U)$ is good, the inclusion $k\colon (X \setminus U, A \setminus U) \to (X,A)$ induces an isomorphism $k_*\colon \mathrm{H}_*(X \setminus U, A \setminus U) \to \mathrm{H}_*(X,A)$.*

2.4 The Conley index for flows

From here throughout the rest of this thesis, the letter X denotes a locally compact Hausdorff space. In our examples, X often has the extra structure of a metric space or a manifold, but these properties are not needed to define the Conley index. We recall results from [Con78].

Let φ be a global flow on X (but note Remark 2.1.3). A set $S \subset X$ is called *invariant* if $\varphi(S,\mathbb{R}) = S$. Note that this is equivalent to $\varphi(S,\mathbb{R}) \subset S$ since $S = \varphi(S \times \{0\})$. The Conley index is not defined for arbitrary invariant sets of a dynamical system. We require the additional property of being *isolated invariant* as defined below. For $M \subset X$, let

$$\operatorname{Inv}(M,\varphi) = \{\, x \in M \mid \varphi(x,\mathbb{R}) \subset M \,\}$$

be the *invariant part of* M. This set is obviously invariant. It contains all invariant subsets of M, i.e., $\operatorname{Inv}(M,\varphi)$ is the union of all invariant subsets of M.

Definition 2.4.1. A set $S \subset X$ is an *isolated invariant set* if there is a compact set $M \subset X$ such that

$$S = \operatorname{Inv}(M,\varphi) \subset \operatorname{int} M.$$

In this situation, M is called an *isolating neighborhood* of S. Note that M is closed because X is Hausdorff. It is also not difficult to see that S is closed in M. Since S is a closed subset of the compact set M, it is also compact.

Note that the empty set $\varnothing$ is an isolated invariant set. We can consider the isolating neighborhood $\varnothing$, but any compact set M with $\operatorname{Inv}(M,\varphi) = \varnothing$ is an isolating neighborhood of the empty set.

Definition 2.4.2. Let S be an isolated invariant set for φ. A compact pair (N,L) is an *index pair* for (S,φ) if it has the following properties:

(i) $S = \operatorname{Inv}(\operatorname{cl}(N \setminus L),\varphi) \subset \operatorname{int}(N \setminus L)$.

(ii) If $x \in L$, $t > 0$ and $\varphi(x,[0,t]) \subset N$, then $\varphi(x,[0,t]) \subset L$.

(iii) If $x \in N$, $t > 0$ and $\varphi(x,t) \notin N$, then there is a $t' \in [0,t]$ such that $\varphi(x,t') \in L$ and $\varphi(x,[0,t']) \subset N$.

Definition 2.4.3. Let (N,L) be an index pair for (S,φ).

- The *homotopy type Conley index* of (S,φ) is the pointed homotopy type $[N/L]$.
- The *homological Conley index* is $\operatorname{CH}_*(S,\varphi) := \widetilde{\operatorname{H}}_*(N/L)$.

The homotopy type Conley index is well-defined, i.e., index pairs (N,L) exist for any given isolated invariant set S and the resulting pointed homotopy type of N/L is the same for any index pair [Con78]. The homological Conley index is well-defined because homology only depends on the homotopy type of the space (Section 2.3).

Also note that N/L is a compact Hausdorff space because it is a compact pair and X is Hausdorff. More precisely: The quotient N/L is Hausdorff because the compact subset L of the Hausdorff space N can be separated from any point $x \in N \setminus L$. The quotient is compact because N is compact and N/L is its image under a continuous map.

It is also possible to make statements about the Conley index for a parametrized flow, i.e., for a continuous map

$$\varphi\colon [0,1] \times X \times \mathbb{R} \to X,$$

where each $\varphi^\lambda := \varphi(\lambda,\cdot,\cdot)$ is a flow.

Proposition 2.4.4 ([MM02]). *If M is an isolating neighborhood for φ^λ for some $\lambda \in [0,1]$, then there is a set U open in $[0,1]$ with $\lambda \in U$ such that M is an isolating neighborhood for φ^μ for every $\mu \in U$.*

Moreover, if M is an isolating neighborhood for all $\lambda \in [0,1]$, then the Conley indices of $(\mathrm{Inv}(M,\varphi^\lambda),\varphi^\lambda)$ are the same for all $\lambda \in [0,1]$.

Later in this thesis, this *continuation property* is not directly mentioned. But it explains why it is a good idea to represent isolating neighborhoods numerically: They are still isolating neighborhoods under small perturbations of φ^λ, even though the isolated invariant set inside usually changes. The continuation property is also important to see that the methods in Chapter 5 can be applied to parametrized flows. But parametrized flows are not the main focus of this thesis, hence we do not go into details.

We mention only one important example for a Conley index here: The Conley index of the empty set is $[*]$, the homotopy type of the one-point space. This can be seen by considering the index pair $(N,L) = (\varnothing,\varnothing)$. Then $\varnothing/\varnothing = *$ by the definition of quotient space. In homology, this yields $\mathrm{CH}_*(\varnothing,\varphi) = \widetilde{\mathrm{H}}_*(*) = 0$.

2.5 The Conley index for maps

In this section, we let $f : X \to X$ be a continuous map. There are several ways of defining the Conley index for such a map (a discrete dynamical system). Each definition uses a certain kind of index pair and an equivalence relation on a map on this index pair (cf. [RS88, Mro90b, Szy95, FR00]). This abundance of definitions is not as diverse as one might think from a quick look at the literature. There is one definition of index pair and one equivalence relation generalizing all the other definitions (cf. Definition 2.5.14).

In addition to the general Definition 2.5.2 of an index pair, we introduce two special kinds of index pairs: strong index pairs and weak index pairs. These concepts are useful numerically – they can be constructed with our methods from rigorous numerics – and theoretically: They give criteria for detecting an index pair. Whether a compact pair is an index pair in the general sense can be hard to check.

In Subsection 2.5.2, we recall the concept of an index map and present the most general definition of the Conley index for a map. We also recall the classical Theorem 2.5.12

which is essential to prove that the Conley index is well-defined. In Subsections 2.5.3 and 2.5.4, we present algebraic versions using homology. These definitions are equivalent in all our numerical examples in this thesis because we work with finite-dimensional vector spaces.

In Subsection 2.5.5, we present a new approach to defining the Conley index for maps. Even though it is not used later in the thesis, it seems quite interesting from a theoretical perspective: It shows that the index map can be used to define the Conley index as the homotopy type of a space via a simple gluing construction. The author hopes that this makes the Conley index more accessible to topologists, who are usually familiar with this type of construction.

Definition 2.5.1.

(i) A *solution* of f through a point $x \in X$ is a map $\gamma\colon \mathbb{Z} \to X$ such that $\gamma(0) = x$ and $\gamma(n+1) = f(\gamma(n))$ for all $n \in \mathbb{Z}$.

(ii) For $M \subset X$, let $\operatorname{Inv}(M, f) := \{\, x \in M \mid \text{there is a solution } \gamma\colon \mathbb{Z} \to M \text{ through } x \,\}$.

(iii) A compact set $M \subset X$ is an *isolating neighborhood* of the *isolated invariant set* S if $S = \operatorname{Inv}(M, f) \subset \operatorname{int} M$.

2.5.1 Index pairs and index maps

There are several definitions of index pairs for maps in the literature. We list three definitions here which are used throughout this thesis. For defining the Conley index, the following most general one suffices.

Definition 2.5.2. Given an isolated invariant set $S \subset X$, a compact pair (N, L) is an *index pair* for (S, f) if

(i) $S = \operatorname{Inv}(\operatorname{cl}(N \setminus L), f) \subset \operatorname{int}(N \setminus L)$ and

(ii) the pointed map

$$f_{(N,L)}\colon N/L \to N/L,$$
$$x \mapsto \begin{cases} f(x) & \text{if } x, f(x) \in N \setminus L, \\ * & \text{otherwise,} \end{cases}$$

is continuous.

In this case, we call $f_{(N,L)}$ the *index map*.

Example 2.5.3. Let $X = \mathbb{R}$, $f(x) = 2x$ and $S = \{0\}$. Choosing $N = [-2, 2]$ and $L = [-2, -1] \cup [1, 2]$ gives an index pair (even in the sense of Definition 2.5.6) for (S, f). Then $N/L = S^1$, the pointed circle, and $f_{(N,L)} \simeq \mathrm{id}_{S^1}$.

The following criterion is useful to detect an index pair.

Proposition 2.5.4 ([RS88, Theorem 4.3]). *For a compact pair (N,L), the map $f_{(N,L)}$ as defined above is continuous if and only if the two following conditions are true for every $x_0 \in f^{-1}(N \setminus L)$:*

(i) If $x_0 \in L$, then there is an open set $U \subset X$ such that $x_0 \in U$ and $f(U \cap N \setminus L) \subset X \setminus N$.

(ii) If $x_0 \in N \setminus L$, then there is an open set $U \subset X$ such that $x_0 \in U$ and $f(U \cap N \setminus L) \subset N \setminus L$.

The following two definitions of special index pairs are not needed in this chapter to define the Conley index for maps, but they appear in constructions in later chapters. In both definitions, we let $S \subset X$ be an isolated invariant set for f.

Definition 2.5.5 ([KMM04, Def. 10.76], [Mro06, Def. 4.2]). A compact pair (N,L) is a *weak index pair for* (S,f) if

(i) $S = \operatorname{Inv}(\operatorname{cl}(N \setminus L), f) \subset \operatorname{int}(N \setminus L)$,

(ii) $f(L) \cap N \subset L$, and

(iii) $\operatorname{cl}(f(N) \setminus N) \cap N \subset L$.

Definition 2.5.6 ([MM02, Def. 3.24]). A compact pair (N,L) is a *strong index pair for* (S,f) if

(i) $S = \operatorname{Inv}(\operatorname{cl}(N \setminus L), f) \subset \operatorname{int}(N \setminus L)$,

(ii) $f(L) \cap N \subset L$, and

(iii) $f(N \setminus L) \subset N$.

Note that both definitions only differ in condition (iii). This condition is usually described by saying that L is an *exit set*. Condition (ii) means that L is *forward invariant in N*.

A strong index pair is a weak index pair. And a weak index pair is an index pair in the sense of Definition 2.5.2. Both these properties are shown in [Mro06].

Condition (iii) of Definition 2.5.6 is sometimes formulated differently as condition (iii') in the following lemma (cf. [Mro06, Definition 4.1] and [MSW15]), but this yields the same definition as we see in the following simple lemma.

Lemma 2.5.7. *Condition (iii) in the definition of a strong index pair is equivalent to the following subset relation, where the given equality is obvious:*

(iii') $N \setminus f^{-1}(N) = N \cap f^{-1}(X \setminus N) \subset L$.

Proof. (iii) $\implies$ (iii'): Let $x \in N \setminus f^{-1}(N)$. Assume $x \notin L$. Then $f(x) \in N$ because of (iii) in Definition 2.5.6. But this contradicts $f(x) \notin N$. Overall $x \in L$.

(iii') $\implies$ (iii): Let $x \in N \setminus L$. Assume that $f(x) \notin N$. Then $x \in L$ by condition (iii'). A contradiction. Overall $f(x) \in N$. □

Given an isolated invariant set S of f, there is always a strong index pair for (S,f) ([Mro90b, Mro94]). The index map induces a linear map $\widetilde{\mathrm{H}}_*(f_{(N,L)})$ on $\widetilde{\mathrm{H}}_*(N/L) = \mathrm{H}_*(N/L,*)$. For numerical purposes, we are also interested in a corresponding map induced in relative homology $\mathrm{H}_*(N,L)$. Assume that (N,L) is a strong index pair. Then f induces a map of pairs

$$\begin{aligned} f_P \colon (N,L) &\to (N\cup f(L), L\cup f(L)),\\ x &\mapsto f(x). \end{aligned}$$

Following [Mro06], Section 5, this yields a commutative diagram on pairs of spaces, where i is an inclusion of pairs and q is the quotient map:

$$\begin{tikzcd} (N,L) \arrow[r, "f_P"] \arrow[d, "q"] & (N\cup f(L), L\cup f(L)) & (N,L) \arrow[l, "i"'] \arrow[d, "q"] \\ (N/L,*) \arrow[rr, "f_{(N,L)}"] & & (N/L,*). \end{tikzcd}$$

Now assume that (N,L) is good. Then $q_* = \mathrm{H}_*(q)$ is an isomorphism by Proposition 2.3.2, and $i_* = \mathrm{H}_*(i)$ is an isomorphism by Proposition 2.3.3 (we remove $f(L)\setminus N$ from the large pair and this set is open in $N\cup f(L)$). Applying the homology functor H_* to the diagram above yields the commutative diagram

$$\begin{tikzcd} \mathrm{H}_*(N,L) \arrow[r, "f_{P*}"] \arrow[d, "\cong"', "q_*"] & \mathrm{H}_*(N\cup f(L), L\cup f(L)) & \mathrm{H}_*(N,L) \arrow[l, "i_*"', "\cong"] \arrow[d, "\cong"', "q_*"] \\ \mathrm{H}_*(N/L,*) \arrow[rr, "\widetilde{\mathrm{H}}_*(f_{(N,L)})"] & & \mathrm{H}_*(N/L,*). \end{tikzcd} \tag{2.2}$$

We usually use the lower linear map in definitions, but when working with good pairs, we often work with f_{P*} instead. A similar diagram exists for weak index pairs. The homological index map $\mathrm{H}_*(f_{(N,L)})$ is used in Subsections 2.5.3 and 2.5.4 to define algebraic versions of the Conley index. For our numerical computations in Chapter 4, these two algebraic index definitions coincide as we show in Theorem 2.5.23.

2.5.2 Homotopy shift equivalence

In this subsection, we recall a very general definition of the Conley for maps. We present it here as introduced in [FR00]. The only difference is that we use general index pairs and not the special version used therein (called filtration pairs), but our generalization does not make the proofs more complicated in any way.

Definition 2.5.8. We call homotopy classes $[f]$ and $[g]$ of continuous maps $f : P \to P$ and $g\colon Q \to Q$ *shift equivalent* if there are continuous maps $r\colon P \to Q$ and $s\colon Q \to P$ such that $gr \simeq rf$, $sg \simeq fs$, $sr \simeq f^n$ and $rs \simeq g^n$ for some $n \in \mathbb{N}$.

The goal of the following discussion is Theorem 2.5.13. The ideas presented until then are not used elsewhere in this thesis.

We use the following abbrevation for a forward trajectory: For $0 \leq n < m$, we define the sets

$$f^{[n,m]}(x) := \{f^n(x), f^{n+1}(x), \dots, f^m(x)\}$$

and, for $U \subset X$,

$$f^{[n,m]}(U) := \bigcup_{x \in U} f^{[n,m]}(x) \subset X.$$

We want to define a map between two index pairs for the same invariant set. First, define for an arbitrary set $M \subset X$:

$$\mathrm{Inv}^n(M, f) := \{ x \in M \mid \text{there is a } y \in M \text{ with } f^n(y) = x \text{ and } f^{[0,2n]}(y) \subset M \}.$$

Obviously, $\mathrm{Inv}^{n+1}(M, f) \subset \mathrm{Inv}^n(M, f)$. If M is compact, then [FR00, Proposition 2.2]:

$$\mathrm{Inv}(M, f) = \bigcap_{n \geq 0} \mathrm{Inv}^n(M, f).$$

Lemma 2.5.9 ([RS88, Proposition 3.1])**.** *Let M and M' be isolating neighborhoods (compact by definition) for (S, f). Then there is an $n \geq 0$ such that*

$$\mathrm{Inv}^n(M, f) \subset \mathrm{int}\, M'.$$

Proof. First note that $\mathrm{Inv}^n(M, f)$ is closed because M is closed. Therefore, for each $n \geq 0$, $V_n := \mathrm{Inv}^n(M, f) \setminus \mathrm{int}\, M'$ is a closed and hence compact set with $V_{n+1} \subset V_n$. Now

$$\bigcap_{n \in \mathbb{N}} V_n = \bigcap_{n \in \mathbb{N}} \mathrm{Inv}^n(M, f) \setminus \mathrm{int}\, M' = \varnothing.$$

Since the intersection of any nested sequence of non-empty compact sets is non-empty, there must be an n such that $V_m = \varnothing$ for $m \geq n$. □

Here we denote index pairs by Greek letters. We let $\alpha = (N_\alpha, L_\alpha)$ and $\beta = (N_\beta, L_\beta)$ be index pairs for (S, f). The index maps are now denoted by $f_\alpha = f_{(N_\alpha, L_\alpha)}$ and $f_\beta = f_{(N_\beta, L_\beta)}$, respectively. Then, using Lemma 2.5.9 and the fact that $\mathrm{cl}(N_\alpha \setminus L_\alpha)$ and $\mathrm{cl}(N_\beta \setminus L_\beta)$ are isolating neighborhoods of (S, f), there is a number $n \geq 0$ such that

$$\mathrm{Inv}^n(N_\alpha \setminus L_\alpha, f) \subset N_\beta \setminus L_\beta \text{ and } \mathrm{Inv}^n(N_\beta \setminus L_\beta, f) \subset N_\alpha \setminus L_\alpha.$$

Let $u = u(\alpha, \beta)$ be the smallest $n \geq 0$ with this property. Obviously, $u(\alpha, \beta) = u(\beta, \alpha)$, and we get the following property right from the definition of $u = u(\alpha, \beta)$.

Lemma 2.5.10. *For any $x \in N_\alpha \setminus L_\alpha$: If $f^{[0,2u]}(x) \subset N_\alpha \setminus L_\alpha$, then $f^u(x) \in N_\beta \setminus L_\beta$.*

Now we define

$$C_{\alpha\beta} := \{\, x \in N_\alpha \setminus L_\alpha \mid f^{[0,2u]}(x) \subset N_\alpha \setminus L_\alpha \text{ and } f^{[u+1,3u+1]}(x) \subset N_\beta \setminus L_\beta \,\}$$

and the map (not necessarily continuous)

$$\begin{aligned} f_{\beta\alpha}\colon N_\alpha/L_\alpha &\to N_\beta/L_\beta, \\ x &\mapsto \begin{cases} f^{3u+1}(x) & \text{if } x \in C_{\alpha\beta}, \\ * & \text{otherwise.} \end{cases} \end{aligned}$$

A special case is $\alpha = \beta$. Then $u(\alpha,\alpha) = 0$ and $f_\alpha = f_{\alpha\alpha}$. The following important theorem is shown in [RS88], where it is stated for a diffeomorphism f on a manifold. The proof therein works for every continuous map $f : X \to X$ on a Hausdorff space X. In order to see this, we sketch some details of that proof here.

Remark 2.5.11. Note that the set $N \setminus L$ is open in N because L is closed in N by assumption. Therefore, the image of $N \setminus L$ in the quotient N/L is also open. Even more: $N \setminus L$ as a subspace of N is homeomorphic to $N \setminus L$ as a subspace of N/L, by the definition of quotient topology. Hence, there is no need to distinguish these two spaces. A set $U \subset N \setminus L$ is open in N/L if and only if U is open in $N \setminus L$.

Theorem 2.5.12 ([RS88, Theorem 6.3])**.**

(i) $f_{\beta\alpha}$ *is continuous.*

(ii) $f_{\alpha\beta} \circ f_{\beta\alpha} = f_\alpha^{6u(\alpha,\beta)+2}$

(iii) $f_{\beta\alpha} \circ f_\alpha = f_\beta \circ f_{\beta\alpha}$

Proof. The idea for the proof of (i) is to consider five cases depending on where x_0 lies within N_α/L_α, and to show for each case that $f_{\beta\alpha}$ is continuous in x_0. We do not recall all cases here, but only present one difficult case from the proof in [RS88] slightly adapted to our needs here. The proofs of the other four cases are similar or shorter. We mainly use Lemma 2.5.10 and the continuity of the index maps f_α and f_β.

Case 5: Suppose that $x_0 \in C_{\alpha\beta}$. We want to show that there is an open set $U \subset X$ such that $x_0 \in U$ and $U \cap N_\alpha \setminus L_\alpha \subset C_{\alpha\beta}$.

For $i \in \{0, \dots, 2u-1\}$, we know that $f^i(x_0) \in f^{-1}(N_\alpha \setminus L_\alpha)$. By Proposition 2.5.4(ii), there are open sets $U_i \subset X$ such that $f^i(x_0) \in U_i$ and

$$f(U_i \cap N_\alpha \setminus L_\alpha) \subset N_\alpha \setminus L_\alpha.$$

Applying Lemma 2.5.10, we have $f^u(x_0) \in N_\beta \setminus L_\beta$. Applying Proposition 2.5.4(ii) to the index pair (N_β, L_β) yields: For each $i \in \{u, \dots, 3u\}$, there is an open set $V_i \subset X$ such that $f^i(x_0) \in V_i$ and

$$f(V_i \cap N_\beta \setminus L_\beta) \subset N_\beta \setminus L_\beta.$$

For $0 \le i \le 3u$, we define open sets

$$W_i := \begin{cases} U_i & \text{if } 0 \le i \le u-1, \\ U_i \cap V_i & \text{if } u \le i \le 2u-1, \\ V_i & \text{if } 2u \le i \le 3u. \end{cases}$$

Now we let

$$U := \bigcap_{i=0}^{3u} f^{-i}(W_i) \subset X,$$

an open set. Let $x \in U \cap N_\alpha \setminus L_\alpha$. Then we argue via induction over increasing i. For $0 \le i \le 2u-1$, we have the implication

$$f^i(x) \in N_\alpha \setminus L_\alpha \text{ and } f^i(x) \in U_i \implies f^{i+1}(x) \in N_\alpha \setminus L_\alpha.$$

Overall, $f^{[0,2u]}(x) \subset N_\alpha \setminus L_\alpha$, and therefore, using Lemma 2.5.10, $f^u(x) \in U_u \cap N_\beta \setminus L_\beta$. From this, we get $f^{[u,3u+1]}(x) \in N_\beta \setminus L_\beta$ by induction as before.

We have shown that $U \cap N_\alpha \setminus L_\alpha \subset C_{\alpha\beta}$. Therefore, $f_{\beta\alpha}(x) = f^{3u+1}(x)$ for $x \in U \cap N_\alpha \setminus L_\alpha$. One should note that $U \cap N_\alpha \setminus L_\alpha$ is open in $N_\alpha \setminus L_\alpha$ and therefore open in in N_α/L_α (Remark 2.5.11). Since f is continuous, $f_{\beta\alpha}$ is continuous in x_0. This finishes the proof of case 5.

The statements (ii) and (iii) are special cases of [RS88, Theorem 6.3(iii)], which states that given three index pairs for (S, f), one has that

$$f_{\gamma\beta} \circ f_{\beta\alpha} = f_\gamma^{3(u(\beta,\gamma)+u(\alpha,\beta)-u(\alpha,\gamma))+1} f_{\gamma\alpha} = f_{\gamma\alpha} f_\alpha^{3(u(\beta,\gamma)+u(\alpha,\beta)-u(\alpha,\gamma))+1}$$

□

For us, the main purpose of Theorem 2.5.12 is now the following observation.

Theorem 2.5.13. *If S is an isolated invariant set for f, and (N, L) and (N', L') are index pairs for (S, f), then there are $n \in \mathbb{N}$ and maps $r\colon N/L \to N'/L'$ and $s\colon N'/L' \to N/L$ such that $rf_{(N,L)} = f_{(N',L')}r$, $sf_{(N',L')} = f_{(N,L)}s$, $sr = f^n_{(N,L)}$ and $rs = f^n_{(N',L')}$.*

This theorem shows that the following definition does not depend on the choice of an index pair. This definition was first proposed in [FR00] using a special kind of index pair and proving the theorem above only for this special kind of index pairs (Theorem 4.3 therein). But as we have just shown, this extra assumption is not necessary for the definition to make sense.

Definition 2.5.14 ([FR00, Definition 4.8])**.** The *shift equivalence Conley index* of (S, f) is the shift equivalence class of $[f_{(N,L)}]$ for an arbitrary index pair (N, L).

The author of this thesis could not find any publication claiming our result that this definition makes sense with the most general Definition 2.5.2 of an index pair. This seems to be implicit knowledge among scientists working with the Conley index for maps.

The other definitions we present for the discrete time Conley index are special cases of this definition. The shift equivalence definition is even universal in a certain sense described in [Szy95, Section 6], where a definition equivalent to Definition 2.5.14 is used [FR00, Section 8].

2.5.3 Leray functor

Given an endomorphism $\alpha\colon V \to V$ of graded vector spaces (e.g., homology groups), the *Leray functor* $\mathbf{L}$ assigns to α an automorphism $\mathbf{L}(\alpha)$. We recall its construction introduced in [Mro90b, Section 4]. Let **End** be the category with objects all the endomorphisms of graded vector spaces. A morphism in **End** from α to β is a commutative diagram

$$\begin{tikzcd} V \arrow[r, "r"] \arrow[d, "\alpha"] & W \arrow[d, "\beta"] \\ V \arrow[r, "r"] & W \end{tikzcd}$$

in the category of graded vector spaces. This is written $\vec{r}\colon \alpha \to \beta$. Let **Aut** be the full subcategory of **End** with objects the automorphisms. For an arbitrary endomorphism $\alpha\colon V \to V$, let

$$\begin{aligned} \operatorname{gker}(\alpha) &:= \{\, v \in V \mid \text{there is an } n \in \mathbb{N} \text{ such that } \alpha^n(v) = 0 \,\} \text{ and} \\ \operatorname{gim}(\alpha) &:= \{\, v \in V \mid \text{for all } n \in \mathbb{N} : v \in \operatorname{im}(\alpha^n) \,\}. \end{aligned}$$

We call $\operatorname{gker}(\alpha)$ the *generalized kernel* and $\operatorname{gim}(\alpha)$ the *generalized image* of α. An object $\alpha\colon V \to V$ of **End** induces a monomorphism

$$\begin{aligned} \alpha'\colon V / \operatorname{gker}(\alpha) &\to V / \operatorname{gker}(\alpha), \\ v + \operatorname{gker}(\alpha) &\mapsto \alpha(v) + \operatorname{gker}(\alpha). \end{aligned}$$

From this we define an automorphism by restricting domain and codomain:

$$\begin{aligned} \alpha''\colon \operatorname{gim}(\alpha') &\to \operatorname{gim}(\alpha'), \\ v + \operatorname{gker}(\alpha) &\mapsto \alpha(v) + \operatorname{gker}(\alpha). \end{aligned}$$

Then for an object $\alpha\colon V \to V$ of **End**, we let $\mathbf{L}(\alpha) := (\operatorname{gim}(\alpha'), \alpha'')$. For a morphism $\vec{r}\colon \alpha \to \beta$ in **End**, we let $\mathbf{L}(\vec{r})$ be the commutative diagram

$$\begin{tikzcd} \operatorname{gim}(\alpha') \arrow[r, "r''"] \arrow[d, "\alpha''"] & \operatorname{gim}(\beta') \arrow[d, "\beta''"] \\ \operatorname{gim}(\alpha') \arrow[r, "r''"] & \operatorname{gim}(\beta'), \end{tikzcd}$$

where r'' is defined analogously to f''. Showing that this yields a well-defined functor $\mathbf{L}\colon \mathbf{End} \to \mathbf{Aut}$ is straightforward. The proofs are given in [Mro90b, Section 4]. If $\alpha \in \mathbf{End}$ is already an automorphism, then $\mathbf{L}(\alpha) = \alpha$. Obviously, two objects α, β of $\mathbf{Aut}$ are isomorphic if and only if there is an isomorphism r of graded vector spaces such that $r\alpha = \beta r$. In this situation, we write $\alpha \cong \beta$.

In the numerical situations considered in this thesis, taking the generalized image in the Leray functor definition does not do anything because of the following simple observation.

Lemma 2.5.15. *If $V' = V/\operatorname{gker}(\alpha)$ is a finite dimensional vector space, then $\alpha'' = \alpha'$ in the definitions above.*

Proof. The relation
$$\dim(V') = \dim(\ker(\alpha')) + \dim(\operatorname{im}(\alpha'))$$
from linear algebra together with $\dim(\ker(\alpha')) = 0$ yields that $\operatorname{im}(\alpha') = V'$. □

A useful criterion is the following Proposition 2.5.17. We give a direct proof here specializing the proof presented in [MSW15]. For the proof, we first recall a simple fact that holds for morphisms in any category.

Lemma 2.5.16. *Let $r\colon V \to W$ be a linear map. If there are linear maps $s, s'\colon W \to V$ such that $sr = \mathrm{id}_V$ and $rs' = \mathrm{id}_W$, then r is an isomorphism.*

Proof. An inverse of r is given by srs' since $(srs')r = sr$ and $r(srs') = rs'$. □

Proposition 2.5.17 ([MSW15, Proposition 2.1]). *Let $\alpha\colon V \to V$ and $\beta\colon W \to W$ be linear endomorphisms of graded vector spaces, and let $\vec{r}\colon \alpha \to \beta$ be a morphism in* **End**. Assume that there is a number $n \geq 1$ and a linear map $s\colon W \to V$ such that the diagram

$$\begin{tikzcd} V \arrow[r, "r"] \arrow[d, "\alpha^n"'] & W \arrow[dl, "s"'] \arrow[d, "\beta^n"] \\ V \arrow[r, "r"] & W \end{tikzcd} \tag{2.3}$$

commutes. Then $\mathbf{L}(\vec{r})\colon \mathbf{L}(\alpha) \to \mathbf{L}(\beta)$ is an isomorphism in **Aut**.

Proof. First note that α^n induces a morphism $\overrightarrow{\alpha^n}\colon \alpha^n \to \alpha^n$, similarly for β^n. By (2.3), the linear map $s\colon W \to V$ induces a morphism $\vec{s}\colon \beta^n \to \alpha^n$, and r induces a morphism $\bar{r}\colon \alpha^n \to \beta^n$. Note that $\bar{r}$ and $\vec{r}$ are both induced by r, but they have different domains and codomains, therefore we denote them differently. We get the following commutative diagram in **End**:

$$\begin{tikzcd} \alpha^n \arrow[r, "\bar{r}"] \arrow[d, "\overrightarrow{\alpha^n}"'] & \beta^n \arrow[dl, "\vec{s}"'] \arrow[d, "\overrightarrow{\beta^n}"] \\ \alpha^n \arrow[r, "\bar{r}"] & \beta^n \end{tikzcd}.$$

Applying the functor $\mathbf{L}\colon \mathbf{End} \to \mathbf{Aut}$ yields the commutative diagram

$$\begin{tikzcd}
\mathbf{L}(\alpha^n) \arrow[r, "\mathbf{L}(\bar{r})"] \arrow[d, "\mathbf{L}\left(\overrightarrow{\alpha^n}\right)"'] & \mathbf{L}(\beta^n) \arrow[dl, "\mathbf{L}(\vec{s})"'] \arrow[d, "\mathbf{L}\left(\overrightarrow{\beta^n}\right)"] \\
\mathbf{L}(\alpha^n) \arrow[r, "\mathbf{L}(\bar{r})"] & \mathbf{L}(\beta^n),
\end{tikzcd}$$

in which the underlying linear map of $\mathbf{L}(\overrightarrow{\alpha^n})$ is $\mathbf{L}(\alpha^n)$, an automorphism of vector spaces. Similary for β^n. Therefore, $\mathbf{L}(\overrightarrow{\alpha^n})$ and $\mathbf{L}(\overrightarrow{\beta^n})$ are automorphisms as morphisms in **Aut**. The upper triangle shows that $\mathbf{L}(\bar{r})$ has a left inverse, the lower triangle shows that it has a right inverse. It is therefore an isomorphism by Lemma 2.5.16.

Now $\operatorname{dom}\mathbf{L}(\alpha^n) = \operatorname{dom}\mathbf{L}(\alpha)$. This can be seen by observing that for any endomorphism γ, $\operatorname{gker}(\gamma^n) = \operatorname{gker}(\gamma)$ and $\operatorname{gim}(\gamma^n) = \operatorname{gim}(\gamma)$. Similarly, $\operatorname{dom}\mathbf{L}(\beta^n) = \operatorname{dom}\mathbf{L}(\beta)$. The underlying linear maps of $\mathbf{L}(\bar{r})$ and $\mathbf{L}(\vec{r})$ are therefore equal. This shows that $\mathbf{L}(\vec{r})$ is also an isomorphism in **Aut**. □

This proposition together with Theorem 2.5.13 shows that the following numerically accessible version of the Conley index does not depend on the choice of an index pair as was first shown in [Mro90b, Theorem 2.6].

Definition 2.5.18. Given an isolated invariant set S of a discrete dynamical system f and an index pair (N,L) of S, we let the *homological Conley index* of (S,f) be

$$\operatorname{CH}_*(S,f) := \mathbf{L}\widetilde{\operatorname{H}}_*(f_{(N,L)});$$

more precisely, its isomorphism class in **Aut**.

2.5.4 Shift equivalence in homology

One can also use the analog of the relation from Definition 2.5.8 on the linear map $\widetilde{\operatorname{H}}_*(f_{(N,L)})$.

Definition 2.5.19. Let $\alpha\colon V \to V$ and $\beta\colon W \to W$ be endomorphisms of graded vector spaces. They are *shift equivalent* if there are linear maps $r\colon V \to W$ and $s\colon W \to V$ and an $n \in \mathbb{N}$ such that $r\alpha = \beta r$, $s\beta = \alpha s$, $rs = \beta^n$ and $sr = \alpha^n$ for some $n \in \mathbb{N}$.

Theorem 2.5.20 ([MM02, Theorem 3.29], [Szy95, Lemma 4.3])**.** *If (N,L) and (N',L') are two index pairs around the same isolated invariant set for a discrete dynamical system, then $\widetilde{\operatorname{H}}_*(f_{(N,L)})$ and $\widetilde{\operatorname{H}}_*(f_{(N',L')})$ are shift equivalent.*

Proof. This follows by applying the homology functor to the relations in Theorem 2.5.13. □

In the later chapters, we only apply these definitions to endomorphisms of finite-dimensional vector spaces. In this special situation, the definition using shift equivalence coincides with the definition using the Leray functor as is shown in [MM02]. We recall the results from there, giving some more details.

Proposition 2.5.21 ([MM02, Prop. 3.30])**.** *If V is finite-dimensional and $\alpha\colon V \to V$, then α and $\mathbf{L}(\alpha)$ are shift equivalent.*

Proof. Since V is finite-dimensional, $\operatorname{gker}(\alpha) = \ker(\alpha^n)$ for some $n > 0$. Thus, the domain of $\mathbf{L}(\alpha)$ is $V / \ker(\alpha^n)$. The map $\alpha^n\colon V \to V$ induces a unique map $a\colon V / \ker(\alpha^n) \to V$ making the upper triangle in the following diagram commute, where each horizontal arrow is the quotient map:

$$\begin{tikzcd} V \arrow[r] \arrow[d, "\alpha^n"'] & V / \ker(\alpha^n) \arrow[dl, "a"'] \arrow[d, "\mathbf{L}(\alpha)^n"] \\ V \arrow[r] & V / \ker(\alpha^n) \end{tikzcd}$$

The commutativity of the lower triangle follows directly from the definition of the Leray functor. □

Proposition 2.5.22 ([MM02, Prop. 3.31])**.** *Two automorphisms are isomorphic in* **Aut** if and only if they are shift equivalent.

Proof. An isomorphism $\vec{r}\colon \alpha \to \beta$ obviously yields a shift equivalence. For the opposite direction, let $\alpha\colon V \to V$ and $\beta : W \to W$ be shift equivalent automorphisms. Then there are $r\colon V \to W$ and $s\colon W \to V$ such that

$$\begin{tikzcd} V \arrow[r, "r"] \arrow[d, "\alpha^n"', "\cong"] & W \arrow[dl, "s"'] \arrow[d, "\beta^n", "\cong"'] \\ V \arrow[r, "r"] & W \end{tikzcd}$$

commutes. Since $sr = \alpha^n$ is an automorphism, r has a left inverse. Similarly, r has a right inverse. By Lemma 2.5.16, r is an isomorphism, and hence $\vec{r}\colon \alpha \to \beta$ is an isomorphism in **Aut**. □

Theorem 2.5.23 ([MM02, Theorem 3.32])**.** *Given two endomorphisms α and β of finite-dimensional vector spaces, they are shift equivalent if and only if $\mathbf{L}(\alpha) \cong \mathbf{L}(\beta)$.*

Proof. Assume that α and β are shift equivalent. Then, by Proposition 2.5.21, $\mathbf{L}(\alpha)$ and $\mathbf{L}(\beta)$ are shift equivalent. They are isomorphic in **Aut** because of Proposition 2.5.22. The opposite implication follows from the same propositions. □

2.5.5 Digression: New approach via mapping torus

This subsection deals with an alternative definition of the Conley index for maps which is not yet published. This definition is not used anywhere in the rest of this thesis. Instead of using an equivalence relation on the index map, we propose defining the Conley index for a map as the homotopy type of a certain space. This is also done in the usual construction of the Conley index for a flow, cf. Definition 2.4.3. Here we introduce such a definition hoping that it enriches our understanding of the Conley index. A publication with more results than presented here is in preparation [Wei].

For a pointed continuous map $\rho\colon P \to P$ on some pointed topological space P, let the *(pointed) mapping torus* be

$$\mathrm{T}_*(\rho) := \frac{P \rtimes [0,1]}{(x,1) \sim (\rho(x),0)}.$$

Its homotopy type depends only on the homotopy class of ρ [Ran87, Proposition 6.1(i)]. For an element $(x,\theta) \in P \rtimes [0,1]$, its equivalence class in the mapping torus is denoted by $[x,\theta] \in \mathrm{T}_*(\rho)$. We show in Theorem 2.5.26 that the following definition is independent of the choice of an index pair. Let $f : X \to X$ be a continuous map, and let $S \subset X$ be an isolated invariant set for f.

Definition 2.5.24. The *mapping torus index* of (S, f) is the pointed homotopy type of $\mathrm{T}_*(f_{(N,L)})$ for an index pair (N, L) of (S, f).

For a pointed map $\rho\colon P \to P$, consider the map

$$\begin{aligned}(P \rtimes [0,1]) \rtimes [0,\infty) &\to \mathrm{T}_*(\rho),\\ ((x,\theta),t) &\mapsto [\rho^{\lfloor t+\theta \rfloor}(x), t+\theta-\lfloor t+\theta \rfloor],\end{aligned}$$

where $\lfloor\cdot\rfloor\colon \mathbb{R}_{\geq 0} \to \mathbb{N}$ denotes the floor function, i.e., given $t \geq 0$, the number $\lfloor t \rfloor \in \mathbb{N}$ is the largest natural number less than or equal to t. This map is continuous and induces the continuous *suspension semiflow*

$$\begin{aligned}\varphi_\rho\colon\ \mathrm{T}_*(\rho) \rtimes [0,\infty) &\to \mathrm{T}_*(\rho),\\ ([x,\theta],t) &\mapsto [\rho^{\lfloor t+\theta \rfloor}(x), t+\theta-\lfloor t+\theta \rfloor]\end{aligned}$$

on the quotient space because

$$[\rho^{\lfloor t \rfloor+1}(x), t+1-\lfloor t \rfloor-1] = [\rho^{\lfloor t \rfloor}(\rho(x)), t-\lfloor t \rfloor].$$

The mapping torus is functorial in the following sense. Define the map

$$j_\rho\colon P \to \mathrm{T}_*(\rho),\ x \mapsto [x,0].$$

Given pointed maps $\rho\colon P \to P$ and $\sigma\colon Q \to Q$ and a pointed map $r\colon P \to Q$ such that $\sigma r = r\rho$, let the induced map $r_\#\colon\ \mathrm{T}_*(\rho) \to \mathrm{T}_*(\sigma)$ be given by $r_\#[x,\theta] = [r(x),\theta]$.

This definition makes the following diagram commute.

$$\begin{tikzcd} P \arrow[r, "\rho"] \arrow[d, "r"] & P \arrow[r, "j_\rho"] \arrow[d, "r"] & \mathrm{T}_*(\rho) \arrow[d, dashed, "r_\#"] \\ Q \arrow[r, "\sigma"] & Q \arrow[r, "j_\sigma"] & \mathrm{T}_*(\sigma) \end{tikzcd}$$

Lemma 2.5.25. *If $P = Q$, $\rho = \sigma$ and $r = \rho^n$ for some $n \in \mathbb{N}$, then the induced map $\rho^n_\#$ is homotopic to the identity on $\mathrm{T}_*(\rho)$.*

Proof. The suspension semiflow defines a homotopy because

$$\begin{aligned} \mathrm{id}_{\mathrm{T}_*(\rho)}[x,\theta] = [x,\theta] = \varphi_\rho([x,\theta],0) \text{ and} \\ \rho^n_\#[x,\theta] = [\rho^n(x),\theta] = \varphi_\rho([x,\theta],n). \end{aligned}$$

□

Theorem 2.5.26. *The mapping torus index of (S,f) is independent of the choice of an index pair (N,L).*

Proof. Let (N,L) and (N',L') be index pairs for (S,f). By Theorem 2.5.13, there are maps r,s and a number $n \in \mathbb{N}$ such that

(i) $rf_{(N,L)} = f_{(N',L')}r$ and $sf_{(N',L')} = f_{(N,L)}s$,
(ii) $sr = f^n_{(N,L)}$ and $rs = f^n_{(N',L')}$.

Then, using Lemma 2.5.25,

$$s_\# r_\# = (sr)_\# = (f^n_{(N,L)})_\# \simeq \mathrm{id},$$

similarly for $r_\# s_\#$. Therefore $\mathrm{T}_*(f_{(N,L)}) \simeq \mathrm{T}_*(f_{(N',L')})$. □

The mapping torus index of the empty set is the one-point space because $(\varnothing,\varnothing)$ is an index pair and $\mathrm{T}_*(\mathrm{id}_{\{*\}}) = \{*\}$. The mapping torus index for Example 2.5.3 is $S^1 \rtimes S^1$.

A very similar approach using the unpointed suspension semiflow of f and then the flow version of the Conley index has been presented in [Flo90, Section 2]. This approach does not use the index map. How to use it for numerical computations is unclear. Definition 2.5.24 seems to offer a link between this approach and the shift equivalence definition in Subsection 2.5.2.

The numerical usefulness of our mapping torus definition is not yet clear. But since the mapping torus is well-studied in algebraic topology, there are results about how to compute invariants like homology and homotopy groups. A long exact sequence describing its homology groups via the homological index map is given in [Hat02, Example 2.48]. The mapping torus is also a special kind of homotopy colimit. This property seems helpful because results are often published in this more general context.

Chapter 3

Rigorous numerics for dynamical systems

In this chapter, we recall and apply established ideas for numerically finding the Conley index of a flow inside an isolating neighborhood. Similar ideas were already used in [Pil99]. The theoretical ideas are not difficult, but the practical implementation requires choosing a discretization in time and in space as well as software for integrating flows (we use [CAPD]). Whereas a finer discretization of the space is usually better, the strategy for choosing the time step is less clear, as we show with an example.

3.1 Rigorous numerics for maps

We use some kind of time discretization of the flow. The main reason is that we can numerically construct (weak) index pairs for a map, but not directly for a flow. It is therefore very useful that we can apply algorithms developed for maps in order to find properties of the given flow. Here we recall **basic notions** for combinatorially representing a discrete dynamical system on $\mathbb{R}^d$ using interval arithmetic.

An *elementary interval* I is an interval of the form $I = [i] := [i, i]$ or $I = [i, i+1]$ for some $i \in \mathbb{Z}$. An *elementary cube* is a product of elementary intervals $Q = \prod_{i=1}^{d} I_i = I_1 \times \ldots \times I_d \subset \mathbb{R}^d$. The *dimension* $\dim Q$ is the number of intervals $[i, i+1]$ in this product and $\operatorname{emb} Q := d$ its *embedding number*. The word *cube* usually refers to an elementary cube in this thesis. For a set $\mathcal{A}$ of cubes having the same embedding number d, we let $|\mathcal{A}| = \bigcup_{Q \in \mathcal{A}} Q \subset \mathbb{R}^d$ be their *geometric realization*.

A k-dimensional elementary cube is called a *k-cube*. We often use special names depending on dimension: a *vertex* is a 0-cube, an *edge* is a 1-cube, a *square* is a 2-cube, and Q is a *full cube* if $\dim Q = \operatorname{emb} Q$.

For a set $\mathcal{X}$ of full cubes, a *multivalued combinatorial map* $\mathcal{F}$ on $\mathcal{X}$ is a map from $\mathcal{X}$ to its power set, i.e., for each $Q \in \mathcal{X}$, $\mathcal{F}(Q) \subset \mathcal{X}$. This is written as $\mathcal{F}\colon \mathcal{X} \rightrightarrows \mathcal{X}$. For $\mathcal{A} \subset \mathcal{X}$, let $\mathcal{F}(\mathcal{A}) = \bigcup_{Q \in \mathcal{A}} \mathcal{F}(Q)$. For $n \in \mathbb{N}$, inductively define $\mathcal{F}^n(\mathcal{A})$ via $\mathcal{F}^{n+1}(\mathcal{A}) := \mathcal{F}(\mathcal{F}^n(\mathcal{A}))$, and let $\mathcal{F}^{-n}(\mathcal{A}) := \{Q \in \mathcal{X} \mid \mathcal{F}^n(Q) \cap \mathcal{A} \neq \varnothing\}$. One can also think of $\mathcal{F}$ as a directed graph with vertex set $\mathcal{X}$ and an arrow from Q to P if $P \in \mathcal{F}(Q)$.

Let $I \subset \mathbb{Z}$ be an interval of integers containing 0. For a combinatorial multivalued map $\mathcal{F}\colon \mathcal{X} \rightrightarrows \mathcal{X}$, a *solution* through $Q \in \mathcal{X}$ is a map $\Gamma\colon I \to \mathcal{X}$ such that

(i) $\Gamma(0) = Q$, and

(ii) $\Gamma(k+1) \in \mathcal{F}(\Gamma(k))$ whenever $k, k+1 \in I$.

Definition 3.1.1. For a set $\mathcal{M} \subset \mathcal{X}$, let

(i) $\operatorname{Inv}(\mathcal{M}, \mathcal{F}) = \{Q \in \mathcal{M} \mid \text{there is a solution } \mathbb{Z} \to \mathcal{M} \text{ through } Q\}$,

(ii) $\operatorname{Inv}^{\pm}(\mathcal{M}, \mathcal{F}) = \{Q \in \mathcal{M} \mid \text{there is a solution } \mathbb{Z}^{\pm} \to \mathcal{M} \text{ through } Q\}$.

Obviously, $\operatorname{Inv}(\mathcal{M}, \mathcal{F}) = \operatorname{Inv}^-(\mathcal{M}, \mathcal{F}) \cap \operatorname{Inv}^+(\mathcal{M}, \mathcal{F})$. The dynamics of a map $f : X \to X$ with $X = |\mathcal{X}|$ can be captured as follows.

Definition 3.1.2. A map $\mathcal{F}\colon \mathcal{X} \rightrightarrows \mathcal{X}$ is a *global combinatorial enclosure* of $f : X \to X$ if $f(Q) \subset \operatorname{int}|\mathcal{F}(Q)|$ for all $Q \in \mathcal{X}$.

We do not require $\mathcal{F}$ to be optimal in any sense. The image $\mathcal{F}(Q)$ could be a very coarse enclosure for $f(Q)$. But a finer enclosure contains more precise information about f. Variants of this definition like Definition 3.4.1 are suitable for numerical representations. The following is a simple observation from these definitions.

Lemma 3.1.3. *Suppose $\mathcal{F}$ is a combinatorial enclosure of f.*

(i) Given a solution $\gamma\colon \mathbb{Z} \to X$ of f through $x \in Q$, then $\gamma(n) \in \mathcal{F}^n(Q)$ for all $n \in \mathbb{Z}$.

(ii) For any $\mathcal{N} \subset \mathcal{X}$, $\operatorname{Inv}(|\mathcal{N}|, f) \subset |\operatorname{Inv}(\mathcal{N}, \mathcal{F})|$.

We want to apply the rigorous numerics presented here to flows. Given a flow φ, we analyze the map

$$\begin{aligned} \varphi_h\colon X &\to X, \\ x &\mapsto \varphi(x, h), \end{aligned}$$

for a time step parameter $h > 0$.

3.2 Cubical homology and cubical singular homology

Our algorithms use cubical homology, a homology theory on sets of cubes. We recall some notation and basic properties [KMM03, KMM04].

For $d \geq 1$ and $0 \leq n \leq d$, let C_n^d be the set of formal (finite) linear combinations of the form $c = \sum_i \alpha_i Q_i$, where $\alpha_i \in \mathbb{F}$ and Q_i are cubes such that $\dim Q_i = n$ and $\operatorname{emb} Q_i = d$. These cubes form the basis of C_n^d. Elements of C_n^d are called *cubical chains*. For such a cubical chain $c \in C_n^d$, we let $\dim c = n$ and $\operatorname{emb} c = d$ be the *dimension* and the *embedding number* of c, respectively.

- For $c = \sum_i \alpha_i Q_i \in C_n^d$ with $Q_i \neq Q_j$ whenever $i \neq j$ and Q an elementary cube, let $c(Q) := \begin{cases} \alpha_i \text{ if } Q = Q_i \text{ for some } i, \\ 0 \text{ otherwise.} \end{cases}$
- The *support* of c is $|c| := \bigcup \{ Q \mid Q \text{ is a cube with } c(Q) \neq 0 \} \subset \mathbb{R}^d$.

An *n-chain* is an element of C_n^d for some $d \geq 1$, a *chain* is an element of any C_n^d for arbitrary n and d. The *cubical product* of two chains c_1 and c_2 with arbitrary dimensions and embedding numbers is

$$c_1 \diamond c_2 := \sum_{P,Q} c_1(P) c_2(Q)\, P \times Q,$$

where the formal sum is taken over all cubes P and Q. For example, $P \diamond Q = P \times Q$, where the cubes on the left are considered as chains and the product on the right is the Cartesian product of two cubes.

We construct the boundary operator ∂ on chains by induction over the embedding number of cubes. The cubes with embedding number 1 are precisely the intervals. To define ∂ on C_0^1, let $\partial[i] = 0$. To define ∂ on C_1^1, let $\partial[i, i+1] = [i+1] - [i]$. For embedding numbers greater than 1, we first use that given a cube Q with $\operatorname{emb} Q > 1$, there are a unique interval J and a unique cube P such that $\operatorname{emb} P + 1 = \operatorname{emb} Q$ and $Q = J \times P$. We consider Q as a cubical chain and define its boundary operator as

$$\partial Q := \partial J \diamond P + (-1)^{\dim J} J \diamond \partial P. \tag{3.1}$$

This yields a definition of ∂ on the basis elements of C_n^d for all $0 \leq n \leq d$. Extending linearly, this yields linear maps $\partial_n \colon C_n^d \to C_{n-1}^d$, where $\partial_0 \colon C_0^d \to 0$. One can show that $\partial_{n-1}\partial_n = 0$. A useful implication of (3.1) is the following formula valid for arbitrary chains c_1 and c_2:

$$\partial(c_1 \diamond c_2) := \partial c_1 \diamond c_2 + (-1)^{\dim c_1} c_1 \diamond \partial c_2. \tag{3.2}$$

Definition 3.2.1. A finite set of cubes $\mathcal{X}$ is a *cubical complex* if all cubes in $\mathcal{X}$ have the same embedding number (denoted $\operatorname{emb} \mathcal{X}$) and it is downward closed under inclusion in the following sense: If $Q \in \mathcal{X}$, then every cube $P \subset Q$ is an element of $\mathcal{X}$. Given a finite set $\mathcal{Y}$ of cubes, we call the smallest cubical complex containing $\mathcal{Y}$ the *cubical complex generated by* $\mathcal{Y}$.

Remark 3.2.2. We use this notion here in analogy to the notion of a simplicial complex. The space $|\mathcal{X}|$ is refered to as a cubical set in [KMM03], but we avoid this notion here because it is already used with a different meaning in combinatorial homotopy theory where a cubical set is the cubical analog of a simplicial set (cf. [Kan55], [BHS11, Chapter 11]).

Let $\mathcal{A} \subset \mathcal{X}$ be cubical complexes with $\operatorname{emb} \mathcal{A} = \operatorname{emb} \mathcal{X} = d$. Let $\mathcal{X}^n := \{ Q \in \mathcal{X} \mid \dim Q = n \}$ be the n-cubes in $\mathcal{X}$, and let $C_n(\mathcal{X})$ be the subspace of C_n^d generated by $\mathcal{X}^n$. The restriction of the boundary from above yields a well defined linear map

$\partial_n^{\mathcal{X}}\colon C_n(\mathcal{X}) \to C_{n-1}(\mathcal{X})$. Let the vector space of *relative n-chains of $\mathcal{X}$ modulo $\mathcal{A}$* be the quotient vector space

$$C_n(\mathcal{X},\mathcal{A}) := C_n(\mathcal{X})/C_n(\mathcal{A}),$$

on which the boundary operator on $\mathcal{X}$ induces a boundary operator

$$\begin{aligned}\partial_n^{(\mathcal{X},\mathcal{A})}\colon C_n(\mathcal{X},\mathcal{A}) &\to C_{n-1}(\mathcal{X},\mathcal{A}),\\ z + C_n(\mathcal{A}) &\mapsto \partial_n^{\mathcal{X}}(z) + C_{n-1}(\mathcal{A}).\end{aligned}$$

The vector space $Z_n(\mathcal{X},\mathcal{A}) := \ker \partial_n^{(\mathcal{X},\mathcal{A})}$ is the space of *relative n-cycles*, its subspace $B_n(\mathcal{X},\mathcal{A}) := \operatorname{im} \partial_{n+1}^{(\mathcal{X},\mathcal{A})}$ contains the *relative n-boundaries*.

We call the quotient $\mathrm{H}_n(\mathcal{X},\mathcal{A}) = Z_n(\mathcal{X},\mathcal{A})/B_n(\mathcal{X},\mathcal{A})$ the *n-th cubical homology* of $(\mathcal{X},\mathcal{A})$. It fulfills the axioms of homology from Section 2.3, as is shown in [KMM04].

Cubical singular homology

In the proofs of Section 4.1, we need to work explicitly with the chains representing singular homology. We recall the singular homology theory which is closest to cubical homology: *Cubical singular homology* as presented in [Mas91]. We let $I^n := [0,1]^n \subset \mathbb{R}^n$ with $I^0 := [0]$. For a topological space X, a *singular n-cube* is a continuous map $T\colon I^n \to X$. We define the following inclusions $f_i, b_i\colon I^{n-1} \to I^n$ for each $1 \le i \le n$. For $x \in I^{n-1}$, let

$$\begin{aligned}f_i(x_1,\dots,x_{n-1}) &= (x_1,\dots,x_{i-1},0,x_i,\dots,x_{n-1}) \quad \text{(front)},\\ b_i(x_1,\dots,x_{n-1}) &= (x_1,\dots,x_{i-1},1,x_i,\dots,x_{n-1}) \quad \text{(back)}.\end{aligned}$$

We denote the vector space with basis consisting of all cubical singular n-cubes by $C_n^S(X)$, the *cubical singular n-chains*. We let $\partial_n^S\colon C_n^S(X) \to C_{n-1}^S(X)$ be defined on this basis by:

$$\partial_n^S(T) = \sum_{i=1}^{n} (-1)^i (T \circ f_i - T \circ b_i).$$

From this, we construct the *cubical singular homology* $\mathrm{H}_*(X)$ using quotient spaces analogously to the definition of cubical homology above.

Given a cubical complex $\mathcal{X}$, we let $X = |\mathcal{X}|$ and $d = \operatorname{emb} \mathcal{X}$. A chain in $C_n(\mathcal{X})$ is a singular chain in $C_n^S(X)$ in the following way. For an elementary cube $Q = \prod_{i=1}^d [l_i, r_i]$ with $\dim Q = n$, let $\sigma\colon \{1,\dots,n\} \to \{1,\dots,d\}$ be the increasing map such that $r_i = l_i + 1$ if and only if $i \in \operatorname{im}\sigma$. Then $r_{\sigma(j)} = l_{\sigma(j)} + 1$ for all j, and $r_i = l_i$ if $i \notin \operatorname{im}\sigma$. We get a map $\alpha_n(Q)\colon I^n \to X \subset \mathbb{R}^d$ via

$$(\alpha_n(Q)(t_1,\dots,t_n))_i = \begin{cases} l_i & \text{if } r_i = l_i,\\ l_i + t_{\sigma^{-1}(i)} & \text{if } r_i = l_i + 1.\end{cases}$$

We get a chain map $\alpha_*\colon C_*(\mathcal{X}) \to C^S_*(X)$, i.e., $\partial^S_n \circ \alpha_n = \alpha_{n-1} \circ \partial^{\mathcal{X}}_n$ for each n. For example, $\alpha_1([0,1])(t) = t$ and then $\partial^S_1 \alpha_1([0,1]) = \alpha_0([1]) - \alpha_0([0])$. This chain map induces a linear map $\mathrm{H}_*(\alpha_*)\colon \mathrm{H}_*(\mathcal{X}) \to \mathrm{H}_*(X)$.

It is a classical result of algebraic topology that cubical singular homology is isomorphic (for every topological space) to the more commonly used simplicial singular homology [EML53].

A comparison between singular and cubical homology is sketched in [KMM04, Chapter 11]. Here we only cite that given cubical complexes $\mathcal{A} \subset \mathcal{X}$, there is a natural isomorphism of graded vector spaces between cubical homology and singular homology, i.e.,

$$\mathrm{H}_*(\mathcal{X}, \mathcal{A}) \cong \mathrm{H}_*(|\mathcal{X}|, |\mathcal{A}|).$$

3.3 Isolating neighborhoods

The following theorem is essential for our approach using a discrete dynamical system as a tool to analyze the flow φ.

Theorem 3.3.1 ([Mro90a, Theorem 1]). *Let $S \subset X$ be compact. Then the following three conditions are equivalent:*

(i) S is an isolated invariant set with respect to φ.

(ii) For every $h > 0$, S is an isolated invariant set with respect to φ_h.

(iii) There is a number $h > 0$ such that S is an isolated invariant set with respect to φ_h.

This theorem tells us that given $h > 0$, the dynamical systems φ and φ_h have the same isolated invariant sets. More precisely, if S is isolated invariant for φ, then it is isolated invariant for φ_h due to $(i) \implies (ii)$. And if S is isolated invariant for φ_h, then it is isolated invariant for φ because of $(iii) \implies (i)$.

The following observation about isolating neighborhoods is interesting because isolating neighborhoods can often be represented numerically.

Corollary 3.3.2 ([Pil99, Lemma 6]). *Given an $h > 0$ and an isolating neighborhood $M \subset X$ for φ_h, then*

$$\operatorname{Inv}(M, \varphi) = \operatorname{Inv}(M, \varphi_h).$$

Proof. The inclusion $\operatorname{Inv}(M, \varphi) \subset \operatorname{Inv}(M, \varphi_h)$ holds trivially for any set $M \subset X$. For the opposite inclusion, we use that the set $\operatorname{Inv}(M, \varphi_h)$ is isolated invariant for φ_h by assumption. Using $(iii) \implies (i)$ from the theorem, it is invariant for φ, hence $\operatorname{Inv}(M, \varphi_h) \subset \operatorname{Inv}(M, \varphi)$. □

Remark 3.3.3. If M is just an isolating neighborhood for the flow φ, the invariant sets $\operatorname{Inv}(M, \varphi)$ and $\operatorname{Inv}(M, \varphi_h)$ need not be equal. For example, let $X = \mathbb{R}^2$, and let φ be the

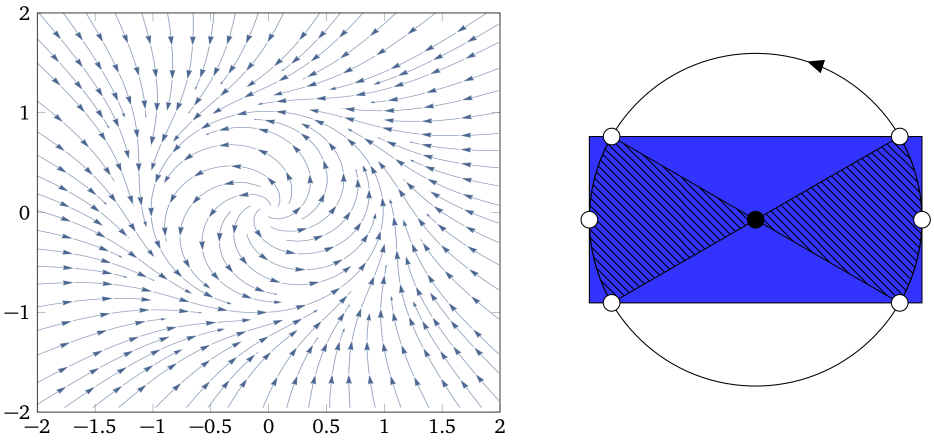


FIGURE 3.1: The left subfigure shows a phase portrait of the dynamical system in Remark 3.3.3. The right subfigure shows the limit cycle, the rectangle M (blue region) and the set $\operatorname{Inv}(M, \varphi_h)$, the union of two closed sectors (hatched region). The black point marks $\operatorname{Inv}(M, \varphi) = \{0\}$, the points filled white constitute the set $\operatorname{Inv}(M, \varphi_\pi) \cap \operatorname{bd} M$.

flow generated by the following ordinary differential equation:

$$\dot{x}_1 = -x_2 + x_1(1 - x_1^2 - x_2^2)$$
$$\dot{x}_2 = x_1 + x_2(1 - x_1^2 - x_2^2)$$

The dynamical system has an equilibrium at the origin. The unit circle is an attracting limit cycle. All points except the equilibrium $(0,0)$ are attracted by it. This can be seen directly by using polar coordinates (r, θ) with $r^2 = x_1^2 + x_2^2$, $\tan\theta = x_2/x_1$. This yields $\dot{r} = r(1 - r^2)$ and $\dot{\theta} = 1$. A solution on the limit cycle is given by $t \mapsto (\cos t, \sin t)$. Figure 3.1 shows some trajectories. If we choose $M = [-1, 1] \times [-0.5, 0.5]$ and $h = \pi$, then $\operatorname{Inv}(M, \varphi_\pi)$ intersects the boundary of M, and $\operatorname{Inv}(M, \varphi) = \{0\} \subsetneq \operatorname{Inv}(M, \varphi_\pi)$; details are shown in the figure.

3.4 Showing isolation numerically

We reuse and slightly adapt notions from Section 3.1 for the problem of showing that a given set of full cubes forms an isolating neighborhood for φ_h.

Given $d > 0$ and a tuple $s \in \mathbb{R}^d_{>0}$ representing a box size, the space $X = \mathbb{R}^d$ is the union of the following full cubes:

$$\mathcal{X} = \left\{ \prod_{i=1}^{d} [m_i s_i, (m_i + 1)s_i] \;\middle|\; m \in \mathbb{Z}^d \right\}.$$

Let $\mathcal{K} \subset \mathcal{X}$ be a finite subset. Our goal is to describe φ_h (and in a certain sense φ) within $K = |\mathcal{K}|$.

For subsets $Z \subset K \subset \mathbb{R}^d$, we let $\operatorname{int}_K(Z)$ be the interior of Z with respect to the subspace topology of $K \subset \mathbb{R}^d$, the union of all sets which are open in K and subsets of Z. We use the following restricted version of a combinatorial enclosure (cf. Definition 3.1.2).

Definition 3.4.1. A map $\mathcal{F}\colon \mathcal{K} \rightrightarrows \mathcal{K}$ is a *restricted combinatorial enclosure* of $f : \mathbb{R}^d \to \mathbb{R}^d$ if for every $Q \in \mathcal{K}$: $f(Q) \cap K \subset \operatorname{int}_K |\mathcal{F}(Q)|$.

Note that $\mathcal{F}(Q) = \varnothing$ is allowed if $f(Q) \cap K = \varnothing$. For a global combinatorial enclosure $\mathcal{G}$, we would have $\mathcal{G}(Q) \neq \varnothing$ for all cubes Q. For a set $\mathcal{A} \subset \mathcal{K}$, define

$$\text{WRAP}(\mathcal{A}) := \{Q \in \mathcal{K} \mid Q \cap |\mathcal{A}| \neq \varnothing\}. \tag{3.3}$$

We construct a restricted combinatorial enclosure $\mathcal{F}\colon \mathcal{K} \rightrightarrows \mathcal{K}$ of $\varphi_h\colon \mathbb{R}^d \to \mathbb{R}^d$. For each $Q \in \mathcal{K}$, we use the software library [CAPD] to find a set $V \subset \mathbb{R}^d$ such that $\varphi(Q,h) \subset \operatorname{int}(V)$. We use higher-order Taylor methods. The algorithms used by us are described in [NJ01], [WZ11], and [Zgl08]. We let $\mathcal{F}(Q) := \{\, P \in \mathcal{K} \mid P \cap V \neq \varnothing \,\}$.

It is now straightforward to check that $\mathcal{F}$ fulfills Definition 3.4.1: First note that $\varphi_h(Q) \cap K \subset \operatorname{int}(V) \cap K$. The set $\operatorname{int}(V) \cap K$ is open in K and a subset of $|\mathcal{F}(Q)|$. It is therefore a subset of $\operatorname{int}_K |\mathcal{F}(Q)|$.

Let $\mathcal{M} \subsetneq \mathcal{K}$ be such that $M = |\mathcal{M}| \subset \operatorname{int} K$. In order to show that M is an isolating neighborhood, we want to show algorithmically that

$$\text{WRAP}(\operatorname{Inv}(\mathcal{M},\mathcal{F})) \subset \mathcal{M},$$

which then implies $|\operatorname{Inv}(\mathcal{M},\mathcal{F})| \subset \operatorname{int} M$. This suffices because $\operatorname{Inv}(M,\varphi_h) \subset |\operatorname{Inv}(\mathcal{M},\mathcal{F})|$ by Lemma 3.1.3.

In order to compute $\operatorname{Inv}(\mathcal{M},\mathcal{F})$, we proceed as in [Mro06]: We consider the multivalued map $\mathcal{F}_\mathcal{M}\colon \mathcal{M} \rightrightarrows \mathcal{M}$ defined by $\mathcal{F}_\mathcal{M}(Q) := \mathcal{F}(Q) \cap \mathcal{M}$. Then

$$\mathcal{M} \supset \mathcal{F}_\mathcal{M}(\mathcal{M}) \supset \mathcal{F}^2_\mathcal{M}(\mathcal{M}) \supset \dots \text{ and } \mathcal{M} \supset \mathcal{F}^{-1}_\mathcal{M}(\mathcal{M}) \supset \mathcal{F}^{-2}_\mathcal{M}(\mathcal{M}) \supset \dots \tag{3.4}$$

Since we work with finite sets of cubes, there is a smallest $m \in \mathbb{N}$ with $\mathcal{F}^m_\mathcal{M}(\mathcal{M}) = \mathcal{F}^{m+1}_\mathcal{M}(\mathcal{M})$. We construct the left descending sequence of subsets in (3.4) by applying $\mathcal{F}_\mathcal{M}$ repeatedly until the set of cubes does not change. Now, [Mro06, Theorem 6.13] tells us that $\mathcal{F}^m_\mathcal{M}(\mathcal{M}) = \operatorname{Inv}^-(\mathcal{M},\mathcal{F})$, cf. Definition 3.1.1. Similarly, $\operatorname{Inv}^+(\mathcal{M},\mathcal{F}) = \mathcal{F}^{-n}_\mathcal{M}(\mathcal{M})$ for the smallest $n \in \mathbb{N}$ with $\mathcal{F}^{-n}_\mathcal{M}(\mathcal{M}) = \mathcal{F}^{-n-1}_\mathcal{M}(\mathcal{M})$. After computing these combinatorial versions of positive and negative invariant sets, we use the fact that $\operatorname{Inv}(\mathcal{M},\mathcal{F}) = \operatorname{Inv}^-(\mathcal{M},\mathcal{F}) \cap \operatorname{Inv}^+(\mathcal{M},\mathcal{F})$.

There is often a wide range of good choices for the parameter h in the algorithm. But the runtime of the algorithm can become quite large if either h is very small or h is close to a value where M is not an isolating neighborhood for φ_h.

Example 3.4.2. Using a good choice for h, one can analyze the van der Pol oscillator, a well-known second order differential equation in one dimension, here considered as a

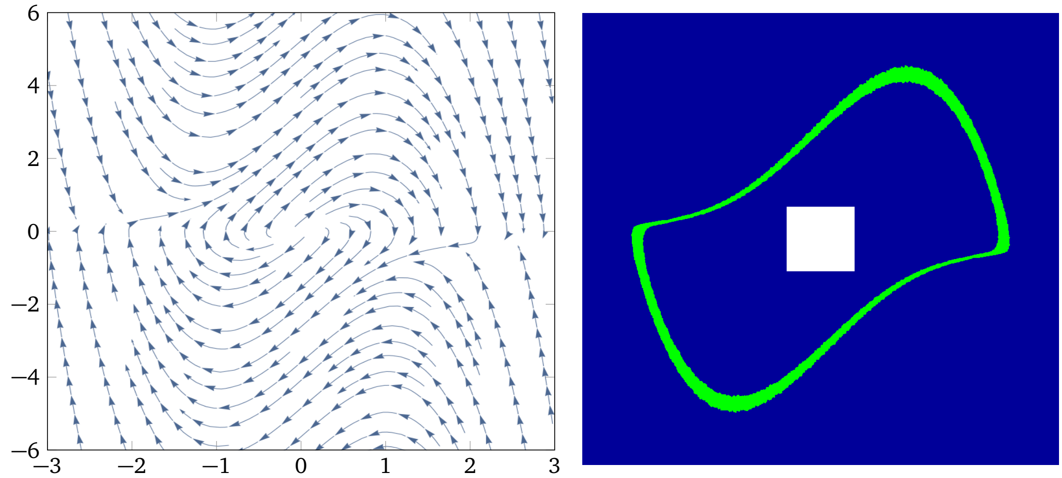


FIGURE 3.2: The left subfigure shows a phase portrait of the van der Pol equations for $\mu = 2$. The right subfigure shows the construction of $\mathrm{Inv}(\mathcal{M}, \mathcal{F})$ given $\mathcal{M}$ as described in Example 3.4.2. Both colored areas together constitute $\mathcal{M}$ after several iterations. It consists of $(14^2 - 4) \cdot 4^5 = 196\,608$ cubes. The green area is the subset $\mathrm{Inv}(\mathcal{M}, \mathcal{F})$.

two-dimensional ordinary differential equation:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \mu(1 - x_1^2)x_2 \end{aligned}$$

We applied the method described to these equations with $\mu = 2$, using $h = 0.1$ as a parameter. A plot of some trajectories is shown in Figure 3.2. It is well known that the van der Pol equations have a limit cycle for $\mu > 0$ [Tes12, Chapter 7]. We started with a candidate $\mathcal{K}$ that was a subdivision of the rectangle $[-3, 3] \times [-6, 6]$ into $16 \cdot 16$ rectangles. We let $\mathcal{M}$ be $\mathcal{K}$ without the cubes touching the boundary. Then we removed the four rectangles around $(0, 0)$ from $\mathcal{M}$ in order to remove this equilibrium.

After subdividing this $\mathcal{M}$ five times (every time subdividing each box into 4 boxes and computing $\mathrm{Inv}(\mathcal{M}, \mathcal{F})$ for the subdivided set $\mathcal{M}$), the statement $\mathrm{WRAP}(\mathrm{Inv}(\mathcal{M}, \mathcal{F})) \subset \mathcal{M}$ was true for the first time, as can be seen in Figure 3.2. Our implementation took 215 seconds to run. Most of the time was used for building the map $\mathcal{F}$ and then the set $\mathrm{Inv}(\mathcal{M}, \mathcal{F})$ at each subdivision step. Note that $\mathcal{M}$ was only a very coarse enclosure of the invariant set. We also used $\mathcal{F}$ to construct a weak index pair as in Section 3.6. Letting $S = \mathrm{Inv}(M, \varphi)$, we got a Conley index of $\mathrm{CH}_0(S, \varphi) = \mathbb{F}$, $\mathrm{CH}_1(S, \varphi) = \mathbb{F}$ and $\mathrm{CH}_n(S, \varphi) = 0$ for $n > 1$.

3.5 An example with good and bad time step choices

We tried to verify that a given set M is an isolating neighborhood using the differential equation from Remark 3.3.3. As the set $\mathcal{M}$ we chose the 12 boxes depicted in Figure 3.3.

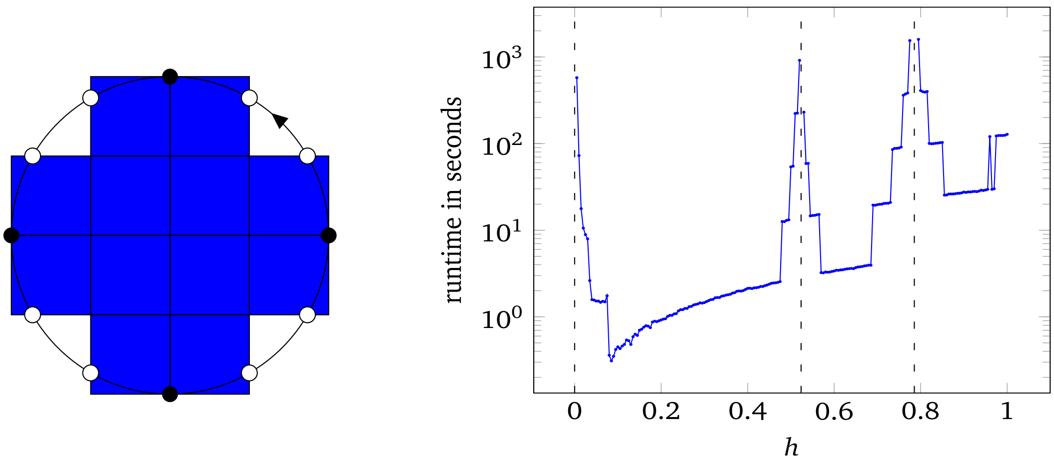


FIGURE 3.3: Good and bad choices of h as described in Section 3.5. The set $\mathcal{M}$ consists of the filled blue boxes on the left. For $h = \pi/6$, the marked points lying on the boundary of M constitute an orbit of period 12 for φ_h. The points marked white also lie in $\mathrm{Inv}(M, \varphi_h)$ for $h = \pi/4$ (with period 8, their orbits are not drawn).

Each cube has a size of $s = (0.5, 0.5)$. Our algorithm builds $\mathcal{F}$ repeatedly on finer subdivisions of $\mathcal{M}$. It was run for h from 0.005 to 1.000 incrementing by 0.005. The set $M = |\mathcal{M}|$ is an isolating neighborhood for φ because each point in $M \setminus \{(0,0)\}$ is attracted by the limit cycle and hence leaves M in forward time. It is not, however, an isolating neighborhood for φ_h for every $h > 0$.

For certain time steps h, there can be points on the boundary of M whose orbits do not leave M. In our example, M is not an isolating neighborhood for $\varphi_{\pi/6}$. Hence, when h is close to $\pi/6$, the combinatorial enclosure $\mathcal{F}$ of φ_h has to be very fine to recognize the isolation. This leads to high runtimes around $h = \pi/6$. Another problem that can occur for any $\mathcal{M}$: For very small h, the map φ_h is close to the identity. This implies that the combinatorial map $\mathcal{F}$ has a lot of cubes Q with $Q \in \mathcal{F}(Q)$, therefore $Q \in \mathrm{Inv}(\mathcal{M}, \mathcal{F})$. If this happens for a box Q on the boundary of $\mathcal{M}$, then

$$\textsc{wrap}(\mathrm{Inv}(\mathcal{M}, \mathcal{F})) \subset \mathcal{M} \text{ is false,}$$

hence the algorithm subdivides further. This leads to the jumps in runtimes visible in Figure 3.3. An example where all choices for h lead to bad output is presented in Section 5.4.

3.6 Computing the Conley index

In this section, we summarize how we can use the combinatorial enclosure $\mathcal{F}$ of φ_h constructed above to compute the Conley index of the isolated invariant set inside $|\mathcal{M}|$. These established ideas are later modified:

- In Chapter 4, we work with a map for which we usually cannot construct a combinatorial enclosure: the Poincaré map P.
- In Chapter 5, we consider another way of discretizing φ in time. This helps in finding finer (smaller) isolating neighborhoods and is motivated by examples like the one in Section 3.5.

The following theorem explains that given an isolated invariant set S for φ_h, the Conley index of (S,φ_h) immediately gives us the Conley index of (S,φ). A more general version, Theorem 5.2.3, is proved in Chapter 5.

Theorem 3.6.1 ([Mro90a, Theorem 2]). *Let S be an isolated invariant set for φ_h. Let (N,L) be an index pair for (S,φ_h) and $I_{(N,L)} = \widetilde{\mathrm{H}}_*((\varphi_h)_{(N,L)})$ its homological index map. Then there is an isomorphism*

$$\mathrm{CH}_*(S,\varphi) \cong \operatorname{dom}\mathbf{L}(I_{(N,L)}).$$

This theorem was already used with the algorithmic ideas presented here in [Pil99]; a recent survey is [AKP09].

There are several ways for constructing an index pair given a combinatorial enclosure $\mathcal{F}$. Our approach presented in Example 3.4.2 uses the isolating neighborhood $\mathcal{M}$ as input (as we also do in Chapter 4). In this case, the approach from [Mro06] seems most useful.

Assume we have a restricted combinatorial enclosure $\mathcal{F}\colon \mathcal{K} \to \mathcal{K}$ of f and we have already shown numerically that $\mathrm{WRAP}(\mathrm{Inv}(\mathcal{M},\mathcal{F})) \subset \mathcal{M}$.

Then we let $\mathcal{N} = \mathrm{Inv}^-(\mathcal{M},\mathcal{F})$ and $\mathcal{L} = \mathcal{N} \setminus \mathrm{Inv}^+(\mathcal{M},\mathcal{F})$ (cf. Definition 3.1.1). By [Mro06, Theorems 7.3 and 8.1], the pair of spaces $(|\mathcal{N}|,|\mathcal{L}|)$ is a weak index pair for $(\mathrm{Inv}(M,f),f)$.

In order to compute the homological index map, we need an extra assumption: We assume $f(M) \subset \operatorname{int} K$. This can be assured by making M much smaller than K and finding a rigorous numerical enclosure of $f(M)$. Then we get $f(Q) \subset \operatorname{int}|\mathcal{F}(Q)|$ for each $Q \in \mathcal{M}$, and therefore $f(N) \subset \operatorname{int}|\mathcal{F}(\mathcal{N})|$. Now let $\overline{\mathcal{N}} = \mathcal{N} \cup \mathcal{F}(\mathcal{N})$ and $\overline{\mathcal{L}} = \mathcal{L} \cup (\mathcal{F}(\mathcal{N}) \setminus \mathcal{N})$. We get $\mathcal{F}(\mathcal{N}) \subset \overline{\mathcal{N}}$ and $\mathcal{F}(\mathcal{L}) \subset \overline{\mathcal{L}}$. Define $\overline{N} := |\overline{\mathcal{N}}|$, $\overline{L} := |\overline{\mathcal{L}}|$. By [Mro06, Theorem 8.1], the map

$$f_P\colon (N,L) \to (\overline{N},\overline{L}), \quad x \mapsto f(x),$$

and the inclusion $i\colon (N,L) \hookrightarrow (\overline{N},\overline{L})$ induce the following analog of Diagram (2.2) (note that we only have a weak index pair this time):

$$\begin{tikzcd}
\mathrm{H}_*(N,L) \arrow[r, "f_{P*}"] \arrow[d, "q_*", "\cong"'] & \mathrm{H}_*(\overline{N},\overline{L}) & \mathrm{H}_*(N,L) \arrow[l, "i_*"', "\cong"] \arrow[d, "q_*", "\cong"'] \\
\widetilde{\mathrm{H}}_*(N/L) \arrow[rr, "\widetilde{\mathrm{H}}_*(f_{(N,L)})"] & & \widetilde{\mathrm{H}}_*(N/L).
\end{tikzcd} \tag{3.5}$$

We have descriptions of f_p and i as multivalued combinatorial maps, and use the approach from [MMP05] to compute the induced maps in relative homology.

Our numerical experiments show that beginning with a given flow φ, after constructing the weak index pair (N, L) for $f = \varphi_h$, the homological index map $\widetilde{\mathrm{H}}_*(f_{(N,L)})$ is often the identity. In Lemma 5.2.2, we show that such a pair (N, L) exists for every isolated invariant set. But, unfortunately, there is no obvious way to ensure this extra property during construction. Therefore, computing $I_{(N,L)}$ cannot easily be avoided, even if f has the special form $f = \varphi_h$.

In Chapter 4, this approach is not used. There we construct an index pair for some φ_h, but it is not an index pair for the Poincaré map (the one we are interested in). Computing the Conley index the way described here is used in Chapter 5.

Chapter 4

The Conley index for Poincaré maps

In this chapter, we describe a rigorous numerical way of finding the Conley index for a special kind of Poincaré map. Assume we are given a function $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ such that the resulting non-autonomous ordinary differential equation (ODE)

$$\dot{x} = f(t, x) \tag{4.1}$$

has unique solutions in the following sense. We assume that the ordinary differential equation

$$\begin{aligned} \dot{t} &= 1, \\ \dot{x} &= f(t, x). \end{aligned}$$

on the *extended phase space* $\Omega := \mathbb{R} \times \mathbb{R}^d = \mathbb{R}^{d+1}$ induces a flow

$$\psi : \Omega \times \mathbb{R} \to \Omega.$$

This situation occurs if $f \in C^1(\mathbb{R}^{d+1}, \mathbb{R}^d)$. We only assume that ψ is a global flow for the ease of presentation, cf. Section 2.1. Additionally, we assume that there is a real number $T > 0$ such that

$$f(t + T, x) = f(t, x) \text{ for all } x \in \mathbb{R}^d.$$

We divide $\mathbb{R}$ by the action of the additive subgroup $T\mathbb{Z} \subset \mathbb{R}$, identifying t and t' if $t - t' = nT$ for some $n \in \mathbb{Z}$. We let $\langle t \rangle := t + T\mathbb{Z}$ be the orbit of $t \in \mathbb{R}$ under this action. Let

$$\Sigma := \mathbb{R}/T\mathbb{Z} \times \mathbb{R}^d$$

and

$$\begin{aligned} q : \Omega &\to \Sigma, \\ (t, x) &\mapsto (\langle t \rangle, x). \end{aligned}$$

Since f is periodic, we get the following well-defined flow on the quotient space:

$$\begin{aligned} \varphi : \Sigma \times \mathbb{R} &\to \Sigma, \\ \big((\langle t \rangle, x), s\big) &\mapsto q\big(\psi((t,x),s)\big). \end{aligned}$$

This flow has a *Poincaré map*

$$\begin{aligned} P' \colon \langle 0 \rangle \times \mathbb{R}^d &\to \langle 0 \rangle \times \mathbb{R}^d, \\ (\langle 0 \rangle, x) &\mapsto \varphi\big((\langle 0 \rangle, x), T\big), \end{aligned}$$

Sometimes, we want to consider this a map on $\mathbb{R}^d$, so we let

$$\begin{aligned} P : \mathbb{R}^d &\to \mathbb{R}^d, \\ x &\mapsto \pi_d P'(\langle 0 \rangle, x), \end{aligned}$$

where we let

$$\begin{aligned} \pi_d : \Sigma &\to \mathbb{R}^d, \\ (\langle t \rangle, x) &\mapsto x \end{aligned}$$

be the projection to the last d factors.

We refer to both P and P' as the Poincaré map, and from now we also simply use the letter P to denote P' without fear of confusion. This is the standard definition of Poincaré maps when analyzing differential equations of this form [Tes12, Section 1.6]. The Poincaré map contains information about solutions of the differential equation. For example, if we find an $n \in \mathbb{N}$ with $P^n(x) = x$, then $(\langle 0 \rangle, x) = \varphi((\langle 0 \rangle, x), nT)$, i.e., Equation (4.1) has a periodic orbit through x with period nT.

Isolated invariant sets for φ and P are deeply related as the following proposition shows. The equivalence of the first three conditions follows simply from Theorem 3.3.1.

Proposition 4.0.2 ([MSW15, Proposition 4.1])**.** *Let $S \subset \Sigma$. For $t \in [0, T)$, let*

$$S_t := S \cap (\langle t \rangle \times \mathbb{R}^d).$$

Then the following statements are equivalent:

(i) S is an isolated invariant set for φ.

(ii) S is an isolated invariant set for φ_h for all $h > 0$.

(iii) S is an isolated invariant set for φ_h for some $h > 0$.

(iv) S_0 is an isolated invariant set for the Poincaré map P and $S_t = \varphi(S_0)$ for every $t \in [0, T)$.

The mathematical object we want to compute in this chapter is $\mathrm{CH}_*(S_0, P)$ as in Definition 2.5.18 for an isolated invariant set S of φ.

The theoretical results needed for its rigorous algorithmic computation are formulated and proved in [MSW15]. In Section 4.1, we introduce the basic notions which allow us to encode dynamical information as geometrical properties of cubes and cubical chains. Section 4.2 presents the central theoretical result from this article and we sketch its proof. Beginning with Section 4.3, these theoretical results are applied to develop the central algorithm. This was done by the author of this thesis and is described in detail.

4.1 Some notation and definitions

Given a subdivision parameter $m \geq 1$, the extended phase space $\Sigma = \mathbb{R}/T\mathbb{Z} \times \mathbb{R}^d$ is covered as follows. Let

$$\mathcal{X} = \left\{ [j, j+1] \times \prod_{i=1}^{d} [k_i, k_i+1] \;\middle|\; 0 \leq j < 2^m, k_i \in \mathbb{Z} \right\}.$$

To cover a bounded region, we choose $s_1, \ldots, s_d$ with each $s_i > 0$ and define

$$\begin{aligned} \alpha \colon |\mathcal{X}| &\to \Omega = \mathbb{R} \times \mathbb{R}^d, \\ (x_0, \ldots, x_d) &\mapsto \left(T \cdot \frac{x_0}{2^m},\ s_1\left(\frac{x_1}{2^{m-1}} - 1\right), \ldots,\ s_d\left(\frac{x_d}{2^{m-1}} - 1\right)\right). \end{aligned} \tag{4.2}$$

We let $p := q \circ \alpha$. Note that $\alpha([0, 2^m]^{d+1}) = [0, T] \times [-s_1, s_1] \times \ldots \times [-s_d, s_d]$ and $p(|\mathcal{X}|) = \Sigma$.

$$\begin{tikzcd} {|\mathcal{X}|} \arrow[r, "\alpha"] \arrow[dr, "p"'] & \Omega \arrow[d, "q"] \\ & \Sigma \end{tikzcd}$$

Definition 4.1.1. For a set $\mathcal{A}$ of cubes, let

$$[\![\mathcal{A}]\!] := p(|\mathcal{A}|) = \bigcup_{Q \in \mathcal{A}} p(Q) \subset \Sigma$$

be its *geometric realization*.

Observe that $[\![\mathcal{A} \cup \mathcal{B}]\!] = [\![\mathcal{A}]\!] \cup [\![\mathcal{B}]\!]$ and $[\![\mathcal{A} \cap \mathcal{B}]\!] \subset [\![\mathcal{A}]\!] \cap [\![\mathcal{B}]\!]$ for arbitrary sets $\mathcal{A}, \mathcal{B}$ of cubes.

The *geometric realization* of a chain c is $[\![c]\!] := p(|c|)$.

We call a 1-chain $c \in C_1(\mathcal{A})$ a *path* if there are $x, y \in \mathcal{A}^0, x \neq y$, such that $\partial c = x - y$. For an elementary cube $Q = I_0 \times I_1 \times \ldots \times I_d$, we let $I_i(Q) := I_i$ for $0 \leq i \leq d$. Removing the first factor is denoted by π_d , i.e.,

$$\pi_d(Q) := I_1(Q) \times \ldots \times I_d(Q) \subset \mathbb{R}^d.$$

This enables us to denote subsets and subchains with the time component lying inside a given interval.

- For an elementary interval I, let
$$\mathcal{A}_I := \{Q \in \mathcal{A} \mid I_0(Q) \subset I\}.$$
Based on this, we introduce similar notation for intervals which are not closed: Given $i, j, k \in \mathbb{Z}$, we let $\mathcal{A}_{[j,k)} := \mathcal{A}_{[j,k]} \setminus \mathcal{A}_{[k]}$, $\mathcal{A}_{(j,k]} := \mathcal{A}_{[j,k]} \setminus \mathcal{A}_{[j]}$ and $\mathcal{A}_{(j,k)} := \mathcal{A}_{[j,k)} \setminus \mathcal{A}_{[j]}$.
- For a chain $c = \sum_i \alpha_i Q_i \in C_k(\mathcal{A})$ and I be of the form $[j,k]$, $[j,k)$, $(j,k]$ or (j,k), we let $c_I := \sum_{Q \in \mathcal{A}_I} c(Q)Q \in C_k(\mathcal{A}_I)$.
- Define $\pi_d(c) := \sum_i \alpha_i \pi_d(Q)$.

From here, assume that $\mathcal{L}^{d+1} \subset \mathcal{N}^{d+1}$ are finite sets of full cubes such that their geometric realization $(\llbracket \mathcal{N}^{d+1} \rrbracket, \llbracket \mathcal{L}^{d+1} \rrbracket)$ is a weak index pair. Now let $\mathcal{N}$ be the cubical complex generated by $\mathcal{N}^{d+1}$, cf. Definition 3.2.1. Let $N = \llbracket \mathcal{N}^{d+1} \rrbracket = \llbracket \mathcal{N} \rrbracket \subset \Sigma$. In an analogous way, we define $\mathcal{L}$ and L. We are interested in the relative homology of (N_0, L_0), where $N_0 = \{(\langle 0 \rangle, x) \in N\}$ and analogously for L. But not necessarily $N_0 = \llbracket \mathcal{N}_{[0]} \rrbracket$. We only know that $\llbracket N_{[0]} \cup N_{[2^m]} \rrbracket = N_0$.

In order to compute $\mathrm{H}_*(N_0, L_0)$ correctly, we add sets of cubes
$$\left\{ [0] \times \prod_{i=1}^{d} I_i \;\middle|\; [2^m] \times \prod_{i=1}^{d} I_i \in \mathcal{N} \right\} \text{ and } \left\{ [2^m] \times \prod_{i=1}^{d} I_i \;\middle|\; [0] \times \prod_{i=1}^{d} I_i \in \mathcal{N} \right\}$$
to $\mathcal{N}$. Then $\llbracket \mathcal{N}_{[0]} \rrbracket = \llbracket \mathcal{N}_{[2^m]} \rrbracket = N_0$ and still $\llbracket \mathcal{N} \rrbracket = N$. We do the same for $\mathcal{L}$.

We want to describe the relative chains from Section 3.2 in terms of "absolute" chains. Let
$$Z_n^i := \{\, u \in C_n(\mathcal{N}_{[i]}) \mid \partial_n u \in C_{n-1}(\mathcal{L}_{[i]}) \,\}.$$
A chain $u \in Z_n^i$ represents a class of chains $u + C_n(\mathcal{L}_{[i]}) \in Z_n(\mathcal{N}_{[i]}, \mathcal{L}_{[i]})$. We also call the elements of Z_n^i *cycles*. In the following two definitions, let $i, j \in \{0, \ldots, 2^m\}$, $u \in Z_n^i$ and $v \in Z_n^j$.

Definition 4.1.2. The pair (u, v) is a *pair of contiguous cycles* if there are chains $w \in C_{n+1}(\mathcal{N}_{[i,j]})$ and $z \in C_n(\mathcal{L}_{[i,j]})$ such that
$$\partial w = u - v + z.$$

A simple example of contiguous cycles in the case $d = 1$ is shown in Figure 4.1. The following property cannot be seen from the figure because it requires knowledge about the behavior of φ over a whole time interval.

Definition 4.1.3. A pair (u, v) of contiguous cycles is called *h-movable* if there are w and z as in Definition 4.1.2 such that $\varphi(\llbracket w \rrbracket, [0, h]) \subset \llbracket \mathcal{N} \rrbracket$ and $\varphi(\llbracket z \rrbracket, [0, h]) \subset \llbracket \mathcal{L} \rrbracket$.

A simple observation is the following lemma.

Lemma 4.1.4. *Let (u, v) and (v, v') be pairs of contiguous cycles. Then:*

(i) (u, v') *is a pair of contiguous cycles.*

(ii) If (u, v) and (v, v') are h-movable, then (u, v') is h-movable.

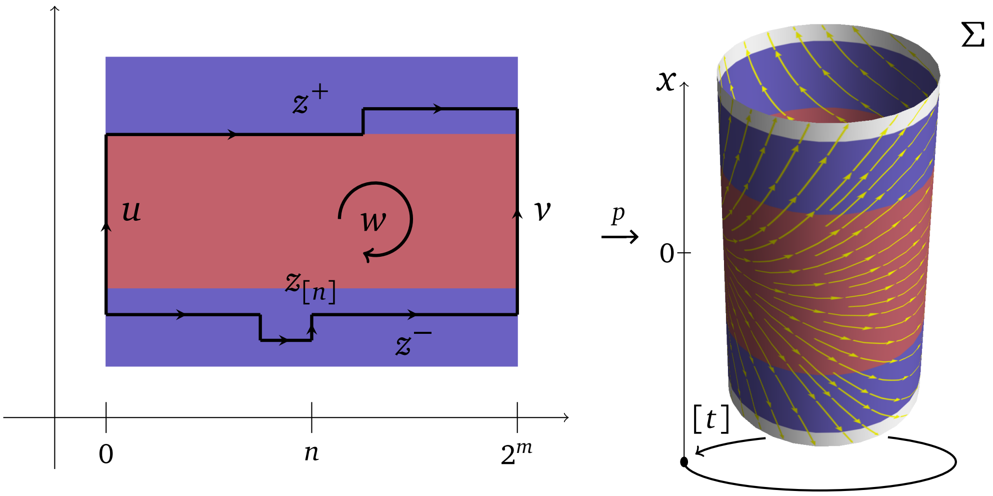


FIGURE 4.1: A simple example where $d = 1$ and the flow φ is induced by a vector field $f : \mathbb{R} \times \mathbb{R}^1 \to \mathbb{R}^1$ with $f(t,0) = 0$ and $f(t,x) \cdot x > 0$ for $x \neq 0$. The invariant set S of φ is the periodic orbit at $x = 0$. In the left figure, the blue set is $|\mathcal{L}| \subset \mathbb{R}^2$; the blue set in the right figure is $[\![\mathcal{L}]\!] \subset \Sigma = \mathbb{R}/T\mathbb{Z} \times \mathbb{R}^1$, similarly for $\mathcal{N}$, the union of the red and blue cubes. The left figure shows cubical chains with $\partial w = u - v + z_+ - z_-$.

4.2 The main theorem and a sketch of its proof

Given a finite-dimensional vector space V, a basis $\mathcal{B}$ of V and a linear map $\alpha\colon V \to V$, let $M_{\mathcal{B}}(\alpha)$ be the matrix (a_{ij}) such that $\alpha(v_j) = \sum_i a_{ij} v_i$ for all $v_j \in \mathcal{B}$. In the context of the Conley index, the choice of a basis $\mathcal{B}$ often does not matter. We let $M(\alpha)$ be the set of all matrices A such that there is a basis $\mathcal{B}$ of V with $A = M_{\mathcal{B}}(\alpha)$. It is a central result in linear algebra that given $A, A' \in M(\alpha)$, there is an invertible matrix T such that $A = T^{-1}A'T$. Two linear maps α, β are isomorphic in **End** if and only if $M(\alpha) \cap M(\beta) \neq \varnothing$.

We need to shift a cubical chain by increasing the time coordinate in the following way. For $u \in C_n(\mathcal{N}_{[0]})$, let

$$\bar{u} := [2^m] \times \pi_d(u) \in C_n(\mathcal{N}_{[2^m]}).$$

Let $S := \mathrm{Inv}(\mathrm{cl}(N \setminus L), \varphi_h)$. By Proposition 4.0.2, S_0 is an isolated invariant set for the Poincaré map P.

Theorem 4.2.1 ([MSW15, Theorem 4.5]). *Let $T/h \in \mathbb{Q}$ and $n \in \mathbb{N}$. Given a basis $\mathcal{B} = ([u_1], \ldots, [u_k])$ of $\mathrm{H}_n(\mathcal{N}_{[0]}, \mathcal{L}_{[0]})$ and a $(k \times k)$-matrix $A = (a_{ij})$ such that*

$$\left(u_j, \sum_{i=1}^{k} a_{ij} \bar{u}_i \right) \text{ for } j = 1, \ldots, k$$

are h-movable pairs of contiguous cycles, and let α be an endomorphism with $A \in M(\alpha)$. Then $\mathrm{CH}_n(S_0, P) \cong \mathbf{L}(\alpha)$.

For example, if A in Theorem 4.2.1 is an invertible matrix, then the Conley index is represented by any matrix similar to A.

Remark 4.2.2. A similar theorem is [MS10, Theorem 6.3], where one assumes that N is an isolating block for the flow φ. Isolating blocks for flows are hard to find numerically, whereas there are good algorithms constructing index pairs for maps. The cost we pay here is that the proof becomes harder.

Also note that $\mathrm{H}_n(\mathcal{N}_{[0]}, \mathcal{L}_{[0]}) = 0$ for $n > d$. Therefore, the *Lefschetz number* of $\mathrm{CH}_*(S_0, P)$ in the following corollary is a finite sum. Let $\operatorname{tr} A$ denote the trace of a matrix A.

Corollary 4.2.3 ([MSW15, Corollary 4.8])**.** *If each matrix A_n represents the Conley index $\mathrm{CH}_n(S_0, P)$ as in Theorem 4.2.1 and*

$$\sum_{n=0}^{d} (-1)^n \operatorname{tr} A_n \neq 0,$$

then P has a fixed point.

We sketch the proof presented in [MSW15, Section 5.2]. The purpose of this sketch is to give an idea where the assumptions of Theorem 4.2.1 and Proposition 2.5.17 are used. Additionally, it might help the reader decide whether or not to read the detailed proof. The rest of this section is not used anywhere else in this thesis.

From the given weak index pair (N, L), a certain strong index pair (N^*, L^*) (usually not a pair of cubical complexes) for (S, φ_h) is defined. Since $T/h \in \mathbb{Q}$, there are $p, q \in \mathbb{N}$ such that $ph = qT$.

Then another pair of spaces (N^{**}, L^{**}) is constructed with $N^{**} \setminus L^{**} = N^* \setminus L^*$ and $N^{**}/L^{**} = N^*/L^*$. It is a weak index pair for (S, φ_τ) for every $\tau \in (0, T]$ and, as shown in [MSW15, Lemma 5.11],

$$(\varphi_\tau)_{(N^{**}, L^{**})}(\langle t \rangle, x) = \begin{cases} \varphi_\tau(\langle t \rangle, x) & \text{if } \varphi((\langle t \rangle, x), [0, \tau]) \subset N^{**} \setminus L^{**}, \\ * & \text{otherwise.} \end{cases} \tag{4.3}$$

Additionally, (N_0^{**}, L_0^{**}) is a weak index pair for (S, P) and, using (4.3) for $\tau = T$, its index map with respect to the Poincaré map P is described by

$$P_{(N_0^{**}, L_0^{**})}(x) = \begin{cases} P(x) & \text{if } \varphi((\langle 0 \rangle, x), [0, T]) \subset N^* \setminus L^*, \\ * & \text{otherwise.} \end{cases}$$

In order to get a description of $\mathbf{L}(P_{(N_0^{**}, L_0^{**})})$, we are working towards applying Proposition 2.5.17. For $i \in \mathbb{N}$ and a subset $A \subset \Sigma$, let

$$A_i := \{(\langle ih \rangle, x) \in A\}.$$

This enables us to define

$$\Psi_i \colon N_i^*/L_i^* \to N_{i+1}^*/L_{i+1}^*,$$
$$(\langle ih\rangle, x) \mapsto \begin{cases} \varphi((\langle ih\rangle, x), h) & \text{if } (\langle ih\rangle, x),\ \varphi((\langle ih\rangle, x), h) \in N^* \setminus L^*, \\ * & \text{otherwise.} \end{cases}$$

Now we compose these maps to reach $\langle ph\rangle = \langle qT\rangle = \langle 0\rangle \in \mathbb{R}/T\mathbb{Z}$. We let

$$\Psi := \Psi_{p-1} \circ \ldots \circ \Psi_0 \colon N_0^*/L_0^* \to N_0^*/L_0^*.$$

From (4.3) for $\tau = h$, we get that

$$\Psi_i(\langle ih\rangle, x) = \begin{cases} \varphi((\langle ih\rangle, x), h) & \text{if } \varphi((\langle ih\rangle, x), [0,h]) \subset N^* \setminus L^*, \\ * & \text{otherwise.} \end{cases}$$

This shows:

Lemma 4.2.4 ([MSW15, Lemma 5.15])**.**

$$\Psi = P^q_{(N_0^{**}, L_0^{**})}.$$

We want to describe Ψ in terms of A. Let

$$\kappa \colon N_0^*/L_0 \to N_0^*/L_0^*$$

be induced by the inclusion $L_0 \subset L_0^*$. One defines Ψ' similarly to Ψ, but on N_0^*/L_0, and gets a commutative diagram

$$\begin{array}{ccc} N_0^*/L_0 & \xrightarrow{\kappa} & N_0^*/L_0^* \\ \big\downarrow{\scriptstyle \Psi'} & & \big\downarrow{\scriptstyle \Psi} \\ N^*/L_0 & \xrightarrow{\kappa} & N_0^*/L_0^*. \end{array} \tag{4.4}$$

Without going into detail, we use that the support of each cubical n-chain u_i is contained in N_0^*. Now we consider each u_i a cubical singular chain (as in Section 3.2), and by composing this map $u_i \colon [0,1]^n \to N_0^*$ with the quotient map $N_0^* \to N_0^*/L_0$, it represents an element $\widetilde{u}_i \in \widetilde{\mathrm{H}}_n(N_0^*/L_0)$. One can show that these chains are linearly independent. Let V be the vector space with basis $\mathcal{B} = (\widetilde{u}_1, \ldots, \widetilde{u}_k)$. Now let $\alpha \colon V \to V$ be such that $A = M_{\mathcal{B}}(\alpha)$. We skip the proofs of the following two lemmas, where the movability assumption is used.

Lemma 4.2.5 ([MSW15, Lemma 5.14])**.** *For* $1 \leq i \leq k$*,*

$$\widetilde{\mathrm{H}}_n(\Psi')(\widetilde{u}_i) = \alpha^q(\widetilde{u}_i).$$

Let r be the restriction of $\widetilde{\mathrm{H}}_n(\kappa)$ to the domain V.

Lemma 4.2.6 ([MSW15, Lemma 5.13]). *The diagram*

$$\begin{tikzcd}
V \arrow[r, "r"] \arrow[d, "\alpha"] & \widetilde{\mathrm{H}}_n(N_0^*/L_0^*) \arrow[d, "\widetilde{\mathrm{H}}_n(P_{(N_0^{**},L_0^{**})})"] \\
V \arrow[r, "r"] & \widetilde{\mathrm{H}}_n(N_0^*/L_0^*)
\end{tikzcd}$$

commutes.

Now one can show that there are a linear map s and a number $k^* \in \mathbb{N}$ such that the diagram

$$\begin{tikzcd}
V \arrow[r, "r"] \arrow[d, "\alpha^{qk^*}"] & \widetilde{\mathrm{H}}_n(N_0^*/L_0^*) \arrow[dl, "s"'] \arrow[d, "\widetilde{\mathrm{H}}_n(\Psi^{k^*})"] \\
\widetilde{\mathrm{H}}_n(N_0^*/L_0) \arrow[r, "\widetilde{\mathrm{H}}_n(\kappa)"] & \widetilde{\mathrm{H}}_n(N_0^*/L_0^*)
\end{tikzcd} \tag{4.5}$$

commutes. We apply the reduced homology functor $\widetilde{\mathrm{H}}_n$ to (4.4). There is a number k for which $\mathrm{im}(\widetilde{\mathrm{H}}_n(\Psi')^k) \subset V$ [MSW15, Lemma 5.16]. We put the resulting diagram below Diagram 4.5 to see that the following diagram commutes, where we use Lemma 4.2.5 on the left and Lemma 4.2.4 on the right.

$$\begin{tikzcd}
V \arrow[r, "r"] \arrow[d, "\alpha^{qk^*}"] \arrow[dd, bend right=60, "\alpha^{qk^*+qk}"'] & \widetilde{\mathrm{H}}_n(N_0^*/L_0^*) \arrow[dl, "s"'] \arrow[d, "\widetilde{\mathrm{H}}_n(\Psi)^{k^*}"] \arrow[dd, bend left=60, "\widetilde{\mathrm{H}}_n(P_{(N_0^{**},L_0^{**})})^{qk^*+qk}"] \\
\widetilde{\mathrm{H}}_n(N_0^*/L_0) \arrow[r, "\widetilde{\mathrm{H}}_n(\kappa)"] \arrow[d, "\widetilde{\mathrm{H}}_n(\Psi')^k"] & \widetilde{\mathrm{H}}_n(N_0^*/L_0^*) \arrow[d, "\widetilde{\mathrm{H}}_n(\Psi)^k"] \\
V \arrow[r, "r"] & \widetilde{\mathrm{H}}_n(N_0^*/L_0^*)
\end{tikzcd}$$

We can finally apply the criterion from Proposition 2.5.17 and get Theorem 4.2.1.

4.3 The numerical representation

In this section and the following one, we present the algorithm used to check the prerequisites of Theorem 4.2.1. The algorithm is based on the rigorous construction of a weak index pair (N, L) of φ_h as in [Mro06]. The index pair is represented by elementary cubes. An implementation is available on the author's homepage [web].

The algorithm constructs 1-chains z and v, and a 2-chain w as linear combinations of elementary cubes. In the course of their construction, the algorithm has to ensure the movability condition. We go on using the notation from Section 4.1.

Discretizing the generator of the dynamical system

We start with an appropriate variation of Definition 3.1.2.

Definition 4.3.1. Given a continuous map $f : \Sigma \to \Sigma$, the map $\mathcal{F}\colon \mathcal{X} \rightrightarrows \mathcal{X}$ is a *combinatorial enclosure* of f if, for every $Q \in \mathcal{X}$, $f(p(Q)) \subset \operatorname{int}[\![\mathcal{F}(Q)]\!]$.

Given a product $V \subset \Omega$ of closed intervals (an *interval vector*) and an interval $J \subset [0,\infty)$, we use the software library [CAPD] to find an interval vector $V' \subset \Omega$ such that $\psi(V,J) \subset \operatorname{int}(V')$. We use the same methods as mentioned in Section 3.4.

Given an elementary cube Q (not necessarily full) with $Q \subset |\mathcal{X}|$, $\alpha(Q)$ as defined in (4.2) is an interval vector in Ω. Given a time interval $J \subset [0,\infty)$, the aforementioned methods for rigorous numerics compute an interval vector $E \subset \Omega$ such that $\psi(\alpha(Q),J) \subset \operatorname{int}(E)$. Then we represent $q(E) \subset \Sigma$ as a set of cubes in $\mathcal{X}$ as follows:

$$\Phi(Q,J) := \{Q' \in \mathcal{X} \mid p(Q') \cap q(E) \neq \varnothing\} \subset \mathcal{X},$$

which could be infinite because E does not need to be bounded. Then $\varphi(p(Q),J) \subset \operatorname{int}[\![\Phi(Q,J)]\!]$. Let $\mathcal{F}^J$ denote the restriction of $\Phi(-,J)$ to full cubes in $\mathcal{X}$. Then $\mathcal{F}^J : \mathcal{X} \rightrightarrows \mathcal{X}$ is a combinatorial enclosure of φ_t for every $t \in J$. When checking the conditions of Theorem 4.2.1, we use the intervals $J = [h,h]$, having length zero, and $J = [0,h]$. Our algorithm operates on this finite set of full cubes:

$$\mathcal{K} := \{[i,i+1] \times [j,j+1] \times [k,k+1] \mid i,j,k \in \{0,1,\dots,2^m-1\}\} \subset \mathcal{X}.$$

The dynamics of $\varphi(\cdot,J)$ which we are interested in is represented numerically by the function COVER, defined for a cube Q of arbitrary dimension:

$$\text{COVER}(Q,J) := \Phi(Q,J) \cap \mathcal{K}.$$

Remark 4.3.2. The function COVER is the only method through which our algorithm receives information about the dynamical system. This means:

(i) Our algorithm does not only work with the CAPD software library. One could use any numerical integrator in the function Φ as long as $\varphi(p(Q),J) \subset \operatorname{int}[\![\Phi(Q,J)]\!]$ for each cube Q. Since the precise formula of the ODE is only used within Φ (we parse it from a string), one could even work without an explicit formula as long as one is able to construct such an enclosure $\Phi(Q,J)$.

(ii) The algorithm does not get any information about $\varphi([\![Q]\!],J) \setminus [\![\mathcal{K}]\!]$. This information is not necessary.

(iii) One might also implement COVER for certain cubes Q without using Φ, e.g. if it was known from separate considerations that $\varphi([\![Q]\!],h) \cap [\![\mathcal{K}]\!] = \varnothing$.

For a set $\mathcal{A} \subset \mathcal{K}$ of full cubes, let (analogous to Equation (3.3))

$$\text{WRAP}(\mathcal{A}) := \{Q \in \mathcal{K} \mid Q \cap |\mathcal{A}| \neq \varnothing\},$$

where we consider the right edge at $[2^m]$ glued to the left edge at $[0]$.

Constructing the weak index pair

We define the following subset of $\mathcal{K}$:

$$\mathcal{M} := \{[i,i+1] \times [j,j+1] \times [k,k+1] \mid 0 \le i < 2^m, j,k \in \{1,\dots,2^m-2\}\}\,.$$

The geometric realization $[\![\mathcal{M}]\!]$ is our candidate for an isolating neighborhood of φ_h. The property $[\![\mathcal{M}]\!] \subset \operatorname{int}[\![\mathcal{K}]\!]$ is crucial for the proof of Proposition 4.4.1. For a combinatorial enclosure $\mathcal{F}\colon \mathcal{X} \rightrightarrows \mathcal{X}$ of φ_h, we let its restriction to $\mathcal{M}$ be defined by

$$\mathcal{F}_{\mathcal{M}}\colon \mathcal{M} \rightrightarrows \mathcal{M}, \quad Q \mapsto \mathcal{F}(Q) \cap \mathcal{M} = \textsc{cover}(Q,[h,h]) \cap \mathcal{M}.$$

Now we use the approach described in Section 3.6. We subdivide $\mathcal{M}$ into smaller cubes and compute a finer enclosure $\mathcal{F}_{\mathcal{M}}$ until

$$\textsc{wrap}(\operatorname{Inv}(\mathcal{M},\mathcal{F})) \subset \mathcal{M}. \tag{4.6}$$

Then, as in Section 3.6, we construct a pair $(\mathcal{N}^3,\mathcal{L}^3)$ of sets of full cubes such that $([\![\mathcal{N}^3]\!],[\![\mathcal{L}^3]\!])$ is a weak index pair for $(\operatorname{Inv}(M,\varphi_h),\varphi_h)$.

Remark 4.3.3. After finding the weak index pair, it is also possible to replace $\mathcal{M}$ by a set $\mathcal{M}' \subset \mathcal{M}$ such that $\textsc{wrap}(\operatorname{Inv}(\mathcal{M},\mathcal{F})) \subset \mathcal{M}'$. Then $\operatorname{Inv}(\mathcal{M}',\mathcal{F}) = \operatorname{Inv}(\mathcal{M},\mathcal{F})$. Using $\mathcal{M}'$, we can reduce the thickness of the exit set $\mathcal{L}^3$.

4.4 Algorithms

The algorithms presented in this section compute the matrix A appearing in the special case $n = 1$ of Theorem 4.2.1.

Our calculations use the following **input**:

(i) a formula describing the vector field f (alternatively, use Remark 4.3.2),
(ii) the period T such that $f(t+T,x) = f(t,x)$,
(iii) lengths $a,b > 0$ describing the region $K = [0,T] \times [-a,a] \times [-b,b] \subset [0,T] \times \mathbb{R}^2$,
(iv) a time step h and a subdivision parameter m;

and return the following **output** (if the calculations do not fail):

(i) matrix A fulfilling the conditions of Theorem 4.2.1
(ii) optionally: cubical chains u_j, z_j and w_j for $j = 1,\dots,k$ showing that $(u_1,\dots,u_k)$ and A fulfill the requirements of Theorem 4.2.1 (cf. Definitions 4.1.2 and 4.1.3).

The approach proceeds as follows: After computing a pair $(\mathcal{N},\mathcal{L})$ of cubical complexes representing a weak index pair and a basis $\{[u_j]\}$ of $\mathrm{H}_1(\mathcal{N}_{[0]},\mathcal{L}_{[0]})$, the computer constructs a 1-chain v_j for each cubical 1-chain u_j such that (u_j,v_j) is an h-movable pair of contiguous cycles. The final step then consists in finding a 1-chain homologous to v_j which is a linear combination of the chains $(\bar{u}_1,\dots,\bar{u}_k)$.

In our examples, $\mathrm{H}_0(\mathcal{N}_{[0]}, \mathcal{L}_{[0]}) = 0$. However, calculating the Conley index in 0-th homology can be done as follows. A generator $[u] \in \mathrm{H}_0(\mathcal{N}_{[0]}, L_{[0]})$ can be represented by a vertex x in a component of $\mathcal{N}_{[0]}$ that has empty intersection with $\mathcal{L}_{[0]}$. The required 1-chain w is a path along edges of $\mathcal{N}$ such that $\varphi([\![w]\!], [0,h]) \subset N$ and $\partial w = y - x$ with $y \in \mathcal{N}^0_{[2^m]}$.

4.4.1 Checking usability of a cube

We are ready to formulate the functions used by our algorithm for computing $\mathrm{CH}_1(S_0, P)$. Function MOVABLE in Algorithm 1 is the function which executes the integration of φ over the time interval $[0,h]$.

Algorithm 1

```
1: function MOVABLE(cube Q, set of full cubes 𝒜)
2:     set of full cubes ℬ := COVER(Q, [0,h])
3:     if ℬ ⊂ 𝒜 then return TRUE
4:     else return FALSE
```

Proposition 4.4.1.

(i) If MOVABLE$(Q, \mathcal{N}^3)$ *is* TRUE *for a cube* $Q \in \mathcal{N}$*, then* $\varphi(p(Q), [0,h]) \subset N$.

(ii) If MOVABLE$(Q, \mathcal{L}^3)$ *is* TRUE *for a cube* $Q \in \mathcal{L}$*, then* $\varphi(p(Q), [0,h]) \subset L$.

Proof.

(i) By assumption, COVER$(Q, [0,h]) = \Phi(Q, [0,h]) \cap \mathcal{K} \subset \mathcal{N}^3$. This yields

$$\varphi(p(Q), [0,h]) \subset \operatorname{int}[\![\Phi(Q, [0,h]) \cap (\mathcal{K} \cup (\mathcal{X} \setminus \mathcal{K})]\!] \subset [\![\mathcal{N}^3]\!] \cup [\![\mathcal{X} \setminus \mathcal{K}]\!],$$

which is a disjoint union since WRAP$(\mathcal{N}^3) \subset \mathcal{K}$. The claim follows from the connectedness of $\varphi(p(Q), [0,h])$ and $p(Q) \subset [\![\mathcal{N}^3]\!]$.

(ii) This is analogous to (i).

□

The functions USABLEN and USABLEL in Algorithm 2 confirm that a cube can be used as a summand of w or z, respectively.

Lemma 4.4.2.

(i) If USABLEN(Q) *is* TRUE*, then* $\varphi(p(Q), [0,h]) \subset N$.

(ii) If USABLEL(e) *is* TRUE*, then* $\varphi(p(e), [0,h]) \subset L$.

(iii) If USABLEN(c) *is* TRUE*, then* $\varphi([\![c]\!], [0,h]) \subset N$.

Algorithm 2

function USABLEN(square Q)
  if $Q \notin \mathcal{N}^2$ **then return** FALSE
  for $F \in$ COBOUNDARY$(Q) \cap \mathcal{N}^3$ **do**
    boolean FGOOD := TRUE
    for $Q' \in$ BOUNDARY(F) **do**
      if not MOVABLE$(Q', \mathcal{N}^3)$ **then**
        FGOOD := FALSE
        break ▷ stop checking squares in boundary of F
    if FGOOD **then return** TRUE
  return FALSE

function USABLEL(edge e)
  if $e \notin \mathcal{L}^1$ **then return** FALSE
  if MOVABLE$(e, \mathcal{L}^3)$ **then**
    for $Q \in$ COBOUNDARY(e) **do**
      if USABLEN(Q) **then**
        return TRUE
  return FALSE

function USABLEN(2-chain c)
  set of squares $\mathcal{A} := \{Q \mid c(Q) \neq 0\}$
  if not $\mathcal{A} \subset \mathcal{N}^2$ **then return** FALSE
  for $Q \in \mathcal{A}$ **do**
    if not USABLEN(Q) **then return** FALSE
  return TRUE

Proof. This follows from Proposition 4.4.1. □

Remark 4.4.3. The checks in functions USABLEN and USABLEL are slightly more restrictive than one might expect. Note that USABLEN(Q) is TRUE if and only if Q is in the boundary of a full cube F for which all boundary cubes are movable. In practice, this helps avoid some dead ends in Algorithm 5.

4.4.2 Constructing a contiguous pair of cycles

The symbol VAR in Algorithms 3 and 4 means that the following variable is passed by reference to the function.

For a vertex $y = [i] \times [j] \times [k] \in \mathcal{L}^0$, define the following 5-tuple $S(y)$ of oriented edges

$$\begin{aligned} S(y) = ([i,i+1]\times[j]\times[k],\quad &[i]\times[j,j+1]\times[k],\quad [i]\times[j]\times[k,k+1],\\ -[i]\times[j-1,j]\times[k],\quad &-[i]\times[j]\times[k-1,k]), \end{aligned}$$

which is used in Algorithm 3.

Algorithm 3

1: **function** PATHBACKTRACKING(VAR path c, vertex x)
2:     **if** $c=0$ **then** $y := x$
3:     **else**
4:         let e be the last edge of c
5:         **if not** USABLEL(e) **then return** FALSE
6:         let y be the endpoint of c
7:     **if** $y \in \mathcal{L}^0_{[2^m]}$ **then return** TRUE
8:     **for** $d \in S(y)$ **do**
9:         1-chain $s := c + d$
10:         **if** PATHBACKTRACKING(s, x) **then**
11:             $c := s$; **return** TRUE   ▷ success
12:     **return** FALSE   ▷ c remains unchanged

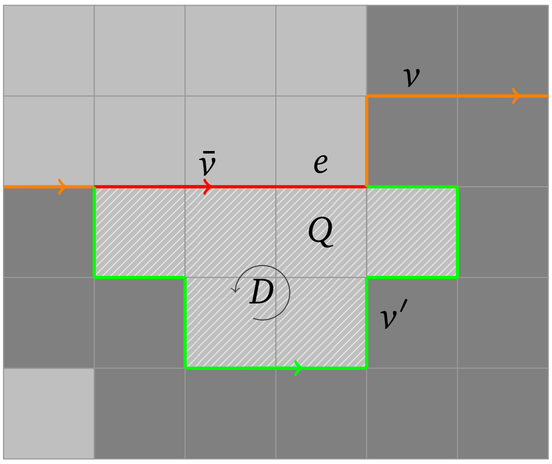
13: **function** PATH(vertex x)
14:     1-chain $c_x := 0$
15:     **if** PATHBACKTRACKING(c_x, x) **then return** c_x
16:     **else return** FAILURE

Proposition 4.4.4. *Assume that function* PATH *from Algorithm 3 is called with a vertex $x \in \mathcal{L}^0_{[0]}$ as input. If it terminates successfully, then it returns a path c_x satisfying $\partial c_x = y - x$ for some $y \in \mathcal{L}^0_{[2^m]}$ and $\varphi([\![c_x]\!],[0,h]) \subset L$.*

Additionally, for every $n \in \{0,\dots,2^m-1\}$ there is exactly one edge e such that $\pi_1(e) = [n,n+1]$ and $c_x(e) \neq 0$. This edge has coefficient $c_x(e) = 1$.

Proof. The algorithm performs a depth-first search using backtracking. A new candidate path s is rejected in line 10 if USABLEL(d) $=$ FALSE. Therefore $\varphi([\![c_x]\!],[0,h]) \subset L$ follows from Lemma 4.4.2(ii). The last property follows from the definition of $S(y)$. □

First we use cubical homology software in order to construct a finite basis $\{[u_j]\}$ of $\mathrm{H}_1(\mathcal{N}_{[0]},\mathcal{L}_{[0]})$, where each $u_j \in C_1(\mathcal{N}_{[0]})$ is a path with $\partial u_j \in C_0(\mathcal{L}_{[0]})$. From here on we drop the index j for readability and fix some 1-chain $u = u_j$. Then $\partial u = x^+ - x^-$ with $x^+, x^- \in \mathcal{L}^0_{[0]}$. The 2-chain w is constructed by successively adding oriented squares. If necessary, squares within a layer $\mathcal{N}_{[n]}$ are added using the function FLOODFILL in



The squares drawn are contained in $\{Q' \in \mathcal{N}_{[n]} \mid \text{USABLEN}(Q') = \text{TRUE}\}$. One can imagine the time axis (n axis) perpendicular to the figure. Each square Q' is dark gray if $\text{USABLEN}([n, n+1] \diamond \partial \pi_d(Q'))$ is TRUE and light gray otherwise. Line 7 of Alg. 4 shows that FLOODFILL adds light gray squares to D (hatched area). The red 1-chain $\bar{v}$ is a part of v (red and orange) that is replaced by the homologous 1-chain v' (green).

FIGURE 4.2: A typical state of the variables of Algorithm 5 in line 15

Algorithm 4. We use the index n during construction even though the functor H_n appears in Theorem 4.2.1. Since we only consider H_1 in our computations, this should not lead to confusion.

Algorithm 4

```
1: function FLOODFILL(square Q ∈ 𝒩_[n], 1-chain c, VAR 2-chain D)
2:     if (D(Q) ≠ 0 or not USABLEN(Q)) then return
3:     D := D + Q
4:     for edges e′ with (∂Q)(e′) ≠ 0 do            ▷ add neighboring squares of Q
5:         if c(e′) ≠ 0 then continue                    ▷ do not cross c
6:         for Q′ ∈ COBOUNDARY(e′) ∩ 𝒩_[n] do
7:             if not USABLEN([n, n+1] ⋄ ∂π_d(Q′)) then
8:                 if Q′ ≠ Q then
9:                     FLOODFILL(Q′, c, D)
```

Proposition 4.4.5. *The function* FLOODFILL *from Algorithm 4 with input $Q \in \mathcal{N}^2_{[n]}$, a chain $c \in C_1(\mathcal{N}_{[n]})$ and $D = 0 \in C_2$ terminates. After execution, D is a 2-chain in $\mathcal{N}_{[n]}$ with $\varphi([\![D]\!], [0, h]) \subset N$.*

Proof. The recursion terminates because $\mathcal{N}^2_{[n]}$ is finite, hence the search tree is finite. The property $\varphi([\![D]\!], [0, h]) \subset N$ is guaranteed by the check in line 2 and Lemma 4.4.2(i). $\square$

We are ready to formulate Algorithm 5 which constructs v using the given 1-chain u. The idea is sketched in Figure 4.2. Note that the lines containing w could be removed without changing the behavior of the algorithm.

Algorithm 5

1: **function** FINDPARTNER(1-chain u)
2: 0-chain $x^+ - x^- := \partial u$
3: 1-chain $z := \text{PATH}(x^+) - \text{PATH}(x^-)$
4: 2-chain $w := 0$
5: 1-chain $v := u + z_{[0]}$
6: **for** $n := 0$ **to** $2^m - 1$ **do**
7: OUTERLOOPLABEL:
8: **for** e with $v(e) \neq 0$ **do**
9: **if not** $\text{USABLEN}([n, n+1] \diamond \pi_d(e))$ **then**
10: **for** $Q \in \text{COBOUNDARY}(e) \cap \mathcal{N}_{[n]}$ **do** ▷ try both sides of e
11: 2-chain $D := 0$
12: $\text{FLOODFILL}(Q, v, D)$
13: **if** $(\partial Q)(e) \cdot v(e) = 1$ **then** $D := -D$ ▷ switch orientation
14: 1-chain $\bar{v} := \sum_{v(e')(\partial D)(e') \neq 0} v(e')e'$
15: 1-chain $v' := \bar{v} + \partial D$
16: **if** $\text{USABLEN}\big([n, n+1] \diamond \pi_d v'\big)$ **then**
17: $w := w - D$
18: $v := v - \bar{v} + v'$
19: **goto** OUTERLOOPLABEL
20: **return** FAILURE ▷ give up if adding squares in $\mathcal{N}_{[n]}$ did not help
21: $w := w - [n, n+1] \diamond \pi_d v$
22: $v := [n+1] \diamond \pi_d v + z_{[n+1]}$
23: **return** v

Proposition 4.4.6. *When function* FINDPARTNER *from Algorithm 5 is run with input a path $u \in C_1(\mathcal{N}_{[0]}, \mathcal{L}_{[0]})$ and it returns v, then (u, v) is a contiguous and h-movable pair of cubical cycles. Additionally, $\partial w = u - v + z$.*

Proof. Proposition 4.4.4 and the definition of z in line 3 show that $\varphi([\![z]\!], [0, h]) \subset L$. Note that right after the initialization of v in line 5, the pair (u, v) is contiguous and movable because $u - v + z_{[0]} = 0$. Then for every change of the variable v, let v_{old} be its old value and v_{new} its new value. The proposition is proven by showing that the pair $(v_{\text{old}}, v_{\text{new}})$ is contiguous and movable at every change (cf. Lemma 4.1.4). There are two kinds:

(i) The change in line 18: Observe that $\partial D = v_{\text{new}} - v_{\text{old}}$ and $\varphi([\![D]\!], [0, h]) \subset N$ because it was constructed using FLOODFILL (cf. Proposition 4.4.5)

(ii) The change in line 22: The successful termination of the for-loop in line 8 together with the check in line 9 ensures that $\varphi([\![[n, n+1] \diamond \pi_d v_{\text{old}}]\!], [0, h]) \subset N$

by Lemma 4.4.2(i). Additionally, using (3.2),

$$\begin{aligned}\partial\left([n,n+1]\diamond\pi_d v_{\mathrm{old}}\right) &= \partial[n,n+1]\diamond\pi_d v_{\mathrm{old}} - [n,n+1]\diamond\partial\pi_d v_{\mathrm{old}}\\ &= [n+1]\diamond\pi_d v_{\mathrm{old}} - [n]\diamond\pi_d v_{\mathrm{old}} - z_{(n,n+1)} = v_{\mathrm{new}} - v_{\mathrm{old}} - z_{(n,n+1]}.\end{aligned}$$

The property $\partial w = u - v + z$ follows from adding these equations over all changes of w. □

4.4.3 Finding a matrix describing the index map

Algorithm 6

function FINDMATRIX(cubical complexes $\mathcal{N},\mathcal{L}$)
  construct $\{u_1,\dots,u_k\}\subset C_1(\mathcal{N}_{[0]},\mathcal{L}_{[0]})$ representing a basis of $\mathrm{H}_1(\mathcal{N}_{[0]},\mathcal{L}_{[0]})$
  $\bar{u}_i := [2^m]\diamond\pi_d(u_i)$ for each i
  $(k\times k)$-matrix $A=[a_{ij}]$, every $a_{ij}:=0\in\mathbb{F}$.
  for $j:=1$ **to** k **do**
    $v:=$ FINDPARTNER(u_j)
    construct $w'\in C_2(\mathcal{N}_{[2^m]})$ such that: ▷ analogous to Alg. 5, lines 7 to 20
      (i) MOVABLE$(Q,\mathcal{N}^3)$ whenever $w'(Q)\neq 0$;
      (ii) if $(v-\partial w')(e)\neq 0$, then MOVABLE$(e,\mathcal{L}^3)$ or $u_i(e)\neq 0$ for some i; and
      (iii) there is an edge e such that $v(e)w'(e)=1$.
    $v' := v-\partial w'$
    for $i:=1$ **to** k **do** ▷ fill j-th column of matrix A
      find $e\in\mathcal{N}^1_{[2^m]}$ such that $\bar{u}_i(e)\cdot v'(e)\neq 0$
      if such an e was found **then** $a_{ij}:=v'(e)/\bar{u}_i(e)$
    while $c:=v'-\sum_i a_{ij}\bar{u}_i\neq 0$ **do** ▷ check if j-th column of A is correct
      let e be an edge with $c(e)\neq 0$
      if MOVABLE$(e,\mathcal{L})$ **then** $v':=v'-c(e)$
      else return FAILURE
  return A

Proposition 4.4.6 and Theorem 4.2.1 now yield:

Theorem 4.4.7. *Let $T/h\in\mathbb{Q}$ and let $(\mathcal{N},\mathcal{L})$ be cubical complexes constructed as above, in particular $(\llbracket\mathcal{N}\rrbracket,\llbracket\mathcal{L}\rrbracket)$ is a weak index pair for (S,φ_h).*

When the function FINDMATRIX *from Algorithm 6 does not fail and returns A, then A fulfills the requirements of Theorem 4.2.1. Therefore,* $\mathrm{CH}_1(S_0,P)\cong\mathbf{L}(\alpha)$ *for any α with $A\in M(\alpha)$.*

Proof. Observe that each pair $(v_j,\sum_i a_{ij}\bar{u}_i)$ is contiguous and movable by construction in Algorithm 6. Since all pairs (u_j,v_j) are contiguous and movable by Proposition 4.4.6, all pairs $(u_j,\sum_i a_{ij}\bar{u}_i)$ are contiguous and movable by Lemma 4.1.4. □

Potential failure

There are several places where the algorithm can fail. The construction of the weak index pair can only fail due to relation (4.6) at the end of Section 4.3 being false. If this happens, one can increase m until running out of memory. If this does not help, choosing a different time step h might. Time steps similar to each cube's width $T \cdot 2^{-m}$ in time direction proved useful in experiments.

Algorithm 3 might not find a path from $\mathcal{L}_{[0]}$ to $\mathcal{L}_{[2^m]}$ even though one exists. The reason is that we only construct paths which go forward in time direction as shown in Proposition 4.4.4. Algorithm 5 can only handle this subset of paths because we are always increasing the index n in time direction, but never decreasing (loop called in line 6). This suffices in our examples presented in Section 4.5.

Runtime and memory complexity

The time step h and the size of a cube Q do not significantly influence the runtime or the memory used. Both the runtime and the memory for the construction of $\mathcal{F}_{\mathcal{M}}$ grow proportionally to 8^m, the number of full cubes in $\mathcal{K}$.

The construction of z, w and v mainly requires runtime. The memory used is smaller than for representing $\mathcal{F}_{\mathcal{M}}$. Since the number of edges and squares grows proportionally to 8^m, the memory used also does. But the runtimes for PATH could be high in the worst case.

For finding the path in Algorithm 3, we use a depth-first search which stops when a usable path is found. In the worst case, the algorithm tries every path starting from x exactly once. For each endpoint $y \in \mathcal{L}^0$ of the existing path, there are up to 5 different oriented edges which can be added and where integration must be performed. The number of edges in $\mathcal{L}^1$ rises with $O(8^m)$. If trying to add a new edge e cost the same each time, the worst case time complexity would be $O(5^{8^m})$, an upper bound for our algorithm.

We use strategies to stay away from this worst case. Even though not mentioned in the pseudocode, we explicitly store edges which were already checked to avoid additional integration, which yields practical runtimes around $O(8^m)$. If possible, the algorithm first goes forward along the time direction in the loop starting at line 8. Then, typically, backtracking is only used within one layer $\mathcal{L}_{[n]}$, so the algorithm was sufficiently fast in practice.

The function FLOODFILL in Algorithm 4 has a time complexity of $O(4^m)$ because we run the integration only once on each square and then save the result. Each layer $\mathcal{N}_{[n]}$ constains at most 4^m squares. Due to the loop in Algorithm 5, line 6, constructing w_j and v_j for each generator u_j has a time complexity of $O(2^m \cdot 4^m) = O(8^m)$. The construction of v' and the matrix A in Algorithm 6 requires only negligible resources.

4.5 Examples

Example 4.5.1 (One-dimensional first relative homology)**.** We applied Algorithm 6 using the weak index pair constructed as in Subsection 4.3 to the differential equation

$$\dot{z} = (1 + e^{i\eta t}|z|^2)\overline{z},$$

which shows chaotic behavior for $\eta \in (0,1]$ (cf. [MS10] and references therein). This equation has period $T = 2\pi/\eta$ in t. We analyzed the equation for $\eta = 2.0$ using the parameter $h = 1/64 = 0.015625$. More precisely, since π is irrational, the algorithm used $\pi' \in [\pi - \varepsilon, \pi + \varepsilon]$, where ε is the machine precision. Therefore $T' = 2\pi'/\eta \in \mathbb{Q}$, i.e., the numerical proof is found for $\eta' = 2\pi'/T'$ instead of η. We covered the candidate $M = S^1 \times [-3,3] \times [-3,3] = [\![\mathcal{M}]\!]$ for an isolating neighborhood using cubes of equal size as described above. Our algorithm found a combinatorial index pair $(\mathcal{N},\mathcal{L})$ inside $\mathcal{M}$ at a subdivision depth of $m = 6$. The chains constructed by our algorithm are shown in Figure 4.3. The output was the matrix $A = (-1)$.

Theorem 4.4.7 applies. We conclude that the generator $[u] \in H_1(N_0, L_0)$ is sent to $-[u] \in H_1(N_0, L_0)$ under the relative homology endomorphism induced by the Poincaré map P.

Using our implementation, finding the combinatorial index pair $(\mathcal{N},\mathcal{L})$ took 330 seconds seconds. At this stage, the program used 466 MB of RAM, which was almost completely used for storing $\mathcal{F}_{\mathcal{M}}$. After finding $(\mathcal{N},\mathcal{L})$, $\mathcal{F}_{\mathcal{M}}$ is deleted and less memory is used. The construction of all the chains in Algorithm 6 took 149 seconds, out of which 18 seconds were used to find z. The rest for finding w, v and v'. Most of the time was used for the rigorous integrations. The set $\mathcal{K}$ consists of $(2^6)^3 = 262\,144$ cubes, the set $\mathcal{N}$ of 132 728 cubes.

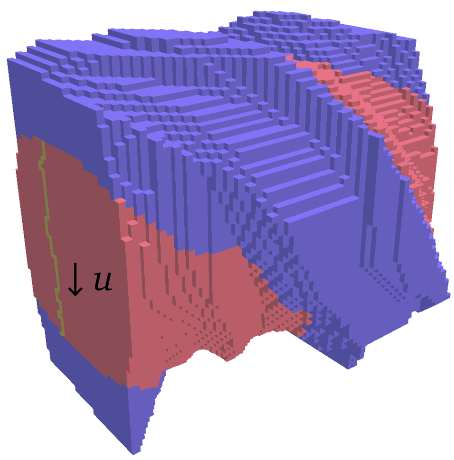
Example 4.5.2 (Two-dimensional first relative homology)**.** We applied the same algorithms as above to the equation

$$\dot{z} = e^{i\eta t}\overline{z}^2 + \overline{z},$$

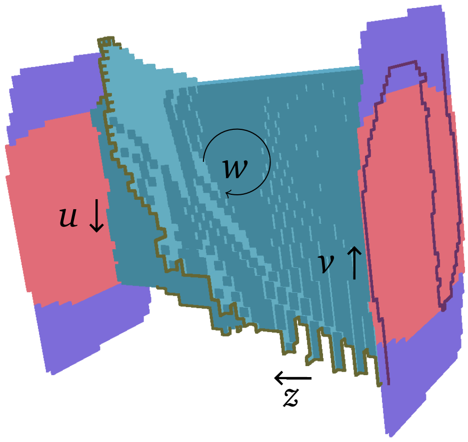
This equation has period $T = 2\pi/\eta$ in t. We analyzed the equation for $\eta = 2.0$. Again, we used the candidate $M = S^1 \times [-3,3] \times [-3,3] = [\![\mathcal{M}]\!]$ for an isolating neighborhood. Our software found a combinatorial index pair $(\mathcal{N},\mathcal{L})$ inside $\mathcal{M}$ at a refinement depth of $m = 7$ using the parameter $h = 1/64$. The index pair with the resulting chains is shown in Figure 4.4. Let α be an endomorphism with

$$A = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \in M(\alpha),$$

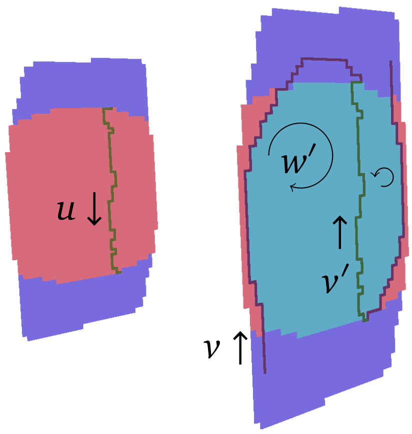
Now α is an automorphism (the determinant of A is 1), therefore $\mathbf{L}(\alpha) = \alpha$. Using Theorem 4.4.7, the Conley index of the Poincaré map is given up to conjugacy by $\mathrm{CH}_1(S_0, P) \cong \alpha$, i.e., $A \in M(\mathrm{CH}_1(S_0, P))$.



(a) The full cubes in $(\mathcal{N}, \mathcal{L})$ (blue: $\mathcal{L}^3$, red: $\mathcal{N}^3 \setminus \mathcal{L}^3$) and u on the left-hand side.

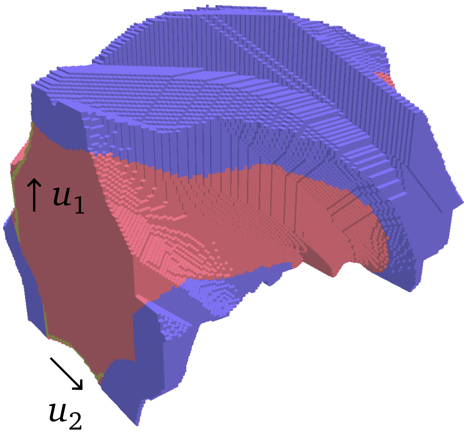


(b) The blue squares are $\mathcal{L}^2_{[0]} \cup \mathcal{L}^2_{[2^m]}$. Similarly for $\mathcal{N}$. The chains after executing FINDPARTNER(u) are shown.

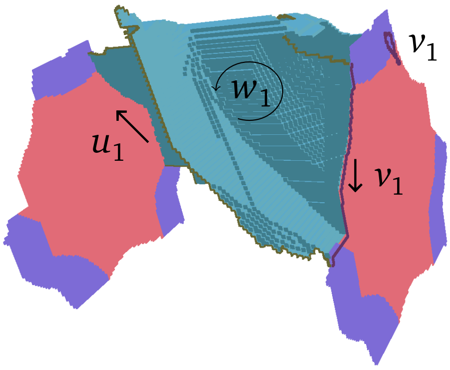


(c) Overall results. In this special example, the support of v' is contained in the support of v.

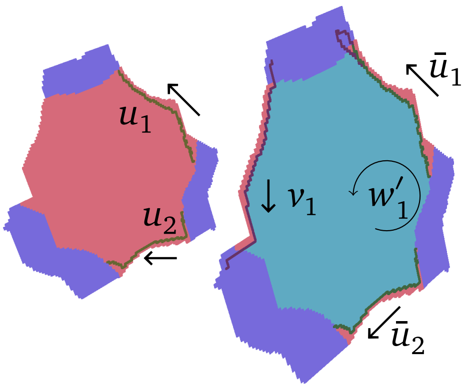
FIGURE 4.3: The intermediate results for Example 4.5.1. The algorithm makes sure that $\partial w = u - v + z$ as shown in Subfigure (b). Subfigure (c) shows the construction of v' from v. Note that $\partial w' - v + v'$ lies in $\mathcal{L}$. The movability properties of the chains cannot be seen in the figure, but they are checked numerically. The output is $A = (-1)$, which can be seen from Subfigure (c) because u points down and v' points up.



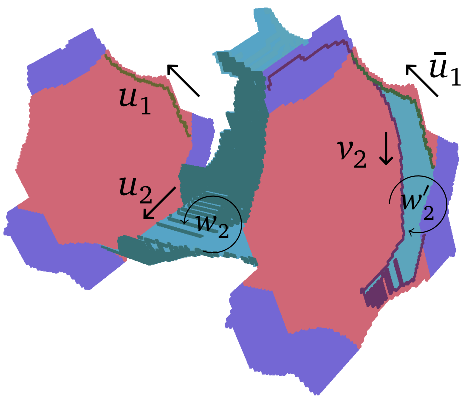
(a) Index pair $(\mathcal{N}, \mathcal{L})$ and 1-chains u_1, u_2



(b) The construction of v_1. Note that v_1 consists of two connected 1-chains, one of which has boundary 0.



(c) The 2-chain w'_1 fills most of $\mathcal{N}_{[2^m]} \setminus \mathcal{L}_{[2^m]}$. Here $v'_1 = -\bar{u}_1 + \bar{u}_2$.



(d) Computing the second column of A. Here $v'_2 = -\bar{u}_1$.

FIGURE 4.4: Outputs for Example 4.5.2, using the same colors as in Figure 4.3

Finding $(\mathcal{N}, \mathcal{L})$ took 2 256 seconds on the same hardware as before, using 3.4 GB of memory. Then Algorithm 6 required further 545 seconds, out of which 99 seconds were used to construct z_1 and z_2.

In contrast to this result, when starting with $M = S^1 \times [-0.1, 0.1] \times [-0.1, 0.1]$, the algorithm yields a different Conley index because the output in this case is $A = (1)$. Since the Conley index is a function of the invariant sets, $\mathrm{Inv}([-0.1, 0.1]^2, P) \neq \mathrm{Inv}([-3, 3]^2, P)$.

Chapter 5

Time discretization for finding Morse decompositions of a flow

In this chapter, we return to the question of choosing the time parameter h to discretize a given flow φ on $\mathbb{R}^d$. In Section 3.5, we hinted at potential difficulties. In this chapter, we propose the following time discretization. For a flow φ on X and a continuous function $\tau\colon X \to \mathbb{R}_{>0}$, we let

$$\begin{aligned}\varphi_\tau\colon X &\to X,\\ x &\mapsto \varphi(x, \tau(x)).\end{aligned}$$

In addition to computing Conley indices, we would also like to compute Morse decompositions; these tell us which isolated invariant sets can be connected by an orbit.

We first show that the information we get numerically about φ_τ also tells us something about φ. Then we present a numerical example, where the computations for φ_τ yield more information than for φ_h. The idea of using a time step varying in the phase space was also proposed in [CMLZ08, Definition 3.1]. However, the necessary theoretical background is not covered therein. The theory is not fully analogous to the case of a constant function τ, i.e., $\tau(x) = h \in \mathbb{R}_{>0}$ for all $x \in X$. This theoretically easier situation is discussed in Chapter 3. Most of the material covered in this chapter is also presented in [MMW15]. An example where Morse decompositions for φ are computed using φ_h is presented in [PGCL12].

5.1 Isolating neighborhoods

Even though there is an inverse φ_{-h} of φ_h for any flow φ, there need not be an inverse for φ_τ. We formulate a simple criterion for the existence of backward solutions of φ_τ.

Lemma 5.1.1. *Let $x \in X$ and suppose that $\tau(\varphi(x,\mathbb{R})) \subset \mathbb{R}_{>0}$ is bounded. Then there is a solution $\gamma\colon \mathbb{Z} \to X$ of the discrete system $\varphi_\tau\colon X \to X$ such that $\gamma(0) = x$ and $\gamma(\mathbb{Z}) \subset \varphi(x,\mathbb{R})$.*

Proof. Define $\gamma(n) = \varphi_\tau^n(x)$ for $n \geq 0$. Let $n < 0$ and assume that $\gamma(n+1)$ is already constructed. Then there is an $s \leq 0$ such that $\gamma(n+1) = \varphi(x,s)$. Define the function

$$g\colon \mathbb{R} \to \mathbb{R}, \quad t \mapsto \tau(\varphi(x,t)) - s + t.$$

We have $g(s) > 0$, and since $t \mapsto \tau(\varphi(x,t)) - s$ is bounded, $g(t) < 0$ for t sufficiently small. Hence, by the intermediate value theorem, there is a $t' < s$ such that $g(t') = 0$. Choose $\gamma(n) := \varphi(x,t')$. Then

$$\varphi(\gamma(n), \tau(\gamma(n))) = \varphi(\varphi(x,t'), s - t') = \varphi(x,s) = \gamma(n+1).$$

□

The following theorem slightly generalizes Theorem 3.3.1.

Theorem 5.1.2 ([MSW15])**.** *Let $S \subset X$ be compact. Then the following three conditions are equivalent:*

(i) S is an isolated invariant set with respect to φ.

(ii) For every continuous map $\tau\colon X \to \mathbb{R}_{>0}$, S is an isolated invariant set with respect to φ_τ.

(iii) There is a number $h > 0$ such that S is an isolated invariant set with respect to φ_h.

Proof. Assume condition (i) holds and fix a continuous map $\tau\colon X \to \mathbb{R}_{>0}$. Obviously $\varphi_\tau(S) \subset S$. Using Lemma 5.1.1, for every $x \in S$ there is an $x' \in S$ such that $\varphi_\tau(x') = x$. This means that $S \subset \varphi_\tau(S)$. Hence S is invariant with respect to φ_τ. Choose N, an isolating neighborhood for S with respect to φ. Let $T := \sup\{\, \tau(x) \mid x \in N \,\} < \infty$. To see that S is an isolated invariant set with respect to φ_τ, consider the (in general not continuous) map

$$\begin{aligned} \sigma\colon N &\to [0,\infty], \\ x &\mapsto \sup\{\, t \in \mathbb{R}^+ \mid \varphi(x,[0,t]) \subset N \,\}. \end{aligned}$$

First we show that each $x \in S$ has an open neighborhood V_x such that $\sigma(V_x) \subset [T,\infty]$: Assume not. Then there is a sequence of points x_n converging to x with $\sigma(x_n) < T$. Since the times $\sigma(x_n)$ lie in the compact set $[0,T]$, we can take a subsequence of $\{x_n\}$ (again called $\{x_n\}$) such that $\sigma(x_n)$ converges to some $T^* \in [0,T]$ for $n \to \infty$. Let $y_n = \varphi(x_n, \sigma(x_n))$, which yields a sequence of points in $\operatorname{bd} N$. Using the continuity of φ, y_n converges to $\varphi(x,T^*)$. Since $\operatorname{bd} N$ is closed, $\varphi(x,T^*) \in \operatorname{bd} N$. But also $\varphi(x,T^*) \in S$ because $x \in S$. A contradiction.

Since S is compact, a finite union of the sets $\operatorname{cl} V_x$ constitute a compact neighborhood $M \subset N$ of S such that $\sigma(x) \geq T$ for $x \in M$. We will show that $S = \operatorname{Inv}(M, \varphi_\tau)$. Obviously we have

$$S = \operatorname{Inv}(N,\varphi) = \operatorname{Inv}(M,\varphi) \subset \operatorname{Inv}(M,\varphi_\tau).$$

To show the opposite inclusion, take $x \in \operatorname{Inv}(M, \varphi_\tau)$ and let $\gamma\colon \mathbb{Z} \to X$ be a solution of φ_τ through x. Let $x_n := \gamma(n)$ and $t_n := \tau(x_n)$. Then

$$x_{n+1} = \varphi_\tau^{n+1}(x) = \varphi(\varphi_\tau^n(x), t_n) = \varphi(x_n, t_n).$$

By definition of T, we have $t_n \leq T$. Since $x_n \in M$, we have $\sigma(x_n) \geq T$. It follows that $\varphi(x_n, [0, t_n]) \subset N$ for all n, and consequently $x \in \operatorname{Inv}(N, \varphi) = S$. Thus implication *(i)* $\Longrightarrow$ *(ii)* is proven. Implication *(ii)* $\Longrightarrow$ *(iii)* is obvious because we can always take τ to be a constant positive function. Implication *(iii)* $\Longrightarrow$ *(i)* is part of Theorem 3.3.1. □

Remark 5.1.3. There is no full analogue of Corollary 3.3.2 since we cannot replace condition *(iii)* in the theorem above by the statement

(iii') There is a continuous function $\tau\colon X \to \mathbb{R}^+$ such that S is isolated invariant with respect to φ_τ.

An example for which $(iii') \implies (i)$ is wrong can be constructed by considering a system with a limit cycle and using a time step function τ which sends a certain point x' on the limit cycle to itself under φ_τ by letting $\tau(x')$ be the period of the orbit. With an appropriate choice of τ for x near x', this yields an isolated fixed point x' for φ_τ, but the set $S = \{x'\}$ would not be invariant for φ. We defer a detailed discussion to Example 5.3.7, where the example is presented in the context of Morse decompositions.

5.2 Comparing Conley indices

The main ingredient for comparing the Conley index of φ with the Conley index of φ_h for some $h > 0$ is the existence of a common index pair for both dynamical systems. The following lemma was verified in the proof of [Mro90a, Theorem 2].

Lemma 5.2.1. *Let S be an isolated invariant set for φ_h (hence for φ). Then there is a compact pair (N', L') which is an index pair for (S, φ) and a strong index pair for (S, φ_h).*

We also have a slightly more general statement for a time-step function τ, but we need an extra assumption about the equality of the invariant sets with respect to φ and φ_τ.

Lemma 5.2.2. *Let S be an isolated invariant set for φ_τ and also for φ. Then there is a compact pair (N', L') such that*

(i) (N', L') is an index pair for (S, φ) and a strong index pair for (S, φ_τ); and

(ii) the index map $(\varphi_\tau)_{(N', L')}$ is homotopic to the identity on N'/L'.

Proof. We prove part (i) analogously to the proof of Lemma 5.2.1 as given in [Mro90a, Theorem 2]: The proof of [Ryb87, Chapter I, Theorem 5.1] shows the existence of an open neighborhood $V \subset X$ of S such that $\operatorname{cl} V$ is an isolating neighborhood of S and of continuous functions $\kappa, \lambda\colon V \to [0, \infty)$ such that $S = \kappa^{-1}(0) \cap \lambda^{-1}(0)$, and

(a) If $\kappa(x) > 0, t > 0$ and $\varphi(x,t) \in V$, then $\kappa(x) < \kappa(\varphi(x,t))$;

(b) if $\lambda(x) > 0, t > 0$ and $\varphi(x,t) \in V$, then $\lambda(x) > \lambda(\varphi(x,t))$.

For an arbitrary $\zeta > 0$, define subsets of V:

$$G(\zeta) := \{ x \in V \mid \kappa(x) < \zeta, \lambda(x) < \zeta \},$$
$$H(\zeta) := \{ x \in V \mid \kappa(x) \le \zeta, \lambda(x) \le \zeta \}.$$

Let $T := \max\{ \tau(x) \mid x \in \operatorname{cl} V \}$. The local compactness of X can be used to observe that for any open neighborhood U of S there is a $\zeta > 0$ such that $\varphi(H(\zeta), [0, T]) \subset U$.

Applying this observation to $U = V$, we conclude the existence of an $\varepsilon > 0$ such that $\varphi(H(\varepsilon), [0, T]) \subset V$. Let $M := H(\varepsilon)$. Applying the observation to $U = G(\varepsilon)$ shows the existence of a $\delta > 0$ such that $\varphi(H(\delta), [0, T]) \subset G(\varepsilon)$. Define

$$N' := \{ x \in V \mid \kappa(x) \le \varepsilon, \lambda(x) \le \delta \},$$
$$L' := \{ x \in N' \mid \delta \le \kappa(x) \}.$$

The pair (N', L') is an index pair for (S, φ) and a strong index pair for (S, φ_τ). It is straightforward to check the properties in Definitions 2.4.2 and 2.5.6. As an example, we verify property 2.5.6(iii) for φ_τ. Let $x \in N' \backslash L'$. The property $x \in N'$ means that $\kappa(x) \le \varepsilon$ and $\lambda(\varphi_\tau(x)) \le \lambda(x) \le \delta$. We also have $\kappa(x) < \delta$. Now assume $\varphi_\tau(x) \notin N'$. Then $\kappa(\varphi_\tau(x)) > \varepsilon$. This contradicts $\varphi(H(\delta), [0, T]) \subset G(\varepsilon)$. Overall, this shows $\varphi_\tau(x) \in N'$ and therefore property 2.5.6(iii).

To see part (ii) of the proposition, we consider the semiflow $\psi \colon N'/L' \times [0, \infty) \to N'/L'$ given by

$$\psi(x,t) = \begin{cases} \varphi(x,t) & \text{if } \varphi(x,[0,t]) \in N' \setminus L', \\ * & \text{otherwise.} \end{cases}$$

It is continuous by [RS88, Theorem 4.2]. Similarly, we could use the homotopy constructed in the proof of [Con78, III.4.2B].

Now observe that given a point $x \in N' \setminus L'$,

$$\varphi_\tau(x) \in N' \setminus L' \text{ if and only if } \varphi(x, [0, \tau(x)]) \subset N' \setminus L'. \tag{5.1}$$

One impliciation is trivial. We only need to show that $\varphi_\tau(x) \in N' \setminus L'$ implies that the flow trajectory from x to this point lies in $N' \setminus L'$. First note that $x \in H(\delta)$, and therefore $\varphi(x, [0, \tau(x)]) \subset G(\varepsilon) \subset V$. Now for $y \in \varphi(x, [0, \tau(x)])$, we get $\kappa(y) < \varepsilon$. We also have $\lambda(y) < \delta$ because of (b). Therefore, $y \in N'$. Additionally, $\kappa(y) < \delta$ because of the assumption $\kappa(\varphi_\tau(x)) < \delta$ and (a). This shows $y \in N' \setminus L'$.

The equivalence (5.1) enables us to describe the index map:

$$(\varphi_\tau)_{(N',L')}(x) = \psi(x, \tau(x)).$$

Therefore

$$\begin{aligned} H\colon N'/L' \rtimes [0,1] &\to N'/L', \\ (x,s) &\mapsto \psi(x, s\tau(x)) \end{aligned}$$

is a pointed homotopy from the identity to the index map of φ_τ on N'/L'. □

Now we show how the Conley index of (S,φ) can be computed from the Conley index of (S,φ_τ), where we let dom denote the domain of a homomorphism.

Theorem 5.2.3. *Let S be an isolated invariant set for φ and φ_τ. Let (N,L) be an index pair for (S,φ_τ) and $I_{(N,L)} = \widetilde{\mathrm{H}}_*((\varphi_\tau)_{(N,L)})$ its homological index map. Then there is an isomorphism*

$$\mathrm{CH}_*(S,\varphi) \cong \operatorname{dom} \mathbf{L}(I_{(N,L)}).$$

Proof. Consider a common index pair (N',L') for (S,φ) and (S,φ_τ) as in Lemma 5.2.2. Property (ii) therein shows that $I_{(N',L')} = \widetilde{\mathrm{H}}(\varphi_\tau)_{(N',L')}$ is the identity in homology (Section 2.3), and it still is the identity after applying the Leray functor. The automorphisms $\mathbf{L}(I_{(N,L)})$ and $\mathbf{L}(I_{(N',L')})$ are isomorphic in **Aut** because (N,L) and (N',L') are index pairs for (S,φ_τ). This shows that also $\mathbf{L}(I_{(N,L)}) = \mathrm{id}_{\widetilde{\mathrm{H}}(N'/L')}$.

Since (N',L') is an index pair for (S,φ), we have

$$\mathrm{CH}_*(S,\varphi) = \widetilde{\mathrm{H}}_*(N'/L') = \operatorname{dom} \mathbf{L}(I_{(N,L)}).$$

□

The following theorem is similar to, but slightly stronger than Theorem 5.2.3. Even if homology is infinite-dimensional, taking the generalized image is not necessary in Theorem 5.2.3.

Theorem 5.2.4 (M. Mrozek, [MSW15])**.** *Let S be an isolated invariant set for φ and φ_τ. Let (N,L) be an index pair for (S,φ_τ) and $I_{(N,L)} = \widetilde{\mathrm{H}}(\varphi_\tau)_{(N,L)}$ its homological index map. Then there is an isomorphism*

$$\mathrm{CH}_*(S,\varphi) \cong \widetilde{\mathrm{H}}_*(N/L)/\operatorname{gker}(I_{(N,L)}).$$

Proof. There is a common index pair (N',L') for (S,φ) and (S,φ_τ) as in Lemma 5.2.2. Thus, by Theorem 2.5.20, there are homomorphisms r,s and an $n \in \mathbb{N}$ such that the following diagram commutes.

$$\begin{tikzcd} \widetilde{\mathrm{H}}_*(N/L) \arrow[r, "I^n_{(N,L)}"] \arrow[d, "r"'] & \widetilde{\mathrm{H}}_*(N/L) \arrow[d, "r"] \\ \widetilde{\mathrm{H}}_*(N'/L') \arrow[r, "I^n_{(N',L')}=\mathrm{id}"'] \arrow[ur, "s"] & \widetilde{\mathrm{H}}_*(N'/L') \end{tikzcd}$$

The lower right triangle shows that s is injective and r is surjective. This yields

$$\mathrm{CH}_*(S,\varphi) \cong \widetilde{\mathrm{H}}_*(N'/L') \cong \mathrm{im}(s) = \mathrm{im}(I^n_{(N,L)}) \cong \widetilde{\mathrm{H}}_*(N/L)/\ker(I^n_{(N,L)}).$$

Additionally, $\ker(I^n_{(N,L)}) = \mathrm{gker}(I_{(N,L)})$ because $I^{kn}_{(N,L)} = I^n_{(N,L)}$ for any $k > 0$. □

5.3 Morse decompositions

Here we deal with another important invariant for a dynamical system which can be extracted from a restricted combinatorial enclosure in some isolating neighborhood $M \subset X$. We are interested in possible trajectories inside $S = \mathrm{Inv}(M,\varphi)$.

In order to describe what happens to a trajectory in S at times going to $\pm\infty$, we recall the notions of α- and ω-limit sets.

Definition 5.3.1. Let $x \in X$, and let φ be a global flow on X.

(i) The *ω-limit set* $\omega(x,\varphi)$ is the set of accumulation points of $\varphi(x,[0,\infty))$.

(ii) The *α-limit set* $\alpha(x,\varphi)$ is the set of accumulation points of $\varphi(x,(-\infty,0])$.

Definition 5.3.2. Let $x \in X$ and $f : X \to X$.

(i) The *ω-limit set* $\omega(x,f)$ is the set of accumulation points of $\{ f^k(x) \mid k > 0 \}$.

(ii) A point $y \in X$ is in the *α-limit set* $\alpha(x,f)$ if and only if there exists a solution $\gamma : \mathbb{Z} \to X$ with $\gamma(0) = x$ such that y is an accumulation point of $\gamma(\mathbb{Z}^-)$.

Definition 5.3.3. Given a dynamical system (i.e., a flow φ or a map f) and an isolating neighborhood M with $S = \mathrm{Inv}(M)$ (refering to $\mathrm{Inv}(M,\varphi)$ or $\mathrm{Inv}(M,f)$, respectively), we call a set of disjoint isolated invariant sets $\{S_p \mid p \in \mathcal{P}\}$ together with an acyclic directed graph MG with vertices $\mathcal{P}$ a *Morse decomposition in M* if for every $y \in S$ one of the following holds:

(i) $y \in S_p$ for some $p \in \mathcal{P}$; or

(ii) there are $p, q \in \mathcal{P}$ and a path from p to q in MG, such that $\alpha(y) \subset S_p$ and $\omega(y) \subset S_q$.

In order to show that a given Morse decomposition for φ_τ is also a Morse decomposition for φ, we can apply the following two lemmas.

For $A \subset X$, let

$$\varphi_{[0,\tau]}(A) := \bigcup_{x \in A} \varphi(x,[0,\tau(x)]) \subset X.$$

Lemma 5.3.4. *Let $\{S_p \mid p \in \mathcal{P}\}$ be a Morse decomposition in M for φ_τ with isolating neighborhoods M_p, i.e., $M_p \cap M_q = \emptyset$ for $p \neq q$ and $S_p = \mathrm{Inv}(M_p,\varphi_\tau) \subset \mathrm{int}\, M_p$ for all $p \in \mathcal{P}$. Let $p \in \mathcal{P}$. If*

$$(*) \qquad \varphi_{[0,\tau]}(M_p) \subset M \text{ and } \varphi_{[0,\tau]}(M_p) \cap M_q = \emptyset \text{ for all } q \in \mathcal{P} \setminus \{p\},$$

then $\mathrm{Inv}(M_p,\varphi) = \mathrm{Inv}(M_p,\varphi_\tau)$.

Proof. Let $p \in \mathcal{P}$ and $M'_p = \operatorname{cl}\left(M \setminus \bigcup_{q \in \mathcal{P}\setminus\{p\}} M_q\right)$. First we show

$$\operatorname{Inv}(M_p, \varphi_\tau) \subset \operatorname{Inv}(M'_p, \varphi). \tag{5.2}$$

Let $\gamma \colon \mathbb{Z} \to M_p$ be a solution to φ_τ in M_p. Then for each $n \in \mathbb{Z}$: $\varphi(\gamma(n), [0, \tau(\gamma(n))]) \subset M'_p$ by assumption $(*)$ in the lemma. Gluing these pieces together yields a trajectory for φ in M'_p.

We also show

$$\operatorname{Inv}(M'_p, \varphi_\tau) \subset \operatorname{Inv}(M_p, \varphi_\tau). \tag{5.3}$$

Assume there is a point $x \in \operatorname{Inv}(M'_p, \varphi_\tau) \setminus \operatorname{Inv}(M_p, \varphi_\tau)$. Hence, $x \notin \bigcup_{q \in P} S_q$. Now either $\alpha(x, \varphi_\tau)$ or $\omega(x, \varphi_\tau)$ must lie in some S_q with $q \neq p$ because they cannot lie within the same Morse set. But this means that any solution of φ_τ through x has to contain points in $\operatorname{int} M_q$, which is disjoint from M'_p. We conclude $x \notin \operatorname{Inv}(M'_p, \varphi_\tau)$, a contradiction.

Overall, we get the following inclusions, where the middle one is trivial.

$$S_p \overset{(5.2)}{\subset} \operatorname{Inv}(M'_p, \varphi) \subset \operatorname{Inv}(M'_p, \varphi_\tau) \overset{(5.3)}{\subset} S_p.$$

Each set is the same subset of M_p. Therefore also $\operatorname{Inv}(M'_p, \varphi) = \operatorname{Inv}(M_p, \varphi)$. □

Lemma 5.3.5. *Let $M \subset X$ be an isolating neighborhood for φ_τ (and hence for φ). Let $\{S_p \mid p \in \mathcal{P}\}$ be a Morse decomposition for φ_τ in M with Morse graph* MG *and assume that each S_p is invariant also for φ. Then $\{S_p \mid p \in \mathcal{P}\}$ is also a Morse decomposition for φ in M with Morse graph* MG.

Proof. We only need to show that the Morse graph MG is preserved. From here, consider a point $y \in \operatorname{Inv}(M, \varphi) \subset \operatorname{Inv}(M, \varphi_\tau)$. There are $p, q \in \mathcal{P}$ and a directed path from p to q in MG such that $\alpha(y, \varphi_\tau) \subset S_p$ and $\omega(y, \varphi_\tau) \subset S_q$. We show that $\omega(y, \varphi) \subset S_q$ by contradiction. The analogous statement for the α-limit is proven similarly.

Let M_q be an isolating neighborhood for φ_τ and therefore for φ around S_q, hence $\operatorname{Inv}(M_q, \varphi) = \operatorname{Inv}(M_q, \varphi_\tau) = S_q \subset \operatorname{int} M_q$. We continue similarly to the proof of Theorem 5.1.2. We define a function

$$\sigma \colon M_q \ni x \mapsto \sup\{\, t \in \mathbb{R}^+ \mid \varphi(x, [0, t]) \subset M_q \,\} \in [0, \infty].$$

Let $T = \max \tau(M_q)$. Now there is a compact neighborhood $\widetilde{M_q}$ of S_q such that $\sigma(x) \geq T$ for all $x \in \widetilde{M_q}$. For all $n \in \mathbb{N}$, let $\gamma(n) := \varphi_\tau^n(y)$.

Assume that $\omega(y, \varphi)$ contains a point y' outside of M_q. We construct a subsequence of γ as follows: Since $\widetilde{M_q}$ is a neighborhood of $\omega(y, \varphi_\tau)$ and $\gamma(\mathbb{N}) \cap S_q = \varnothing$, there is an $\tilde{n} \geq 0$ such that $\gamma(\tilde{n}) \in \widetilde{M_q} \setminus S_q$. Let $s > 0$ be such that $\gamma(\tilde{n}) = \varphi(y, s)$. There is an $s' \geq s$ such that $\sigma(\varphi(y, s')) = T$, and then $\varphi(x, [s', s' + T]) \subset \operatorname{cl}(M_q \setminus \widetilde{M_q})$. Hence, there is an $n_0 \geq \tilde{n}$ such that $\gamma(n_0) \in \operatorname{cl}(M_q \setminus \widetilde{M_q})$.

Moving forward, we construct a subsequence $\gamma(n_0), \gamma(n_1), \ldots$ of points in $\operatorname{cl}(M_q \setminus \widetilde{M_q})$. It has a converging subsequence whose limit is also in $\omega(y, \varphi_\tau) \subset S_q$. A contradiction.

Overall, $\omega(y, \varphi) \subset M_q$, and therefore $\omega(y, \varphi) \subset \operatorname{Inv}(M_q, \varphi) = S_q$ because ω-limits are invariant. □

We are ready to show

Theorem 5.3.6. *Let M be an isolating neighborhood for φ_τ for an arbitrary continuous function $\tau: X \to \mathbb{R}_{>0}$ and $\{S_p \mid p \in \mathcal{P}\}$ a Morse decomposition for φ_τ in M with isolating neighborhoods $\{M_p\}$.*

Suppose that either

(A) the function τ is constant, i.e., $\tau(x) = h$ for all $x \in X$; or

(B) for all $p \in \mathcal{P}$: $\varphi_{[0,\tau]}(M_p) \subset M$ and for all $q \neq p$ holds $\varphi_{[0,\tau]}(M_p) \cap M_q = \emptyset$.

Then $\{S_p\}$ is also a Morse decomposition for φ in M with the same Morse graph.

Proof. Case (A) follows from Corollary 3.3.2 and Lemma 5.3.5, case (B) from Lemmas 5.3.4 and 5.3.5. □

The following example shows that when (A) is not fulfilled, we need some kind of check that a Morse set S_p for φ_τ is indeed invariant under φ.

Example 5.3.7. Let $X = S^1 = \mathbb{R}/2\pi\mathbb{Z}$ be the phase space. Then $M := X$ is an isolating neighborhood for any flow because it is compact and open (in X). Consider the flow φ with $\varphi(\langle\theta\rangle, t) = \langle\theta + t\rangle$, where $\langle\theta\rangle = \theta + 2\pi\mathbb{Z}$.

For every point $\langle\theta\rangle \in S^1$, its limit sets are $\omega(\langle\theta\rangle, \varphi) = S^1$ and $\alpha(\langle\theta\rangle, \varphi) = S^1$. The sets $\emptyset$ and S^1 are the only subsets invariant for φ. Define the function

$$\begin{aligned} \mathcal{T}: \mathbb{R} &\to \mathbb{R}_{>0}, \\ \theta &\mapsto \sin\theta + 2\pi, \end{aligned}$$

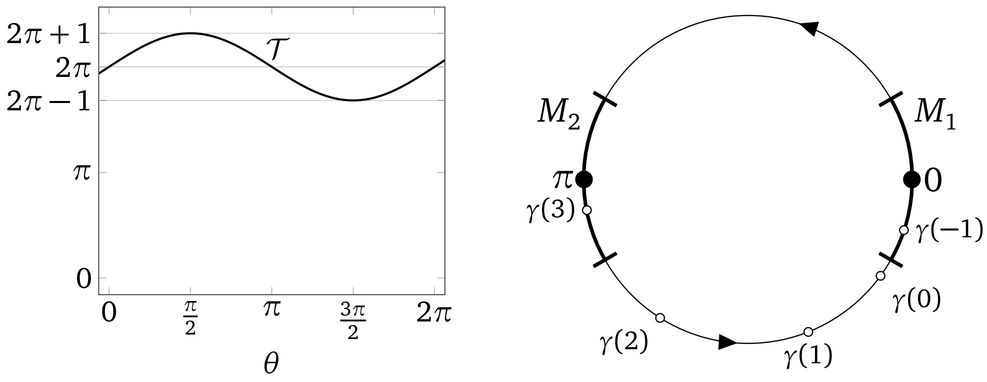
which has period 2π and therefore induces a continuous time step function

$$\tau: S^1 \ni \langle\theta\rangle \mapsto \mathcal{T}(\theta) \in \mathbb{R}_{>0}.$$

Figure 5.1 shows a plot of $\mathcal{T}$ and a trajectory of φ_τ.

By definition, $\mathcal{T}(0) = \mathcal{T}(\pi) = 2\pi$, $2\pi < \mathcal{T}((0,\pi)) < 2\pi+1$ and $2\pi-1 < \mathcal{T}((\pi, 2\pi)) < 2\pi$. Additionally $|\mathcal{T}'(\theta) = |\cos\theta| < 1$ for $\theta \neq 0, \pi$. These properties suffice to see that for any $0 < \varepsilon < \pi$ the subsets $M_1 = [-\varepsilon, +\varepsilon]$ and $M_2 = [\pi - \varepsilon, \pi + \varepsilon]$ are isolating neighborhoods for φ_τ with isolated invariant sets $S_1 = \{0\}$ and $S_2 = \{\pi\}$. Whenever $\theta \notin \{0, \pi\}$, then $\alpha(\langle\theta\rangle, \varphi_\tau) = S_1$ and $\omega(\langle\theta\rangle, \varphi_\tau) = S_2$. We do therefore get a Morse graph $1 \to 2$ for φ_τ, an attractor-repeller pair. But there is no attractor-repeller pair for φ in which both invariant sets are non-empty.

The example shows that the assumption that S_p be invariant also for the flow is necessary in Lemma 5.3.5. Such a function τ can always be constructed when the flow has a limit cycle L. One simply has to replace 2π by the period of the limit cycle and extend τ from L to the whole phase space X using Tietze's extension theorem. It would be interesting to find out if similar problems can also occur in absence of periodic orbits. But in any case: It is hard to exclude the existence of periodic orbits for a given flow.



(a) Time step function $\mathcal{T}$ on S^1 (b) Attractor-repeller pair for φ_τ

FIGURE 5.1: Example 5.3.7. The time step function on the left admits the attractor-repeller pair for φ_τ in the right figure with M_1 isolating the repelling fixed point 0 and M_2 isolating the attracting fixed point π. The arrows represent the direction of the flow φ. A part of a typical trajectory $\gamma\colon \mathbb{Z} \to S^1$ of φ_τ in the lower half of the circle is drawn.

5.4 A numerical example

We applied the algorithms and software described in [AKK$^+$09] and [BGH$^+$12] for finding Morse decompositions (Definition 5.3.3) as follows. The software constructs a restricted combinatorial enclosure $\mathcal{F}\colon \mathcal{M} \rightrightarrows \mathcal{M}$ of a given discrete dynamical system $f : \mathbb{R}^d \to \mathbb{R}^d$.

We call $\mathcal{N} \subset \mathcal{F}$ a *strongly connected path component* of $\mathcal{F}$ if $\mathcal{N} \neq \varnothing$ and, for any $Q, Q' \in \mathcal{N}$, there is a directed path from Q to Q', i.e., $Q' \in \mathcal{F}^n(Q)$ for some $n \geq 1$.

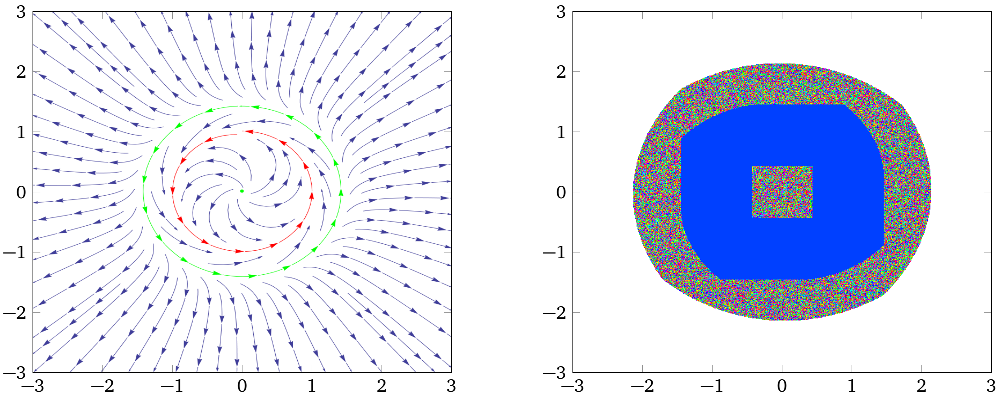
The algorithm constructs the strongly connected path components $\{\mathcal{M}(p) \mid p \in \mathcal{P}\}$ of the directed graph $\mathcal{F}$ and builds the directed graph $\mathrm{MG}(\mathcal{F})$ with vertices $\mathcal{P}$ and a directed edge from p to q if there are $G \in \mathcal{M}(p)$, $H \in \mathcal{M}(q)$ and a path from G to H in $\mathcal{F}$. This approach describes the dynamics of the discrete dynamical system f as follows.

Theorem 5.4.1 ([KMV05, Theorem 4.1])**.** *Suppose that $M = |\mathcal{M}|$ is an isolating neighborhood for f and $S = \mathrm{Inv}(M, f)$. Then:*

(i) Each set $|\mathcal{M}(p)|$ is an isolating neighborhood for f.

(ii) The isolated invariant sets $S_p = \mathrm{Inv}(|\mathcal{M}(p)|, f)$, $p \in \mathcal{P}$, form a Morse decomposition for f in the invariant set S with Morse graph $\mathrm{MG}(\mathcal{F})$.

In the rest of this section, we give an example flow φ and show that this algorithm yields a finer output when using $f(x) = \varphi(x, \tau(x))$ than when using $f(x) = \varphi(x, h)$. The justification that our outputs are indeed Morse decompositions for φ is formulated in Theorem 5.3.6.



(a) Streamplot for Equation (5.4) (b) The combinatorial Morse sets

FIGURE 5.2: Visualization of the flow from Equation (5.4) for $\mu = 2$ in $M = [-3,3] \times [-3,3]$ and outputs when using the fixed time step $h = 0.0006$ as described in Section 5.4.1. There are 57 673 spurious combinatorial Morse sets. Most of them consist of just one box. Calculating the Morse graph was impossible because of memory problems.

5.4.1 Fixed time step

The following ordinary differential equation is particularly challenging to analyze with a fixed time step h.

$$\begin{aligned} \dot{x}_1 &= v_1(x) = -x_2 + x_1(x_1^2 + x_2^2 - \mu)(x_1^2 + x_2^2 - 1) \\ \dot{x}_2 &= v_2(x) = x_1 + x_2(x_1^2 + x_2^2 - \mu)(x_1^2 + x_2^2 - 1) \end{aligned} \tag{5.4}$$

The equation has a fixed point $(0,0)$ and limit cycles with radius 1 and $\sqrt{\mu}$ around the fixed point. This can be seen by its representation in polar coordinates:

$$\begin{aligned} \dot{r} &= r(r^2 - \mu)(r^2 - 1) \\ \dot{\theta} &= 1 \end{aligned} \tag{5.5}$$

The norm of the vector field v increases quickly away from the origin because

$$\|v(x)\| = \sqrt{r^2(r^2-\mu)^2(r^2-1)^2 + r^2} \approx r^5 \text{ for large } \|x\|.$$

Far away from the origin, the solutions behave like the solutions of $\dot{r}(t) = r(t)^5$. This equation is solved by

$$r(t) = \frac{1}{\sqrt{2}\sqrt[4]{\left(\sqrt{2}\cdot r_0\right)^{-4} - t}},$$

where $r_0 = r(0)$. Note that the function r is only defined for $t < (\sqrt{2}\cdot r_0)^{-4} = (x_1^2 + x_2^2)^{-2}/4$ and that, strictly speaking, v induces only a local flow φ. For a point x with $\|x\| > \sqrt{\mu}$, the solution for the system (5.4) is defined on a maximal open interval I_x

with right bound $T_+(x) < \infty$. For $h > T_+(x)$, the value $\varphi_h(x)$ is undefined. Hence, also our integration algorithm fails when the input is a box containing x and we ask it to integrate until time $h > T_+(x)$. The fixed time step strategy can only be used with a parameter $h < \min\{\, T_+(x) \mid x \in M \,\}$. Therefore, when $M = [-3,3]^2$, we have to choose

$$h < T_+((3,3)) \approx (3^2+3^2)^{-2}/4 = 1/1296 \approx 0.0007716.$$

If h is chosen larger, each box $Q \in \mathcal{X}$ near the boundary of M is assigned an arrow in $\mathcal{F}_h$ to all of the other boxes because the algorithm fails to find an enclosure of $\varphi_h(Q)$ (which does not even exist). This would lead to a very large invariant part $\operatorname{Inv}(\mathcal{X}, \mathcal{F}_h)$ and should therefore be avoided. But when choosing h small enough, the directed graph $\mathcal{F}_h$ has a lot of small strongly connected components whose corresponding invariant set of φ is empty (so-called *spurious Morse sets*). They have to occur when the time step h is so small that $\varphi(Q,h) \cap Q \neq \emptyset$. But this happens easily near the origin where $\|v(x)\|$ is small.

The outputs for $\mu = 2$ are shown in Figure 5.2(b). The algorithm as described in [BGH+12] subdivided each dimension of $|\mathcal{M}|$ into 2^9 intervals of equal length. Each colored region is a combinatorial Morse set $\mathcal{M}(p), p \in \mathcal{P}$. The index set $\mathcal{P}$ had 57 675 elements. All but two of the combinatorial Morse sets found are empty.

It took 27 seconds to find the combinatorial Morse sets, but the Morse graph could not be computed because $\mathcal{P}$ was too large for the memory. Also subdividing each dimension into 2^{12} did give similarly bad outputs (after 2396 seconds).

5.4.2 Variable time step

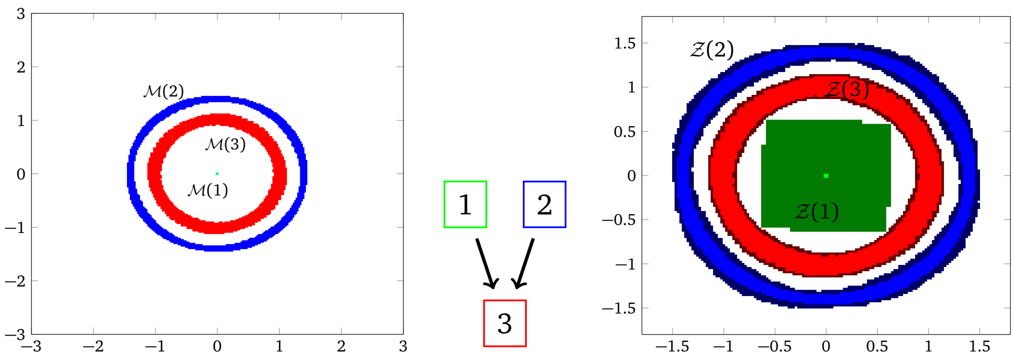
We use the following heuristic for the time step function τ. We describe it for a 2-dimensional example, but it is also applicable in higher dimensions. For each subdivision level, the norm $\|s\| = \sqrt{s_1^2+s_2^2}$ is the diagonal of each of the congruent boxes $Q \in \mathcal{M}$. Using parameters $D > 1$ and $\delta > 0$, we define the continuous function

$$\tau \colon X \to \mathbb{R}_{>0}, \quad x \mapsto \frac{D\|s\|}{\|v(x)\| + \delta}. \tag{5.6}$$

The idea is that the distance between x and $\varphi(x,\tau(x))$ should be around D box diagonals – using a first-order approximation $\varphi(x,\tau(x)) \approx x + \tau(x)v(x)$. The number δ ensures that τ is also defined when $v(x) = 0$. Since τ is usually not constant within a box Q, the value $\tau(Q)$ is an interval. We can find an enclosure of this interval by considering τ as a function on intervals and replacing x by the box Q when calculating in interval arithmetic. The software library [CAPD] is used to construct a restricted combinatorial enclosure $\mathcal{F}_\tau \colon \mathcal{M} \rightrightarrows \mathcal{M}$ of φ_τ. This means that (cf. Definition 3.4.1)

$$\varphi(Q,\tau(Q)) \cap M \subset \operatorname{int}_M |\mathcal{F}_\tau(Q)|.$$

We apply this strategy with $D = 4$ and $\delta = 0.1$ to Equation (5.4). Our implementation returned the finest Morse decomposition of the flow in $S = \operatorname{Inv}([-3,3]^2, \varphi) =$



(a) The combinatorial Morse sets (b) MG (c) Check of criterion (B).

FIGURE 5.3: Output for the same system as in Figure 5.2, but using time step function τ from Equation (5.6) as proposed in Section 5.4.2.

$\{x \mid \|x\| \leq \sqrt{\mu}\}$ (more precisely, the finest one which does not contain empty invariant sets). The output is shown in Figure 5.3. Since τ is not constant, the algorithm has to verify the criterion (B) of Theorem 5.3.6.

For each $p \in \mathcal{P}$, we need to construct an enclosure $\mathcal{Z}(p)$ of all the trajectories $\varphi(x, [0, \tau(x)])$, $x \in |\mathcal{M}(p)|$, such that $\mathcal{Z}(p) \subset \mathcal{M} \setminus \bigcup_{q \neq p} \mathcal{M}(q)$. Our algorithm uses [CAPD] to construct $\mathcal{Z}(p)$ such that

$$\bigcup_{Q \in \mathcal{M}(p)} \varphi(Q, [0, \max \tau(Q)]) \subset |\mathcal{Z}(p)|,$$

In Figure 5.3(c), each set $\mathcal{M}(p)$ is shown with a darker collar around it such that together they form $\mathcal{Z}(p)$. In our example, the algorithm successfully checked $\mathcal{Z}(p) \subset \mathcal{M}$ and $\mathcal{Z}(p) \cap \mathcal{M}(q) = \varnothing$ whenever $p \neq q$. The computed enclosure of $\varphi_{[0,\tau]}(|\mathcal{M}(1)|)$ is close to $|\mathcal{M}(3)|$, but the distance increases when finer resolutions are used.

Finding the Morse decomposition and the Morse graph took 29 seconds, with each dimension subdivided into 2^8 intervals. The additional checks for criterion (B) took only about a second.

Chapter 6

Conclusion

The theory we presented in Chapters 4 and 5 allows us to analyze flows via numerical methods developed for discrete dynamical systems. But in both cases, we could not avoid integrating numerically over a time interval $[0,h]$ or $[0,\max\tau(M_p)]$, respectively. This seems to be the most useful perspective: We analyze a time discretization if we can, but then we make sure that some extra condition is fulfilled.

In Chapter 4, the advantage compared to the existing methods is obvious: We do not need to integrate the Poincaré map for the time T and we can construct an index pair (N,L) without the need to manually construct an index pair for the flow φ, as is done in [MS10]. The proposed algorithm can run fully automatic. But some kind of intelligent strategy to choose the parameter h would still be useful. We would like to find the index pair with as few subdivisions as possible.

We would also like to apply these ideas to autonomous ODEs. This raises problems because not all points require the same time T until they return to the hyperplane on which the Poincaré map is defined. The theoretical generalization is not straightforward. Even if we had a theorem similar to Theorem 4.2.1, we would still need to choose a subdivision of the space. If it is possible to do so using a cubical complex as the combinatorial model, then it should be possible to reuse the algorithm presented here.

It is hard to estimate how helpful the varying time step strategy proposed in Chapter 5 is. It leads to a very intuitive heuristic for choosing a time step function. But it can also coarsen the combinatorial map $\mathcal{F}$ because $\varphi(Q,h)\subset\varphi(Q,\tau(Q))$ for any $h\in\tau(Q)$. This strategy seems only useful if the norm of the vector field varies a lot.

Another general problem when analyzing flows this way: The applications of the methods proposed here are still limited due to computational resources. Numerical integration of ordinary differential equations consumes a lot of memory and runtime compared to simply covering the image of a map f using interval arithmetic. Additionally, this integration requires heavy numerical machinery.

In addition, other methods taking into account properties of ODEs could be used for numerical preprocessing. For example, simply computing the value $v(Q)$ of the vector field can already tell us a lot about possible flow trajectories passing through a box

Q. For example, one could show for a lot of the spurious combinatorial Morse sets in Subsection 5.4.1 that they do not contain invariant subsets.

If one is interested in analyzing one specific equation, using as much prior knowledge as possible should be a good idea before using the very generic machinery we apply here.

But this generality is also an advantage: We do not need a differential equation describing our flow. Since a lot of processes in nature seem continuous, assuming an underlying flow is plausible. Then enclosures of the behavior of a system could be found by observation. The author of this thesis hopes that the ideas and algorithms presented here contribute to the understanding of systems which can be modelled in this way.

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