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Stratified-algebraic vector bundles of small rank

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Abstract. We investigate vector bundles on real algebraic varieties. Our goal is to construct rank 2 real and complex stratified-algebraic vector bundles with prescribed Stiefel–Whitney and Chern classes, respectively. We obtain a partial solution of this problem and present two applications.

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1. Introduction. In the recent joint paper with Kurdyka [14], we introduced and investigated stratified-algebraic vector bundles on real algebraic varieties. They occupy an intermediate position between algebraic and topological vector bundles. A hard problem is to find a characterization of topological vector bundles admitting a stratified-algebraic structure, cf. [12,15]. In the present paper we study rank 2 real and complex stratified-algebraic vector bundles with prescribed Stiefel–Whitney and Chern classes, respectively. As an application, we obtain a criterion for a topological complex vector bundle of rank 2 and a topological quaternionic line bundle to admit a stratified-algebraic structure. This paper fits into the new direction of research in real algebraic geometry developed by several authors [1,3,4,8–15].

We use the term *real algebraic variety* to mean a locally ringed space isomorphic to an algebraic subset of $\mathbb{R}^n$, for some n, endowed with the Zariski topology and the sheaf of real-valued regular functions (such an object is called an affine real algebraic variety in [2]). The class of real algebraic varieties is identical with the class of quasiprojective real algebraic varieties, cf. [2, Proposition 3.2.10, Theorem 3.4.4]. Morphisms of real algebraic varieties are called

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regular maps. Each real algebraic variety carries also the Euclidean topology, which is induced by the usual metric on $\mathbb{R}$. Unless explicitly stated otherwise, all topological notions relating to real algebraic varieties refer to the Euclidean topology.

Let X be a real algebraic variety. By a *stratification* of X we mean a finite collection $\mathcal{S}$ of pairwise disjoint Zariski locally closed subvarieties whose union is X. Each subvariety in $\mathcal{S}$ is called a stratum.

Let $\mathbb{F}$ stand for $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ (the quaternions). When convenient, $\mathbb{F}$ will be identified with $\mathbb{R}^{d(\mathbb{F})}$, where $d(\mathbb{F}) = \dim_{\mathbb{R}} \mathbb{F}$.

For any nonnegative integer n, let $\varepsilon^n_X(\mathbb{F})$ denote the standard trivial $\mathbb{F}$-vector bundle on X with total space $X \times \mathbb{F}^n$, where $X \times \mathbb{F}^n$ is regarded as a real algebraic variety.

An algebraic $\mathbb{F}$-vector bundle on X is an algebraic $\mathbb{F}$-vector subbundle of $\varepsilon^n_X(\mathbb{F})$ for some n (cf. [2, Chapters 12 and 13] for various characterizations of algebraic $\mathbb{F}$-vector bundles).

We now recall the fundamental notion introduced in [14]. A *stratified-algebraic* $\mathbb{F}$*-vector bundle on* X is a topological $\mathbb{F}$-vector subbundle ξ of $\varepsilon^n_X(\mathbb{F})$, for some n, such that for some stratification $\mathcal{S}$ of X, the restriction $\xi|_S$ of ξ to each stratum S in $\mathcal{S}$ is an algebraic $\mathbb{F}$-vector subbundle of $\varepsilon^n_S(\mathbb{F})$.

A topological $\mathbb{F}$-vector bundle on X is said to *admit a stratified-algebraic structure* if it is isomorphic to a stratified-algebraic $\mathbb{F}$-vector bundle on X.

As usual, we denote the ith Stiefel–Whitney class of an $\mathbb{R}$-vector bundle η by $w_i(\eta)$.

Problem 1.1. Given a positive integer k, characterize the cohomology classes u_i in $H^i(X;\mathbb{Z}/2)$ for which there exists a rank k stratified-algebraic $\mathbb{R}$-vector bundle ξ on X with $w_i(\xi) = u_i$ for $1 \le i \le k$.

For a $\mathbb{C}$-vector bundle η, let $c_i(\eta)$ denote its ith Chern class.

Problem 1.2. Given a positive integer k, characterize the cohomology classes u_i in $H^{2i}(X;\mathbb{Z})$ for which there exists a rank k stratified-algebraic $\mathbb{C}$-vector bundle ξ on X with $c_i(\xi) = u_i$ for $1 \le i \le k$.

In [14] we solved Problems 1.1 and 1.2 for $k = 1$. Presently, we address these problems for $k = 2$ (Theorems 2.3 and 2.6). The case $k \ge 3$ seems to be out of reach. Our partial solution of Problem 1.2 leads to a criterion for a topological $\mathbb{C}$-vector bundle of rank 2 and a topological $\mathbb{H}$-line bundle to admit a stratified-algebraic structure (Theorems 2.7 and 2.8).

It would be interesting to obtain at least a partial solution of the counterparts of Problems 1.1 and 1.2 for $k = 2$, where stratified-algebraic vector bundles are replaced by algebraic vector bundles. However, this would require some new approach, different from the methods used here.

2. Constructions of rank 2 vector bundles. For the convenience of the reader, we first review some classical topological constructions.

Convention Working with smooth (of class $\mathcal{C}^\infty$) manifolds, we always assume that submanifolds are closed subsets of the ambient manifold.

Let X be a smooth manifold, and let M be a smooth codimension k submanifold of X. Suppose that the normal bundle to M in X is oriented, and denote by τ_M^X the Thom class of M in the cohomology group $H^k(X, X \setminus M; \mathbb{Z})$, cf. [16, p. 118]. The image of τ_M^X by the restriction homomorphism $H^k(X, X \setminus M; \mathbb{Z}) \to H^k(X; \mathbb{Z})$, induced by the inclusion map $X \hookrightarrow (X, X \setminus M)$, will be denoted by $[\![M]\!]^X$ and called the cohomology class represented by M. If X is compact and oriented, and M is endowed with the compatible orientation, then $[\![M]\!]^X$ is up to sign Poincaré dual to the homology class in $H_*(X; \mathbb{Z})$ represented by M, cf. [16, p. 136]. Similarly, without any orientability assumption, we define the cohomology class $[M]^X$ in $H^k(X; \mathbb{Z}/2)$ represented by M. The cohomology class $[M]^X$ is Poincaré dual to the homology class in $H_*(X; \mathbb{Z}/2)$ represented by M.

Let Y be a smooth manifold, and let N be a smooth submanifold of Y. Let $f\colon X \to Y$ be a smooth map transverse to N. If the normal bundle to N in Y is oriented and the normal bundle to the smooth submanifold $M := f^{-1}(N)$ of X is endowed with the orientation induced by f, then $\tau_M^X = f^*(\tau_N^Y)$, where f is regarded as a map from $(X, X \setminus M)$ into $(Y, Y \setminus N)$ (this follows from [5, p. 117, Theorem 6.7]). In particular,

$$[\![M]\!]^X = f^*([\![N]\!]^Y).$$

Without any orientability assumption,

$$[M]^X = f^*([N]^Y).$$

Let ξ be a rank k smooth $\mathbb{R}$-vector bundle on X. A smooth section $s\colon X \to \xi$ is said to be *transverse regular* if it is transverse to the zero section of ξ. In that case, the zero locus of s,

$$Z(s) := \{x \in X \mid s(x) = 0\},$$

is a smooth codimension k submanifold of X. We identify the normal bundle to $Z(s)$ in X with $\xi|_{Z(s)}$ via the isomorphism induced by s. In particular, if the vector bundle ξ is oriented, then so is the normal bundle to $Z(s)$ in X and

$$e(\xi) = [\![Z(s)]\!]^X,$$

where $e(\xi)$ stands for the Euler class of ξ. Indeed, let E be the total space of ξ and $p\colon E \to X$ the bundle projection. Identify X with the image of the zero section of ξ. The section s is transverse to X and $Z(s) = s^{-1}(X)$. Consequently, $[\![Z(s)]\!]^X = s^*([\![X]\!]^E)$. Hence

$$p^*([\![Z(s)]\!]^X) = p^*(s^*([\![X]\!]^E)) = (s \circ p)^*([\![X]\!]^E) = [\![X]\!]^E,$$

where the last equality holds since $s \circ p\colon E \to E$ is homotopic to the identity map. On the other hand, $p^*(e(\xi)) = [\![X]\!]^E$, cf. [16, p. 98]. It follows that $e(\xi) = [\![Z(s)]\!]^X$ since p^* is an isomorphism. Similarly, if ξ is not necessarily orientable, we get

$$w_k(\xi) = [Z(s)]^X.$$

Recall that on a smooth manifold each topological vector bundle is isomorphic to a smooth vector bundle, which is uniquely determined up to smooth isomorphism, cf. [5, p. 101].

For any rank k $\mathbb{F}$-vector bundle η, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, let $\det \eta$ denote the kth exterior power of η. Thus $\det \eta$ is an $\mathbb{F}$-line bundle. Furthermore,

$$\begin{aligned} w_1(\det \eta) &= w_1(\eta) \quad \text{if } \mathbb{F} = \mathbb{R}, \\ c_1(\det \eta) &= c_1(\eta) \quad \text{if } \mathbb{F} = \mathbb{C}, \end{aligned}$$

cf. [6, p. 64].

Given a subset A of X and a cohomology class u in $H^i(X; G)$, where $G = \mathbb{Z}/2$ or $G = \mathbb{Z}$, we denote by $u|_A$ the image of u by the homomorphism $H^i(X; G) \to H^i(A; G)$ induced by the inclusion map $A \hookrightarrow X$.

Proposition 2.1. *Let X be a smooth manifold and let θ be a rank 2 topological $\mathbb{R}$-vector bundle on X. Then there exist smooth submanifolds M_i of X such that* $\operatorname{codim}_X M_i = i$,

$$w_1(\det \nu) = [M_1]^X|_{M_2} \quad \textit{in } H^1(M_2; \mathbb{Z}/2),$$

where ν is the normal bundle to M_2 in X, and

$$w_i(\theta) = [M_i]^X \quad \textit{for } i = 1, 2.$$

Proof. We may assume without loss of generality that the vector bundle θ is smooth. Let

$$s_1 \colon X \to \det \theta, \quad s_2 \colon X \to \theta$$

be smooth transverse regular sections. Setting $M_i = Z(s_i)$, we get

$$w_1(\theta) = w_1(\det \theta) = [M_1]^X, \quad w_2(\theta) = [M_2]^X.$$

Furthermore, since we identify ν with $\theta|_{M_2}$, we obtain $\det \nu = (\det \theta)|_{M_2}$ and

$$w_1(\det \nu) = w_1((\det \theta)|_{M_2}) = [M_1]^X|_{M_2},$$

as required. □

Proposition 2.1 provides motivation for the following result.

Proposition 2.2. *Let X be a smooth manifold and let M_i be a smooth submanifold of X such that* $\operatorname{codim}_X M_i = i$ *for $i = 1, 2$, and*

$$w_1(\det \nu) = [M_1]^X|_{M_2} \quad \textit{in } H^1(M_2; \mathbb{Z}/2),$$

where ν is the normal bundle to M_2 in X. Then there exists a rank 2 smooth $\mathbb{R}$-vector bundle θ on X with

$$w_i(\theta) = [M_i]^X \quad \textit{for } i = 1, 2.$$

Furthermore, the vector bundle θ can be chosen so that there exist smooth transverse regular sections

$$s_1 \colon X \to \det \theta, \quad s_2 \colon X \to \theta$$

satisfying $Z(s_i) = M_i$ for $i = 1, 2$.

Proof. Let λ be a smooth $\mathbb{R}$-line bundle on X with

$$w_1(\lambda) = [M_1]^X. \tag{1}$$

It is well known that there exists a smooth transverse regular section $u\colon X \to \lambda$ satisfying

$$Z(u) = M_1. \tag{2}$$

Let $\rho\colon T \to M_2$ be a tubular neighborhood of M_2 in X. There exists a smooth transverse regular section $\sigma\colon T \to \rho^*\nu$ such that $Z(\sigma) = M_2$. In particular,

$$(\rho^*\nu)|_{T\setminus M_2} = \mu \oplus \varepsilon_\sigma, \tag{3}$$

where ε_σ is the trivial $\mathbb{R}$-line subbundle of $(\rho^*\nu)|_{T\setminus M_2}$ generated by $\sigma|_{T\setminus M_2}$, and μ is a smooth $\mathbb{R}$-line bundle on $T \setminus M_2$.

We assert that the $\mathbb{R}$-line bundles μ and $\lambda|_{T\setminus M_2}$ are isomorphic. Indeed, we have

$$w_1(\nu) = w_1(\det \nu) = [M_1]^X|_{M_2} = w_1(\lambda)|_{M_2}.$$

Consequently,

$$w_1(\rho^*\nu) = w_1(\lambda|_T),$$

the map $\rho\colon T \to M_2$ being a homotopy inverse of the inclusion map $M_2 \hookrightarrow T$. Hence

$$w_1((\rho^*\nu)|_{T\setminus M_2}) = w_1(\lambda|_{T\setminus M_2}). \tag{4}$$

On the other hand, in view of (3),

$$w_1(\mu) = w_1((\rho^*\nu)|_{T\setminus M_2}). \tag{5}$$

By combining (4) and (5), we get

$$w_1(\mu) = w_1(\lambda|_{T\setminus M_2}),$$

which implies the assertion, cf. [7, p. 234].

Let ε be the standard trivial $\mathbb{R}$-line bundle on X with total space $X \times \mathbb{R}$ and let $\tau\colon X \to \lambda \oplus \varepsilon$ be the smooth section defined by $\tau(x) = (0, (x, 1))$ for all x in X. By the assertion, there exists a smooth isomorphism

$$\varphi\colon (\rho^*\nu)|_{T\setminus M_2} \to (\lambda \oplus \varepsilon)|_{T\setminus M_2}$$

such that $\varphi \circ \sigma = \tau$ on $T \setminus M_2$.

Let θ be the smooth $\mathbb{R}$-vector bundle on X obtained by gluing $\rho^*\nu$ and $(\lambda \oplus \varepsilon)|_{X\setminus M_2}$ over $T \setminus M_2$ using φ. Similarly, let $s_2\colon X \to \theta$ be the smooth section obtained by gluing σ and $\tau|_{X\setminus M_2}$ over $T\setminus M_2$ using φ. By construction, θ is a rank 2 smooth $\mathbb{R}$-vector bundle on X, the section s_2 is transverse regular, and

$$Z(s_2) = M_2. \tag{6}$$

Consequently,

$$w_2(\theta) = [M_2]^X. \tag{7}$$

In view of (1), (2), (6), (7), it remains to prove that the $\mathbb{R}$-line bundles $\det \theta$ and λ are isomorphic. To this end it suffices to show the equality

$$w_1(\det \theta) = w_1(\lambda), \tag{8}$$

cf. [7, p. 234]. Note that

$$w_1((\det\theta)|_{X\setminus M_2}) = w_1(\theta|_{X\setminus M_2}) = w_1((\lambda\oplus\varepsilon)|_{X\setminus M_2}) = w_1(\lambda|_{X\setminus M_2})$$

and hence

$$w_1(\det\theta)|_{X\setminus M_2} = w_1(\lambda)|_{X\setminus M_2}.$$

As a portion of the long exact cohomology sequence of the pair $(X, X\setminus M_2)$, we get

$$H^1(X, X\setminus M_2;\mathbb{Z}/2) \to H^1(X;\mathbb{Z}/2) \xrightarrow{e^*} H^1(X\setminus M_2;\mathbb{Z}/2),$$

where $e\colon X\setminus M_2 \hookrightarrow X$ is the inclusion map. Since $H^1(X, X\setminus M_2;\mathbb{Z}/2) = 0$ (cf. [16, pp. 106, 117], it follows that e^* is a monomorphism. Hence (8) holds, as required. □

If X is a smooth manifold, by combining Propositions 2.1 and 2.2, we obtain a characterization of the cohomology classes u_i in $H^i(X;\mathbb{Z}/2)$ for which there exists a rank 2 topological $\mathbb{R}$-vector bundle θ on X with $w_i(\theta) = u_i$ for $i = 1, 2$. Our partial solution of Problem 1.1 is of a similar nature. We first recall a well-known phenomenon specific to real algebraic geometry. Namely, if X is a nonsingular real algebraic variety, it can happen that a nonsingular Zariski locally closed subvariety of X is Euclidean closed but not Zariski closed.

Theorem 2.3. *Let X be a compact nonsingular real algebraic variety, and let M_i be a smooth submanifold of X such that* $\operatorname{codim}_X M_i = i$ *for $i = 1, 2$, and*

$$w_1(\det\nu) = [M_1]^X|_{M_2} \quad \textit{in } H^1(M_2;\mathbb{Z}/2),$$

where ν is the normal bundle to M_2 in X. Assume that M_i is a nonsingular Zariski locally closed subvariety of X for $i = 1, 2$. Then there exists a rank 2 stratified-algebraic $\mathbb{R}$-vector bundle ξ on X with

$$w_i(\xi) = [M]_i^X \quad \textit{for } i = 1, 2.$$

Proof. It suffices to make use of Proposition 2.2 and [14, Theorem 1.9]. □

Let $\mathbb{K}$ be a subfield of $\mathbb{F}$, where $\mathbb{K}$ (as $\mathbb{F}$) stands for $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Any $\mathbb{F}$-vector bundle η can be regarded as a $\mathbb{K}$-vector bundle, which is indicated by $\eta_{\mathbb{K}}$. In particular, $\eta_{\mathbb{K}} = \eta$ if $\mathbb{K} = \mathbb{F}$.

Suppose now that ξ is a rank k smooth $\mathbb{C}$-vector bundle on a smooth manifold X. Recall that

$$c_k(\xi) = e(\xi_{\mathbb{R}}),$$

where $\xi_{\mathbb{R}}$ is endowed with the orientation induced by the complex structure, cf. [16, p. 158]. If $s\colon X \to \xi$ is a smooth transverse regular section, then we regard the normal bundle to $Z(s)$ in X as a $\mathbb{C}$-vector bundle, identifying it with $\xi|_{Z(s)}$. In particular, the normal bundle to $Z(s)$ in X is canonically oriented, the cohomology class $[\![Z(s)]\!]^X$ in $H^{2k}(X;\mathbb{Z})$ is defined, and

$$c_k(\xi) = [\![Z(s)]\!]^X.$$

If M is a smooth codimension $2k$ submanifold of X and the normal bundle ν to M in X is endowed with a complex structure, then the $\mathbb{R}$-vector bundle

$\nu_{\mathbb{R}}$ is canonically oriented and the cohomology class $[\![M]\!]^X$ in $H^{2k}(X;\mathbb{Z})$ is defined.

For $\mathbb{C}$-vector bundles, Proposition 2.1 takes the following form.

Proposition 2.4. *Let X be a smooth manifold, and let θ be a rank 2 topological $\mathbb{C}$-vector bundle on X. Then there exist smooth submanifolds M_i of X such that $\operatorname{codim}_X M_i = 2i$, the normal bundle ν_i to M_i in X is endowed with a complex structure,*

$$c_1(\det \nu_2) = [\![M_1]\!]^X|_{M_2} \quad \textit{in } H^2(M_2;\mathbb{Z}),$$

and $c_i(\theta) = [\![M_i]\!]^X$ for $i = 1, 2$.

Proof. We may assume without loss of generality that the vector bundle θ is smooth. Let

$$s_1\colon X \to \det \theta, \quad s_2\colon X \to \theta$$

be smooth transverse regular sections. Setting $M_i = Z(s_i)$, we get

$$c_1(\theta) = c_1(\det \theta) = [\![M_1]\!]^X, \quad c_2(\theta) = [\![M_2]\!]^X.$$

Furthermore, since we identify ν_2 with $\theta|_{M_2}$, we obtain $\det \nu_2 = (\det \theta)|_{M_2}$ and

$$c_1(\det \nu_2) = c_1((\det \theta)|_{M_2}) = [\![M_1]\!]^X|_{M_2},$$

as required. □

The following is a counterpart of Proposition 2.2 for $\mathbb{C}$-vector bundles.

Proposition 2.5. *Let X be a smooth manifold, and let M_i be a smooth submanifold of X such that $\operatorname{codim}_X M_i = 2i$, the normal bundle ν_i to M_i in X is endowed with a complex structure for $i = 1, 2$, and*

$$c_1(\det \nu_2) = [\![M_1]\!]^X|_{M_2} \quad \textit{in } H^2(M_2;\mathbb{Z}).$$

Then there exists a rank 2 smooth $\mathbb{C}$-vector bundle θ on X with

$$c_i(\theta) = [\![M_i]\!]^X \quad \textit{for } i = 1, 2.$$

Furthermore, the vector bundle θ can be chosen so that there exist smooth transverse regular sections

$$s_1\colon X \to \det \theta, \quad s_2\colon X \to \theta$$

satisfying $Z(s_i) = M_i$ for $i = 1, 2$.

Proof. The argument is analogous to that in the proof of Proposition 2.2. Let λ be a smooth $\mathbb{C}$-line bundle on X with

$$c_1(\lambda) = [\![M_1]\!]^X. \tag{1}$$

By [14, Lemma 8.20], there exists a smooth transverse regular section $u\colon X \to \lambda$ satisfying

$$Z(u) = M_1. \tag{2}$$

Let $\rho\colon T \to M_2$ be a tubular neighborhood of M_2 in X. There exists a smooth transverse regular section $\sigma\colon T \to \rho^*\nu_2$ such that $Z(\sigma) = M_2$. In particular,

$$(\rho^*\nu_2)|_{T\setminus M_2} = \mu \oplus \varepsilon_\sigma, \tag{3}$$

where ε_σ is the trivial $\mathbb{C}$-line subbundle of $(\rho^*\nu_2)|_{T\setminus M_2}$ generated by $\sigma|_{T\setminus M_2}$, and μ is a smooth $\mathbb{C}$-line bundle on $T \setminus M_2$.

We assert that the $\mathbb{C}$-line bundles μ and $\lambda_{T\setminus M_2}$ are isomorphic. Indeed, we have

$$c_1(\nu_2) = c_1(\det \nu_2) = [\![M_1]\!]^X|_{M_2} = c_1(\lambda)|_{M_2}.$$

Consequently,

$$c_1(\rho^*\nu_2) = c_1(\lambda|_T),$$

the map $\rho\colon T \to M_2$ being a homotopy inverse of the inclusion map $M_2 \hookrightarrow T$. Hence

$$c_1((\rho^*\nu_2)|_{T\setminus M_2}) = c_1(\lambda|_{T\setminus M_2}). \tag{4}$$

On the other hand, in view of (3),

$$c_1(\mu) = c_1((\rho^*\nu_2)|_{T\setminus M_2}). \tag{5}$$

By combining (4) and (5), we get

$$c_1(\mu) = c_1(\lambda|_{T\setminus M_2}),$$

which implies the assertion, cf. [7, p. 234].

Let ε be the standard trivial $\mathbb{C}$-line bundle on X with total space $X \times \mathbb{C}$, and let $\tau\colon X \to \lambda \oplus \varepsilon$ be the smooth section defined by $\tau(x) = (0, (x, 1))$ for all x in X. By the assertion, there exists a smooth isomorphism

$$\varphi\colon (\rho^*\nu_2)|_{T\setminus M_2} \to (\lambda \oplus \varepsilon)|_{T\setminus M_2}$$

such that $\varphi \circ \sigma = \tau$ on $T \setminus M_2$.

Let θ be the smooth $\mathbb{C}$-vector bundle on X obtained by gluing $\rho^*\nu_2$ and $(\lambda \oplus \varepsilon)|_{X\setminus M_2}$ over $T \setminus M_2$ using φ. Similarly, let $s_2\colon X \to \theta$ be the smooth section obtained by gluing σ and $\tau|_{X\setminus M_2}$ over $T\setminus M_2$ using φ. By construction, θ is a rank 2 smooth $\mathbb{C}$-vector bundle on X, the section s_2 is transverse regular, and

$$Z(s_2) = M_2. \tag{6}$$

Consequently,

$$c_2(\theta) = [\![M_2]\!]^X. \tag{7}$$

In view of (1), (2), (6), (7), it remains to prove that the $\mathbb{C}$-line bundles $\det\theta$ and λ are isomorphic. To this end, it suffices to show the equality

$$c_1(\det\theta) = c_1(\lambda), \tag{8}$$

cf. [7, p. 234]. Note that

$$c_1((\det\theta)|_{X\setminus M_2}) = c_1(\theta|_{X\setminus M_2}) = c_1((\lambda \oplus \varepsilon)|_{X\setminus M_2}) = c_1(\lambda|_{X\setminus M_2})$$

and hence

$$c_1(\det\theta)|_{X\setminus M_2} = c_1(\lambda)|_{X\setminus M_2}.$$

As a portion of the long exact cohomology sequence of the pair $(X, X \setminus M_2)$, we get

$$H^2(X, X \setminus M_2; \mathbb{Z}) \to H^2(X; \mathbb{Z}) \xrightarrow{e^*} H^2(X \setminus M_2; \mathbb{Z}),$$

where $e\colon X \setminus M_2 \hookrightarrow X$ is the inclusion map. Since $H^2(X, X \setminus M_2; \mathbb{Z}) = 0$ (cf. [16, pp. 110, 117]), it follows that e^* is a monomorphism. Hence (8) holds, as required. □

If X is a smooth manifold, then Propositions 2.4 and 2.5 yield a characterization of the cohomology classes u_i on $H^{2i}(X;\mathbb{Z})$ for which there exists a rank 2 topological $\mathbb{C}$-vector bundle θ on X with $c_i(\theta) = u_i$ for $i = 1, 2$.

Our partial solution of Problem 1.2 is the following.

Theorem 2.6. *Let X be a compact nonsingular real algebraic variety, and let M_i be a smooth submanifold of X such that $\operatorname{codim}_X M_i = 2i$, the normal bundle ν_i to M_i in X is endowed with a complex structure for $i = 1, 2$, and*

$$c_1(\det \nu_2) = [\![M_1]\!]^X|_{M_2} \quad \text{in } H^2(M_2;\mathbb{Z}).$$

Assume that M_i is a nonsingular Zariski locally closed subvariety of X for $i = 1, 2$. Then there exists a rank 2 stratified-algebraic $\mathbb{C}$-vector bundle ξ on X with

$$c_i(\xi) = [\![M_i]\!]^X \quad \text{for } i = 1, 2.$$

Proof. It suffices to make use of Proposition 2.5 and [14, Theorem 1.9]. □

We conclude this paper by giving two applications of Theorem 2.6.

Theorem 2.7. *Let X be a compact nonsingular real algebraic variety, and let θ be a rank 2 topological $\mathbb{C}$-vector bundle on X. Let M_i be smooth submanifolds of X such that $\operatorname{codim}_X M_i = 2i$, the normal bundle ν_i to M_i in X is endowed with a complex structure,*

$$c_1(\det \nu_2) = [\![M_1]\!]^X|_{M_2} \quad \text{in } H^2(M_2;\mathbb{Z}),$$

and $c_i(\theta) = [\![M_i]\!]^X$ for $i = 1, 2$. Assume that M_i is a nonsingular Zariski locally closed subvariety of X for $i = 1, 2$. If for each integer $k \geq 3$ the only torsion in the cohomology group $H^{2k}(X;\mathbb{Z})$ is relatively prime to $(k-1)!$, then θ admits a stratified-algebraic structure.

Proof. According to Theorem 2.6, there exists a rank 2 stratified-algebraic $\mathbb{C}$-vector bundle ξ on X with $c_i(\xi) = c_i(\theta)$ for $i = 1, 2$. Consequently,

$$c_k(\xi) = c_k(\theta) \quad \text{for all } k \geq 0.$$

Hence, if the condition on the torsion in the cohomology groups $H^{2k}(X;\mathbb{Z})$ is satisfied, then the $\mathbb{C}$-vector bundles ξ and θ are stably equivalent, cf. [17, Theorem 3.2]. This implies, in view of [14, Corollary 3.14], that θ admits a stratified-algebraic structure. □

The second application concerns $\mathbb{H}$-line bundles.

Theorem 2.8. *Let X be a compact nonsingular real algebraic variety, and let λ be a topological $\mathbb{H}$-line bundle on X. Let M be a smooth submanifold of X such that $\operatorname{codim}_X M = 4$, the normal bundle ν to M in X is endowed with a complex structure, $c_1(\det \nu) = 0$ in $H^2(M;\mathbb{Z})$, and $c_2(\lambda_{\mathbb{C}}) = [\![M]\!]^X$. Assume that M is a nonsingular Zariski locally closed subvariety of X. If for each* *integer $k \geq 3$ the only torsion in the cohomology group $H^{2k}(X;\mathbb{Z})$ is relatively prime to $(k-1)!$, then λ admits a stratified-algebraic structure.*

Proof. Suppose that the condition on the torsion in the cohomology groups $H^{2k}(X;\mathbb{Z})$ is satisfied. By Theorem 2.7, the $\mathbb{C}$-vector bundle $\lambda_{\mathbb{C}}$ admits a stratified-algebraic structure. Hence, in view of [14, Theorem 1.7], the $\mathbb{H}$-line bundle λ admits a stratified-algebraic structure. $\square$

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