Maximally Entangled Multipartite States: A Brief Survey

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Abstract. The problem of identifying maximally entangled quantum states of a composite quantum systems is analyzed. We review some states of multipartite systems distinguished with respect to certain measures of quantum entanglement. Numerical results obtained for 4-qubit pure states illustrate the fact that the notion of maximally entangled state depends on the measure used.

1. Introduction

Multipartite entangled states represent a potential tool in several tasks of quantum information, quantum teleportation, error correction codes, among others [1, 2]. Therefore, the study of entanglement properties of such states is essential and recently has been a field of intense research [1, 3–7]. In the case of a two qubit systems the famous Bell state $|\Psi_2^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is *maximally entangled*, as the reduced state is maximally mixed, $\mathrm{Tr}_A|\Psi_2^+\rangle\langle\Psi_2^+| = \mathbb{1}_2/2$. The same property is typical to the entire orbit of locally equivalent states, $(U_A \otimes U_B)|\Psi_2^+\rangle$, with $U_A, U_B \in U(2)$. In a similar way, for any bipartite quantum system consisting of two subsystems with N levels each there exists a generalized Bell state,

$$|\Psi_N^+\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |i\rangle_A \otimes |i\rangle_B, \tag{1}$$

such that the reduced density matrix is maximally mixed, so its entanglement entropy is maximal, $E(|\Psi_N^+\rangle) = \log N$. It is also well justified to call it the *maximally entangled* state, as also other measures of bipartite entanglement achieve its maximum for this very state.

The situation changes for systems consisting of three or more subsystems. There exist several measures of multipartite entanglement and it happens that a given state is most entangled with respect to one measure, but other entanglement measure achieves its maximum for some other state. For instance, among all three-qubit pure states the genuine three-party entanglement measured by three tangle [8] is largest for the state $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$

while the two-tangle τ_2 and the persistence of entanglement [9] is has its maximum for the state $|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$.

The aim of this work is to present a brief review of multipartite entanglement and to discuss possible options of selecting distinguished states which maximize some entanglement measures. The problem of identifying multipartite maximally entangled states can be approached from different perspectives. For instance, we can analyze the entanglement entropy of a multipartite state decomposition in a product basis minimized over all local unitary transformations. Another way to describe entanglement in a composite systems is considering some of its bipartitions and then take the average of certain entanglement measure. This last notion is related to concepts like maximally multipartite-entangled state (MMES) [10], k-uniform states [2] and absolutely maximally entangled (AME) states [6, 7].

This contribution is organized as follows, in Section 2 the minimal decomposition entropy is introduced as a multipartite entanglement measure. Besides, a numerical search for three-qubit maximally entangled states with respect this measure is accomplished. Section 3 is devoted to describe entanglement properties by analyzing some bipartite reductions of a composite system. The notion of absolutely maximally entangled states is given in Section 4. Its connection with orthogonal arrays, multiunitary matrices, mutually orthogonal Latin squares is briefly discussed as well. In Section 5 a list of some maximally entangled states and their minimal decomposition entropy is presented. Besides, we discuss some numerical results obtained for four-qubit random pure states and compare several entropy-based measures of entanglement. Finally, some conclusions are given in Section 6.

2. Multipartite quantum states and its decomposition entropy

Consider the general case of a pure state $|\psi\rangle$ of a quantum system composed of K subsystems with N levels each. Let us write it in a product basis

$$|\psi\rangle = \sum_{i_1=1}^{N} \cdots \sum_{i_K=1}^{N} C_{i_1,i_2,\ldots,i_K} \, |i_1\rangle \otimes |i_2\rangle \otimes \cdots \otimes |i_K\rangle \tag{2}$$

and discuss, how to quantify its entanglement? The tensor C is described by N^K complex entries normalized as usual, $\sum_{i_1=1}^{N} \cdots \sum_{i_n=1}^{N} |C_{i_1,i_2,\ldots,i_K}|^2 = 1$. If all of them but one are equal to zero, the state is separable. Of course the converse does not hold, since any separable state written in a generic product basis has several non-zero components. Thus it is natural to use local unitary transformations, $U_{\text{loc}} = V_1 \otimes \cdots \otimes V_K \in U(N)^{\otimes K}$, and to look for a distinguished product basis [11].

It will be convenient to use a multi-index $\mu = (i_1, i_2, \ldots, i_K)$, with $\mu \in \{1, \ldots, N^K\}$, and to introduce a normalized probability vector, $p_\mu = |C_\mu|^2 = |C_{i_1,i_2,\ldots,i_K}|^2$ of length N^K. As usual it can be characterized by the Rényi entropy

$$S_q(\vec{p}) = \frac{1}{1-q} \log \left(\sum_{\mu=1}^{N} p_\mu^q \right). \tag{3}$$

For $q \to 1$ this quantity reduces to the standard Shannon entropy

$$S_1(p) = -\sum_{\mu=1}^{N} p_\mu \log p_\mu, \tag{4}$$

which in the context of the decomposition of a state $|\psi\rangle$, is called the *Ingarden–Urbanik* entropy [12, 13] and written $S^{\text{IU}}(|\psi\rangle) = S\big(p(|\psi\rangle)\big)$. From this point on log stands for the natural logarithm.

Making use of the generalized entropies S_q for any multipartite state $|\psi\rangle \in H_N^{\otimes K}$ we define the minimal *Rényi–Ingarden–Urbanik* entropy [5],

$$S_q^{\rm RIU}(\psi) \;=\; \min_{U_{\rm loc}} S_q\left[p(U_{\rm loc}|\psi\rangle)\right], \tag{5}$$

where the minimum is taken over all local unitaries $U_{\rm loc}$. Such an approach with $q = 1$ was applied by Bravyi [14], who showed that the *minimal decomposition entropy* $S_1^{\rm RIU}(\psi)$ characterizes entanglement of any pure state $|\psi\rangle$ and determines the minimal information gained by performing projective measurements in each local basis. For any separable state one can find a product basis such that the probability vector p_μ is pure, so one has $S_q^{\rm RIU}(\psi_{sep}) = 0$ for any $q \geq 0$. In the case of a bipartite state the optimal basis is given by the singular value decomposition of the matrix C, so the familiar Schmidt vector λ determines the Rényi entropy of entanglement, $S_q^{\rm RIU}(\psi) = S_q(\lambda)$. However, for $K \geq 3$ subsystems one works with a K-index tensor (2) and in general there are no analytic techniques to find the minimum (5), so one has to rely on numerical methods [5].

The minimal decomposition entropy $S_q^{\rm RIU}$ is by construction constant along the orbit of locally unitary states,

$$|\psi\rangle \underset{LU}{\longleftrightarrow} |\phi\rangle \quad \Rightarrow \quad S_q^{\rm RIU}(|\psi\rangle) = S_q^{\rm RIU}(|\phi\rangle) \tag{6}$$

for any $q \geq 0$. The problem of determining whether two multipartite pure states belong to the same LU orbit was solved for qubits by Kraus [15] and later generalized for higher dimensions [16].

Discussing the generalized decomposition entropy $S_q^{\rm RIU}$ let us distinguish some other values of the Rényi parameter, apart of the Shannon value, $q = 1$. The case $q = 0$, corresponding to the Hartley entropy, is related to the rank R of the tensor C –the minimal number of the components in the product decomposition [17],

$$C_{i_1,i_2,\ldots,i_K} = \sum_{\nu=1}^{R} \gamma_\nu \, a_{i_1}^\nu \otimes b_{i_2}^\nu \otimes \ldots k_{i_K}^\nu, \tag{7}$$

involving R products of K vectors combined with arbitrary coefficients γ_ν. The minimal Hartley entropy, $S_0^{\rm RIU}(\psi) = \log R$, was introduced in [18] to characterize multipartite entanglement and called *Schmidt measure*. The name refers to the bi-partite case, as then R is equal to the Schmidt rank – the number of positive components of the Schmidt vector.

It is known that for any three-qubit state a five term representation exists [19, 20], so in this case $R \leq 5$. In the general case of $|\psi\rangle \in H_N^{\otimes K}$ Carteret, Higuchi and Sudbery have shown [11], that a suitable choice of each of K local unitary matrices $V_j \in U(N)$ allows one to bring $N(N-1)/2$ entries of the tensor C to zero. This implies a simple upper bound for the rank of the tensor,

$$R \leq R_{\max} = N^K - KN(N-1)/2, \tag{8}$$

and for the minimal Rényi decomposition entropy,

$$S_q^{\rm RIU}(\psi) \;\leq\; S_0^{\rm RIU}(\psi) \;\leq\; \log R_{\max}. \tag{9}$$

The first inequality is a consequence of monotonicity of the Rényi entropy with respect to the Rényi parameter q [21].

Minimal decomposition entropy $S_2^{\rm RIU}$, characterizing the maximal purity of the vector p_μ, was studied in [22] in the context of the coding theory. However, even more important is the

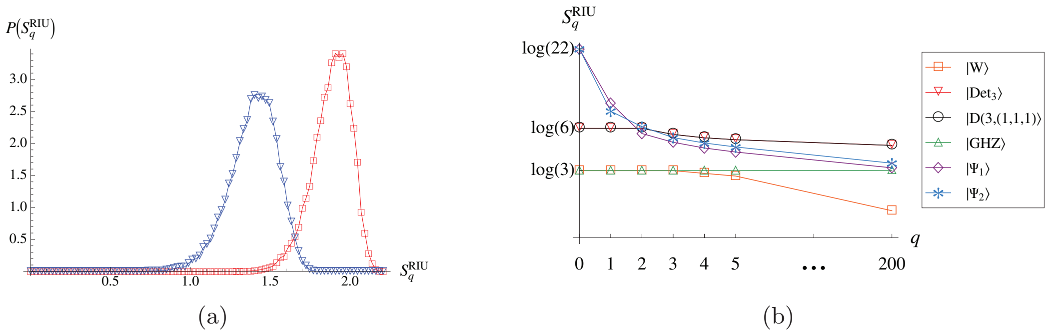

Figure 1: (a) Distributions of the minimal decompositon entropy for random states with $q = 1$ (□) and $q = 2$ (▽) (b) The minimal RIU entropy computed for representative three-qutrit states.

limiting case $q \to \infty$ of the Chebyshev entropy, $S_\infty = -\log p_{\max}$. This entropy characterizes the size of the largest product component, $S_\infty^{\mathrm{RIU}}(|\psi\rangle) = -\log F_{\max}$, where

$$F_{\max}(\psi) \;=\; \max_{U_{\mathrm{loc}}} |\langle\psi|U_{\mathrm{loc}}|0\rangle^{\otimes K}\rangle|^2 \tag{10}$$

denotes the maximal fidelity of $|\psi\rangle$ with respect to any separable state. Thus its function $D_{\mathrm{FS}} = \arccos(\sqrt{F_{\max}})$ represents the Fubini–Study distance to the closest separable state $|\psi_{\mathrm{sep}}\rangle$, often used to quantify entanglement [23, 24]. For this purpose one uses also related quantities [25–29], including infidelity, $E_G = 1-F_{\mathrm{sep}}$, and $E_L = -\log(F_{\mathrm{sep}}) = S_\infty^{\mathrm{RIU}}$, respectively called linear and logarithmic *geometric measures* of quantum entanglement.

Note also that the closest separable state $|\psi_{\mathrm{sep}}\rangle$, provides the principal component [17, 30] of the analyzed tensor (7) distinguished by the coefficient γ_ν with the largest modulus.

For any bipartite case the maximal fidelity (10) is explicitly given by the largest coefficient $\lambda_{\max}$ of the Schmidt vector [26, 31], and can be also calculated for a large class of three-qubit states [32–34]. In other cases one has to rely on numerical optimization, which becomes cumbersome if the number K of subsystems becomes large. Hence it is not easy to to identify the K-party state for which the minimal decomposition entropy S_q^{RIU} is maximal for $q = 1$ or any other value of the Rényi parameter q.

2.1. Searching for three-qutrit maximally entangled states

We are interested in identifying three-qutrit states for which the minimal decomposition entropy (5) is the largest. Some results in the case of three and four-qubit states were obtained by analyzing the distributions of S_q^{RIU} for an ensemble of random states [5]. For three-qutrits the corresponding distributions are shown in Figure 1 a) with $q = 1, 2$. Moreover, we take $q = 200$ as a bound for the case $q = \infty$. To optimize the quantity (5) we perform a random walk over the space of unitary matrices. We found two candidate random states to be maximally entangled with respect the minimal RIU entropy, which in the canonical basis read

$$|\Psi_1\rangle = \sum_{j=0}^{26} c_j^1 |j\rangle, \quad |\Psi_2\rangle = \sum_{j=0}^{26} c_j^2 |j\rangle, \tag{11}$$

where the index j is written in base 3 and the coefficients are give by

$$c^1 = (0.193e^{1.7i}, 0.323e^{-2.01i}, 0.16e^{-2.16i}, 0.229e^{-2.22i}, 0.232e^{-3.12i}, 0.186e^{-2.5i}, 0.239e^{-2.34i}, 0.141e^{-0.411i}, 0.159e^{-0.512i}, 0.099e^{1.54i}, 0.144e^{-2.43i}, 0.148e^{2.13i}, 0.263e^{-1.62i}, 0.322e^{0.475i}, 0.216e^{-1.95i}, 0.068e^{-1.39i}, 0.030e^{-2.89i}, 0.171e^{1.91i}, 0.253e^{-2.82i}, 0.022e^{-0.225i}, 0.06e^{-1.2i}, 0.003e^{2.64i}, 0.133e^{-1.52i}, 0.202e^{2.2i}, 0.194e^{1.08i}, 0.207e^{1.13i}, 0.274e^{-2.29i})$$

$$c^2 = (0.245e^{0.074i}, 0.024e^{2.49i}, 0.248e^{1.66i}, 0.069e^{1.55i}, 0.256e^{0.114i}, 0.118e^{-2.88i}, 0.313e^{-1.24i}, 0.076e^{2.77i}, 0.149e^{0.208i}, 0.208e^{2.56i}, 0.227e^{-2.88i}, 0.157e^{2.27i}, 0.072e^{3.08i}, 0.2e^{-1.07i}, 0.199e^{-1.87i}, 0.13e^{-1.95i}, 0.133e^{1.5i}, 0.218e^{-1.68i}, 0.244e^{-1.84i}, 0.191e^{-3.05i}, 0.049e^{2.61i}, 0.144e^{1.22i}, 0.226e^{2.14i}, 0.278e^{-2.46i}, 0.227e^{0.773i}, 0.186e^{-2.11i}, 0.218e^{-1.52i})$$

One can compare the minimal decomposition entropy of the former states with some representative three-qutrit states. For instance, the K-qudit Dicke state, invariant with respect to permutations, reads

$$|D[K,\vec{k}]\rangle = \sqrt{\frac{\prod_i k_i!}{K!}} \sum_{\pi\in S_K} |\underbrace{0\cdots0}_{k_0}\underbrace{1\cdots1}_{k_2}\cdots\underbrace{(d-1)\cdots(d-1)}_{k_{d-1}}\rangle. \tag{12}$$

The well-known K-qudit W-state and GHZ-state can be represented as linear combinations of some particular cases of the Dicke states (12)

$$\begin{aligned} |\mathrm{W_K}\rangle &= |D[K,(1,K-1,0,\ldots,0)]\rangle, \\ \sqrt{K}|\mathrm{GHZ}^d_K\rangle &= |D[K,(K,\ldots,0)]\rangle + |D[K,(0,K,\ldots,0)]\rangle + \ldots + |D[K,(0,\ldots,K)]\rangle. \end{aligned} \tag{13}$$

For the states (12) the maximal fidelity with respect the closest separable state (10) is known [29], hence their decomposition entropy with $q=\infty$ can be computed straightforward

$$S^{\mathrm{RIU}}_\infty(D[K,\vec{k}]) = -\log\left[\frac{K!}{\prod_i k_i!}\prod_{i=0}^{d-1}\left(\frac{k_i}{K}\right)^{k_i}\right]. \tag{14}$$

On the other hand, for the determinant state of N subsystems with N levels each,

$$|\mathrm{det}_N\rangle = \frac{1}{\sqrt{N!}}\sum_{i_1\cdots i_N=1}^{N}\epsilon_{i_1,\ldots,i_N}|i_1\cdots i_N\rangle, \tag{15}$$

Bravyi [35] has reported that $S^{\mathrm{RIU}}_1(\mathrm{det}_N) = \log N!$. On the other hand, it is not so difficult to show that the states $|\mathrm{det}_3\rangle$ and $|D[3,(1,1,1)]\rangle$ are LU-equivalent by making the transformations $|2\rangle_3 \to -|2\rangle_3$, $|0\rangle_1 \to -|0\rangle_1$, $|2\rangle_1 \to -|2\rangle_1$ and $|1\rangle_3 \to -|1\rangle_3$, and hence $S^{\mathrm{RIU}}_1(D[3,(1,1,1)]) = \log 6$. Fig. 1 b) depicts a comparison between the states (11) with some other three-qutrit states.

3. Partition into two subsystems

Is there any way to describe entanglement of a multipartite state an an explicit way, without performing any awkward optimizations? One simple option is to analyze, under what possible transformations a given state remains entangled. The *persistence* p of entanglement of a K-party pure state $|\phi\rangle$ is defined [9] as the minimal number of local measurements which have to be performed to destroy the entanglement completely. As any measurements of a single qubit transforms the K qubit GHZ state to a product state we have $p(|GHZ_K\rangle) = 1$. On the other hand after a measurement of single qubit the W state reamins entangled, so $p(|W_3\rangle) = 2$ and $p(|W_K\rangle) = K - 1$. Hence from the persistence perspective, the W-state is more entangled than GHZ.

Another manageable way to handle multipartite entanglement is to restrict our attention to the much simpler case of bipartite entanglement. For instance, one can split the K-partite system into M and $K - M$ parties, and use standard techniques to characterize bipartite entanglement, and possibly average over different splittings. This approach enables us to write down a reasonable definition of multipartite entanglement,

$$E_x(|\psi_K\rangle) = \frac{1}{L_K} \sum_A S_x(\rho_A), \tag{16}$$

where $\sigma_A = \mathrm{Tr}_{\bar{A}}|\psi\rangle\langle\psi|$ and the summation goes over all L_K selected partitions of the system into two parties $\{A, \bar{A}\}$. The measure S_x of the bipartite entanglement can be arbitrary, and one often takes the standard von Neumann entropy of the reduced state $S_1(\sigma)$, the Rényi entropy of order two, $S_2 = -\log \mathrm{Tr}\sigma^2$, its function called linear entropy, $S_2^{HC} = 1 - \mathrm{Tr}\sigma^2$, equivalent to the Tsallis entropy of order two, or the Chebyshev (minimal) entropy S_∞.

The simplest choice is to fix the number M of subsystems in the distinguished set A to unity, so then the number of possible bipartite splittings taken into account scales linearly, $L_K = K$. Such a quantity, characterizing the average entanglement of a single subsystem with the remaining part of the system, is called the Mayer–Wallach measure, as these authors [36] applied (16) for any K-qubit state. The measure S_x they used to quantity entanglement occurred to be equivalent [14] to the linear entropy S_2^{HC}. This observation allows us to use this measure also for N-level subsystems and write $Q_1(\phi) = \frac{N}{N-1}\langle S_2^{HC}\rangle$, where the average is taken over all K different splittings, for which $K-1$ subsystems are traced out and the reduced state describes a single subsystem. A suitable prefactor assures that this quantity is bounded by unity, $Q_1^{\max} = 1$, and the maximum is achieved e.g. for the generalized GHZ state,

$$|GHZ_K^N\rangle = \frac{1}{\sqrt{N}}\Big(|1\rangle^{\otimes K} + |2\rangle^{\otimes K} + \cdots + |N\rangle^{\otimes K}\Big). \tag{17}$$

To take into account also entanglement between larger subsystems it is convenient to fix the size of the smaller subsystem, writting $M = |A| \leq K - M$, and to average a suitable entanglement measure E_x over all $L_{K,M} = \binom{K}{M}$ splittings into bipartitions of this size. This approach leads, for instance, to a natural generalization of the Mayer–Wallach measure,

$$Q_M\big(|\phi\rangle\big) \;:=\; \frac{N^M}{N^M - 1}\,\Big\langle 1 - \big(\mathrm{Tr}_{\bar{A}}|\phi\rangle\langle\phi|\big)^2\Big\rangle_M, \tag{18}$$

where the average is taken over all $\binom{K}{M}$ partial traces with respect to environments $\bar{A}$ consisting of $K - M$ parties. By definition this quantity achieves the upper bound $Q_M \leq 1$ if there exists a pure state $|\psi_K\rangle$ such that its M-party reduced density matrix $\mathrm{Tr}_{\bar{A}}|\psi_K\rangle\langle\psi_K|$ with respect to *any* such partition $\bar{A}$ is maximally mixed. Such states are called M-*uniform* [2] or maximally multipartite entangled [10].

An alternative way to characterize the degree of mixing of reduced density matrices can be based on quantum analogue of the classical Rényi entropy (3),

$$S_q(\sigma) := \frac{1}{1-q} \log\, \mathrm{Tr}(\sigma^q), \tag{19}$$

defined for $q \geq 0$ and $q \neq 1$. It can be considered as a natural generalization of the von Neumann entropy, $-\mathrm{Tr}\sigma\log\sigma$, obtained for $q \to 1$. We are going to distinguish three cases, $q = 1$, $q = 2$

and $q \longrightarrow \infty$, and rewrite the corresponding expressions in terms of the eigenvalues of λ_i of the reduced state σ,

$$S_1(\sigma) = -\sum_i \lambda_i \log\, \lambda_i \qquad \text{the von Neumann entropy,} \tag{20a}$$

$$S_2(\sigma) = -\log\, \mathrm{Tr}(\sigma^2) \qquad \text{the Rényi entropy with q=2,} \tag{20b}$$

$$S_\infty(\sigma) = -\log \lambda_{\max} \qquad \text{the Chebyshev entropy.} \tag{20c}$$

Here $\lambda_{\max}$ denotes the largest eigenvalue.

In particular, to characterize the degree of mixing of reductions of states consisting of four subsystems we will use a quantity which depends on the Rényi parameter q and characterizes reductions to one and two-party systems

$$\widetilde{S}_q(\rho) := \sum_{j=1}^{4} \frac{S_q(\rho_j)}{\log 2} + \sum_{k=2}^{4} \frac{S_q(\rho_{1k})}{2\log 2}. \tag{21}$$

Here ρ_j and ρ_{1k} denote reduced density matrices containing one and two particles respectively. Thus the above quantity represents a weighted average over all relevant splittings of the four-party system, and the prefactors $\log 2$ and $2\log 2$ are added to assure that both terms have the same weight, so the value of $\widetilde{S}_q(\rho)$ for any 4-qubit state belongs to the interval $[0, 7]$. For any separable pure state one has $\widetilde{S}_q(|\psi_{\rm sep}\rangle\langle\psi_{\rm sep}|) = 0$, while the extremal value $\widetilde{S}_q = 7$ would correspond to a (non-existing) absolutely maximally entangled state of four qubits. Note that for $q = 2$ the quantity (21) is related to (16) as it corresponds to the sum $Q_1 + Q_2$ of quantities describing one- and two-party reductions defined in (18).

Let us return back to the general case of system composed of an arbitrary number of K systems. Any M-uniform state with $M \geq 2$ is by construction $(M-1)$-uniform, since partial trace of a maximally mixed state is maximally mixed, but the converse does not hold. We already know that the generalized GHZ state (17) is 1-uniform as it maximizes Q_1, but it is not two-uniform. Thus families of M-uniform states form a hierarchy of entanglement. Note that an M-uniform state of a system with K parties may exist if $M \leq K/2$, since any partial trace on a larger subsystem cannot be maximally mixed [2]. It is thus justified to distinguish M-uniform states of a system with $K = 2M$ subsystems, which are called *absolutely maximally entangled* (AME) [37].

To understand implications of these conditions on the general tensor form (2), let us first consider a bipartite state $|\psi\rangle = \sum_{\mu,\nu} C_{\mu,\nu}|\mu,\nu\rangle$ of $N \times N$ system. This state is maximally entangled, and thus 1-uniform, if the partial trace of the projector is equal to the maximally mixed state, $\rho_* = \frac{1}{N}\mathbb{1}$,

$$\mathrm{Tr}_B|\psi\rangle\langle\psi| = \sum_\nu C_{\mu,\nu} C^\dagger_{\nu,\mu'} |\mu\rangle\langle\mu'| = \rho_* \;\Leftrightarrow\; CC^\dagger = \frac{1}{N}\mathbb{1}. \tag{22}$$

This condition is satisfied if the matrix $U = \sqrt{N}C$ is unitary.

4. Arbitrary number of partitions

Consider now a general state (2) of $K = 2M$ subsystems and use composite indices, $\mu = (i_1, \ldots, i_M)$ and $\nu = (i_{M+1}, \ldots, i_{2M})$. Repeating the above argument for the splitting A versus $\bar{A}$ we see that the partial trace with respect to the partition $\bar{A}$ consisting of subsystems $i_{M+1}, \ldots, i_{2M}$ is maximally mixed, if the matrix $C_{\mu,\nu}$ obtained by a suitable reshaping of the rescaled tensor C is unitary. The same argument works for any other choice of M indices defining the composite index μ and the corresponding partition A, so we arrive at the following conclusion [6].

Proposition. A state (2) of $K = 2M$ subsystems with N levels each is M-uniform, and thus *absolutely maximally entangled*, if matrices $C_{\mu,\nu}$ obtained by reshaping the tensor $N^{M/2}C$ into a square matrix of size N^M are unitary for all $\binom{2M}{M}$ choices of M indices out of $2M$, which determine the composite indices μ and ν.

This is the case if the unitary matrix U obtained by reshaping the analyzed tensor remains unitary after performing suitable reorderings of their elements. Such matrices are called *multiunitary*. The corresponding tensors, called *perfect tensors*, are used to construct holographic codes and propose toy models for the bulk/boundary correspondence (anti de Sitter space and conformal field theory) [38].

In the case of a four-party system one has to check condition (22) for three splittings, 12, 13 and 14, which is equivalent to verifying if three matrices, $C_{(1,2)(3,4)}$, $C_{(1,3)(2,4)}$ and $C_{(1,4)(2,3)}$ multiplied by a constant are unitary. Denoting the first matrix U we need to check that its partial transpose U^{T_2} and the reshuffled matrix U^R – see [21] – are unitary. Although there are $\binom{4}{2} = 6$ choices of two subsystems out of four, three other cases will also be covered since as U is unitary so is U^T.

Any familiar Bell state $|\psi_2^+\rangle$ is one-uniform, so it forms the simplest AME state of two qubits. Interestingly, there are no two-uniform, AME states for four qubits – see [39, 40]. In short, the assumption that partions AB, AC and AD of the system $ABCD$ are maximally mixed implies so many constaints that no 4-qubit state can satisfy all of them. In other words there are no multiunitary matrices of order four: if U and $U^{T_2} \in U(4)$ then $U^R \notin U(4)$. This effect can be compared to the *frustration* in a spin system [41]. Larger systems offer more parameters to play with and indeed, for system composed of 4 subsystems with $N = 3$ levels each there exists an AME state.

$$3|\Psi_3^4\rangle = |0000\rangle + |0112\rangle + |0221\rangle + |1011\rangle + |1120\rangle + |1202\rangle + |2022\rangle + |2101\rangle + |2210\rangle. \quad (23)$$

Such a state can be constructed with help of orthogonal arrays [7] or mutually orthogonal Latin squares [6]. The corresponding multiunitary matrix U_9 of order 3^2 is related to a sudoku design, see [6]. There exist also 2-uniform state of 5 qubits [42] and 3-uniform AME state of six qubits [4] and they can be used to design quantum error correction codes. It would be also interesting to study such states for larger systems, as a general question, for what K-partite systems with N levels M-uniform state exists remains open [1, 6, 41].

5. A list of selected highly entangled states

In Table 1 we present a list of some multipartite quantum states that are identified as maximally entangled with respect certain entanglement measures. The catalog itemizes states related to the entanglement criteria previously discussed. In addition, the states are organized with respect their minimal decomposition entropy (5).

5.1. Two qubits

In the case of two qubits each one of the Bell states

$$|\Psi_2^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle), \quad |\Phi_2^{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle), \quad (24)$$

is 1-uniform as the reduced density matrix with respect any of the subsystems is maximally mixed. Moreover, it is not difficult to show that these states are mutually LU-equivalent. On the other hand, for two-qudits the maximally entangled state is given by (1) (including the LU-equivalent states). The minimal RIU entropy (5) in both cases can be computed by means the singular value decomposition of the corresponding coefficient matrices.

State	S_1^{RIU}	S_2^{RIU}	S_∞^{RIU}
Two qubits			
Bell states: $\|\Psi^\pm\rangle$, $\|\Phi^\pm\rangle$	$\log 2$	$\log 2$	$\log 2$
Two qudits			
GHZ-state: $\|\text{GHZ}_2^d\rangle$	$\log d$	$\log d$	$\log d$
Three qubits			
GHZ-state: $\|\text{GHZ}_2^3\rangle$	$\log 2$	$\log 2$	$\log 2$
W-state: $\|\text{W}_3\rangle$	$\log 3$	$\log 3$	$\log(9/4)$
Random "maximal" state: $\|\Phi_1^{\max}\rangle$	1.277	1.020	0.610
Random "maximal" state: $\|\Phi_2^{\max}\rangle$	1.141	1.108	0.727
Four qubits			
Higuchi and Sudbery state: $\|\text{HS}\rangle$	$^\star \log 6$	$^\star \log 6$	$^\star \log(9/2)$
Cluster states: $\|\text{C}_k\rangle$	$^\star \log 4$	$^\star \log 4$	$^\star \log 4$
Random "maximal" state: $\|\Psi_1^{\max}\rangle$	1.934	1.573	0.934
Symmetric state: $\|D[4,(2,0)]\rangle$	$\log(8/\sqrt{3})$	$\log(24/7)$	$\log(8/3)$
$BSSB_4$ state: $\|BSSB_4\rangle$	1.733	$^\star \log(16/3)$	1.393
Ψ_4 state: $\|\Psi_4\rangle$	1.733	$^\star \log(16/3)$	$^\star \log 4$
Hyperdeterminant state: $\|HD\rangle$	1.561	$^\star \log(9/2)$	1.110
Yeo and Chua state: $\|YC\rangle$	1.602	1.410	1.002
L state: $\|L\rangle$	1.561	$^\star \log(9/2)$	1.110
GHZ state: GHZ_4	$\log 2$	$\log 2$	$\log 2$
Five qubits			
1-uniform state $\|\Psi_5\rangle$	$^\star \log 8$	1.716	1.199
AME state $\|\Omega_{5,2}\rangle$	$^\star \log 8$	$^\star \log 8$	2.03
Symmetric state: $\|D[5,(2,0)]\rangle$	2.263	$^\star \log 5$	$\log(216/625)$
Three qutrits			
Random "maximal" state: $\|\Psi_1\rangle$	2.205	1.707	1.1452
Random "maximal" state: $\|\Psi_2\rangle$	2.072	1.82	1.221
Symmetric state $\|D[3,(1,1,1)]\rangle$	$^\star \log 6$	$^\star \log 6$	$\log(9/2)$
Four qutrits			
AME state: $\|\Omega_{4,3}\rangle$	$^\star \log 9$	$^\star \log 9$	$^\star \log 9$
Symmetric state: $\|D[4,(2,1,1,0)]\rangle$	$^\star \log 12$	$^\star \log 12$	$\log(16/3)$

Table 1: Some multipartite maximally entangled states with respect different measures and their minimal RIU entropy (5). The star indicates that the value has been conjectured from numeric calculations.

5.2. Three qubits

It is known that the 1-uniform 3-qubit GHZ-state is maximally entangled with respect the three-tangle but it does not maximizes the minimal decomposition entropy (5) for none value of the Rényi parameter q [5]. In fact, there exist two candidate random states to be maximal with respect this measure with $q=1$ and $q=2$, which in the computational basis read

$$|\Phi_1^{\max}\rangle = 0.27|000\rangle + 0.377|100\rangle + 0.326|010\rangle + 0.363|001\rangle + 0.740e^{-0.79\pi i}|111\rangle, \tag{25a}$$
$$|\Phi_2^{\max}\rangle = 0.438|000\rangle + 0.29|100\rangle + 0.371|010\rangle + 0.316|001\rangle + 0.698e^{-0.826\pi i}|111\rangle. \tag{25b}$$

The minimal decomposition entropy S_∞^{RIU} attains its largest value value for the W-state, however such a state is not 1-uniform.

5.3. Four qubits

It has been conjectured that for the Rényi parameter $q \geq 2$ the 1-uniform state

$$|\text{HS}\rangle = \frac{1}{\sqrt{6}}\big[|0011\rangle + |1100\rangle + w(|0101\rangle + |1010\rangle) + w^2(|0110\rangle + |1001\rangle)\big], \quad w = \exp(2i\pi/3), \tag{26}$$

is maximally entangled with respect the minimal decomposition entropy [5]. This state also gives the maximal average von Neumann entropy of partial traces averaged over all possible 3 splittings of 4 qubit system into two bipartite systems [40]. For the special case of $q = 1$ the following state obtained numerically with a random walk procedure,

$$\begin{aligned}|\Psi_1^{\max}\rangle = 0.630|0000\rangle + 0.281|1100\rangle + 0.202|1010\rangle + 0.24|0110\rangle + 0.232e^{0.494\pi i}|1110\rangle \\ +0.059|1001\rangle + 0.282|0101\rangle + 0.346e^{-0.362\pi i}|1101\rangle + 0.218e^{0.626\pi i}|1011\rangle \\ +0.304|0011\rangle + 0.054e^{-0.725\pi i}|0111\rangle + 0.164e^{0.372\pi i}|1111\rangle,\end{aligned} \tag{27}$$

is a candidate to be maximally entangled with respect the minimal RIU entropy [5]. On the other hand, the three cluster states [43]

$$|\mathrm{C}_1\rangle = \frac{1}{2}\left(|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle\right), \tag{28a}$$

$$|\mathrm{C}_2\rangle = \frac{1}{2}\left(|0000\rangle + |0110\rangle + |1001\rangle - |1111\rangle\right), \tag{28b}$$

$$|\mathrm{C}_3\rangle = \frac{1}{2}\left(|0000\rangle + |0101\rangle + |1010\rangle - |1111\rangle\right), \tag{28c}$$

maximize the Rényi entropy of partial trace for $q \geq 2$ [44]. Numerical computations support the conjecture that for these states the RIU entropy does not depend on the Rényi parameter q [5]. As another example we consider the ground state of the 4-qubit XXX Heisenberg model. This state is locally equivalent to the symmetric Dicke state $D[4,(2,0)]$ and yields $Q_1 = 1$ [36]. Finally, we emphasize the fact that there exist no 4-qubit AME states [6, 39, 40].

The state introduced by Gour and Wallach [44]

$$\begin{aligned}|L\rangle = \frac{1}{2\sqrt{3}}((1+\omega)(|0000\rangle + |1111\rangle) + (1-\omega)(|0011\rangle + |1100\rangle) + \\ \omega^2(|0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle)), \quad \omega = \exp(2i\pi/3),\end{aligned} \tag{29}$$

maximizes the measure Q_2 defined in (18), i.e. the linear entropy $\langle S_2^{HC}\rangle$ of the reduced state averaged over three different splittings into two pairs of bi-partite states. Another state introduced in [44] and called $|M\rangle$ there, which is conjectured to yield the maximal value of the average von Neumann entropy of partial traces, is shown [45] to be locally equivalent to the HS state (26).

Another distinguished state

$$|HD\rangle = \frac{1}{\sqrt{6}}(|1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle + \sqrt{2}|1111\rangle), \tag{30}$$

was called the 'hyperdeterminant state', as it maximizes the four-qubit hyperdeterminant $\mathrm{Det}_4(|\varphi\rangle)$ [45]. This important invariant with respect to local transformations is maximized for the above two states, $\mathrm{Det}_4(|L\rangle) = \mathrm{Det}_4(|HD\rangle) = 1$. However, there exist highly entangled states, $|D_{4,1}\rangle, |D_{4,2}\rangle, |C_k\rangle, |HS\rangle, |GHZ_4\rangle$, for which $\mathrm{Det}_4(|\varphi\rangle) = 0$.

The state $|BSSB_4\rangle$ found by Brown et.al [46] by a numerical search,

$$|BSSB_4\rangle = \frac{1}{2\sqrt{2}}(|0110\rangle + |1011\rangle + i(|0010\rangle + |1111\rangle) + (1+i)(|0101\rangle + |1000\rangle)), \tag{31}$$

is highly entangled with respect to measure based on the average linear entropy of bi-partite reductions. The following state introduced by Yeo and Chua [47] $|YC\rangle$ can be used to perform a faithful teleportation of an arbitrary two-qubit entangled state,

$$|YC\rangle = \frac{1}{2\sqrt{2}}(|0000\rangle - |0011\rangle - |0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle + |1100\rangle + |1111\rangle). \tag{32}$$

It is conjectured to maximize entanglement measures based on Rényi entropy of partial traces for $q \geq 2$. Let us also distinguish the tensor product of the Bell states,

$$|\psi_4\rangle = |\Psi_+\rangle \otimes |\Psi_+\rangle = \frac{1}{2}(|0000\rangle + |0011\rangle + |1100\rangle + |1111\rangle). \tag{33}$$

It is worth mentioning that Verstraete et al. [48] divide all 4-qubit pure states in nine different classes, to which any state can be transformed by SLOCC operations. One of these classes called *generic* contains states of the form [44],

$$|\varphi\rangle = z_0|\Phi^+\rangle|\Phi^+\rangle + z_1|\Phi^-\rangle|\Phi^-\rangle + z_2|\Psi^+\rangle|\Psi^+\rangle + z_3|\Psi^-\rangle|\Psi^-\rangle, \tag{34}$$

where $|\Phi^\pm\rangle$ and $|\Psi^\pm\rangle$ denote the standard Bell states (24), while $z_0, z_1, z_2, z_3 \in \mathbf{C}$ represent complex coefficients normalized as $\sum_{i=0}^{3}|z_i|^2 = 1$. A state belonging to this class represented in the computational basis reads

$$\begin{aligned}|\psi\rangle =& \frac{z_0+z_1}{2}|0000\rangle + \frac{z_0-z_1}{2}|0011\rangle + \frac{z_2+z_3}{2}|0101\rangle + \frac{z_2-z_3}{2}|0110\rangle + \\ & \frac{z_2-z_3}{2}|1001\rangle + \frac{z_2+z_3}{2}|1010\rangle + \frac{z_0-z_1}{2}|1100\rangle + \frac{z_0+z_1}{2}|1111\rangle.\end{aligned} \tag{35}$$

It is easy to see that several of the states mentioned above belong to this class. For instance, the states $|\psi_4\rangle$, $|GHZ_4\rangle$ and $|L\rangle$ correspond to real vectors of coefficients, equal to $(1,0,0,0)$; $(1,1,0,0)/\sqrt{2}$ and $(1,\omega,\omega^2,0)/\sqrt{3}$ respectively.

To illustrate how the above states are distinguished with respect to the measures of entanglement based on degree of mixing of reduced density matrices we use the weighted sum (21) of Renyi entropies, which takes into account all possible one and two-party reductions. Fig. 2 presents numerical data of average Rényi entropies of reduced density matrices for random pure states of four-qubit systems. The data are shown in the planes spanned by two Rényi entropies chosen from the triple $\widetilde{S}_1$, $\widetilde{S}_2$, $\widetilde{S}_\infty$. In other words we see three projections of the 3D body formed in the space spanned by these three quantities. The data obtained for the ensemble of states belonging to the generic class (34) presented in the right panels show that these 1-uniform states are characterized by a high degree of entanglement.

Our numerical results confirm that the states listed above are distinguished indeed: the state HS (26) gives the maximal average entropy $\widetilde{S}_1$, while the entropies $\widetilde{S}_2$ and $\widetilde{S}_\infty$ are maximized by the YC state (32) and the cluster states (28a). It is worth to note that the state $|\Psi_1^{max}\rangle$, defined in (11) and distinguised by the minimal decomposition entropy, is not characterized by a high value of the entropies $\widetilde{S}_q$ of reductions, as it falls outside the area covered by the states from the generic class. Let us emphasize once more that there are no 2-uniform states for 4-qubit systems: the dots corresponding to states analyzed are located far away from the bounds represented in the plots by red dotted lines.

5.4. *Five qubits*

We first consider the 1-uniform state

$$2\sqrt{2}|\Psi_5\rangle = |00000\rangle + |10011\rangle + |00101\rangle + |11001\rangle + |10110\rangle + |01111\rangle + |11100\rangle, \tag{36}$$

that was introduced by Brown *et al.* as a highly entangled state with respect the negative partial transpose criterion [46]. This can be also obtained by means orthogonal arrays [7]. On the other hand, the 5-qubit AME state

$$2\sqrt{2}|\Omega_{5,2}\rangle = |00000\rangle + |00011\rangle + |01100\rangle - |01111\rangle + |11010\rangle + |11001\rangle + |10110\rangle - |10101\rangle, \tag{37}$$

fulfils $Q_1 = 1$ and has applications in quantum teleportation, quantum-state sharing, superdense coding and orthogonal arrays, as well [7, 49]. Interestingly, numerical computations show that the symmetric state $|D(5,(2,0))\rangle$ has larger S_1^{RIU} than these two.

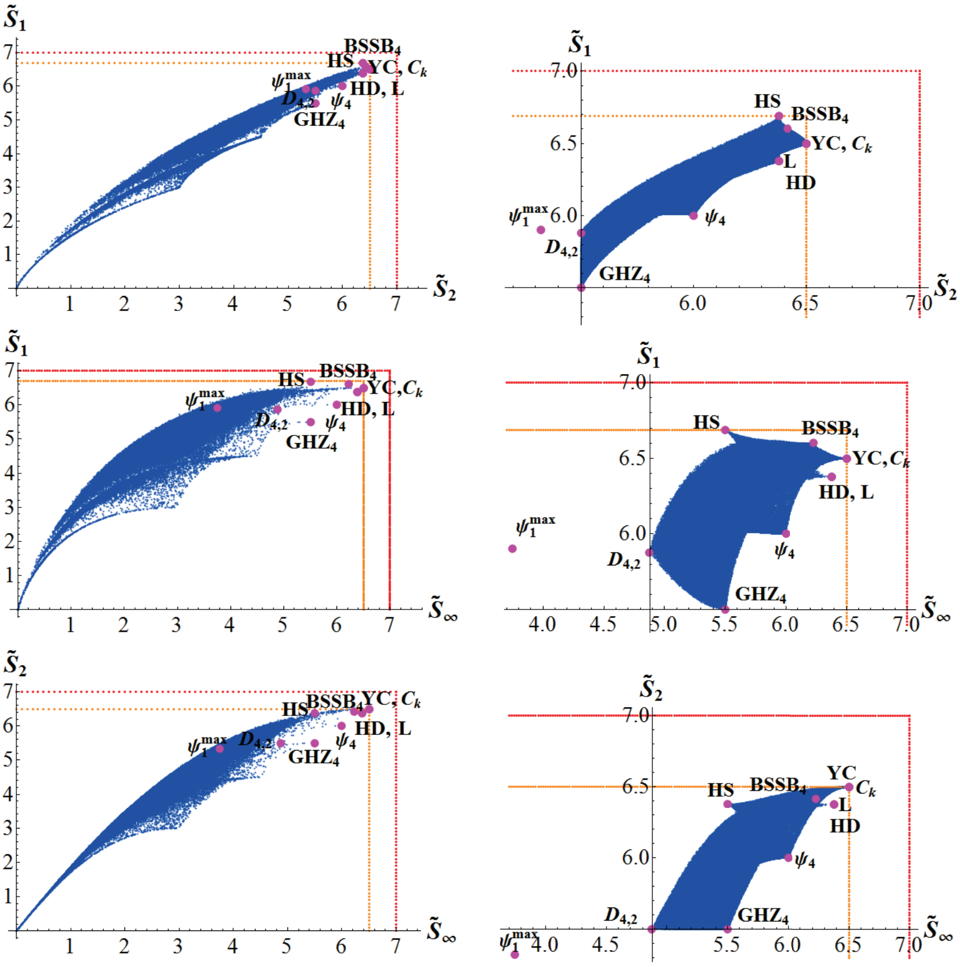

Figure 2: Average Rényi entropies $\widetilde{S}_1$, $\widetilde{S}_2$, $\widetilde{S}_\infty$ of the reduced density matrices for random four-qubit states for three choices of the axes. Left plots show values for 10^6 random states, while right plots present data for random states of the class (34). Red dotted lines indicate upper bounds for the entanglement measures, while yellow line represents the maximal values obtained numerically. Distinguished states are labeled by their names, while $D_{4,2}$ denotes the state $|D[4,(2,0)]\rangle$ defined in (12).

5.5. *Three qutrits*

We focus on the random states (11) as well as the symmetric state $|D(3,(1,1,1))\rangle$. Among several three-qutrit states this last attains the highest value for S_∞^{RIU} while the former states have the largest value of S_1^{RIU} and S_2^{RIU}, respectively.

5.6. *Four qutrits*

As an example of maximally entangled state we consider the state (23) for which our numeric calculations allow us to conjecture that the minimal decomposition entropy is the same regardless the value of the Rényi parameter. However, the symmetric state $|D(4,(2,1,1,0))\rangle$ attains a larger value of (5) with $q = 1, 2$.

6. Conclusions

We have reviewed some entanglement properties of various multipartite states. In general, characterization of multipartite entanglement can be approached from different perspectives as one can study various measures of entanglement related to different aspects of non-locality. In the case of bipartite systems one identifies the class of states locally equivalent to the generalized Bell state, for which several measures of entanglement achieve their maximal values. However, for systems with more components the situation changes: in general, a highly entangled state with respect to a single measure of entanglement can display average, low or even vanishing degree of entanglement, if another measure is used. This can be seen in Fig. 2, where several four-qubit states are plotted according to the average Rényi entropy. The problem of distinguishing states for which the different entanglement measures attain their maximum value remains open even for some measures of three and four-parties entanglement. This fact is evident from Table 1 where several multipartite states are listed with respect their minimal RIU entropy.

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