



Hydrodynamics of the Dirac spectrum



Yizhuang Liu^a, Piotr Warchoł^b, Ismail Zahed^{a,*}

^a Department of Physics and Astronomy, Stony Brook University, Stony Brook, NY 11794-3800, USA

^b M. Smoluchowski Institute of Physics, Jagiellonian University, PL-30348 Krakow, Poland

ARTICLE INFO

Article history:

Received 11 July 2015

Received in revised form 11 November 2015

Accepted 12 December 2015

Available online 15 December 2015

Editor: A. Ringwald

ABSTRACT

We discuss a hydrodynamical description of the eigenvalues of the Dirac spectrum in even dimensions in the vacuum and in the large N (volume) limit. The linearized hydrodynamics supports sound waves. The hydrodynamical relaxation of the eigenvalues is captured by a hydrodynamical (tunneling) minimum configuration which follows from a pertinent form of Euler equation. The relaxation from a phase of unbroken chiral symmetry to a phase of broken chiral symmetry occurs over a time set by the speed of sound.

© 2015 The Authors. Published by Elsevier B.V. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>). Funded by SCOAP³.

1. Introduction

QCD with light quarks breaks spontaneously chiral symmetry. As a result, the light quarks transmute to massive constituents which make up for most of the visible mass in our universe. Empirical evidence for the spontaneous breaking of chiral symmetry is in the form of light pions and kaons in nature [1]. Dedicated first principle QCD lattice simulations with light quarks have established that chiral symmetry is spontaneously broken with a finite chiral condensate and an octet of light mesons [2].

The spontaneous breaking of chiral symmetry is characterized by a large accumulation of eigenvalues of the Euclidean Dirac operator near zero virtuality [3]. This phenomenon signals the onset of an ergodic regime in the chiral Dirac spectrum. Other regimes where the light quarks diffuse or undergo ballistic motion can also be identified [4]. In some ways light quarks interacting via colored Yang–Mills fields behave like disordered electrons in metallic grains.

The essentials of the ergodic regime are captured by a chiral random matrix model [5]. In short, the QCD Dirac spectrum near zero virtuality only retains that the QCD Dirac operator is chiral or block off-diagonal, with random entries that are sampled from the three universal Dyson ensembles [6] with a Gaussian weight. The spectrum corrections are also generic and follow from the neighboring diffusive regime with the light quark return probability falling like a power law for large proper times [4]. Both regimes are separated by the Thouless energy [4,7].

In this letter we develop a hydrodynamical description of the eigenvalues of the QCD Dirac operator in the ergodic regime [4], much along the lines of our recent studies of the eigenvalues of the Polyakov line at large number of colors [8]. We will use this derivation to obtain the following new results: 1/ a hydrodynamical (tunneling) minimum that captures the relaxation of the eigenvalues of the Dirac operator for low virtuality; 2/ a dynamical relaxation time for restoring and/or breaking spontaneously chiral symmetry directly from the Dirac spectrum.

2. Dirac spectrum

The original chiral random matrix model partition function for the eigenvalues of the Dirac spectrum was initially suggested as a null dynamical assumption for the generic analysis of the powers of the chiral condensate in the instanton liquid model [9]. However and more importantly, it was noted later that this assumption provides a universal description of these powers in the microscopic limit [5,10] as detailed in [11]. For QCD in even dimensions with N_f quarks of equal masses m in the complex representation, and the topological charge zero sector [9,5]

$$Z_2[m] = \int dT \prod_{f=1}^{N_f} \det(m^2 + T^\dagger T) e^{-\frac{g}{2} N \text{Tr}(T^\dagger T)} \quad (1)$$

Here T is a symmetric random $C^{N \times N}$ complex matrix capturing the random hopping between N -left and N -right zero modes. (1) was generalized to all Dyson ensembles with $\beta = 1, 2, 4$ corresponding to quarks in different representations and asymmetric ensembles in [6].

* Corresponding author.

E-mail addresses: yizhuang.liu@stonybrook.edu (Y. Liu), piotr.warchol@uj.edu.pl (P. Warchoł), ismail.zahed@stonybrook.edu (I. Zahed).

$$Z_\beta[m] = \int \prod_{i<j}^N |\lambda_i^2 - \lambda_j^2|^\beta \prod_i^N \lambda_i^\alpha (\lambda_i^2 + m^2)^{N_f} e^{-\frac{a\beta N}{2} \lambda_i^2} d\lambda_i \quad (2)$$

with now T a rectangular random $C^{N \times N + \nu}$ complex matrix with ν exact and unpaired zero modes. Here $\alpha = \beta(\nu + 1) - 1$ and λ_i^2 is the eigenvalue of the squared Wishart matrix $W = T^\dagger T$. The number of Dirac eigenvalues is $2N + \nu$. The overall normalizations in (1)–(2) are omitted.

The Gaussian measure is generic at large N , with the parameter a fixed by the chiral condensate. In the microscopic limit whereby the Dirac spectrum near zero-virtuality is magnified so that $N\Delta\lambda = 1$, the interactions between the eigenvalues as mediated by the gauge-fields appear overall as random. As a result only the chiral structure of the Dirac Hamiltonian and the symmetry of the matrix entries under anti-unitary symmetries are relevant [12]. This is the universal regime of ergodicity shared by most disordered electronic systems in the mesoscopic limit [13].

Alternatively, (2) can be regarded as the normalization of the squared and real many-body wave-function

$$Z_\beta[m] = \int \prod_{i=1}^N d\lambda_i |\Psi_0[\lambda_i]|^2 \quad (3)$$

which is the zero-mode solution to the Schrödinger equation $H_0\Psi_0 = 0$ with the self-adjoint squared Hamiltonian [14]

$$H_0 \equiv \sum_{i=1}^N (-\partial_i + \mathbf{a}_i)(\partial_i + \mathbf{a}_i) \quad (4)$$

with $\partial_i \equiv \partial/\partial\lambda_i$ and the potential $\mathbf{a}_i \equiv -\partial_i \ln |\Psi_0|^2$. The mass parameter is $1/2$. The Vandermonde contribution $\Delta = \prod_{i<j} |\lambda_i^2 - \lambda_j^2|^\beta$ induces a diverging 2-body part. It is convenient to re-absorb it using a similarity transformation by defining $\Psi = \Psi_0/\sqrt{\Delta}$ and the new Hamiltonian

$$H = (1/\sqrt{\Delta})H_0\sqrt{\Delta} \quad (5)$$

3. Hydrodynamics

We can use the collective coordinate method in [14] to re-write (5) in terms of the density of paired eigenvalues as a collective variable $\rho(\lambda) = \sum_{i=1}^N \delta(\lambda - \lambda_i)$. The result is a hydrodynamical description of the Dirac spectrum much along the idea pioneered by Dyson [15] and others [16]. After some algebra we obtain

$$H = \int d\lambda (\partial_\lambda \pi \rho \partial_\lambda \pi + \rho \mathbf{u}[\rho]) \quad (6)$$

with the potential-like contribution $\mathbf{u}[\rho] \equiv \mathbf{A}^2$

$$\mathbf{A} = \frac{a\beta N}{2} \lambda - \frac{\beta\pi}{2} [\rho_H(\lambda) - \rho_H(-\lambda)] + \partial_\lambda \ln \sqrt{\rho} - \frac{\alpha}{2\lambda} - \frac{N_f \lambda}{\lambda^2 + m^2} \quad (7)$$

Here ρ_H is the Hilbert transform of ρ

$$[\rho]_H \equiv \rho_H(\lambda) = \frac{1}{\pi} \text{P} \int d\lambda' \frac{\rho(\lambda')}{\lambda - \lambda'} \quad (8)$$

The canonical pair $\pi(\lambda)$ and $\rho(\lambda)$ satisfies the standard commutation rules. Here we will limit our analysis to a classical hydrodynamical description and will use the Poisson bracket $\{\pi(\lambda), \rho(\lambda')\} = \delta(\lambda - \lambda')$. We identify the collective fluid velocity with $v = \partial_\lambda \pi$ and re-write (6) in the more familiar hydrodynamical form

$$H \approx \int d\lambda \rho(\lambda) (v^2 + \mathbf{u}[\rho]) \approx \int d\lambda \rho(\lambda) |v + i\mathbf{A}|^2 \quad (9)$$

ignoring ultra-local terms. The equation of motion for ρ yields the current conservation law $\partial_t \rho = -2\partial_\lambda (\rho v)$, and the equation of motion for v gives the Euler equation

$$\partial_t v = \{H, v\} = -\partial_\lambda [v^2 + \mathbf{A}^2 - \partial_\lambda \mathbf{A} - \mathbf{A} \partial_\lambda \ln \rho + \pi \beta [\mathbf{A} \rho]_H(\lambda) + \pi \beta [\mathbf{A} \rho]_H(-\lambda)] \quad (10)$$

All the relations hold for large but finite N , to allow for a smoothening of the density. The smoothening is over a scale that is larger than the eigenvalue spacing but much smaller than the range of the spectrum.

The dimensionality of the time t is the inverse of the dimensionality of H , both of which are arbitrary. Throughout, the dimension of t is set to the dimension of the squared eigenvalue. When needed, the canonical dimension follows through a pertinent re-scaling using the constant a in (1).

4. Hydrostatic solution

The hydrostatic solution corresponds to the minimum of (9) with $v = 0$ and thus $\mathbf{A}(\lambda) = 0$. In terms of the chirally symmetric combination $\rho^\chi(\lambda) \equiv \rho(\lambda) + \rho(-\lambda)$ we have

$$0 = \frac{a\beta N}{2} \lambda - \frac{\beta\pi}{2} \rho_H^\chi(\lambda) + \frac{1}{2} \partial_\lambda \ln \rho^\chi - \frac{\alpha}{2\lambda} - \frac{N_f \lambda}{\lambda^2 + m^2} \quad (11)$$

In the large N limit only the terms in the first line survive with

$$\rho_0^\chi(\lambda) = \frac{Na}{\pi} \sqrt{\frac{4}{a} - \lambda^2} \quad (12)$$

which is the Wigner semi-circular distribution within the support $|\lambda| \leq 2/\sqrt{a}$.

The hydrostatic equation (11) is classical and resums a large class of $1/N$ effects. The leading $1/N$ correction can be sought in the form $\rho^\chi \approx \rho_0^\chi + \rho_1$ subject to the condition $\int \rho_1 d\lambda = 0$, so that

$$-\beta\pi \rho_{1H}(\lambda) = \frac{1}{\lambda} (\alpha + 2N_f) - \frac{\lambda a}{4 - a\lambda^2} \quad (13)$$

The closed form solution is

$$\rho_1(\lambda) = -\frac{\alpha + 2N_f}{\beta} \delta(\lambda) + \frac{|\lambda|}{\beta} \delta(\lambda^2 - 4/a) + \frac{\alpha + 2N_f - 1}{\beta\pi \sqrt{4/a - \lambda^2}} \quad (14)$$

with the support unchanged. We note that (14) shows a delta-function at zero virtuality with the negative strength $(\alpha + 2N_f)/\beta$.

The general solution to (11) for large but finite N can be sought in the massless case or $m = 0$, by multiplying (11) by $2\rho^\chi \lambda(\lambda^2 + m^2)$ and taking the Hilbert transform. The result is

$$a\beta N \left(\lambda^2 \rho_H^\chi - \lambda \frac{2N}{\pi} \right) - \frac{\beta\pi}{2} \lambda (\rho_H^{\chi 2} - \rho^{\chi 2}) + \lambda \partial_\lambda \rho_H^\chi - (\alpha + 2N_f) \rho_H^\chi = 0 \quad (15)$$

To solve it consider the extension of the resolvent

$$G(z) = \int d\lambda \frac{\rho^\chi(\lambda)}{z - \lambda} \quad (16)$$

to the upper and lower complex plane,

$$G^\pm(z \rightarrow \lambda) = \pi \rho_H^\chi(\lambda) \mp i\pi \rho^\chi(\lambda) \quad (17)$$

so that (15) now reads

$$a\beta N z^2 G - \frac{1}{2}\beta z G^2 + z\partial_z G - (\alpha + 2N_f)G - 2a\beta N^2 z = 0 \quad (18)$$

Taking the large N and small z limits sequentially in (18), yields $G(z \rightarrow 0) \approx -(\alpha + 2N_f)/(\beta z)$ in agreement with the first contribution in (14). However, the limits do not commute since by reversing them we obtain instead $G(z \rightarrow 0) \approx -2(1 + \alpha + 2N_f)/(\beta z)$. We note that this small z -behavior is distinct from the exact result v/z that can be derived from the random matrix model (1). The reason is that this contribution is not part of the hydrodynamical analysis.

A general solution to (18) follows from the mapping $G(z) \equiv -(2/\beta)\partial_z \ln f(z)$ and $z^2 = -w$ and $f(\sqrt{-w}) = g(w)$. Thus

$$2w\partial_w g + (-a\beta N w + 1 - \alpha - 2N_f)\partial_w g - \frac{1}{2}a\beta^2 N^2 g = 0 \quad (19)$$

(19) is Laguerre-like except for the wrong sign in the last contribution. The general solution is a confluent hyper-geometric function U ,

$$g(w) = U\left(\frac{\beta N}{2}, \frac{1 - \alpha - 2N_f}{2}, \frac{a\beta N w}{2}\right) \quad (20)$$

The large z -behavior of (20) is $G(z \rightarrow \infty) \approx 2N/z$. For fixed N , the small z -behavior of (20) corresponds to $G(z \rightarrow 0) \approx -2(1 + \alpha + 2N_f)/(\beta z)$ which is the result we observed earlier in the small z and large N limits taken sequentially. To reverse the limits is more subtle in this case.

In general, the $1/N$ corrections form a trans-series with oscillating contributions that are not accessible by the present method. In particular, the spurious $1/N$ delta-singularities at the edge of the spectrum are likely cancelled by contributions emerging from non-hydrodynamical saddle points [18].

5. Dyson Coulomb gas

We note that (2) can be re-written in terms of 1-dimensional Dyson Coulomb gas [17]. At large N the ensemble described by (1) is sufficiently dense to allow the change in the measure,

$$\prod_{i=1}^N d\lambda_i \approx e^{S[\rho]} D\rho \quad (21)$$

with $S[\rho] = \int d\lambda \rho(\lambda) \ln(\rho_0/\rho(\lambda))$ the Boltzmann entropy [17]. Thus

$$Z_\beta[m] \rightarrow \int D\rho e^{-\Gamma[\beta, m; \rho]} \quad (22)$$

with the effective action

$$\begin{aligned} \Gamma[\beta, m; \rho] = & \int d\lambda \rho(\lambda) \left(\frac{a\beta N}{2} \lambda^2 - \alpha \ln(\lambda) - N_f \ln(\lambda^2 + m^2) \right) \\ & - \frac{\beta}{2} \int d\lambda d\lambda' \rho(\lambda) \rho(\lambda') \ln(\lambda^2 - \lambda'^2) \\ & - \left(\frac{\beta}{2} - 1 \right) \int d\lambda \rho \ln \rho \end{aligned} \quad (23)$$

The β contribution is the self Coulomb subtraction and is consistent with the subtraction in the Hilbert transform. The saddle point equation $\delta\Gamma/\delta\rho = 0$ yields the hydro-static equation (11) using the symmetric density ρ^χ .

6. Hydrodynamical minimum

Following the initial observation in [8], we note that the fixed time zero energy solution to (9) is a minimum with imaginary velocity $v = -i\mathbf{A}$ (tunneling). The conserved current $j \equiv \rho v$ satisfies ($\tau = it$)

$$\begin{aligned} \partial_\tau \rho(\lambda) = \partial_\lambda \left[a\beta N \lambda \rho(\lambda) - \beta \pi \rho(\lambda) [\rho_H(\lambda) - \rho_H(-\lambda)] \right. \\ \left. + \partial_\lambda \rho(\lambda) - \frac{\alpha}{\lambda} \rho(\lambda) - \frac{2N_f \lambda}{\lambda^2 + m^2} \rho(\lambda) \right] \end{aligned} \quad (24)$$

We identify τ with the stochastic time, and (24) describes the stochastic relaxation of the chiral eigenvalue density of the chiral Dirac spectrum (out of equilibrium) to its asymptotic (in equilibrium) hydro-static solution. For $m = 0$, (24) can be rewritten as

$$\begin{aligned} \partial_\tau \rho^\chi = a\beta N \partial_\lambda (\lambda \rho^\chi) - \beta \pi \partial_\lambda (\rho^\chi \rho_H^\chi) \\ + \partial_{\lambda\lambda} \rho^\chi - (\alpha + 2N_f) \partial_\lambda \left(\frac{1}{\lambda} \rho^\chi \right) \end{aligned} \quad (25)$$

After multiplying this equation by λ^2 , we can take its Hilbert transform. The result is

$$\begin{aligned} \partial_\tau \rho_H^\chi = a\beta N \partial_\lambda (\lambda \rho_H^\chi) - \frac{\beta \pi}{2} \partial_\lambda (\rho_H^{\chi 2} - \rho^{\chi 2}) \\ + \partial_{\lambda\lambda} \rho_H^\chi - (\alpha + 2N_f) \partial_\lambda \left(\frac{1}{\lambda} \rho_H^\chi \right) \end{aligned} \quad (26)$$

With the usual definition, this gives an equation for the time dependent Green's function

$$\begin{aligned} \partial_\tau G = a\beta N \partial_z (zG) - \frac{\beta}{2} \partial_z (G^2) \\ + \partial_{zz} G - (\alpha + 2N_f) \partial_z \left(\frac{1}{z} G \right) \end{aligned} \quad (27)$$

which, after a proper rescaling of time τ and G , for $a = 0$, in the large N limit such that $v/N \rightarrow 0$, is an inviscid complex Burgers equation derived in [19] for diffusing chiral matrices.

We note that the stationary solution fulfills

$$a\beta N z G - \frac{\beta}{2} G^2 + \partial_z G - (\alpha + 2N_f) \frac{1}{z} G + C = 0 \quad (28)$$

The constant C is fixed by noting that for large N and large z , $G/N \approx 2/z$ so that $C = 4aN^2$, in agreement with (18).

The general time-dependent solution can be analyzed by first re-scaling $\tau \rightarrow \tau/N$. In the large N limit G solves

$$\partial_\tau G + \beta \left(\frac{1}{N} G - az \right) \partial_z G - a\beta G = 0 \quad (29)$$

which is a Burgers like nonlinear PDE. We can solve it with the method of complex characteristics [20]. For that we introduce curves in the space of z and τ , parametrized by s and labeled by z_0 , along which the nonlinear PDE follows from ordinary ODE

$$\begin{aligned} \frac{dz}{ds} = \beta \left(\frac{1}{N} G - az \right) \\ \frac{dG}{ds} = a\beta G \\ \frac{d\tau}{ds} = 1 \end{aligned} \quad (30)$$

with the condition that $z(\tau = 0) = z_0$ and $\tau(s = 0) = 0$ (which means $\tau = s$). These ODE can be solved for a specific initial condition.

7. Sound waves

To gain some insights to the general time-dependent solutions, we analyze first the hydrodynamical equations in the linearized density approximation. For that, it is convenient to identify the classical hydrodynamical action associated to (6). Using the standard canonical procedure we found

$$\mathbf{S} = \int dt d\lambda \rho(\lambda) (v^2 - \mathbf{u}[\rho]) \quad (31)$$

which is linearized by

$$\rho \approx \rho_0(\lambda) + 2\partial_\lambda \varphi \quad \text{and} \quad \rho v \approx -\partial_t \varphi \quad (32)$$

Inserting (32) into \mathbf{S} yields in the quadratic approximation

$$\mathbf{S}_2 = \int dt \frac{d\lambda}{\rho_0(\lambda)} \left((\partial_t \varphi)^2 - \rho_0^2(\lambda) W^2[\varphi] \right) \quad (33)$$

with the potential

$$W[\varphi] = \beta\pi \partial_\lambda [\varphi_H(\lambda) + \varphi_H(-\lambda)] - \partial_\lambda \left(\frac{\partial_\lambda \varphi}{\rho_0(\lambda)} \right) \quad (34)$$

For $\rho(\lambda) = \rho_0(\lambda)$, after the rescaling $Nt \rightarrow t$, (33) simplifies for large N

$$\begin{aligned} \mathbf{S}_2 \approx N^2 \int dt \frac{d\lambda}{\rho_0(\lambda)} \\ \times \left((\partial_t \varphi)^2 - \frac{(\pi\beta\rho_0(\lambda))^2}{N^2} (\partial_\lambda \varphi_H(\lambda) + \partial_\lambda \varphi_H(-\lambda))^2 \right) \end{aligned} \quad (35)$$

The speed of sound is non-local in the chiral spectrum. At zero virtuality it follows from (35) as

$$v_s(0) = \frac{2\pi\beta\rho_0(0)}{N} = 2\beta\sqrt{a} \quad (36)$$

(36) characterizes the rate of change of the fluid of eigenvalues with the time t . As noted earlier, t is in units of the eigenvalue squared. We note that (35) is extensive with N for $\rho_0(\lambda)/N$ normalized to 1.

8. Relaxation time

An interesting dynamical question regarding the Dirac spectrum is the typical relaxation time associated to the formation or disappearance of the spontaneous breaking of chiral symmetry. We answer this question by focusing on the time it takes for the eigenvalues near zero-virtuality viewed as a hydrodynamical fluid to re-arrange.

For simplicity, consider that at $\tau = 0$ all the eigenvalues are localized at zero virtuality. The relaxation of this phase is characterized by the time it takes the sound wave to fill up the gap or the so-called zero-mode-zone (ZMZ) spanned by Wigner semi-circle. To show this we use the initial condition $G(z = z_0, \tau = 0) = 2N/z_0$ in (30) to obtain the solution

$$G(\tau; z) = e^{a\beta\tau} \frac{2N}{z_0} \quad (37)$$

The remaining equation is now

$$\frac{dz}{d\tau} = \beta \left(\frac{2}{z_0} e^{a\beta\tau} - az \right) \quad (38)$$

which yields

$$z = \frac{1}{a z_0} e^{a\beta\tau} + \left(z_0 - \frac{1}{a z_0} \right) e^{-a\beta\tau} \quad (39)$$

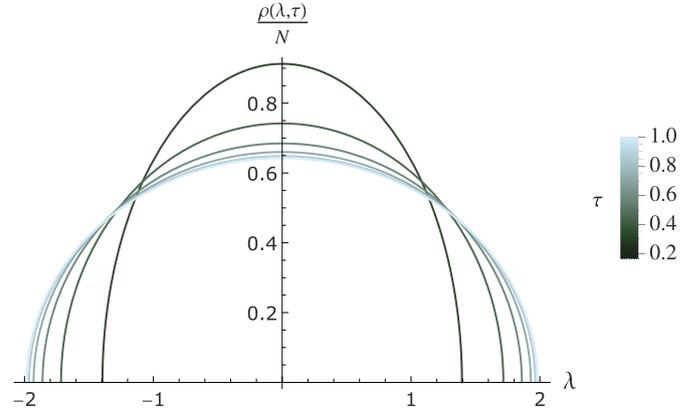


Fig. 1. The spectral density (41) computed for $\beta = 2$, $a = 1$ and $\tau = \{1/6, 1/3, 1/2, 2/3, 5/6, 1\}$.

Solving for z_0 and inserting the answer in (37) yield

$$\begin{aligned} \frac{G(\tau; z)}{N\sqrt{a}} &= \left(1 - e^{-v_s(0)\tau\sqrt{a}} \right)^{-1} \\ &\times \left(z\sqrt{a} - \sqrt{(z\sqrt{a})^2 - 4 \left(1 - e^{-v_s(0)\tau\sqrt{a}} \right)} \right) \end{aligned} \quad (40)$$

The spectral density follows from the discontinuity of (12) (see Fig. 1)

$$\begin{aligned} \rho(\tau, \lambda) &= \frac{aN}{\pi} \left(1 - e^{-v_s(0)\tau\sqrt{a}} \right)^{-1} \\ &\times \left(\frac{4}{a} \left(1 - e^{-v_s(0)\tau\sqrt{a}} \right) - \lambda^2 \right)^{\frac{1}{2}} \end{aligned} \quad (41)$$

which is seen to interpolate between a delta-function $N\delta(\lambda)$ at $\tau = 0$ and a Wigner semi-circle at asymptotic τ at a rate given by the speed of sound $v_s(0)$ at zero virtuality. (41) shows that the relaxation time of the spectral density is

$$\tau_R = \frac{1/\sqrt{a}}{v_s(0)} \quad (42)$$

which is the time it takes the sound to fill up the ZMZ zone.

The time estimate (42) is a characteristic of the re-organization of the fluid of Dirac eigenvalues irrespective of the details of the initial condition. It is non-perturbative. In particular, it characterizes the re-organization of the Dirac eigenvalues from a gapped phase at zero-virtuality with chiral symmetry restored, to an ungapped phase at zero virtuality with chiral symmetry spontaneously broken. For that, consider the initial condition for a gapped distribution of eigenvalues

$$\rho(0, \lambda) = N\delta(\lambda - b) + N\delta(\lambda + b) \quad (43)$$

The pertinent initial condition is now

$$G(z = z_0, \tau = 0) = \frac{2Nz_0}{z_0^2 - b^2} \quad (44)$$

A re-run of the preceding arguments yields a cubic equation for G/N

$$\frac{G(\tau; z)}{N\sqrt{a}} = 2 \frac{z\sqrt{a} - \frac{G(z, \tau)}{2N\sqrt{a}} (1 - e^{-2a\beta\tau})}{(z\sqrt{a} - \frac{G(z, \tau)}{2N\sqrt{a}} (1 - e^{-2a\beta\tau}))^2 - b^2 e^{-2a\beta\tau}} \quad (45)$$

that is solvable exactly for any time. The large time behavior follows by taking the limits $\tau \rightarrow \infty$ and $e^{-a\beta\tau} \rightarrow 0$ in (45). The result

is a static quadratic equation for G/N which yields Wigner semi-circle for the eigenvalue distribution. For large τ but small $e^{-a\beta\tau}$, the explicit solution is

$$\frac{G(\tau; z)}{N\sqrt{a}} \approx \frac{z\sqrt{a} - \sqrt{(z\sqrt{a})^2 - 4(1 - e^{-2a\beta\tau} + ab^2e^{-2a\beta\tau})}}{1 - e^{-2a\beta\tau} + ab^2e^{-2a\beta\tau}} \quad (46)$$

The time evolution is still controlled by the factor $e^{-2a\beta\tau} = e^{-v_s(0)\tau\sqrt{a}}$, and therefore the relaxation time (42).

In so far, the description of the Dirac spectrum as a fluid of eigenvalues is mathematical, with (42) characterizing the rate at which the eigenvalues relax say from an initial distribution with restored chiral symmetry to a final distribution with spontaneous chiral symmetry breaking. We now suggest that this description in eigenvalue space is dual to the relaxation in physical space under the same conditions. The physical relaxation time for the breaking/restoration of chiral symmetry in canonical dimension is then

$$T_R \equiv a^{\frac{3}{2}} \tau_R = \frac{\pi\rho(0)}{2\beta N} = \frac{|(q^\dagger q)|}{2\beta \mathbf{n}} \quad (47)$$

Recall that in the chiral random matrix model (1), the scale a is related to the chiral condensate by the Banks–Casher formula $V_4(q^\dagger q) = -\pi\rho(0) = -N\sqrt{a}$. Thus, the last equality with $\mathbf{n} = N/V_4$.

9. Conclusions

The hydrodynamical description of the chiral Dirac spectrum captures some key aspects of the relaxation of the Dirac eigenvalues in the ergodic regime. The hydrodynamical set-up supports a (tunneling) minimum that describes the hydrodynamical relaxation of the Dirac eigenvalues as a fluid. The fluid exhibits sound waves that can be used to estimate the time it takes for a gapped eigenvalue density with chiral symmetry restored, to a Wigner semi-circle with chiral symmetry spontaneously broken. We have suggested that this time is dual to a physical relaxation time.

Acknowledgements

The work of YL and IZ is supported in part by the U.S. Department of Energy under Contracts No. DE-FG-88ER40388. The work of PW is supported by the DEC-2011/02/A/ST1/00119 grant and the UMO-2013/08/T/ST2/00105 ETIUDA scholarship of the (Polish) National Centre of Science. PW thanks Stony Brook University for its hospitality during the completion of this work.

References

- [1] M.A. Nowak, M. Rho, I. Zahed, World Scientific, Singapore, 1996, 528 pp.
- [2] C. Bernard, C.E. DeTar, L. Levkova, S. Gottlieb, U.M. Heller, J.E. Hetrick, J. Osborn, D.B. Renner, et al., PoS LAT 2007 (2007) 090, arXiv:0710.1118 [hep-lat].
- [3] T. Banks, A. Casher, Nucl. Phys. B 169 (1980) 103.
- [4] R.A. Janik, M.A. Nowak, G. Papp, I. Zahed, Phys. Rev. Lett. 81 (1998) 264, arXiv:hep-ph/9803289; R.A. Janik, M.A. Nowak, G. Papp, I. Zahed, Prog. Theor. Phys. Suppl. 131 (1998) 471; R.A. Janik, M.A. Nowak, G. Papp, I. Zahed, arXiv:hep-ph/9905274.
- [5] E.V. Shuryak, J.J.M. Verbaarschot, Nucl. Phys. A 560 (1993) 306, arXiv:hep-th/9212088; J.J.M. Verbaarschot, I. Zahed, Phys. Rev. Lett. 70 (1993) 3852, arXiv:hep-th/9303012.
- [6] J.J.M. Verbaarschot, Phys. Rev. Lett. 72 (1994) 2531, arXiv:hep-th/9401059.
- [7] J.C. Osborn, J.J.M. Verbaarschot, Nucl. Phys. B 525 (1998) 738, arXiv:hep-ph/9803419.
- [8] Y. Liu, P. Warchol, I. Zahed, arXiv:1505.02107 [hep-ph].
- [9] M.A. Nowak, J.J.M. Verbaarschot, I. Zahed, Phys. Lett. B 217 (1989) 157.
- [10] H. Leutwyler, A.V. Smilga, Phys. Rev. D 46 (1992) 5607.
- [11] R.A. Janik, M.A. Nowak, G. Papp, I. Zahed, Acta Phys. Pol. B 29 (1998) 3957, arXiv:hep-ph/9812376.
- [12] A. Mollgaard, K. Splittorff, Phys. Rev. D 88 (11) (2013) 116007, arXiv:1309.4335 [hep-lat]; P.H. Damgaard, J. Phys. Conf. Ser. 287 (2011) 012004, arXiv:1102.1295 [hep-ph]; J.J.M. Verbaarschot, arXiv:hep-th/0502029; G. Akemann, E. Kanzieper, Nucl. Phys. Proc. Suppl. 94 (2001) 681, arXiv:hep-lat/0010092; K. Takahashi, S. Iida, Nucl. Phys. A 670 (2000) 60; M.E. Berbenni-Bitsch, M. Gockeler, H. Hehl, S. Meyer, P.E.L. Rakow, A. Schafer, T. Wettig, Phys. Lett. B 466 (1999) 293, arXiv:hep-lat/9907014; T. Wettig, arXiv:hep-lat/9905020.
- [13] E. Akkermans, G. Montambaux, Cambridge University Press, 2007, 582 pp.
- [14] A. Jevicki, B. Sakita, Nucl. Phys. B 165 (1980) 511; A. Jevicki, Nucl. Phys. B 376 (1992) 75; A.P. Polychronakos, Phys. Rev. Lett. 74 (1995) 5153, arXiv:hep-th/9411054.
- [15] F.J. Dyson, J. Math. Phys. 13 (1972) 90.
- [16] C.W.J. Beenakker, B. Rejaei, Physica A 203 (1994); C.W.J. Beenakker, Rev. Mod. Phys. 69 (1997) 731.
- [17] F.J. Dyson, J. Math. Phys. 3 (1962) 140; F.J. Dyson, J. Math. Phys. 3 (1962) 157; F.J. Dyson, J. Math. Phys. 3 (1962) 1191.
- [18] F. Kalisch, D. Braak, arXiv:cond-mat/0201585.
- [19] J.-P. Blaizot, M.A. Nowak, P. Warchol, Phys. Lett. B 724 (2013) 170, arXiv:1303.2357 [hep-ph]; J.-P. Blaizot, M.A. Nowak, P. Warchol, Phys. Rev. E 87 (2013) 052134, arXiv:1211.0029 [math-ph].
- [20] S.J. Chapman, C.J. Howls, J.R. King, A.B. Olde Daalhuis, Nonlinearity 20 (2007) 2425; J.-P. Blaizot, M.A. Nowak, Phys. Rev. E 82 (2010) 051115; J.-P. Blaizot, M.A. Nowak, P. Warchol, Phys. Rev. E 87 (2013) 052134.