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On lifting of 2-vector fields to r-jet prolongation of the tangent bundle

Abstract. If $m \geq 3$ and $r \geq 1$, we prove that any natural linear operator A lifting 2-vector fields $\Lambda \in \Gamma(\bigwedge^2 TM)$ (i.e., skew-symmetric tensor fields of type $(2,0)$) on m-dimensional manifolds M into 2-vector fields $A(\Lambda)$ on r-jet prolongation J^rTM of the tangent bundle TM of M is the zero one.

Introduction. All manifolds considered in this paper are assumed to be finite dimensional and smooth. Maps between manifolds are assumed to be smooth (of C^∞).

Let $\mathcal{M}f_m$ be the category of m-dimensional manifolds and their submersions and $\mathcal{VB}$ be the category of vector bundles and their vector bundle homomorphisms.

The r-jet prolongation of the tangent bundle over m-manifolds is the (vector bundle) functor $J^rT : \mathcal{M}f_m \to \mathcal{VB}$ sending any m-manifold M into the vector bundle J^rTM of r-jets j^r_xX at points $x \in M$ of vector fields X on M and every $\mathcal{M}f_m$-map $\varphi : M \to N$ into $J^rT\varphi : J^rTM \to J^rTN$ given by $J^rT\varphi(j^r_xX) = j^r_{\varphi(x)}(T\varphi \circ X \circ \varphi^{-1})$.

An $\mathcal{M}f_m$-natural linear operator $A : \bigwedge^2 T \rightsquigarrow \bigwedge^2 T(J^rT)$ is an $\mathcal{M}f_m$-invariant family of $\mathbf{R}$-linear regular operators (functions)

$$A : \Gamma\Big(\bigwedge^2 TM\Big) \to \Gamma\Big(\bigwedge^2 T(J^rTM)\Big)$$

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for m-manifolds M, where $\Gamma(\bigwedge^2 TN)$ is the vector space of 2-vector fields (i.e., skew-symmetric tensor fields of type $(2,0)$) on a manifold N. The invariance of A means that if $\Lambda \in \Gamma(\bigwedge^2 TM)$ and $\Lambda_1 \in \Gamma(\bigwedge^2 TM_1)$ are φ-related (i.e., $\bigwedge^2 T\varphi \circ \Lambda = \Lambda_1 \circ \varphi$) for a $\mathcal{M}f_m$-map $\varphi : M \to M_1$, then $A(\Lambda)$ and $A(\Lambda_1)$ are $J^rT\varphi$-related.

The main result of the present note can be written as follows.

Theorem 0.1. *If $m \geq 3$ and $r \geq 1$, then any natural linear operator A lifting 2-vector fields $\Lambda \in \Gamma(\bigwedge^2 TM)$ on m-manifolds M into 2-vector fields $A(\Lambda) \in \Gamma(\bigwedge^2 T(J^rTM))$ on J^rTM is the zero one.*

The general concept of natural operators can be found in the fundamental monograph [2]. Natural operators lifting 2-vector fields can be applied in investigations of Poisson structures. That is why, they are studied in many papers, see e.g. [1, 3].

From now on, the usual coordinates on $\mathbf{R}^m$ will be denoted by $x^1, \dots, x^m$. The usual canonical vector fields on $\mathbf{R}^m$ will be denoted by $\partial_1, \dots, \partial_m$.

1. Some lemmas. The proof of Theorem 0.1 will occupy the rest of the note. We start with several lemmas.

Lemma 1.1. *Let $m \geq 3$ and $r \geq 1$ be integers. Consider an $\mathcal{M}f_m$-natural linear operator $A : \bigwedge^2 T \rightsquigarrow \bigwedge^2 T(J^rT)$. Assume that $A((x^1)^q \partial_2 \wedge \partial_3)_{|j_0^r \partial_1} = 0$ for $q = 0, 1, 2, \dots$. Then $A = 0$.*

Proof. To prove that $A = 0$, it is sufficient to show that $A(\Lambda)_{|j_x^r Y} = 0$ for any m-manifold M, any $x \in M$, any $Y \in \mathcal{X}(M)$ and any $\Lambda \in \Gamma(\bigwedge^2 TM)$.

Of course, we may (without loss of generality) assume $Y_{|x} \neq 0$. Then by the invariance of A with respect to charts and the Frobenius theorem we may assume $M = \mathbf{R}^m$, $x = 0$ and $Y = \partial_1$. Since A is linear, we may assume that $\Lambda = f Z_1 \wedge Z_2$, where $f : \mathbf{R}^m \to \mathbf{R}$ and Z_1 and Z_2 are constant vector fields on $\mathbf{R}^m$. Moreover, we may assume that ∂_1, Z_1, Z_2 are $\mathbf{R}$-linearly independent. Then, because of the invariance of A with respect to linear isomorphisms, we may assume that $Z_1 = \partial_2$ and $Z_2 = \partial_3$. Then by the multi-linear Peetre theorem (Theorem 19.9 in [2]) we may assume that $f = (x_1)^{\alpha_1} (x^2)^{\alpha_2} (x^3)^{\alpha_3} \dots (x^m)^{\alpha_m}$ is an arbitrary monomial.

Let $\alpha_1, \dots, \alpha_m$ be arbitrary non-negative integers. There exists a 0-preserving $\mathcal{M}f_m$-map $\varphi = (x^1, \varphi^2(x^2), x^3, \dots, x^m)$ preserving $x^1, x^3, \dots, x^m$, ∂_1, ∂_3 and sending (the germ at 0 of) ∂_2 into (the germ at 0 of) $\partial_2 + (x^2)^{\alpha_2} \partial_2$. Then by the invariance of A with respect to φ, from the assumption $A((x^1)^{\alpha_1} \partial_2 \wedge \partial_3)_{|j_0^r \partial_1} = 0$, we get

$$A((x^1)^{\alpha_1} \partial_2 \wedge \partial_3 + (x^1)^{\alpha_1} (x^2)^{\alpha_2} \partial_2 \wedge \partial_3)_{|j_0^r \partial_1} = 0 \, .$$

Then $A((x^1)^{\alpha_1} (x^2)^{\alpha_2} \partial_2 \wedge \partial_3)_{|j_0^r \partial_1} = 0$. Furthermore, there exists an $\mathcal{M}f_m$-map $\psi = (x^1, x^2, \psi^3(x^3, \dots, x^m), \dots, \psi^m(x^3, \dots, x^m))$ preserving 0, x^1,

x^2, ∂_1, ∂_2 and sending the germ at 0 of ∂_3 into the germ at 0 of $\partial_3 + (x^3)^{\alpha_3} \dots (x^m)^{\alpha_m}\partial_3$. Then by the invariance of A with respect to ψ, from the equality $A((x^1)^{\alpha_1}(x^2)^{\alpha_2}\partial_2 \wedge \partial_3)_{|j_0^r\partial_1} = 0$, we get

$$A((x^1)^{\alpha_1}(x^2)^{\alpha_2}\partial_2 \wedge \partial_3 + (x^1)^{\alpha_1}(x^2)^{\alpha_2}(x^3)^{\alpha_3} \dots (x^m)^{\alpha_m}\partial_2 \wedge \partial_3)_{|j_0^r\partial_1} = 0\,.$$

Then $A((x^1)^{\alpha_1}(x^2)^{\alpha_2}(x^3)^{\alpha_3} \dots (x^m)^{\alpha_m}\partial_2 \wedge \partial_3)_{|j_0^r\partial_1} = 0$. The lemma is complete. □

Lemma 1.2. (Lemma 42.4 in [2]) *Let N be a n-manifold and $x_o \in N$ be a point. Let X and Y be vector fields on a manifold N such that $X_{|x_o} \neq 0$ and $j^r_{x_o}(X) = j^r_{x_o}(Y)$. Then there exists an $\mathcal{M}f_n$-map φ such that $j^{r+1}_{x_o}(\varphi) = j^{r+1}_{x_o}(\mathrm{id})$ and $(\varphi)_* Y = X$ on some neighborhood of x_o.*

Lemma 1.3. *Let $m \geq 3$ and $r \geq 1$ be integers. Consider an $\mathcal{M}f_m$-natural linear operator $A : \bigwedge^2 T \rightsquigarrow \bigwedge^2 T(J^rT)$. Assume that $A((x^1)^q\partial_2 \wedge \partial_3)_{|j_0^r\partial_1} = 0$ for $q = 0, 1, 2, \dots, r$. Then $A = 0$.*

Proof. Let $q \geq r+1$ be an integer. Since $j_0^r\partial_2 = j_0^r(\partial_2 + (x^1)^q\partial_2)$, then (by Lemma 1.2) there exists an $\mathcal{M}f_m$-map

$$\varphi = (\varphi^1(x^1, x^2), \varphi^2(x^1, x^2), x^3, \dots, x^m)$$

preserving ∂_3, sending the germ at 0 of ∂_2 into the germ at 0 of $\partial_2 + (x^1)^q\partial_2$ and such that $j^{r+1}_x\varphi = j^{r+1}_0(\mathrm{id})$. Then φ preserves $j_0^r\partial_1$. Using the invariance of A with respect to φ, from assumption $A(\partial_2 \wedge \partial_3)_{|j_0^r\partial_1} = 0$, we get $A(\partial_2 \wedge \partial_3 + (x^1)^q\partial_2 \wedge \partial_3)_{|j_0^r\partial_1} = 0$. So, $A((x^1)^q\partial_2 \wedge \partial_3)_{|j_0^r\partial_1} = 0$. Then $A((x^1)^q\partial_2 \wedge \partial_3)_{|j_0^r\partial_1} = 0$ for any $q = 0, 1, \dots$. So, $A = 0$ because of Lemma 1.1. The lemma is complete. □

Let $\mathcal{J}^r(X^C)$ be the flow lift of a vector field X on M to J^rTM and $\mathcal{J}^r(X^V)$ be the vertical lift of X to J^rTM given by

$$\mathcal{J}^r(X^V)_{|j^r_xY} = \frac{d}{dt}_{|t=0}(j^r_xY + tj^r_xX)\,.$$

Lemma 1.4. *Let X be a vector field on a manifold M such that $X_{|x_o} = 0$ for some point $x_o \in M$. Let $\rho = j^r_{x_o}Y \in J^rT_{x_o}M$. Then*

$$\mathcal{J}^r(X^C)_{|\rho} = -\frac{d}{d\tau}_{|\tau=0}(\rho + \tau j^r_{x_o}([X, Y])) = -\mathcal{J}^r([X, Y]^V)_\rho\,,$$

where the bracket is the usual one on vector fields.

Proof. Let $\{\varphi_t\}$ be the flow of X. Then $\{J^rT\varphi_t\}$ is the flow of $\mathcal{J}^r(X^C)$ and $\varphi_t(x_o) = x_o$ for any sufficiently small t. Then

$$\begin{aligned}\mathcal{J}^r(X^C)_{|\rho} &= \frac{d}{dt}_{|t=0} J^rT\varphi_t(j^r_{x_o}(Y)) = \frac{d}{dt}_{|t=0} j^r_{x_o}((\varphi_t)_*Y)\\ &= -\frac{d}{dt}_{|t=0} j^r_{x_o}((\varphi_{-t})_*Y)) = -\frac{d}{d\tau}_{|\tau=0}(\rho + \tau j^r_{x_o}([X, Y]))\,. \quad \square\end{aligned}$$

Lemma 1.5. *For any $\lambda \in \mathbf{R}$, the collection consisting of*

$$v_i(\lambda) := \mathcal{J}^r((\partial_i)^C)_{|j^r_0(\lambda\partial_1)} \text{ and } V^\alpha_j(\lambda) := \mathcal{J}^r((x^\alpha\partial_j)^V)_{|j^r_0(\lambda\partial_1)}$$

for all $i, j = 1, \dots, m$ and $\alpha = (\alpha_1, \dots, \alpha_m) \in (\mathbf{N} \cup \{0\})^m$ with $|\alpha| = \alpha_1 + \dots + \alpha_m \leq r$ is the basis in $T_{j^r_0(\lambda\partial_1)} J^r T\mathbf{R}^m$. Of course, $x^\alpha := (x^1)^{\alpha_1} \cdot \dots \cdot (x^m)^{\alpha_m}$.

Proof. We have $V^\alpha_j(\lambda) = \frac{d}{dt}_{|t=0}(j^r_0(\lambda\partial_1) + t j^r_0(x^\alpha\partial_j))$. So, the lemma is clear. □

Lemma 1.6. *Let $m \geq 3$ and $r \geq 1$ be integers. Consider an $\mathcal{M}f_m$-natural linear operator $A : \bigwedge^2 T \rightsquigarrow \bigwedge^2 T(J^r T)$. Denote $v_i := v_i(1)$ and $V^\alpha_i := V^\alpha_i(1)$. Then, given $q = 0, 1, \dots, r-1$, we have*

$$A((x^1)^q \partial_2 \wedge \partial_3)_{|j^r_0(\partial_1)} = a^{(q)} v_2 \wedge v_3$$

for some (unique) real number $a^{(q)}$. Moreover, we have

$$A((x^1)^r \partial_2 \wedge \partial_3)_{|j^r_0(\partial_1)} = a v_2 \wedge v_3 + b v_2 \wedge V_3^{(r,0,\dots,0)} - b v_3 \wedge V_2^{(r,0,\dots,0)}$$

for some (unique) real numbers a and b.

Proof. Let $q \in \{0, 1, \dots, r\}$. Because of Lemma 1.5, we can write

$$\begin{aligned} A((x^1)^q \partial_2 \wedge \partial_3)_{|j^r_0(\lambda\partial_1)} &= \sum_{1 \leq i < j \leq m} a^{i,j}(\lambda) v_i(\lambda) \wedge v_j(\lambda) \\ &+ \sum_{i,j,\alpha} b^{i,j}_\alpha(\lambda) v_i(\lambda) \wedge V^\alpha_j(\lambda) + \sum_{(i,\alpha)<(j,\beta)} c^{i,j}_{\alpha,\beta}(\lambda) V^\alpha_i(\lambda) \wedge V^\beta_j(\lambda) \end{aligned}$$

for some (unique) real numbers $a^{i,j}(\lambda), b^{i,j}_\alpha(\lambda), c^{i,j}_{\alpha,\beta}(\lambda)$ smoothly depending on λ (and depending on q), where $\sum_{i,j,\alpha}$ is the sum over all $i, j \in \{1, \dots, m\}$ and all $\alpha \in (\mathbf{N} \cup \{0\})^m$ with $|\alpha| \leq r$, and $\sum_{(i,\alpha)<(j,\beta)}$ is the sum over all $i, j \in \{1, \dots, m\}$ and all $\alpha, \beta \in (\mathbf{N} \cup \{0\})^m$ with $|\alpha| \leq r$ and $|\beta| \leq r$ and $(i, \alpha) < (j, \beta)$. Here $(\mathbf{N} \cup \{0\}) \times (\mathbf{N} \cup \{0\})^m$ is ordered lexicographically, i.e., $(i, \alpha) \leq (j, \beta)$ iff $i < j$ or ($i = j$ and $\alpha_1 < \beta_1$) or ($i = j$, $\alpha_1 = \beta_1$ and $\alpha_2 < \beta_2$) or … or ($i = j$, $\alpha_1 = \beta_1, \dots, \alpha_{m_1} = \beta_{m_1}, \alpha_m \leq \beta_m$).

If $\alpha_2 + \dots + \alpha_m \geq 1$, using the invariance of A with respect to $(x^1, tx^2, \dots, tx^m)$, we get $t^2 b^{i,j}_\alpha(\lambda) = t^s b^{i,j}_\alpha(\lambda)$ for some integer $s < 2$. Hence $b^{i,j}_\alpha(\lambda) = 0$ if $\alpha_2 + \dots + \alpha_m \geq 1$. If $\alpha_2 + \dots + \alpha_m = 0$ and $(i, j) \notin \{(2, 3), (3, 2)\}$, then (applying the invariance of A with respect to $(x^1, tx^2, \tau x^3, x^4, \dots, x^m)$) we get $b^{i,j}_{(\alpha_1,0,\dots,0)}(\lambda) = 0$. By almost the same arguments, if $\alpha_2 + \dots + \alpha_m + \beta_2 + \dots + \beta_m \geq 1$ or $(i, j) \neq (2, 3)$, then $c^{i,j}_{\alpha,\beta}(\lambda) = 0$.

Similarly, by the invariance of A with respect to $(x^1, tx^2, \tau x^3, x^4, \dots, x^m)$, if $(i,j) \neq (2,3)$, then $a^{i,j}(\lambda) = 0$. Hence

$$
\begin{aligned}
A((x^1)^q \partial_2 \wedge \partial_3)_{|j_0^r(\lambda\partial_1)} &= a(\lambda) v_2(\lambda) \wedge v_3(\lambda) \\
&+ \sum_{l=0}^{r} b_l(\lambda) v_2(\lambda) \wedge V_3^{(l,0,\dots,0)}(\lambda) + \sum_{l=0}^{r} c_l(\lambda) v_3(\lambda) \wedge V_2^{(l,0,\dots,0)}(\lambda) \\
&+ \sum_{l_1,l_2=0}^{r} d_{l_1,l_2}(\lambda) V_2^{(l_1,0,\dots,0)}(\lambda) \wedge V_3^{(l_2,0,\dots,0)}(\lambda)
\end{aligned}
$$

for the (unique) real numbers $a(\lambda), b_l(\lambda), c_l(\lambda), d_{l_1,l_2}(\lambda)$ smoothly depending on λ (and depending on q).

Since $[\partial_2 + x^2\partial_3, \partial_3] = 0$, there exists an $\mathcal{M}f_m$-map

$$\varphi = (x^1, \varphi^2(x^2, x^3), \varphi^3(x^2, x^3), x^4, \dots, x^m)$$

preserving 0 and x^1 and ∂_1 and (the germ at 0 of) ∂_3 and sending (the germ at 0 of) ∂_2 into (the germ at 0 of) $\partial_2 + x^2\partial_3$. One can easily see that such φ preserves (the germ at 0 of) $(x^1)^q \partial_2 \wedge \partial_3$ (as $\partial_2 \wedge \partial_3 = (\partial_2 + x^2\partial_3) \wedge \partial_3$), $j_0^r(\lambda\partial_1)$, $v_2(\lambda)$ (as $\mathcal{J}^r((x^2\partial_3)^C)_{|j_0^r(\lambda\partial_1)} = 0$ because of Lemma 1.4), $v_3(\lambda)$, $V_3^{(l,0,\dots,0)}(\lambda)$ and $V_2^{(r,0,\dots,0)}(\lambda)$, and it sends $V_2^{(l,0,\dots,0)}(\lambda)$ into $V_2^{(l,0,\dots,0)}(\lambda) + V_3^{(l,1,0,\dots,0)}(\lambda)$ for $l = 0, 1, \dots, r-1$. Then using the invariance of A with respect to φ, we get

$$
\begin{aligned}
&\sum_{l=0}^{r-1} c_l(\lambda) v_3(\lambda) \wedge V_3^{(l,1,0,\dots,0)}(\lambda) \\
&\qquad + \sum_{l_1=0}^{r-1} \sum_{l_2=0}^{r} d_{l_1,l_2}(\lambda) V_3^{(l_1,1,0,\dots,0)}(\lambda) \wedge V_3^{(l_2,0,\dots,0)}(\lambda) = 0\,.
\end{aligned}
$$

Then $c_l(\lambda) = 0$ for $l = 0, \dots, r-1$ and $d_{l_1,l_2} = 0$ for $l_1 = 0, \dots, r-1$ and $l_2 = 0, \dots, r$. Quite similarly, replacing 2 by 3 and vice-versa, we get $b_l(\lambda) = 0$ for $l = 0, \dots, r-1$ and $d_{l_1,l_2}(\lambda) = 0$ for $l_2 = 0, \dots, r-1$ and $l_1 = 0, \dots, r$. Moreover, $b_r(\lambda) = -c_r(\lambda)$. Hence

$$
\begin{aligned}
A((x^1)^q \partial_2 \wedge \partial_3)_{|j_0^r(\lambda\partial_1)} &= a(\lambda) v_2(\lambda) \wedge v_3(\lambda) \\
&+ b(\lambda) v_2(\lambda) \wedge V_3^{(r,0,\dots,0)}(\lambda) - b(\lambda) v_3(\lambda) \wedge V_2^{(r,0,\dots,0)}(\lambda) \\
&+ c(\lambda) V_2^{(r,0,\dots,0)}(\lambda) \wedge V_3^{(r,0,\dots,0)}(\lambda)
\end{aligned}
$$

for the (unique) real numbers $a(\lambda)$, $b(\lambda)$, $c(\lambda)$ smoothly depending on λ (and depending on q). Then, using the invariance of A with respect to $(tx^1, x^2, \dots, x^m)$, we get $\frac{1}{t^q} b(t\lambda) = \frac{1}{t^r} b(\lambda)$ and $\frac{1}{t^q} c(t\lambda) = \frac{1}{t^{2r}} c(\lambda)$. Then $c(\lambda) = 0$ for $q = 0, \dots, r$, and $b(\lambda) = 0$ for $q = 0, \dots, r-1$. The lemma is complete. $\square$

Lemma 1.7. *Let $m \geq 3$ and $r \geq 1$ be integers. Consider an $\mathcal{M}f_m$-natural linear operator $A : \bigwedge^2 T \rightsquigarrow \bigwedge^2 T(J^rT)$. Then $A(\partial_2 \wedge \partial_3)_{|j^r_0(\partial_1)} = 0$.*

Proof. Since $j^r_0(\partial_2 + (x^1)^{r+1}\partial_2) = j^r_0(\partial_2)$, then (by Lemma 1.2) there exists an $\mathcal{M}f_m$-map

$$\varphi = (\varphi^1(x^1, x^2), \varphi^2(x^1, x^2), x^3, \dots, x^m)$$

preserving 0 and ∂_3 and sending the germ at 0 of ∂_2 into the germ at 0 of $\partial_2 + (x^1)^{r+1}\partial_2$ and such that $j^{r+1}_0\varphi = j^{r+1}_0(\mathrm{id})$. Then φ preserves v_3, $j^r_0(\partial_1)$ and it sends v_2 into $v_2 + (r+1)V_2^{(r,0,\dots,0)}$. Then by the invariance of A with respect to φ and Lemma 1.6, we get

$$A((x^1)^{r+1}\partial_2 \wedge \partial_3)_{|j^r_0(\partial_1)} = (r+1)a^{(0)}V_2^{(r,0,\dots,0)} \wedge v_3 \, .$$

Similarly, replacing 2 on 3 and vice-versa, we easily get

$$A((x^1)^{r+1}\partial_2 \wedge \partial_3)_{|j^r_0(\partial_1)} = (r+1)a^{(0)}v_2 \wedge V_3^{(r,0,\dots,0)} \, .$$

Then $a^{(0)} = 0$. The lemma is complete. □

Lemma 1.8. *Let $m \geq 3$ and $r \geq 1$ be integers. Consider an $\mathcal{M}f_m$-natural linear operator $A : \bigwedge^2 T \rightsquigarrow \bigwedge^2 T(J^rT)$. Then $A(f(x^1, x^2)\partial_2 \wedge \partial_3)_{|j^r_0(\partial_1)} = 0$ for any smooth map $f : \mathbf{R}^2 \to \mathbf{R}$ with $j^r_0(f) = 0$.*

Proof. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be such that $j^r_0(f) = 0$. Since $j^r_0(\partial_2 + f(x^1, x^2)\partial_2) = j^r_0(\partial_2)$, then (by Lemma 1.2) there exists an $\mathcal{M}f_m$-map

$$\psi = (\psi^1(x^1, x^2), \psi^2(x^1, x^2), x^3, \dots, x^m)$$

preserving 0 and ∂_3, and sending the germ at 0 of ∂_2 into the germ at 0 of $\partial_2 + f(x^1, x^2)\partial_2$ and such that $j^{r+1}_0(\psi) = j^{r+1}_0(\mathrm{id})$ (then ψ preserves $j^r_0(\partial_1)$). Then using Lemma 1.7 and the invariance of A with respect to ψ from $A(\partial_2 \wedge \partial_3)_{|j^r_0(\partial_1)} = 0$, we get $A(\partial_2 \wedge \partial_3 + f(x^1, x^2)\partial_2 \wedge \partial_3)_{|j^r_0(\partial_1)} = 0$. So, $A(f(x^1, x^2)\partial_2 \wedge \partial_3)_{|j^r_0(\partial_1)} = 0$. □

2. Proof of the main result.

Proof of Theorem 0.1. Let $m \geq 3$ and $r \geq 1$ be integers. Consider an $\mathcal{M}f_m$-natural linear operator $A : \bigwedge^2 T \rightsquigarrow \bigwedge^2 T(J^rT)$. We are going to prove that $A = 0$. Because of Lemma 1.3 it is sufficient to prove that $A((x^1)^q\partial_2 \wedge \partial_3)_{|j^r_0(\partial_1)} = 0$ for $q = 0, \dots, r$.

Let $q \in \{0, \dots, r\}$. By Lemma 1.7, we may assume that $q \geq 1$. Since $j^r_0(\partial_2 + (x^1)^{r+1}\partial_2) = j^r_0(\partial_2)$, then (by Lemma 1.2) there exists an $\mathcal{M}f_m$-map

$$\varphi = (\varphi^1(x^1, x^2), \varphi^2(x^1, x^2), x^3, \dots, x^m)$$

preserving 0 and ∂_3, and sending the germ at 0 of ∂_2 into the germ at 0 of $\partial_2 + (x^1)^{r+1}\partial_2$ and such that $j^{r+1}_0\varphi = j^{r+1}_0(\mathrm{id})$. Then φ preserves ∂_3, v_3, $j^r_0(\partial_1)$, $V_2^{(r,0,\dots,0)}$, $V_3^{(r,0,\dots0)}$, and it sends v_2 into $v_2 + (r+1)V_2^{(r,0,\dots,0)}$ (to see this we propose to use Lemma 1.4) and it sends the germ at 0 of $(x^1)^q\partial_2$ into the germ at 0 of $(x^1)^q\partial_2 + f(x^1,x^2)\partial_2$ for some $f:\mathbf{R}^2\to\mathbf{R}$ with $j^r_0(f)=0$.

If $q\le r-1$, then by the invariance of A with respect to φ and Lemma 1.6 and Lemma 1.8, we get

$$0 = A(f(x^1,x^2)\partial_2\wedge\partial_3)_{|j^r_0(\partial_1)} = (r+1)a^{(q)}V_2^{(r,0,\dots,0)}\wedge v_3\,.$$

Then $a^{(q)}=0$, and then $A((x^1)^q\partial_2\wedge\partial_3)_{|j^r_0(\partial_1)}=0$.

If $q=r$, then by the invariance of A with respect to φ and Lemma 1.6 and Lemma 1.8, we get

$$\begin{aligned} 0 &= A(f(x^1,x^2)\partial_2\wedge\partial_3)_{|j^r_0(\partial_1)} \\ &= (r+1)aV_2^{(r,0,\dots,0)}\wedge v_3 + b(r+1)V_2^{(r,0,\dots,0)}\wedge V_3^{(r,0,\dots,0)}\,. \end{aligned}$$

Then $a=0$ and $b=0$, and then $A((x^1)^r\partial_2\wedge\partial_3)_{|j^r_0(\partial_1)}=0$.

Hence $A=0$ because of Lemma 1.3 and Theorem 0.1 is complete. □

References

[1] Debecki, J., *Linear natural lifting p-vectors to tensors of type $(q,0)$ on Weil bundles*, Czechoslovak Math. J. **66(141)** (2) (2016), 511–525.

[2] Kolář, I, Michor, P. W., Slovák, J., *Natural Operations in Differential Geometry*, Springer-Verlag, Berlin, 1993.

[3] Mikulski, W. M., *The linear natural operators lifting 2-vector fields to some Weil bundles*, Note di Math. **19**(2) (1999), 213–217.

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