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Modelling Influence Propagation in Social Networks

Anna Szczepanek
Institute of Mathematics, Jagiellonian University
ul. Łojasiewicza 6, 30-348 Kraków, Poland
e-mail: *anna.szczepanek@im.uj.edu.pl*

Abstract. This paper presents a formalised description of the models of influence propagation in social networks introduced in the classic paper of Kempe et al. The formal framework that we propose clarifies the structure of the most popular propagation models and helps rigorously re-establish the essential results concerning the problem of influence maximisation. We also introduce new models of propagation and show how they fit into the general picture. In particular, we focus on models that capture either positive or negative effects of resisting influence on individual's future resistance.

Keywords: social networks, influence propagation, influence maximisation, viral marketing.

1. Introduction

Influence propagation in social networks is an important area of research due to its applicability in various fields such as technology adoption, spread of ideas or, most recently, viral marketing. This marketing strategy consists in targeting a small group of highly influential individuals in order to trigger a large cascade of recommendations of the product throughout the network. However, the problem of identifying the most influential individuals turned out to be NP-hard even in the most basic models. This problem was addressed in the classic paper of Kempe, Kleinberg and Tardos [1], where the authors proposed a method of finding approximate solutions, laying foundations for further research in the field. The main aim of the present paper is to introduce a formal framework for influence propagation. This framework not only allows us to rigorously re-establish the main results obtained in [1, 2], but it also exposes relations between different models and reveals some yet unexplored ones, thus allowing us to define new meaningful models of propagation.

2. General Framework

Throughout the paper a social network is represented by a directed graph $G = (V, E)$, the vertices $V = \{v_1, \dots, v_N\}$ correspond to individuals and the edges $E \subset (V \times V) \setminus \{(v, v)|v \in V\}$ represent relations between them. We call this kind of graph a *network*. The set $N_v := \{u \in V \mid (u, v) \in E\}$ is called the set of *predecessors* of v. Let us fix a network G and a probability space $(\Omega, 2^\Omega, \mathbb{P})$, where Ω is finite.

Definition 1 *A family of stochastic processes* $\mathcal{P} := \{\mathcal{P}_S\}_{S \in 2^V}$, *where* $\mathcal{P}_S \colon \Omega \times \mathbb{N} \ni (\omega, i) \mapsto S_i(\omega) \in 2^V$, *is called a* propagation *in* G *if for all* $S \in 2^V$ *we have* $S_0(\omega) := S$ *and for* $\omega \in \Omega$, $i \geq 1$ *and* $v \in V$ *it follows from* $v \in S_i(\omega) \setminus S_{i-1}(\omega)$ *that there exists* $u \in S_{i-1}(\omega)$ *such that* $(u, v) \in E$.

A single process $\mathcal{P}_S$ is called a *propagation process* from S and the sets constituting the trajectory $(S, S_1(\omega), S_2(\omega), \dots)$ are called the *active sets* in subsequent time steps. The set S is said to be the *initial set* and it corresponds to the vertices chosen for activation at time $i = 0$. We also set $S_{-1} := \emptyset$ for the sake of consistency. As a model of propagation we understand a way of determining $S_i(\omega)$ for fixed i, ω and S. We say that two propagations are *equivalent* if for every initial set S and for every $W \subset V$ and $i \geq 0$ the value of probability $\mathbb{P}(S_i = W)$ is the same in both propagations. A propagation $\mathcal{P}$ is said to be *finished after time* τ, if τ is the smallest natural number such that $S_\tau(\omega) = S_{\tau+k}(\omega)$ for all ω, S and $k \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of non-negative integers

Lemma 1 *Let* $\mathcal{P}$ *be a propagation in* $G = (V, E)$. *If for all* $S \in 2^V$, $\omega \in \Omega$ *and* $i \in \mathbb{N}$ *we have* $S_{i-1}(\omega) \subset S_i(\omega)$ *and the implication* $S_{i-1}(\omega) = S_i(\omega) \Rightarrow S_i(\omega) = S_{i+1}(\omega)$ *holds, then* $\mathcal{P}$ *is finished after time* $\tau \leq \#V - 1$.

Proof. Let N denote the cardinality of V and suppose, contrary to our claim, that there exist S, ω and $k \in \mathbb{N}$ such that $S_{N-1}(\omega) \neq S_{N-1+k}(\omega)$. Then $S_{i-1}(\omega) \subsetneq S_i(\omega)$ for every $i \in \{0, 1, \dots, N\}$, thus for every such i we can choose $v_i \in S_i(\omega) \setminus S_{i-1}(\omega)$. We obtain $\{v_0, v_1, \dots, v_N\} \subset V$, which is a contradiction. □

Let $\mathcal{P}$ be a propagation finished after time τ. For an initial set S we consider the random variable $\varphi_S \colon \Omega \ni \omega \mapsto S_\tau(\omega) \subset V$. The function $\sigma \colon 2^V \to \mathbb{N}$, $\sigma(S) = \mathbb{E}(\#\varphi_S)$ is then called the *influence function* in $\mathcal{P}$. The problem of finding $S^* \subset V$ such that $S^* \in \operatorname{argmax}\{\sigma(T) \mid T \subset V,\ \#T = k\}$ is called the *influence maximisation problem* with parameter $k \in \mathbb{N}$. The breakthrough within the field of influence maximisation came when a lower bound for the efficiency of the greedy algorithm was established in [1]. The method used was based on a theorem proved in [3]. Before stating a special case of the theorem (see Theorem 1), we first recall some necessary notions. Let us consider a nonempty set X and a function $\sigma \colon 2^X \to \mathbb{N}$. We say that σ is *monotone* if $\sigma(S) \leq \sigma(T)$ holds for all sets S, T satisfying $S \subset T \subset X$. If for all such sets

S, T and for every $v \notin T$ we have $\sigma(S \cup \{v\}) - \sigma(S) \geq \sigma(T \cup \{v\}) - \sigma(T)$ then we say that σ is *submodular*. We also say that S is an *r-approximated solution* to the problem of maximising σ over k-element subsets of X if S satisfies $\sigma(S) \geq r\sigma(S^*)$, where $S^* \in \operatorname{argmax}\{\sigma(T) \mid T \subset X,\ \#T = k\}$ and $r \in (0, 1]$. By the *greedy algorithm* we mean the following steps:

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1. S := ∅
2. for i= 1 to k {
      v_i ← argmax_{v∈X\S} (σ(S ∪ {v}) − σ(S))
      S ← S ∪ {v_i}  }
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Theorem 1 *If X is a nonempty set and a function $\sigma\colon 2^X \to \mathbb{N}$ is monotone and submodular, then the set chosen by the greedy algorithm is a $(1 - 1/\mathrm{e})-$approximated solution to the problem of maximising σ over k-element subsets of X.*

Therefore, if we consider a propagation $\mathcal{P}$ in which the influence function σ is monotone and submodular, then the greedy algorithm finds $(1 - 1/\mathrm{e})-$approximated solution to the influence maximisation problem. In [1] the following strategy of determining the submodularity and monotonicity of σ was proposed. First, for a given $G = (V, E)$ we consider the set of subgraphs of the form $H = (V, F)$, where $F \subset E$. This set is denoted by Γ. For a given $S \subset V$ we define the set of *vertices reachable from S* in a subgraph H, denoted by $\psi_H(S)$: a vertex v is reachable from S if either $v \in S$ or there exist $k \in \mathbb{N}$ and vertices $\{w_0, w_1, \ldots, w_k\} \subset V$ satisfying $w_0 \in S$ and $(w_{i-1}, w_i) \in F$ for $i = 1, \ldots, k$, and also $(w_k, v) \in F$. Then we construct a function that transports probability measure from Ω on Γ in a way that enables us to express σ using the function $S \mapsto \#\psi_H(S)$, which is easily proved to be submodular and monotone. We will see that this method can be successfully applied to two basic models of propagation, but it fails even for slight generalisations.

Theorem 2 *Let $\mathcal{P}$ be a propagation in $G = (V, E)$. If there exists $h : \Omega \to \Gamma$ satisfying $\sigma(S) = \mathbb{E}(\#\psi_{h(\cdot)}(S))$ for every $S \subset V$, then σ is monotone and submodular.*

3. Cascade models

Cascade models describe propagation of influence as a sequence of attempts taken by active vertices in order to influence their inactive neighbours. We re-establish the main results stated in [1] to show how they fit into our framework (see Theorem 3), and we distinguish models that capture the change in individual's susceptibility to influence after having resisted influencers' attempts in the past.

For every edge (u, v) in a given network $G = (V, E)$ we fix a parameter $p_{(u,v)} \in (0, 1]$ and assign every pair $(u, v) \in V \times V$ a random variable $X_{(u,v)}\colon \Omega \to \{0, 1\}$ such that $\mathbb{P}(X_{(u,v)} = 1) = p_{(u,v)}$, where $p_{(u,v)} := 0$ if $(u, v) \notin E$. We assume that the random variables $\{X_{(u,v)} \,|\, (u, v) \in V^2\}$ are independent. We say that a propagation is given by the *Independent Cascade Model* (ICM) if for an initial set S and for a fixed $\omega \in \Omega$ the subsequent sets of active vertices are defined as $S_i(\omega) := S_{i-1}(\omega) \cup A_i(\omega)$, where $A_0(\omega) := S$ and $A_i(\omega) := \{v \in V \backslash S_{i-1}(\omega) |\, \exists_{u \in V}\ u \in A_{i-1}(\omega) \wedge X_{(u,v)}(\omega) = 1\}$

for $i \geq 1$. The set A_i is interpreted as the set of newly activated vertices in time step i. Note that the propagation given by ICM satisfies the assumptions of Lemma 1, thus it is finished after time $\tau \leq \#V - 1$. We also have $\mathbb{P}(v \in S_{i+1} \mid v \notin S_i) = 1 - \prod_{u \in A_i}(1 - p_{(u,v)})$. The influence maximisation problem in ICM is NP-hard [1].

Lemma 2 *If $v \notin S$, then $v \in S_\tau(\omega)$ if and only if there exist vertices $u_0, u_1 \ldots, u_k$ such that $u_0 \in S$ and we have $u_j \in A_j(\omega)$ and $X_{u_{j-1},u_j}(\omega) = 1$ for $j \in \{1, \ldots, k\}$, and also $X_{u_k,v}(\omega) = 1$.*

Proof. Note that $v \in S_\tau(\omega)$ if and only if $v \in A_{k+1}(\omega)$ for some $k \in \{0, \ldots, \tau - 1\}$. This is equivalent to the existence of $u_k \in A_k(\omega)$ such that $X_{u_k,v}(\omega) = 1$. Repeating this reasoning k times completes the proof. □

Theorem 3 *The influence function in ICM is monotone and submodular.*

Proof. Consider a network $G = (V, E)$ with ICM and define $h \colon \Omega \to \Gamma$ by taking $h(\omega) = (V, E_\omega)$ such that $(u, v) \in E_\omega$ if and only if $X_{(u,v)}(\omega) = 1$, where $X_{(u,v)}$ are the random variables associated with ICM. We will show that $\varphi_S(\omega) = \psi_{h(\omega)}(S)$ for all $S \subset V$ and $\omega \in \Omega$. Fix S and ω. It suffices to consider $v \notin S$ and to show that $v \in S_\tau(\omega)$ if and only if $v \in \psi_{h(\omega)}(S)$. This equivalence follows directly from Lemma 2, thus $\sigma(S) = \mathbb{E}(\#\psi_{h(\cdot)}(S))$ and Theorem 2 completes the proof. □

We now generalise ICM as proposed in [1], i.e. by relaxing the assumption of independence of $X_{u,v}$ for $u \in N_v$. We set $X_v := (X_{(u_i,v)} \mid u_i \in V,\ i = 1, \ldots, N)$ for every $v \in V$ and assume that the random vectors $\{X_v\}_{v \in V}$ are independent. We have to consider the probability of activation, conditioning on previous unsuccessful attempts. We construct the parameters setting first $p_v(u, \emptyset) := \mathbb{P}(X_{(u,v)} = 1)$ for every $v \in V$ and $u \in N_v$. If $p_v(u, \emptyset) < 1$, then we set the value $p_v(w, \{u\}) := \mathbb{P}(X_{(w,v)} = 1 \mid X_{(u,v)} = 0)$ for every $w \in N_v \setminus \{u\}$. Analogously, we set $p_v(u, S) := \mathbb{P}(X_{(u,v)} = 1 \mid \forall_{w \in S}\ X_{(w,v)} = 0)$ whenever the probability of the condition is nonzero. Moreover, we require that the parameters satisfy the condition $(1 - p_v(u_i, W))(1 - p_v(u_j, W \cup \{u_i\})) = (1 - p_v(u_j, W))(1 - p_v(u_i, W \cup \{u_j\}))$ for all $v \in V$, $u_i, u_j \in N_v$ and $W \subset N_v \setminus \{u_i, u_j\}$. Since every permutation can be written as a product of transpositions, this condition is sufficient to ensure that $\mathbb{P}(v \in S_{i+1} \mid v \notin S_i)$ does not depend on the order assumed in $A_i(\omega)$, where $A_i(\omega)$ is defined as in ICM. If for a given $\omega \in \Omega$ and $i \geq 1$ the sets of active vertices $S_i(\omega)$ are defined as in ICM, then we say that the propagation is given by the *General Cascade Model* (GCM). Obviously, Lemma 1 applies and the propagation in GCM is finished.

Let us now define two models that are special cases of GCM. If the implication

$$S \subset T \implies p_v(u, S) \leq p_v(u, T)$$

holds for any $v \in V$, $u \in N_v$ and $S, T \subset N_v \setminus \{u\}$ such that $p_v(u, S)$ and $p_v(u, T)$ are defined, then the model is called the *Increasing Cascade Model* (IncrCM). If the reversed inequality holds, namely

$$S \subset T \implies p_v(u, S) \geq p_v(u, T),$$

then the model is called the *Decreasing Cascade Model* (DCM). Observe that DCM expresses the principle that the more times one resists the influence, the more their resistance rises, whereas IncrCM captures exactly the opposite behaviour, namely that influence accumulates, i.e. past attempts, though unsuccessful, have weakened individual's resistance. There are obvious everyday examples of various types of influence that would require one of these models rather than the other. Note also that ICM is a special case of both DCM and IncrCM. DCM was proposed in [1] and we will argue that its influence function is submodular, but, as was shown in [2], a suitable transport of the probability measure from Ω on Γ need not exist.

4. Threshold models

The second most popular model of influence propagation is the Linear Threshold Model (LTM). We describe it not only to complete the picture, but also because it exemplifies the accumulative mechanism of propagation. We reformulate the results from [1] (see Theorem 4) and propose a simple yet meaningful generalisation of LTM.

For every edge in $G = (V, E)$ we fix $b_{(u,v)} \in (0, 1]$ such that $\sum_{u\colon (u,v)\in E} b_{(u,v)} \leq 1$ for all $v \in V$. We set $b_{(u,v)} := 0$ if $(u, v) \notin E$. Each vertex $v \in V$ is assigned a uniformly distributed random variable $\theta_v \colon \Omega \to (0, 1)$ and these random variables are assumed to be independent. A propagation is said to be given by the *Linear Threshold Model* if for an initial set S and a fixed $\omega \in \Omega$ the sets of active vertices are defined as $S_i(\omega) := S \cup \{v \in V \mid \sum_{u\in S_{i-1}(\omega)} b_{(u,v)} \geq \theta_v(\omega)\}$ for $i \geq 1$. Lemma 1 ensures that the propagation given by LTM is finished after time $\tau \leq \#V - 1$. Moreover, we have $\mathbb{P}(v \in S_{i+1} \mid v \notin S_i) = \sum_{u\in S_i \setminus S_{i-1}} b_{(u,v)}/(1-\sum_{u\in S_{i-1}} b_{(u,v)})$. The influence maximisation problem in LTM is NP-hard [1]. In proving submodularity in LTM, we cannot repeat the argument used in the case of ICM, since if there existed $h\colon \Omega \to \Gamma$ satisfying $\varphi_S(\omega) = \psi_{h(\omega)}(S)$ for every ω and S, then the submodularity of $\#\varphi_\bullet(\omega)\colon 2^V \to \mathbb{N}$ would follow. However, Example 1 shows that this function need not be submodular in a propagation given by LTM.

Example 1 *Consider a network $G = (V, E)$, where $V = \{u_1, u_2, v\}$ and $E = \{(u_1, v), (u_2, v)\}$, and LTM with parameters $b_{u_i,v} = 0.3$ for $i \in \{1, 2\}$. Fix ω such that $\theta_v(\omega) = 0.5$. For $S := \emptyset$ and $T := \{u_1\}$ we obtain $\#\varphi_{S\cup\{u_2\}}(\omega) - \#\varphi_S(\omega) = 1$ and $\#\varphi_{T\cup\{u_2\}}(\omega) - \#\varphi_T(\omega) = 2$, hence $\#\varphi_\bullet(\omega)$ is not submodular.*

Theorem 4 *The influence function σ in LTM is monotone and submodular.*

Proof. Consider $G = (V, E)$ with LTM. We fix $v \in V$ and order arbitrarily the set of its predecessors $N_v = \{u_1, \ldots, u_{k_v}\}$. We define $h\colon \Omega \to \Gamma$, $h(\omega) = (V, E_\omega)$ as follows

$$(u_i, v) \in E_\omega \iff \sum_{j=1}^{i-1} b_{u_j,v} < \theta_v(\omega) \leq \sum_{j=1}^{i} b_{u_j,v}\,, \quad i \in \{1, \ldots, k_v\}.$$

Fix $S \subset V$ and $\omega \in \Omega$. In the subgraph $h(\omega)$ we have the following sets of vertices reachable form S in successive time steps: $\psi_0(\omega) := S$ and $\psi_i(\omega) := \psi_{i-1}(\omega) \cup \{v \in V \mid \exists_{u\in\psi_{i-1}(\omega)}\ (u, v) \in E_\omega\}$.

We obtained a propagation in which $\mathbb{P}(v \in \psi_{i+1} \mid v \notin \psi_i) = \sum_{u\in\psi_i\setminus\psi_{i-1}} b_{(u,v)}/(1-\sum_{u\in\psi_{i-1}} b_{(u,v)})$ for every $i \geq 0$. It follows from the independence of $\{\theta_v\}_{v\in V}$ that the propagation given by LTM and the one defined above are equivalent, hence $\mathbb{P}(S_{N-1} = W) = \mathbb{P}(\psi_{N-1} = W)$, where $N := \#V$. Clearly, $\psi_{N-1}(\omega) = \psi_{h(\omega)}(S)$, since in a graph with N vertices there are at most $N-1$ edges between every two vertices. Moreover, from Lemma 1 we get $S_{N-1}(\omega) = \varphi_S(\omega)$ for every ω. Thus $\mathbb{E}(\#\varphi_S) = \mathbb{E}(\#\psi_{h(\cdot)}(S))$ and Theorem 2 completes the proof. □

Following [1], we generalise LTM by relaxing the assumption of influence additivity. For each vertex v we consider a function $f_v : 2^{N_v} \to (0,1]$ which is monotone and satisfy $f_v(\emptyset) = 0$. If for a given $\omega \in \Omega$ and $i \geq 1$ the sets of active vertices are defined as $S_i(\omega) := S \cup \{v \in V \mid f_v(S_{i-1}(\omega)) \geq \theta_v(\omega)\}$, then we say that the propagation is given by the *General Threshold Model* (GTM). The function f_v is called the *activation function* of v. Lemma 1 implies that the propagation in GTM is finished after time $\tau \leq \#V - 1$. Moreover, we have $\mathbb{P}(v \in S_{i+1} \mid v \notin S_i) = (f_v(S_i) - f_v(S_{i-1}))/(1 - f_v(S_{i-1}))$.

A compelling special case of GTM is taken from LTM by relaxing normalisation condition, i.e. we only demand that $b_{(u,v)} \in (0,1]$ for every $(u,v) \in E$. If the sets of active vertices are defined as in LTM, then we say that the propagation is given by the *non-normalised* LTM (nLTM). This model is more natural than LTM in the sense that it shares its parameters with ICM: the parameter corresponding to an edge (u,v) describes the probability that v will be activated by an unassisted attempt taken by u. The influence maximisation problem in nLTM is harder than in LTM. In particular, a probability distribution on Γ that would allow us to use the subgraph method need not exist (see Example 2). It also cannot be solved by uniform scaling down the parameters of nLTM, so that parameters assigned to edges that come into a vertex add up to at most one and the model becomes LTM, since such rescaling may lead to a model with different optimal initial set (see Example 3).

Example 2 *Set the network $G = (V,E)$, where $V = \{u_1,u_2,u_3,v\}$ and $E = \{(u_1,v), (u_2,v),(u_3,v)\}$, and consider nLTM with $b_{u_i,v} = 0.5$ for $i \in \{1,2,3\}$. Assume that there is a probability distribution on Γ induced by $h\colon \Omega \to \Gamma$ as in Theorem 2. Taking $\{u_1,u_2,u_3\}$ as the initial set, we see that the empty subgraph has probability zero. Considering next the initial sets consisting of two vertices taken from $\{u_1,u_2,u_3\}$, we see that the probability of every subgraph with exactly one edge is equal to zero. Finally, taking the initial sets $\{u_i\}$, $i \in \{1,2,3\}$, we conclude that each of the three subgraphs with exactly two edges has probability* 0.5.

Example 3 *Set the network $G = (V,E)$, where $V = \{u_1,u_2,w_1,w_2,v,x,y\}$ and $E = \{(u_1,x),(u_2,x),(x,v),(w_1,y),(w_2,y),(y,v)\}$ and consider nLTM with the following parameters: $b_{u_i,x} = 0.6$ and $b_{w_i,y} = 1$ for $i \in \{1,2\}$ and $b_{x,v} = 1$, $b_{y,v} = 0.25$ (see Figure 1). By direct computation we see that $\{u_1,w_1\}$ is the only optimal 2-elements initial set. However, if we divide all the above parameters by 2, then $\{w_1,w_2\}$ becomes the only optimal 2-elements initial set in the resulting LTM.*

In proving the submodularity of σ in nLTM, we make use of some properties of the special case of GTM in which all the activation functions f_v are required to be submodular. This model, which we call *Locally Submodular Threshold Model* (LocSTM), was first proposed in [1] and the submodularity of its influence function was proved in [4]. In nLTM we have $f_v(S) = \min(1, \sum_{u\in S} b_{u,v})$ and these functions

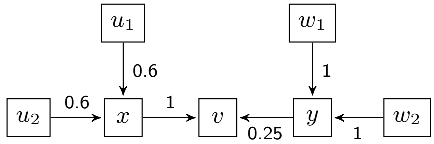

Figure 1. The network from Example 3.

are easily seen to be submodular, hence nLTM is a special case of LocSTM and the influence function in nLTM is indeed submodular.

5. Generalised models

We follow [1, 2] to show that GTM and GCM are in fact two different parametrisations of the same model. Suppose we have GCM and we want to define a corresponding GTM . Since in GTM the value $f_v(U)$ gives the probability of activating v, provided $U \subset N_v$ is active, we set $f_v(U)$ as being equal to the probability of activating v by vertices from $U = \{u_1, \ldots, u_m\}$ in the given GCM, i.e.

$$f_v(U) = 1 - \prod_{i=1}^{m}(1 - p_v(u_i, U^{i-1})), \tag{1}$$

where $U^0 := \emptyset$ and $U^j := \{u_1, \ldots, u_j\}$ for $j \geq 1$. Note that $f_v(U)$ is well defined, since we have already observed that the right-hand side of (1) depends only on the elements of U and not on their order. Suppose now, conversely, that GTM is given. Fix $v \in V$, $w \in N_v$ and $U \subset N_v \setminus \{w\}$ and define $p_v(w, U)$ as being equal to the probability in GTM of the influence of $U \cup \{w\}$ being sufficient to activate v on condition that the influence of U was insufficient, i.e.:

$$p_v(w, U) = \frac{f_v(U \cup \{w\}) - f_v(U)}{1 - f_v(U)}. \tag{2}$$

Note that if $f_v(U) = 1$ then $\mathbb{P}(\forall_{u \in U}\ X_{(u,v)} = 0) = 0$ and we do not set the value $p_v(w, U)$. It is easily seen that (1) and (2) are equivalent. The next theorem, which claims the equivalence of the propagations given by GTM and GCM, was first proved in [2]. In particular, it implies that theorems proved for either of the models are valid for both.

Theorem 5 *If we consider GTM and GCM such that their parameters $f_v(U)$ and $p_v(w, U)$, respectively, satisfy (1) for every $v \in V$, $w \in N_v$ and $U \subset N_v \setminus \{w\}$, then these models give equivalent propagations.*

Proof. Fix $S \subset V$, $v \in V$ and sets $U = \{u_1, \ldots, u_m\}$, $W = U \cup \{u_{m+1}, \ldots, u_{m+n}\}$. It follows easily that in GTM the value $\mathbb{P}(v \in S_{i+1} \,|\, S_i = W, S_{i-1} = U)$ equals

$(f_v(W) - f_v(U))/(1 - f_v(U))$, and in GCM it equals $1 - \prod_{j=m+1}^{m+n}(1 - p_v(u_j, W^{j-1}))$, where $W^m := U$ and $W^j := \{u_1, \ldots, u_j\}$ for $j > m$. The independence of $\{\theta_v\}_{v \in V}$ in GTM and that of $\{X_v\}_{v \in V}$ in GCM along with (1) complete the proof. □

Theorem 6 *The influence function is monotone in a propagation given by GTM.*

Proof. Consider $G = (V, E)$ with GTM. Fix initial sets S and T satisfying $S \subset T$. It follows from the monotonicity of f_v, by induction over time steps, that $S_i(\omega) \subset T_i(\omega)$ for every $i \geq 0$ and $\omega \in \Omega$. Hence $\#S_{N-1}(\omega) \leq \#T_{N-1}(\omega)$ and $\sigma(S) \leq \sigma(T)$. □

Clearly, the influence maximisation problem in the generalised models is NP-hard. Moreover, for every $c > 0$ it is NP-hard to find a n^{-c}-approximated solution to this problem [1]. The equivalence of GCM and GTM helps us analyse the already defined special cases of GCM, namely DCM and IncrCM, since we can interchange between cascade and threshold parameterisations of a model in question.

The submodularity of the influence function in DCM can be proved by introducing generalised propagation processes that allow delayed activation of vertices [2], but it also follows from the fact that DCM is a special case of LocSTM, since the defining condition of DCM is, by (2), equivalent to $(f_v(S \cup \{u\}) - f_v(S))/(1 - f_v(S)) \geq (f_v(T \cup \{u\}) - f_v(T))/(1 - f_v(T))$, where $v \in V$, $u \in N_v$ and $S \subset T \subset N_v \setminus \{u\}$. It follows easily from the monotonicity of f_v that this condition implies submodularity of f_v, hence also that of σ. Thereby we see that DCM fits naturally into the framework build on submodularity. In what follows we investigate more closely IncrCM. Example 4 shows that its influence function need not be submodular, hence IncrCM is not a special case of LocSTM. A simple necessary condition for a model to be LocSTM can be derived as follows. For every $v \in V$, $S \subset N_v$ and $u_1 \in S$ the submodularity of f_v implies that $f_v(S) - f_v(S \setminus \{u_1\}) \leq f_v(\{u_1\}) - f_v(\emptyset)$. Repeating this argument for the sets $S \setminus \{u_1, \ldots, u_k\}$, where $1 \leq k < \#S$, we obtain $f_v(S) \leq \sum_{u \in S} f_v(u)$. Note that as the limit case, i.e. $f_v(S) = \min(\sum_{u \in S} f_v(u), 1)$, we obtained nLTM. Using (2), we conclude that nLTM is indeed a special case of IncrCM, thus it is an example of a model that captures the accumulative mechanism of influence propagation and at the same time fits into the submodular framework.

Example 4 *Consider $G = (V, E)$, where $V = \{u_1, u_2, v\}$, $E = \{(u_1, v), (u_2, v)\}$. Set $p_v(u_1, \emptyset) = p_v(u_2, \emptyset) = 0.1$ and $p_v(u_1, \{u_2\}) = p_v(u_2, \{u_1\}) = 8/9$. Using (1), we obtain the threshold parameters $f_v(\{u_1\}) = f_v(\{u_2\}) = 0.1$ and $f_v(\{u_1, u_2\}) = 0.9$. From these it follows easily that $\sigma(\{u_2\}) - \sigma(\emptyset) = 1.1$ and $\sigma(\{u_1, u_2\}) - \sigma(\{u_1\}) = 1.8$.*

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7. References

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