A remark on the distribution of products of independent normal random variables

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Abstract

We present a proof of the explicit formula of the probability density function of the product of normally distributed independent random variables using the multiplicative convolution formula for Meijer G functions.

Keywords: normal distribution, Meijer G-functions

Article history

Received: 6 January 2021
Received in revised form: 8 March 2021
Accepted: 9 March 2021
Available: 10 March 2021

Introduction

Basic statistic tests are constructed with the aim of sums ξ_1 + ξ_2 + · · · + ξ_n, squares ξ_1^2, ξ_2^2, . . . , ξ_n^2, sum of squares ξ_1^2 + ξ_2^2 + · · · + ξ_n^2, quotients ξ_1/ξ_2 or other algebraic operations on sequences of independent random variables ξ_1, ξ_2, ..., ξ_n (see eg. [6]). While the sum ξ_1 + ξ_2 + · · · + ξ_n has a density that is commonly easy to determine other algebraic operations lead to densities that are more and more complicated and demand special functions in analysis. For example the sum of the independent identically distributed (abbreviated as i.i.d.) random variables ξ_i, i = 1, 2, ..., n with the normal N(m_i, σ_i) probability density function (p.d.f. for abbreviation)

\[ f_i(t) := \frac{1}{\sigma_i \sqrt{2\pi}} \exp \left( -\frac{1}{2} \left( \frac{t - m_i}{\sigma_i} \right)^2 \right) \quad (1) \]

has p.d.f. of the same family of normal distributions N(m, σ)

\[ f(t) := \frac{1}{\sigma \sqrt{2\pi}} \exp \left( -\frac{1}{2} \left( \frac{t - m}{\sigma} \right)^2 \right) \]

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with \( m := m_1 + m_2 + \cdots + m_n \) and \( \sigma^2 = \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2 \). The sum
\[
\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2
\]
of i.i.d. random variables with standard normal p.d.f. (i.e. \( N(m, \sigma) \) with \( m = 0 \) and \( \sigma = 1 \)) is the \( \chi^2 \) distribution with \( n \) degrees of freedom, i.e. has p.d.f. defined by
\[
f(t) := \begin{cases} 
\frac{1}{2^n \Gamma(n/2)} t^{n/2 - 1} e^{-t/2}, & t \geq 0 \\
0, & t < 0 
\end{cases}
\]
which requires the use of the Euler gamma function \( \Gamma \). It is known (see [9], [10]) that the answer to a very simple question about the distribution of the product of normally distributed random variables requires the use of the Meijer G-functions \( G_{m,0}^{0,m} \), see (2). We present a proof of the explicit formula of the probability density function of the product of normally distributed independent random variables in terms of Meijer G functions using the multiplicative convolution formula for Meijer G-functions (8). In some special cases the function \( G_{m,n}^{p,q} \) can be reduced to a simpler form (see [1]), for example standard normal distribution \( N(0,1) \) has the p.d.f. given by
\[
f(t) = \frac{1}{\sqrt{2\pi}} \exp\left( -\frac{t^2}{2} \right) = \frac{1}{\sqrt{2\pi}} G^{1,0}_{0,1}\left( \left| \frac{t^2}{2} \right| \right)
\]
The function \( G^{2,0}_{0,2}(x) \) can be expressed with the modified Bessel function of the second kind \( K_\nu \), see (11).

**Meijer G-functions and their basic properties**

The \( G \)-function was introduced by C.S. Meijer in 1936 and is defined in terms of Mellin-Barnes type integrals (see [1], [2] and references given there)
\[
G_{p,q}^{m,n}\left( \left| \begin{array}{c} a \\ b \end{array} \right| z \right) = \frac{1}{2\pi i} \int_L \frac{\prod_{k=1}^m \Gamma(b_k - \zeta) \prod_{k=1}^n \Gamma(1 - a_k + \zeta)}{\prod_{k=m+1}^q \Gamma(1 - b_k + \zeta) \prod_{k=n+1}^p \Gamma(a_k - \zeta)} z^\zeta \, d\zeta \quad (2)
\]
where \( a = (a_1, \ldots, a_n, a_{n+1}, \ldots, a_p) \) and \( b = (b_1, \ldots, b_m, b_{m+1}, \ldots, b_q) \) are sequences of real or complex parameters, with the contour of integration \( L \) suitably chosen on the complex plane \( \mathbb{C} \). If \( m = 0, n = 0, p = 0 \) or \( q = 0 \) we put 1 in place of the product over an empty set of indices and we put a sign of omission ‘−’ in
an appropriate place in the symbol of the \( G \)-function or omitting the sign ‘−’ if this does not lead to confusion. For example

\[
G_{0,m}^m(\zeta|b_1,b_2,\ldots,b_m) = G_{0,m}^m(\zeta|b_1,b_2,\ldots,b_m)
\]

\[
= \frac{1}{2\pi i} \int_L \Gamma(b_1 - \zeta) \Gamma(b_2 - \zeta) \cdots \Gamma(b_m - \zeta) \zeta^\zeta \, d\zeta
\]

We recall selected basic formulae involving Meijer \( G \)-functions (see [1], [2], [7]). By the definition of \( G \)-function we have

\[
z^\alpha G_{\mu,\nu}^{\sigma,\tau}(\frac{a_1,\ldots,a_p}{b_1,\ldots,b_q}) = G_{\mu,\nu}^{\sigma,\tau}(\frac{a_1 + \sigma,\ldots,a_p + \sigma}{b_1 + \sigma,\ldots,b_q + \sigma})
\]

\[
G_{\mu,\nu}^{\sigma,\tau}(\frac{a_1,\ldots,a_p}{b_1,\ldots,b_q}) = G_{\mu,\nu}^{\sigma,\tau}(\frac{1 - b_1,\ldots,1 - b_q}{1 - a_1,\ldots,1 - a_p}).
\]

The classical integral formula for Meijer \( G \)-functions is of fundamental importance in calculations:

\[
\int_0^\infty G_{\mu,\nu}^{\sigma,\tau}(\eta x|\frac{a_1,\ldots,a_p}{b_1,\ldots,b_q}) G_{\mu,\nu}^{\sigma,\tau}(\omega x|\frac{c_1,\ldots,c_\sigma}{d_1,\ldots,d_\tau}) \, dx = \frac{1}{\eta} G_{\mu,\nu}^{\sigma,\tau}(\frac{\omega}{\eta} - b_1,\ldots,-b_m,c_1,\ldots,c_\sigma,-b_{m+1},\ldots,-b_q)
\]

By (3) and (4) we get a slightly more general integral formula

\[
\int_0^\infty x^{n-1} G_{\mu,\nu}^{\sigma,\tau}(\eta x|\frac{a_1,\ldots,a_p}{b_1,\ldots,b_q}) G_{\mu,\nu}^{\sigma,\tau}(\omega x|\frac{c_1,\ldots,c_\sigma}{d_1,\ldots,d_\tau}) \, dx = \eta^{-1-\alpha} \int_0^\infty G_{\mu,\nu}^{\sigma,\tau}(\eta x|\frac{a_1 + \alpha - 1,\ldots,a_p + \alpha - 1}{b_1 + \alpha - 1,\ldots,b_q + \alpha - 1}) G_{\mu,\nu}^{\sigma,\tau}(\omega x|\frac{c_1 + \alpha - 1,\ldots,c_\sigma + \alpha - 1}{d_1 + \alpha - 1,\ldots,d_\tau + \alpha - 1}) \, dx
\]

By (3) and (5) we have

\[
\int_0^\infty x^{\alpha-1} G_{\mu,\nu}^{\sigma,\tau}(\eta x|\frac{a_1,\ldots,a_p}{b_1,\ldots,b_q}) G_{\mu,\nu}^{\sigma,\tau}(\omega x|\frac{c_1,\ldots,c_\sigma}{d_1,\ldots,d_\tau}) \, dx = \eta^{-\alpha} \int_0^\infty G_{\mu,\nu}^{\sigma,\tau}(\eta x|\frac{a_1 + \alpha,\ldots,a_p + \alpha}{b_1,\ldots,b_q}) G_{\mu,\nu}^{\sigma,\tau}(\omega x|\frac{1 - d_1,\ldots,1 - d_\tau}{1 - c_1,\ldots,1 - c_\sigma}) \, dx
\]

Putting \( \alpha = 0 \) in (6) we get

\[
\int_0^\infty G_{\mu,\nu}^{\sigma,\tau}(\eta x|\frac{a_1,\ldots,a_p}{b_1,\ldots,b_q}) G_{\mu,\nu}^{\sigma,\tau}(\frac{\omega}{\eta} x|\frac{c_1,\ldots,c_\sigma}{d_1,\ldots,d_\tau}) \, dx = G_{\mu,\nu}^{\sigma,\tau}(\frac{\eta}{\omega} x|\frac{a_1,\ldots,a_p}{b_1,\ldots,b_q}) G_{\mu,\nu}^{\sigma,\tau}(\frac{\eta}{\omega} x|\frac{c_1,\ldots,c_\sigma}{d_1,\ldots,d_\tau})
\]
For \( \omega = t^{-1} \) and \( \eta = 1 \) in (7) we obtain the multiplicative convolution formula for Meijer \( G \)-functions:

\[
\{G^{m,n}_{p,q} \ast G^{\mu,\nu}_{\sigma,\tau}\}(t) := \int_0^\infty G^{m,n}_{p,q}(x)G^{\mu,\nu}_{\sigma,\tau}(tx^{-1})\frac{dx}{x} = G^{m+\mu,n+\nu}_{p+\sigma,q+\tau}(t)
\]

or more specifically

\[
\int_0^\infty G^{m,n}_{p,q}\left(x\begin{bmatrix} a_1, \ldots, a_p \end{bmatrix} b_1, \ldots, b_q\right)G^{\mu,\nu}_{\sigma,\tau}\left(t\begin{bmatrix} c_1, \ldots, c_\sigma \end{bmatrix} d_1, \ldots, d_\tau\right)\frac{dx}{x} = G^{m+\mu,n+\nu}_{p+\sigma,q+\tau}\left(t\begin{bmatrix} a_1, \ldots, a_n, c_1, \ldots, c_\sigma, a_{n+1}, \ldots, a_p \end{bmatrix} b_1, \ldots, b_m, d_1, \ldots, d_\tau, b_{m+1}, \ldots, b_q\right)
\]  

(8)

By (6) we get also

\[
\int_0^\infty x^{\alpha-1}G^{m,n}_{p,q}\left(\begin{bmatrix} a_1, \ldots, a_p \end{bmatrix} b_1, \ldots, b_q\right)\frac{\eta x^2}{2} G^{\mu,\nu}_{\sigma,\tau}\left(\begin{bmatrix} c_1, \ldots, c_\sigma \end{bmatrix} d_1, \ldots, d_\tau\right)dx = 2^\alpha \int_0^\infty x^\alpha G^{m,n}_{p,q}\left(\begin{bmatrix} a_1, \ldots, a_p \end{bmatrix} b_1, \ldots, b_q\right)\frac{\eta^2}{\omega} G^{\mu,\nu}_{\sigma,\tau}\left(\begin{bmatrix} c_1, \ldots, c_\sigma \end{bmatrix} d_1, \ldots, d_\tau\right)ds
\]  

\[
= 2^{\alpha-1} \int_0^\infty x^{\alpha-1}G^{m,n}_{p,q}\left(\begin{bmatrix} a_1, \ldots, a_p \end{bmatrix} b_1, \ldots, b_q\right)\frac{\eta}{\omega} G^{\mu,\nu}_{\sigma,\tau}\left(\begin{bmatrix} c_1, \ldots, c_\sigma \end{bmatrix} d_1, \ldots, d_\tau\right)dx
\]  

(9)

Putting \( \alpha = 0 \), \( \eta = 1 \) and \( \omega = t^{-1} \) in (9) we get

\[
\int_0^\infty x^{-1}G^{m,n}_{p,q}\left(\begin{bmatrix} a_1, \ldots, a_p \end{bmatrix} b_1, \ldots, b_q\right)G^{\mu,\nu}_{\sigma,\tau}\left(\begin{bmatrix} c_1, \ldots, c_\sigma \end{bmatrix} d_1, \ldots, d_\tau\right)dx = \frac{1}{2} G^{m+\mu,n+\nu}_{p+\sigma,q+\tau}\left(t\begin{bmatrix} a_1, \ldots, a_n, c_1, \ldots, c_\sigma, a_{n+1}, \ldots, a_p \end{bmatrix} b_1, \ldots, b_m, d_1, \ldots, d_\tau, b_{m+1}, \ldots, b_q\right)
\]  

(10)

We recall a formula for the \( G^{2,0}_{0,2} \)-function

\[
G^{2,0}_{0,2}(x|a, b) = G^{2,0}_{0,2}\left(x\mid \begin{bmatrix} - & -a & b \end{bmatrix}\right) = 2x^{\frac{-a+b}{2}} K_{a-b}(2\sqrt{x})
\]  

(11)

(see 5.6(4) in [1]), where \( K_\nu \) is the modified Bessel function of the second kind of order \( \nu \). Many other formulae to express the Meijer \( G \)-function in other special functions can be found in [1]. Most likely, we should not expect, unfortunately, a reasonable reduction of the Meijer function \( G^{m,0}_{0,m} \) to simpler special functions when \( m > 2 \). In [3] some applications of the Fox \( H \) function

\[
H^{m,n}_{p,q}\left[z\mid \begin{bmatrix} (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \end{bmatrix} (b_1, \beta_1), \ldots, (b_q, \beta_q)\right] := \frac{1}{2\pi i} \int_{L} \prod_{k=1}^{m} \Gamma(b_k - \beta_k \zeta) \prod_{k=1}^{n} \Gamma(1 - ak + \alpha_k \zeta) \prod_{k=m+1}^{n} \Gamma(1 - b_k + \beta_k \zeta) \prod_{k=n+1}^{p} \Gamma(ak - \alpha_k \zeta) \frac{z^\zeta}{\prod_{k=1}^{q} \Gamma(1 - b_k + \beta_k \zeta)} d\zeta
\]  

(12)
were investigated to study the distribution of products, quotients and powers of independent random variables. However the Fox $H$ function generalizes the Meijer $G$ function and we should not expect all the more any simplification of results formulated in terms of the Fox $H$ function to more elementary functions.

Main result

**Theorem 3.1** Let $\xi_1, \xi_2, \ldots, \xi_m$ be i.i.d. random variables with standard normal p.d.f. with $m_i = 0$ and $\sigma_i = 1$, see (1). Then the product $\xi_1 \xi_2 \ldots \xi_m$ has p.d.f. given by

$$f_{\xi_1 \xi_2 \ldots \xi_m}(t) = (2\pi)^{-m/2} G_{0,m,0}^{m,0} \left( \begin{array}{c} 2^m t^2 \\ 0,0,\ldots,0 \end{array} \right)$$

(13)

In the case when $m = 2$ we have

$$f_{\xi_1 \xi_2}(t) = \frac{1}{\pi} K_0(|t|)$$

(14)

where $K_0$ is the modified Bessel function of the second kind of order 0.

**Proof.** By the formula for the p.d.f. of the product $\xi \eta$ of two independent random variables $\xi$ and $\eta$ with p.d.f. $f_\xi$ and $f_\eta$, respectively, we get (see eg. [6]):

$$f_{\xi \eta}(t) = \int_{-\infty}^{\infty} f_\xi(t x^{-1}) f_\eta(x) \frac{dx}{|x|}$$

$$= \int_{-\infty}^{\infty} f_\xi(x) f_\eta(t x^{-1}) \frac{dx}{|x|}$$

Hence for two i.i.d. random variables $\xi_1$, $\xi_2$ with standard normal p.d.f. we have

$$f_{\xi_1 \xi_2}(t) = \int_{-\infty}^{\infty} f_{\xi_1}(t x^{-1}) f_{\xi_2}(x) \frac{dx}{|x|} = 2 \int_{0}^{\infty} f_{\xi_1}(t x^{-1}) f_{\xi_2}(x) \frac{dx}{x}$$

because the function $f_{\xi_2}$ is even. The last integral is the multiplicative convolution of functions $f_{\xi_1}$ and $f_{\xi_2}$:

$$\int_{0}^{\infty} f_{\xi_1}(t x^{-1}) f_{\xi_2}(x) \frac{dx}{x} =: \{f_{\xi_1} \ast f_{\xi_2}\}(t).$$
By the convolution formula for Meijer G-functions, see (8):

\[ G_{p,q}^{m,n} * G_{p',q'}^{m',n'} = G_{p+p',q+q'}^{m+m'+n+n'} \]

for \( f_{\xi_1}(u) = f_{\xi_2}(u) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}u^2) = \frac{1}{\sqrt{2\pi}} G_{0,1}^{1,0}(\frac{u^2}{2} | 0) = \frac{1}{\sqrt{2\pi}} G_{0,1}^{1,0}(\frac{u^2}{2} | 0) \)

and using formulae (10) and (11) we get:

\[ f_{\xi_1,\xi_2}(t) = 2 \int_0^\infty f_{\xi_1}(tx^{-1}) f_{\xi_2}(x) \frac{dx}{x} = \frac{2}{(\sqrt{2\pi})^2} \int_0^\infty G_{0,1}^{1,0}(\frac{1}{2} t^2 | 0) G_{0,1}^{1,0}(\frac{x^2}{2} | 0) dx = \frac{2}{(\sqrt{2\pi})^2} \int_0^\infty G_{0,1}^{1,0}(\frac{t^2}{x^2} | 0) G_{0,1}^{1,0}(\frac{x^2}{2} | 0) dx = \frac{2}{(\sqrt{2\pi})^2} \cdot \frac{1}{2} G_{0,2}^{2,0}(\frac{t^2}{4} | 0, 0) = \frac{1}{2\pi} G_{0,2}^{2,0}(2^{-2} t^2 | 0, 0) = \frac{1}{\pi} K_0(|t|) \]

Assume for \( m \geq 2 \) the product \( \xi_1 \xi_2 \ldots \xi_{m-1} \) has p.d.f. of the form

\[ f_{\xi_1,\xi_2,\ldots,\xi_{m-1}}(t) = \frac{1}{(\sqrt{2\pi})^{m-1} G_{0,m-1}^{m-1,0}(\frac{t^2}{2m-1} | 0, \ldots, 0)} \]

Proceeding as above for two factors \( f_{\xi_1} \) and \( f_{\xi_2} \) we have by (10):

\[ f_{\xi_1,\xi_2,\ldots,\xi_{m-1},\xi_m}(t) = 2 \int_0^\infty f_{\xi_1,\xi_2,\ldots,\xi_{m-1}}(tx^{-1}) f_{\xi_1}(x) \frac{dx}{x} = \frac{2}{(\sqrt{2\pi})^m} \int_0^\infty G_{0,m-1}^{m-1,0}(\frac{1}{2m-1} t^2 | 0, \ldots, 0) G_{0,1}^{1,0}(\frac{x^2}{2} | 0) dx = \frac{2}{(\sqrt{2\pi})^m} \int_0^\infty G_{0,m-1}^{m-1,0}(\frac{t^2}{2m} | 0, \ldots, 0) G_{0,1}^{1,0}(\frac{x^2}{2} | 0) dx = \frac{2}{(\sqrt{2\pi})^m} \cdot \frac{1}{2} G_{0,m}^{m,0}(\frac{t^2}{2m} | 0, \ldots, 0) \]

\[ = \frac{1}{(\sqrt{2\pi})^m} G_{0,m}^{m,0}(2^{-m} t^2 | 0, 0, \ldots, 0). \]

Corollary 3.2 Let \( \xi_1, \xi_2, \ldots, \xi_m \) be independent normally distributed random variables with mean values \( m_i = 0 \) and standard deviations \( \sigma_i > 0 \), i.e.
with p.d.f.
\[ f_{\xi_i}(t) = \frac{1}{\sigma_i \sqrt{2\pi}} \exp\left( -\frac{t^2}{\sigma_i^2} \right), \quad i = 1, 2, \ldots, m. \]

The p.d.f. of the product \( \xi_1\xi_2 \ldots \xi_m \) equals
\[
 f_{\xi_1\xi_2\ldots\xi_m}(t) = \frac{1}{\sigma(\sqrt{2\pi})^m} G_{m,0}^{m,0}(2^{-m}(t/\sigma)^2|0, \ldots, 0) \tag{15}
\]
where \( \sigma := \sigma_1 \sigma_2 \ldots \sigma_m \).

**Remark 3.3** It was noticed in [4], [5] that the density (14) of the product \( \xi_1\xi_2 \) was found in 1932 in [11]. The formula (14) is also an exercise to the reader in [8]. The density (15) of the product \( \xi_1\xi_2 \ldots \xi_m \) was found in [9] in 1966. See also [10].

**Question 3.4** Is it possible to simplify the Meijer function \( G_{0,m}^{m,0}(x|0, \ldots, 0) \) for \( m > 2 \) as in the case of \( m = 2 \) when \( G_{0,2}^{2,0}(t^2|0, 0) = 2K_0(|t|) \)?

**Question 3.5** Is it possible to give an explicit formula for the p.d.f. of the product \( \xi_1\xi_2 \ldots \xi_m, \ m \geq 2, \) of normally distributed independent random variables with different mean values?

We may find a partial answer to the question in [8] in a special case of the product \( \xi_1\xi_2 \) where \( \xi_1 \sim N(0, \sigma) \) and \( \xi_2 \sim N(m, \sigma) \).

**Acknowledgements**

The author would like to thank the reviewers for drawing attention to the results published in [3], [4], [5], [8], [9], [10], [11] and for indicating an error in the formula (5) in the earlier version of the paper.

**References**

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Figure 1: Plots of the p.d.f. of the products $\xi_1 \ldots \xi_m$ for $m = 1$ (red line), $m = 2$ (green line), $m = 3$ (blue line) and $m = 4$ (black line).

Figure 2: Plots of the p.d.f. of the products $\xi_1 \ldots \xi_m$ for $m = 1$ (red line), $m = 2$ (green line), $m = 3$ (blue line) and $m = 4$ (black line).