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Stratified $\mathcal{C}^p$-semialgebraic triviality

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Abstract. We show that a stratified submersion between stratified semialgebraic sets is locally $\mathcal{C}^p$-semialgebraically trivial outside the set of stratified generalized critical values.

1. Introduction

One of the main issues in singularity theory is the study of stability of functions and mappings. It is an important open problem to have a precise description of the set of bifurcation points of a given map $f : X \to \mathbb{R}^k$ i.e. the smallest set B with the property that the map

$$f_{|X \setminus f^{-1}(B)} : X \setminus f^{-1}(B) \to \mathbb{R}^k \setminus B$$

is locally trivial.

In the semialgebraic category, the famous Hardt's theorem [9] asserts that continuous semialgebraic mappings admit local semialgebraic trivializations outside some semialgebraic set of positive codimension.

Recall that a **local semialgebraic trivialization** of the map $f : X \to Y$ at a point $y \in Y$ is a semialgebraic homeomorphism $h : f^{-1}(U_y) \to T \times U_y$, where U_y is a neighbourhood of y in Y and T is a semialgebraic set, such that $\pi \circ h = f$, $\pi : T \times U_y \to U_y$ coincides with the canonical projection onto U_y. If the mapping h is $\mathcal{C}^p$ then we say that f is $\mathcal{C}^p$-semialgebraically trivial.

For a semialgebraic ($\mathcal{C}^p$, $p \geq 2$) mapping $f : \mathbb{R}^n \to \mathbb{R}^k$, it was shown by K. Kurdyka, P. Orro, and S. Simon [10] that f admits local $\mathcal{C}^p$ trivializations (not necessarily semialgebraic) on the preimage of the complement of the set $K_0(f) \cup K_\infty(f)$, where $K_0(f)$ is the set of critical values and $K_\infty(f)$ is the set of asymptotic critical values. This is actually a special case of Rabier's fibration theorem [11]. It is worth mentioning that L.R.G. Dias, M.A.S. Ruas and M. Tibăr, in an interesting paper [7], introduced the so called t-regularity condition and compared it with other regularity conditions. In particular they proved that in the case of $\mathcal{C}^1$ semialgebraic mappings $f : \mathbb{R}^n \to \mathbb{R}^k$ the notions of asymptotic critical point and t-regular point

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are equivalent (see [7, Theorem 3.1]). Moreover they showed that the t-regularity condition implies ρ-regularity (a Milnor-type condition of transversality to the Euclidean distance function) [7, Proposition 5.5], and that the latter condition yields in turn topological triviality at infinity in the case of $\mathcal{C}^2$ semialgebraic mappings $f : X \to \mathbb{R}^k$ defined on a submanifold of $\mathbb{R}^n$ [7, Theorem 5.7]. In the o-minimal setting, J. Escribano considered the case of $\mathcal{C}^p$ functions $f : \mathbb{R}^k \to \mathbb{R}$ [8]. Recently in [13] we proved:

Theorem 1. *Let $X \subset \mathbb{R}^n$ be a closed Nash manifold and let $f : X \to \mathbb{R}^k$ be a Nash mapping. Then for any $y \in \mathbb{R}^k \setminus K(f)$ there is a neighborhood U_y of y such that the restriction map $f|_{f^{-1}(U_y)}$ is Nash trivial.*

We recall that a mapping is Nash if it is both $\mathcal{C}^\infty$ and semialgebraic. The Nash class of functions sits in between the polynomials and the analytic functions. They enjoy many of the properties of the polynomial functions such as Noetherianity and Positivestellensatz as well as some properties of analytic functions such as the Weierstrass Preparation theorem or Theorems A and B of Cartan [3,4].

Let us mention, that earlier M. Coste and M. Shiota showed [5] that any Nash map $f : X \to \mathbb{R}^k$, $X \subset \mathbb{R}^n$ closed Nash manifold, is locally Nash trivial outside some Nash subset.

In this note we study the case when the subset X is semialgabraic, possibly with singularities. Our aim is to prove

Theorem 2. *Let $(X, \Sigma) \subset \mathbb{R}^n$ be a stratified semialgebraic subset, $Y \subset \mathbb{R}^k$ a smooth semialgebraic subset. Every stratified $\mathcal{C}^p$ submersion $f : (X, \Sigma) \to Y$, $p \geq 1$, is locally $\mathcal{C}^p$-semialgebraically stratified trivial at every $y \in Y \setminus K_\Sigma(f)$.*

This theorem asserts that stratified submersions are locally $\mathcal{C}^p$-semialgebraically stratified trivial outside the set of stratified generalized critical values for arbitrary positive integers p.

For the definitions of stratified critical values and stratified trivialization, see the next section. The proof of Theorem 2 is given in Section 3.

2. Basic definitions

Definition 3. A finite partition Σ of a subset $X \subset \mathbb{R}^n$ into manifolds S_i is called a **stratification** of X if it satisfies a frontier condition, i.e. if for all $(\alpha, \beta) \in A \times A$ such that $S_\alpha \cap \overline{S_\beta} \neq \emptyset$, one has $S_\alpha \subset \overline{S_\beta}$. The members of the stratification we call strata, and we say that (X, Σ) is a stratified subset of $\mathbb{R}^n$.

Definition 4. Let $(X, \Sigma) \subset \mathbb{R}^n$ be a stratified subset, $Y \subset \mathbb{R}^k$ a semialgebraic submanifold. We say that a continuous map $f : X \to Y$ is a **stratified map** if the restriction of f to each stratum $S \in \Sigma$ is Nash. We say that a stratified map is a **stratified submersion** if the restriction of f to each stratum $S \in \Sigma$ is a submersion.

Definition 5. We say that a stratification Σ of X satisfies Whitney's condition (b) (or is Whitney $(b)-$regular) if for all pair of strata S_i, S_j one has: for every sequences of points $(x_m) \subset S_i$ and $(y_m) \subset S_j$ converging to x such that the limit of secants $\overline{x_m y_m}$ and limit of tangent spaces $T_{y_m} S_j$ exist then $\lim \overline{x_m y_m} \subset \lim T_{y_m} S_j$.

Let $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^k$ be semialgebraic subsets, Y smooth.

Definition 6. A **semialgebraic trivialization** of a map $f : X \to Y$ at a point $y \in Y$ is a semialgebraic homeomorphism $h : X \to f^{-1}(y) \times Y$ such that $\pi \circ h = f$, where $\pi : f^{-1}(y) \times Y \to Y$ is the canonical projection on Y.

We say that a map $f : X \to Y$ is **locally trivial** if any point $y \in Y$ has a neighbourhood U_y such that the restriction map $f : f^{-1}(U_y) \to Y$ admits a trivialization.

Remark 7. Given a trivialization $h = (h_y, f)$ of a mapping f we may always suppose that $h_y(x) = x$ for any $x \in f^{-1}(y)$.

Indeed, if h_y does not have this property we can set $h'_y = g \circ h_y$, where $g = (h_{y|f^{-1}(y)})^{-1}$ and now, $h := (h'_y, f)$ is a trivialization with the desired property.

Definition 8. We say that the map $f : X \to Y$ is **locally $\mathcal{C}^p$-semialgebraically stratified trivial** if it is locally trivial and if there exists a stratification of X such that each local trivialization $h = (h_y, f) : f^{-1}(U_y) \to f^{-1}(y) \times Y$ induces on every stratum S_j a $\mathcal{C}^p$ semialgebraic diffeomorphism $h_{|S_j} : S_j \to (S_j \cap f^{-1}(y)) \times Y$.

2.1. Stratified generalized critical values

Generalized critical values were considered by many authors and they play an important role in the study of stability of mappings. Let us briefly recall their definition. For a linear operator $A : \mathbb{R}^n \to \mathbb{R}^k$, we set

$$\nu(A) = \inf_{||\varphi||=1} ||A^*\varphi||.$$

This is the so-called Rabier's function, which was originally defined in [11] for continuous linear operators between arbitrary Banach spaces over $\mathbb{R}$ or $\mathbb{C}$. It can be shown ([10], Proposition 2.2) that, for some natural metric d, $\nu(A) = d(A, \Sigma)$, where Σ is the set of singular operators. For the basic properties of this function, we refer the reader to [10].

In the sequel we will write $x_m \to a$ meaning that the sequence $(x_m) \subset X$ is tending to some point $a \in X$ as m goes to ∞.

Let $f : U \to \mathbb{R}^k$ be a semialgebraic mapping, with $U \subset \mathbb{R}^n$ (not necessarily bounded) smooth semialgebraic subset. Recall that the set of critical values of f is

$$K_0(f) := \{y \in \mathbb{R}^k \mid \exists x \in f^{-1}(y)\, , \nu(d_x f) = 0\}$$

and the set of **the asymptotic critical values at infinity** is

$$K_\infty(f) := \{y \in \mathbb{R}^k \mid \exists (x_m) \in U\, , |x_m| \to +\infty\, , f(x_m) \to y, \text{ and } |x_m|\nu(d_{x_m} f) \to 0\}.$$

Definition 9. Let $f : (X, \Sigma) \to \mathbb{R}^k$ be a semi-algebraic stratified map. The set

$$K_\Sigma(f) := \bigcup_{S \in \Sigma} K_\infty(f_{|S})$$

is called the **set of stratified asymptotic critical values**.

As a consequence of the Sard type theorem for asymptotic critical values [10, 14], this set has empty interior in $\mathbb{R}^k$.

3. First Isotopy Lemma

In this section we give a proof of Theorem 2. The proof is patterned on the proof of the main Theorem of [13]. We will also make use of the following semialgebraic version of Thom's first isotopy lemma due to Coste and Shiota:

Theorem 10. [6, Theorem 1] *Let $U \subset \mathbb{R}^n$ be an open subset, $(X, \Sigma) \subset U$ some semialgebraic stratified set. Let $f : U \to \mathbb{R}^k$ be a Nash mapping such that the restriction of f to X is proper, and that the restriction to each stratum $S \in \Sigma$ is a Nash submersion. Then there is a semialgebraic trivialization of f over $\mathbb{R}^k$, compatible with Σ:*

$$h = (h_0, p_{|X}) : X \to f^{-1}(0) \times \mathbb{R}^k$$

such that h restricted to S is a Nash diffeomorphism onto $S \cap f^{-1}(0) \times \mathbb{R}^k$, for any stratum $S \in \Sigma$.

To prove Theorem 2 we will need the following Lemma:

Lemma 11. *Let $S \subset \mathbb{R}^n$ be a Nash submanifold and let $f : S \to \mathbb{R}^k$ be a Nash submersion. For each point $y \in \mathbb{R}^k \setminus K_\infty(f)$ there is a neighborhood U_y of y such that for any real number R large enough the restriction of the map (f, ρ) to the submanifold*

$$S_{R,y} := S \cap (f^{-1}(U_y) \setminus B^n(0, R))$$

is a submersion, where $\rho(x) = \sum_{i=1}^{n} x_i^2$;

Proof. Fix $y \in \mathbb{R}^k \setminus K_\infty(f)$. We proceed by way of contradiction, assuming that for any R, the restriction of f to the submanifold $S_{R,y}$ is not a submersion. Then, by Curve Selection Lemma, there is a semialgebraic arc $\gamma : [c, +\infty) \to S$ for some $c > 0$, tending to infinity such that:

(i) $\lim_{t\to+\infty} f(\gamma(t)) = y$,
(ii) $\gamma(t) \wedge \nabla_{\gamma(t)} f_1 \wedge \cdots \wedge \nabla_{\gamma(t)} f_k = \nabla_{\gamma(t)}\rho \wedge \nabla_{\gamma(t)} f_1 \wedge \cdots \wedge \nabla_{\gamma(t)} f_k = 0.$

We can assume that the arc γ is of the form:

$$\gamma(t) = at + o(t), \quad \text{with } |a| = 1$$

and hence:

$$\gamma'(t) = a + o(1).$$

By (ii) we have that:

$$a = \lim_{t\to+\infty} \frac{\gamma(t)}{|\gamma(t)|} \in \lim_{t\to+\infty} span(\nabla_{\gamma(t)} f_1, \ldots, \nabla_{\gamma(t)} f_k) = \lim_{t\to+\infty} (d_{\gamma(t)} f)^*(\mathbb{R}^k) \tag{3.1}$$

where the latter limits are taken in the Grassmannian, and the $span(v_1, \ldots, v_k)$ denotes a linear space generated by the vectors $v_1, \ldots, v_k$.
By Curve Selection Lemma there is $w(t) \in (d_{\gamma(t)} f)^*(\mathbb{R}^k)$ tending to a. This implies that there exists $v(t) \in \mathbb{R}^k$ for which we have:

$$w(t) = t(d_{\gamma(t)} f)^*(v(t)). \tag{3.2}$$

Observe that $v(t)$ is bounded, for otherwise if $v(t) \to \infty$ then:

$$t(d_{\gamma(t)} f)^* \left(\frac{v(t)}{|v(t)|} \right) = \frac{w(t)}{|v(t)|} \to 0,$$

which means that the point y is an asymptotic critical value of the function f, in contradiction with our assumption on y.

As, by (i), $f(\gamma(t))$ is bounded, we have:

$$\left| \int_c^{+\infty} \frac{df(\gamma(t))}{dt} \right| < \infty. \tag{3.3}$$

Since $f(\gamma(t))$ is a semialgebraic mapping, it has a Puiseux expansion, which, by (3.3), entails that there are some constant C and some $\alpha > 1$ such that:

$$\left| \frac{df(\gamma(t))}{dt} \right| = \left| d_{\gamma(t)} f \cdot \gamma'(t) \right| \leq \frac{C}{t^\alpha}. \tag{3.4}$$

Observe now that we have for some constant C':

$$\begin{aligned} &< \gamma'(t), w(t) >=< \gamma'(t), t(d_{\gamma(t)} f)^*(v(t)) > \\ &= t < \frac{df(\gamma(t))}{dt}, v(t) >\leq t\frac{C'}{t^\alpha} = \frac{C'}{t^{\alpha-1}} \to 0. \end{aligned} \tag{3.5}$$

This fact is impossible since, by definition, both $\gamma'(t)$ and $w(t)$ are tending to a. Hence, for $R > 0$ large enough, the mapping (f, ρ) is a submersion on the set $S_{R,y}$. □

Let us point out that Lemma 11 gives yet another proof, that a point $y \in \mathbb{R}^k \setminus K_\infty(f)$ is a ρ-regular value in the sense of [7].

The following Lemma is a well known fact (see [2]) which we recall for the convenience of the reader.

Lemma 12. *Let (X, Σ) be a Whitney (b) stratified subset of $\mathbb{R}^n$ and $N \subset \mathbb{R}^n$ a $\mathcal{C}^1$ submanifold. If N is transverse to all the strata of Σ then $\Sigma_N := \{S \cap N \ : \ S \in \Sigma\}$ is a Whitney (b) stratification of $N \cap X$.*

Proof. Let S and T be two strata of Σ such that $T \subset \overline{S}$. Let $(x_m) \subset S \cap N$ and $(y_m) \in T \cap N$ be two sequences tending to $y \in T \cap N$. Assume moreover that the sequence of secants $\overline{x_m y_m}$ converges to a line l and that the sequences of tangent spaces $T_{x_m}(S \cap N)$ has a limit τ. Extracting a subsequence, if necessary, we may assume that the sequence $T_{x_m} S$ converges to some τ'. Since $T_y N$ is transverse to $T_y S$, it follows that $\tau = \tau' \cap T_y N$. The Whitney (b) condition of the pair S, T gives $l \subset \tau'$. On the other hand as N is a $\mathcal{C}^1$ submanifold $l \subset T_y N$ as well, which yields $l \subset \tau$. □

Proof of Theorem 2. Let U_y be the neighbourhood of y provided by Lemma 11. The restricted mapping

$$(f, \rho) : f^{-1}(U_y) \setminus \overline{B}^n(0, R) \to \mathbb{R}^k \times (R, +\infty) \tag{3.6}$$

is proper. By Lemma 11, for R large enough, it constitutes a stratified mapping with the stratification induced by Σ.

Moreover, Lemma 11 shows that for each stratum $S \in \Sigma$ the restriction of the map ρ to $S \setminus \overline{B^n(0, R)}$ is a submersion for R large enough. Hence the sphere $S^{n-1}(0, 2R)$ is transverse to each stratum $S \in \Sigma$. Due to Lemma 12, the stratification Σ' given by $S \cap B^n(0, 2R)$ and $S \cap S^{n-1}(0, 2R)$, $S \in \Sigma$, is Whitney (b)-regular and f induces a Nash submersion on every stratum $S' \in \Sigma'$. Hence, the restriction map

$$(f_R, \Sigma') : \overline{B^n(0, 2R)} \cap f^{-1}(U_y) \ni x \mapsto f(x) \in Y \tag{3.7}$$

is a proper stratified submersion for R large enough.

By Theorem 10, there exist two trivializations:

$$H : f^{-1}(U_y) \setminus B^n(0, R) \to (S^{n-1}(0, R) \cap f^{-1}(y)) \times U_y \times [R^2, +\infty), \tag{3.8}$$

$$\Phi : f^{-1}(U_y) \cap \overline{B^n(0, 2R)} \to (f^{-1}(y) \cap \overline{B^n(0, 2R)}) \times U_y \tag{3.9}$$

of type $H(x) = (h(x), f(x), \rho(x))$ and $\Phi(x) = (\varphi(x), f(x))$, for some $R > 0$. We may regard H as a trivialization of the map f which preserves the levels of ρ. Indeed, the mapping

$$A : f^{-1}(U_y) \setminus B^n(0, 2R) \to (f^{-1}(y) \setminus B^n(0, 2R)) \times U_y$$

given by

$$x \mapsto (H^{-1}(h(x), y, \rho(x)), f(y)).$$

is such a trivialization. By Remark 7 we may assume that both mappings A and φ induce the identity on the preimage of y under f. Furthermore, these trivializations preserve the strata.

Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a nonnegative decreasing $\mathcal{C}^2$ semialgebraic function such that:

$$\alpha(t) = \begin{cases} 1 & t < R + \varepsilon \\ 0 & t > 2R - \varepsilon \end{cases} \tag{3.10}$$

for some small $\varepsilon > 0$.

We now define a map

$$\Psi = (\psi, f) : f^{-1}(U_y) \to f^{-1}(y) \times U_y$$

by setting:

$$\psi(x) = \begin{cases} \varphi(x), & |x| \leq R \\ \varphi(H^{-1}(h(x), \alpha(|x|)f(x) + (1-\alpha(|x|))y, \rho(x))), & R < |x| < 2R \\ H^{-1}(h(x), y, \rho(x)), & |x| \geq 2R. \end{cases} \tag{3.11}$$

We claim that this map is a semialgebraic trivialization, shrinking the neighbourhood U_y of y if necessary.

As both maps h and φ preserve the strata so does ψ. It thus suffices to show that ψ induces a diffeomorphism on every stratum. In other words, we can assume that X is a manifold. First, observe that the map (ψ, f) is an injection. Indeed, let $\gamma_1, \gamma_2 : (0, 1) \to f^{-1}(U_y)$ be semialgebraic arcs such that $(\psi, f)(\gamma_1(t)) = (\psi, f)(\gamma_2(t))$ for all $t \in (0, 1)$ and $f(\gamma_1(t)) \to y$ as $t \to 0$.

1^0 CASE. $\gamma_1(t) \to \infty$.

As the map H preserves the distance of the origin and Φ preserves $B^n(0, 2R)$, we see that $\gamma_2(t)$ leaves $B^n(0, 2R)$ and since H is a trivialization itself we conclude that $\gamma_1(t) = \gamma_2(t)$.

2^0 CASE. $\gamma_1(t)$ tends to some point x_1.

In this case, γ_2 also tends to some point, say x_2. Note that, as h and φ induce the identity map on the preimage of y (under (f, ρ) and f respectively), so does ψ. This means that $x_1 = x_2$. Moreover, this implies that the derivative of the map Ψ has maximal rank near this set and thus, by the local inverse mapping theorem, Ψ induces a local diffeomorphism near x_1. This entails that $\gamma_1(t) = \gamma_2(t)$ for t small, which means that the map Ψ is injective.

We now turn to show surjectivity. For z in U_y, let $\psi_z : f^{-1}(z) \to f^{-1}(y)$ denote the map induced by ψ. As ψ_z tends to the identity, it is a submersion for z close to y which entails that it is an open map. But, since it is a proper map, it must be a closed map as well. Hence, the image of ψ_z is the union of some connected components of $f^{-1}(y)$. It means that it is enough to prove that, for z close to y, the image of ψ_z contains at least one point in every connected component. Observe that as Φ and H are both trivializations, $f^{-1}(y)$ lies in the closure of $f^{-1}(U)$, for all neighbourhoods U of y. As ψ_z tends to the identity (since A and φ both induce the identity on $f^{-1}(y)$), it means that the image of ψ_z contains at least one point in every connected component of $f^{-1}(y)$, as required. □

Remark 13. Using the techniques developed by Coste and Shiota in [6] one could obtain a trivialization which is $\mathcal{C}^\infty$ on each stratum. The argument is however fairly technical.

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