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Does Our Universe Prefer Exotic Smoothness?

Torsten Asselmeyer-Maluga [1,2,†], Jerzy Król [2,3,*,†] and Tomasz Miller [2]

1 German Aerospace Center (DLR), Rutherfordstr. 2, 12489 Berlin, Germany; Torsten.Asselmeyer-Maluga@dlr.de
2 Copernicus Center for Interdisciplinary Studies, Jagiellonian University, Szczepańska 1/5, 31-011 Cracow, Poland; tomasz.miller@uj.edu.pl
3 Cognitive Science and Mathematical Modelling Chair, University of Information Technology and Management, ul. Sucharskiego 2, 35-225 Rzeszów, Poland
* Correspondence: iriking@wp.pl
† These authors contributed equally to this work.

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Abstract: Various experimentally verified values of physical parameters indicate that the universe evolves close to the topological phase of exotic smoothness structures on $\mathbb{R}^4$ and K3 surface. The structures determine the α parameter of the Starobinski model, the number of e-folds, the spectral tilt, the scalar-to-tensor ratio and the GUT and electroweak energy scales, as topologically supported quantities. Neglecting exotic R^4 and K3 leaves these free parameters undetermined. We present general physical and mathematical reasons for such preference of exotic smoothness. It appears that the spacetime should be formed on open domains of smooth K3$\#\overline{CP^2}$ at extra-large scales possibly exceeding our direct observational capacities. Such potent explanatory power of the formalism is not that surprising since there exist natural physical conditions, which we state explicitly, that allow for the unique determination of a spacetime within the exotic K3.

Keywords: exotic R^4 and cosmology; space topology changes; exotic K3; spacetime

1. Introduction

The micro-scale of the physical world and the large cosmological scales, when organised into a single cosmological model of the universe, should be finely interrelated. Even though we do not fully understand how these scales might intersect and interact with each other, our partial understanding allows for important insights. In particular, we expect that the complete picture of the domain of their common applicability would be a crucial ingredient of the successful theory of quantum gravity. The reason is simple: The universe at large scales where gravity dominates is described by the theory of general relativity (GR), whereas at the micro-scale the suitable theory is quantum mechanics (QM).

There are many reasons to introduce exotic smoothness. From the physics point of view one natural reason is quantum gravity. In the last years, we developed an approach, smooth quantum gravity, where the quantization procedure is given by a change of the smoothness structure [1]. The approach works only for four-dimensional spacetimes and has many connections to noncommutative geometry. Loosely speaking, the change of the smoothness structure is a quantization of the geometry in the sense of quantum gravity. A direct consequence of this approach is the determination of topology changes. To illustrate, let us consider a spacetime of topology $S^3 \times \mathbb{R}$. In the usual smoothness structure, this spacetime is foliated like $S^3 \times \{t\}$, i.e., the topology of the space S^3 remains constant. In contrast, a spacetime with topology $S^3 \times \mathbb{R}$ but exotic smoothness can also be foliated like $S^3 \times \{t\}$ but not smoothly. The smooth decomposition of an exotic $S^3 \times \mathbb{R}$ is a spacetime where the spatial component changes in a complicated process. Interestingly, the change seen as a process can be very different but the result of the change depends only on the topology of the spacetime. In the presented paper we construct a spacetime from first principles and show that there are two topology changes. Interestingly, this universal feature of spacetime can be understood by considering certain exotic $\mathbb{R}^4$.

The standard smoothness structure of $\mathbb{R}^4$ is the unique structure such that the product $\mathbb{R} \times \mathbb{R}^3$ is smooth. An exotic R^4 is a topological 4-manifold $\mathbb{R}^4$ which, if smooth, is nondiffeomorphic to the standard smooth $\mathbb{R}^4$. In any dimension other than 4, there exists a unique smoothness structure on $\mathbb{R}^n, n \neq 4$, the standard smooth $\mathbb{R}^n$. The existence of exotic R^4 was established in the 1980s and, together with the existence of at least two families of R^4s each containing uncountably infinitely many different nondiffeomorphic R^4s, are highly nontrivial mathematical facts (e.g., [2]). One such family of small exotic R^4s comprises those R^4s that are embeddable in the standard $\mathbb{R}^4$ as open subsets while the large exotic R^4s are not embeddable in $\mathbb{R}^4$ and hence in S^4.

The existence of such smooth exotic 4-manifolds may seem to be a purely mathematical curiosity; however, the application to physics also discussed in this paper shows it is not. On all (known) four-dimensional open manifolds there exist uncountably many different nondiffeomorphic smoothness structures. Compact 4-manifods can be endowed with countably many such structures. The main point advocated here and in our previous works is that one cannot understand the origins of certain values of important physical parameters (cosmology, particle physics) and one cannot understand the common domain of GR and quantum phenomena in the spacetime of dimension 4 without referring to exotic smooth 4-manifolds. Even though the current state of investigation does not support decisively and univocally the above categorical statements, the results collected strongly support them.

The exceptional (though quite direct) feature of exotic R^4s is that they are all Riemann smooth 4-manifolds which cannot be flat, i.e., their Riemann curvature tensors are not vanishing on any exotic R^4s. From the point of view of physics, a nonzero gravitational energy density is assigned to each exotic R^4, contrary to the case of the standard $\mathbb{R}^4$. Recently Gabor Etesi showed that certain smooth four-dimensional manifolds, namely the large exotic R^4s, are precisely the gravitational instantons [3]. Both these facts, being a Ricci-flat gravitational instanton and carrying nonzero gravitational energy, show that R^4s indeed place themselves in the overlapping domain of classical and quantum regimes of gravity. We will discuss the particular role played by the Ricci-flatness in the process of the generation of masses in spacetime. This is one of the first physical effects which has been considered in the context of exotic R^4 and it is known as the Brans conjecture. It states that exotic R^4s serve as sources of an external gravitational field in spacetime [4,5]. Moreover, R^4s determine noncommutative von Neumann algebras which is not the case for the standard $\mathbb{R}^4$ and this is yet another indication that R^4s are properly (though somewhat mysteriously) placed in the common domain of GR and QM (e.g., [1,6]). In recent publications [7–9] we have shown how the appearance of nonstandard smoothness on $\mathbb{R}^4$ and a K3 surface leads to explaining in purely topological terms the extremely tiny value of the cosmological constant and some other cosmological parameters.

This apparent multifaceted role of exotic smoothness on R^4 in physics, especially cosmology, motivates the attempt to understand the exotic smoothness as a consequence of certain, quite general, conditions imposed on physical spacetime of dimension 4. In what follows we explicitly state these conditions and discuss them from the physical and mathematical points of view. Both threads finely meet and intertwine in dimension 4 giving rise to a quite powerful explanatory framework. In particular it appears that considering space as homology 3-spheres (including S^3) is a general fact following from the causal and Lorentzian-metric structure (for a spacetime being a smooth 4-cobordism). Exotic 4-smoothness determines such cobordisms canonically which lies in the core of the presented approach. Finally, we overview and discuss the main results obtained within the framework.

2. Spacetime and Exotic Smoothness

In our previous work [9] we discussed a model with a compact spacetime, the K3 surface, where the cosmic evolution was given by an open submanifold. The important feature of the model is that a certain exotic R^4 is necessarily embedded into (a smooth version of) K3. Let us now reverse the argumentation and consider an evolution of the cosmos which starts with a 3-sphere and allows for spatial topology changes. As a consequence we will obtain the K3 surface with the two transitions as discussed in [9].

Topology describes the global properties of a manifold which are invariant with regard to the local shape or geometry. A local theory based on differential geometry like GR restricts very weakly the topology of spacetime. Because of this ambiguity as a rule we have to set a topology of the cosmos by hand, e.g., Einstein used the 3-sphere S^3 but $\mathbb{R}^3$ is another common choice.

Here we will discuss the topological implications of the assumed spacetime with an exotic smooth structure. We shall also need some further mild conditions to formulate a sufficiently useful cosmological model. The first condition is given by the measurement data of the cosmological background radiation of the COBE, WMAP and PLANCK experiments [10–13]. The analysis of the spectrum by Luminet et al. [14] gives a hint of a cosmos with a finite volume which is compatible with the Einstein cosmos S^3 or any other compact model, but not with $\mathbb{R}^3$. Thus our first condition on the topology of the cosmos is the following

1. The cosmos Σ is a compact 3-manifold without boundary.

Next we concentrate on spacetime. The choice of a spacetime is strongly restricted by two demands: Smoothability and causality (including the existence of a Lorentz metric). Usually the two conditions can be fulfilled if the spacetime M is diffeomorphic to $\Sigma \times \mathbb{R}$ with the (spatial) 3-manifold Σ, i.e., one makes the assumption that the topology of Σ is fixed. However, it is widely believed that the inclusion of quantum-gravitational effects enforces transitions of the (spatial) topology. We discussed in our previous works the possibility of an exotic smoothness structure which leads necessarily to topological transitions. To enable the topological transitions of Σ we have to model the spacetime as a cobordism M with $\partial M = \Sigma_0 \sqcup \Sigma$ describing the nontrivial evolution (i.e., $M \neq \Sigma \times \mathbb{R}$) from the initial state Σ_0 to the cosmos Σ at the epoch t. The cobordism M between a compact 3-manifold is also itself compact for a finite time interval. A compact manifold M possesses a Lorentz metric if (and only if) there exists a nonvanishing vector field, i.e., its Euler characteristic $\chi(M)$ is zero [15,16] or in case of the cobordism the relative Euler characteristic vanishes $\chi(M, \partial M) = 0$. Thus the second condition is:

2a. The relative Euler characteristic $\chi(M, \partial M)$ of the spacetime M is zero.

The topological censorship theorem [17] requires a simply connected spacetime. This is a necessary condition to avoid time-loops (which are contractible in a simply connected spacetime):

2b. The spacetime M is simply connected.

Conditions 2a and 2b imply the vanishing of the relative homology groups $H_k(M, \partial M) = 0$ for $k = 0, 2, 3$.

For let a 4-manifold M be 4-cobordism between two 3-manifolds Σ_1, Σ_2 such that $\partial M = \Sigma_1 \sqcup \Sigma_2$. To determine the homology of M, one has to use the following long exact sequence of homology groups

$$\ldots \to H_k(\partial M) \to H_k(M) \to H_k(M, \partial M) \to H_{k-1}(\partial M) \to H_{k-1}(M) \to \ldots$$

where the maps between the homology groups are induced by the inclusions $\partial M \to M$ and $M \to M/\partial M$. Now let us assume that M is simply connected, i.e., $H_1(M) = 0$ (Condition 2b above). We thus obtain the sequence

$$0 \to H_2(\partial M) \to H_2(M) \to H_2(M, \partial M) \to H_1(\partial M) \to 0 \tag{1}$$

where we used the Poincaré duality $H_3(M, \partial M) = H^1(M) = \mathrm{Hom}(H_1(M), \mathbb{Z}) = 0$. For the other terms of the sequence we get $H_k(M, \partial M) = 0$ for $k = 0, 3$ (Betti numbers $b_0 = b_3 = 0$) and $H_k(M, \partial M) = \mathbb{Z}$ for $k = 1, 4$ (Betti numbers $b_1 = b_4 = 1$). In order to ensure the existence of a

Lorentz metric we need M to admit a nonvanishing time-like vector field which requires the relative Euler characteristics to vanish, $\chi(M, \partial M) = 0$. Since $\chi(M, \partial M) = b_0 - b_1 + b_2 - b_3 + b_4$, we obtain $\chi(M, \partial M) = b_2$. All in all, the demand that a Lorentz metric exists leads to $H_2(M, \partial M) = 0$. Therefore from Sequence (1) we obtain $H_1(\partial M) = 0 = H_2(\partial M)$ and hence the boundary ∂M must be a disjoint union of homology 3-spheres.

Thus we see that the physical conditions of the existence of a Lorentz metric (Condition 2a) and of causality (Condition 2b) are equivalent to the following condition for the topology of the cosmos:

3. The cosmos Σ is a homology 3-sphere.

Let us summarise the points above and draw conclusions for the entire spacetime M. Interestingly, the conditions stated above have a strong and direct connection to the smoothness structure of M. The spacetime M is assumed to be a 4-manifold with a metric fulfilling the Einstein equation and admitting a smoothness structure. The smoothness structure in dimension 4 is characterised by the embedding of a certain four-dimensional submanifold $A \subset M$ – the Akbulut cork. The Akbulut cork is a contractible 4-manifold with the boundary a homology 3-sphere [18]. Now we choose an exotic smoothness structure. This step is motivated by the generation of matter resulting from the exotic smoothness structure (see [19,20] for instance). The smoothness of the exotic M requires that the Akbulut cork of M possesses two homology 3-spheres as boundaries $\partial A = S_0 \sqcup S_1$ and that the initial sphere $S_0 = S^3$ is a simple 3-sphere contained in Σ_0 in agreement with the two physical conditions (2a and 2b) above. This is precisely the point where the exotic R^4 is generated: The neighbourhood of the Akbulut cork $N(A) \subset M$ as embedded in the 4-manifold M is an exotic R^4 if M admits an exotic smoothness structure (or M is exotic). Then, Conditions 1–3 lead us univocally to a simple cosmological model:

4. The spacetime M is a smooth 4-manifold with $\partial M = \Sigma_0 \sqcup \Sigma$, realising a cobordism between two homology 3-spheres.

Initial state: The cosmos begins as a compact 3-manifold Σ_0 without boundary (Condition 1) and possesses the topology of a homology 3-sphere (Condition 2).

Dynamics: The spacetime is a cobordism M with $\partial M = \Sigma_0 \sqcup \Sigma$ (Condition 3). This 4-manifold is simply connected (Condition 2b) and its pseudo-Riemannian metric (Condition 2a) is determined by the Einstein equation. The cosmos expands from Σ_0 to Σ with the scaling factor $a(t)$ determined by the Friedmann equation. It is interesting to note that cobordisms represent properly spacetime in the categorical approach by John Baez [21]. In Baez's representation the entire category of spacetime cobordisms (between 3-space manifolds) is considered leading to a natural connection with quantum mechanics (as in topological quantum field theory, TQFT). Even though in our approach the smoothness structures in dimension 4 determine nontrivial cobordisms and we do not discuss the quantum operator representation, still this would be an interesting nontrivial task to find connections with TQFT.

Topology transition: The homology of the cosmos is an invariant (both Σ_0 and Σ are homology 3-spheres, Conditions 2 and 3). The topology of the initial state Σ_0 may change to Σ by a homology-preserving transition (nontriviality of $M \neq \Sigma \times \mathbb{R}$).

In order to firmly establish the model we now have to choose tangible candidates for Σ_0 and Σ. One can exclude that Σ_0 is a point singularity because in this case we would have $\chi(M) = 1$ (i.e., the time-like vector field vanishes at this singular point). However, we have seen that the Akbulut cork of M is a cobordism between a 3-sphere S^3 and a homology 3-sphere S_1 and that $S^3 \subset \Sigma_0$. Thus, it seems natural to choose $\Sigma_0 = S^3$:

5. The initial state Σ_0 is the Einstein cosmos S^3.

This choice for the initial state is further supported by Ashtekar et al. [22] where the authors described a cosmological model with the big bounce effect (see also [23]). The model does not show a singularity, i.e., there is no big crunch but rather contraction is followed again by expansion.

The cork A with $\partial A = S^3 \sqcup S_1$ is a submanifold of M with $\partial M = S^3 \sqcup \Sigma$. Thus S_1 is in the interior of M and Σ is the boundary. Given Σ as the state at time t one can interpret S_1 as an intermediate state $\Sigma(t_1) = S_1$ with $t_1 < t$. However, according to Donaldson [24] not all homology 3-spheres are smoothly cobordant to S^3 (i.e., M with $\partial M = S^3 \sqcup \Sigma$ is not smooth for all Σ). There is no full classification of such homology 3-spheres but rather a long list of counterexamples. One example shows that there is no smooth cobordism M between S^3 and one or more Poincaré spheres. A large class of homology 3-spheres are Brieskorn spheres described as submanifolds of $\mathbb{C}^3$

$$\Sigma(a,b,c) = \left\{ (z_1, z_2, z_3) \in \mathbb{C}^4 \mid z_1^a + z_2^b + z_3^c = 0, |z_1|^2 + |z_2|^2 + |z_3|^2 = 1 \right\}$$

with a, b, c different prime numbers. The Brieskorn spheres are distinguished from other homology 3-spheres because they are irreducible and any homology 3-sphere is a sum of irreducible homology 3-spheres. Any irreducible 3-manifold Σ is characterized to be not splittable to the connected sum other than $\Sigma \# S^3$ (prime decomposition, see [25]), i.e., irreducible Σ can only be split trivially into $\Sigma \# S^3$ (diffeomorphic to Σ). Secondly, there is another splitting of irreducible 3-manifolds along 2-tori into simpler pieces, the so-called JSJ decomposition (Jaco–Shalen–Johannson decomposition, see [26]). The remaining pieces are called atoroidal irreducible 3-manifolds. Brieskorn spheres are the only nonhyperbolic irreducible homology 3-spheres. As we shall see shortly, these properties are crucial for applications in physics.

The solution of the geometrization conjecture implies that there are two important geometric classes of topological manifolds in dimension 3: Hyperbolic and nonhyperbolic 3-manifolds. The class of nonhyperbolic 3-manifolds is divided into seven subclasses among which there are the spherical and Euclidean geometries. Hyperbolic 3-manifolds are very special with respect to their properties. The main property important in this work is the rigidity of the volume for any diffeomorphism and conformal transformation (Mostow rigidity, see [27]), i.e., the volume is a topological invariant. Any scaling of a hyperbolic 3-manifold is an isometry or a hyperbolic 3-manifold cannot be scaled. This fact is extremely important for the evolution of the spatial component (as given by the cobordism M): If the intermediate state, say $\Sigma(t_1)$ at $t_0 < t_1 < t$, is a hyperbolic homology 3-sphere then the expansion of the spatial component has to stop (because of the Mostow rigidity). Therefore we have to assume that this intermediate state must be a nonhyperbolic 3-manifold. For simplicity reasons we choose an irreducible, nonhyperbolic 3-manifold (otherwise one has a sum of irreducible 3-manifolds as an intermediate earlier state which comprises of these irreducible 3-manifolds). For this reason the Brieskorn spheres are natural building blocks of all nonhyperbolic homology 3-spheres. The counterexample is the Poincaré sphere $\Sigma(2,3,5)$ which is the simplest one but cannot be used in any smooth cobordism with S^3. Moreover, the next one $\Sigma(2,3,7)$ provides another counterexample. The simplest Brieskorn sphere which is smoothly cobordant to S^3 is $\Sigma(2,5,7)$. Thus we look for an exotic M with the Akbulut cork A with $\partial A = S_0 \sqcup S_1$, $S_0 = S^3$ and $S_1 = \Sigma(2,5,7)$:

6. The intermediate state $\Sigma(t_1) = S_1$ at $t_0 < t_1 < t$ is the Brieskorn cosmos $\Sigma(2,5,7)$.

Finally we have to choose the 4-manifold M itself. There are two points of consideration which are important here. At first, in [19] we have shown that the transition of a standard 4-manifold to an exotic one results in non-Ricci-flatness. If we hypothesise that all matter terms in the Einstein–Hilbert action are only caused by exotic smoothness in the above way then the 4-manifold with its standard structure has to be Ricci-flat. However, there are only two compact 4-manifolds with a Ricci-flat metric, the 4-torus and the K3 surface

$$K = \left\{ (x,y,z,t) \in CP^3 \middle| \, x^4 + y^4 + z^4 + t^4 = 0 \right\} . \tag{2}$$

The 4-torus is a flat manifold that is not simply connected and so it contradicts Condition 2b, thus from the physical point of view the K3 surface is the preferred candidate of a spacetime. This is further supported by the second fact of consideration: The proposed 4-manifold A with $\partial A = S_0 \sqcup S_1$, $S_0 = S^3$ and $S_1 = \Sigma(2,5,7)$ is the Akbulut cork of a distinct 4-manifold K which is again the K3 surface.

The K3 surface is a compact 4-manifold with nonvanishing Euler characteristic and thus it admits no Lorentz metric. Therefore, the K3 surface itself cannot be the physical spacetime. However, we can imagine the cobordism M (with $\chi(M) = 0$ and equipped with a Lorentz metric) embedded in K. The submanifold $M \subset K$ is determined by K if one requires that both manifolds have the same Akbulut cork A with $\partial A = S_0 \sqcup S_1$, $S_0 = S^3$ and $S_1 = \Sigma(2,5,7)$. The choice $S_0 = S^3$ (Condition 4) is extended to the cork of the K3 surface if one replaces K by a version of the K3 surface $\mathcal{K} = K \setminus D^4$ with boundary $\partial(K \setminus D^4) = S^3$, i.e., we get $\partial\mathcal{K} = \partial(K \setminus D^4) = S^3 = S_0 = \Sigma_0$. Thus we arrive at the last condition of the model:

7. The K3 surface $\mathcal{K} = K \setminus D^4$ determines the 4-manifold M with $\partial M = S^3 \sqcup \Sigma$ by its common Akbulut cork. M is the physical spacetime.

Then the boundary component S^3 of M agrees with $\partial\mathcal{K}$ and M contains also the Akbulut cork A of $\mathcal{K}$, i.e., the 4-manifold representing the first transition $S_0 = S^3 \to S_1 = \Sigma(2,5,7)$ is the Akbulut cork of $\mathcal{K}$. Let us assume that the matter component in spacetime is caused by the exotic smoothness. However, the exotic smoothness is not determined by the topology of the Akbulut cork A but by the embedding of A into $\mathcal{K}$. Therefore we have to determine the neighbourhood $N(A) \subset \mathcal{K}$ of A in $\mathcal{K}$ to determine the smoothness structure. However, then the remaining part $\mathcal{K} \setminus N(A)$ is obtained purely by its topology. The boundary $\partial\left(\mathcal{K} \setminus N(A)\right) = S^3 \sqcup \Sigma$ contains the second component Σ (as a boundary of $N(A)$) which is also a homology 3-sphere (using the result of Freedman [18,28]). The topology of Σ is partly determined by the topology of $\mathcal{K}$. The reasons are the following.

Topological 4-manifolds are classified by the intersection form σ [18]. In case of our 4-manifold $\mathcal{K}$, one obtains

$$\begin{aligned} \sigma_{\mathcal{K}} &= E_8 \oplus E_8 \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= 2E_8 \oplus 3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2E_8 \oplus 3H \end{aligned}$$

in the usual notation. The intersection form of the Akbulut cork A, as well as of $N(A)$, vanishes. By the splitting theorem in [29] one obtains

$$\sigma_{\mathcal{K} \setminus N(A)} = \sigma_{\mathcal{K}} = 2E_8 \oplus 3H$$

i.e., the same intersection form. Now $\mathcal{K} \setminus N(A)$ has the boundary

$$\partial\left(\mathcal{K} \setminus N(A)\right) = S^3 \sqcup \Sigma$$

and must be a smooth 4-manifold. Especially the block structure of the intersection form is reflected by the splitting of a 4-manifold. With these information we obtain the following general result

$$\Sigma = P\#P\#\left(K_1\#K_2\#K_3\right)\#S^3\,. \tag{3}$$

This 3-manifold Σ is also a homology 3-sphere consisting of three principal parts: The connected sum $P\#P$ of two Poincaré spheres, the connected sum of three irreducible homology 3-spheres $K_1\#K_2\#K_3$ and a 3-sphere. Of course one can omit the last 3-sphere but we keep it here as a reminder that the 3-sphere is always present in the connected sum # not changing the diffeomorphism class.

With Decomposition (3) at hand, we are able to complete our model using all six conditions above. It starts with a 3-sphere (Einstein cosmos), then the first transition to the Brieskorn sphere $\Sigma(2,5,7)$ takes place and finally it changes (second transition) to $\Sigma = P\#P\#\left(K_1\#K_2\#K_3\right)\#S^3$.

The two transitions are interpreted as inflationary phases [7,30] determining also the neutrino masses [8]. The three irreducible homology 3-spheres K_1, K_2, K_3 are identified with hyperbolic, homology 3-spheres inducing the matter part of the universe [19,20] with connections [6] to the models of Furey [31,32], Gresnigt [33], Bilson–Thompson [34,35]. The transition to the $P\#P$ part gives the cosmological constant [9]. Then, following the logic of the cosmological standard model, the remaining part S^3 (appearing as $S^2 \times [0,1]$ in the sum above) must be the dark matter component which will be discussed in a forthcoming paper. Finally we arrive at the picture:

- $P\#P$ causes the cosmological constant (= dark energy)
- K_1, K_2, K_3 is responsible for the matter part (= three generations?)
- S^3 or $S^2 \times [0,1]$ is associated with the dark matter (in the form of a gravitational soliton?)

3. Physical Parameters

Let us collect and discuss the results obtained on the base of our topological model of the evolving cosmos. Exotic smoothness in dimension 4 is the main player in the model. Therefore we have to motivate the appearance of exotic smoothness. The approach in the previous section is based extensively on the concept of cobordism for the spacetime. Thus, one has to consider the bounadry terms of the Einstein–Hilbert action. As starting point, let us discuss the Einstein–Hilbert action for a 4-manifold with boundary following our work [20]. In general, for a manifold M with boundary $\partial M = \Sigma$ one has the expression

$$S_{EH}(M) = \int\limits_M R\sqrt{g}\, d^4x + \int\limits_\Sigma H\,\sqrt{h}\, d^3x$$

where H is the mean curvature of the boundary with metric h. In the following we will discuss the boundary term, i.e., we reduce the problem to the discussion of the action

$$S_{EH}(\Sigma) = \int\limits_\Sigma H\,\sqrt{h}\, d^3x \tag{4}$$

along the boundary Σ (a 3-manifold). Following [20], Equation (4) over a 3-manifold Σ is equivalent to the Dirac action of a spinor over Σ. Main result of [20] is the following relation between the corresponding Dirac operators

$$D^M\Phi = D^\Sigma\psi - H\psi \tag{5}$$

where D^Σ or D^M denote the Dirac operator on the 3-manifold Σ or 4-manifold, respectively. Now Φ must be a parallel spinor, i.e.,

$$D^M\Phi = 0 \tag{6}$$

Finally we get

$$D^\Sigma\psi = H\psi \tag{7}$$

leading to

$$\int\limits_\Sigma H\,\sqrt{h}\, d^3x = \int\limits_\Sigma \bar{\psi}\, D^\Sigma\psi\,\sqrt{h} d^3x \tag{8}$$

In our previous work [1] we discussed a foliation of the 3-manifold which extends to the cobordism representing the topology change of the 3-manifold. There, we introduced the Godbillon–Vey invariant as topological invariant of the foliation. This foliation of codimension one is defined by a one-form ω (the leaves are the constant values) with integrability condition $\omega \wedge d\omega = 0$. Then the Godbillon–Vey invariant is defined by an integral over the 3-form $\eta \wedge d\eta$ with $d\omega = -\eta \wedge \omega$. Clearly, the foliation will also influence the spinor defined by Equation (8). For that purpose we reinterpret the invariant

$gv = \eta \wedge d\eta$ as the abelian Chern–Simons form for the abelian gauge field η. Then a covariant constant 1-form ω such that

$$D_\eta \omega = d\omega + \eta \wedge \omega = 0$$

defines a foliation, because the integrability condition $\omega \wedge d\omega = 0$ is automatically fulfilled. However, here we will use the coupling between the abelian gauge field η and the spinor ψ to the Dirac–Chern–Simons action functional on the 3-manifold

$$S_{DCS} = \int\limits_\Sigma \left(\bar{\psi}\, D^\Sigma_\eta \psi \,\sqrt{h} d^3x + \eta \wedge d\eta \right)$$

with the critical points at the solution

$$D^\Sigma_\eta \psi = 0 \quad d\eta = \tau(\psi, \psi)$$

where $\tau(\psi, \psi)$ is the unique quadratic form for the spinors locally given by $\bar{\psi}\gamma^\mu\psi$. Now we consider a spacetime $\Sigma \times I$, so that the solution is translationally invariant. Expressed differently, we choose a spacetime with foliation induced by the foliation of Σ extended by translation. An alternative description for this choice is by considering the gradient flow of these equations

$$\begin{aligned} \frac{d}{dt}\eta &= d\eta - \tau(\psi, \psi) \\ \frac{d}{dt}\psi &= D^\Sigma_\eta \psi \end{aligned}$$

However, it is known that this system is equivalent to the Seiberg–Witten equation for $\Sigma \times I$ by using an appropriated choice of the so-called $Spin_C$ structure. Then this $Spin_C$ structure is directly related to the foliation. Therefore a nontrivial foliation together with the existence of Fermions induces a nontrivial solution of the gradient system which results in a nontrivial solution of the Seiberg–Witten equations. However, this nontrivial solution (i.e., $\psi \neq 0, \eta \neq 0$) is a necessary condition for the existence of an exotic smoothness structure.

With these arguments we obtained a strong relation between foliations, exotic smothness and our model for a spacetime (with spatial topology change). The origin of this foliation can be traced back to the Einstein cosmos. As discussed in [1], this initial state S^3 of the universe cannot be a smooth S^3 but rather a wild embedded 3-sphere (representing the quantum geometry of the quantum state). It is a direct consequence of exotic smoothness. As shown in the previous section, this initial state determines the stages of all further changes. In particular, it determines the growing of the 3-manifolds within the topology changing process. This process is related to hyperbolic geometry so that the scaling parameter a of the 3-manifold is part of the hyperbolic metric da^2/a^2 (relative to the scaling change $d\vartheta^2$ along the cobordism), i.e., we have the relation

$$\frac{da^2}{a^2} = d\vartheta^2$$

between the foliation of the wild embedded 3-sphere and the foliation of the cobordism (representing the topology change) leading to the formal solution

$$a = a_0 \exp(\vartheta)\,.$$

This relation is at the root of the exponential behaviour for the physical parameters, e.g., the scaling parameter reads [9]

$$\vartheta = \frac{3}{2 \cdot CS(\Sigma)}\,.$$

Here the embedding of the exotic R^4 is important because it is directly related to the wild embedding of the 3-sphere representing the initial state. With the arguments above, one obtains an independent derivation of various results based on the exponential behaviour above. This shows that the model in the previous section is completely consistent with the previous work. For completeness, in what follows, we will present main results of this kind.

The curvature of an exotic R^4 depends on the embedding into a broader manifold. Still one can extract the invariant topological quantity of the curvature which corresponds to the embedding. The deep result of [9] is that one finds that the topological invariant quantity of the embedding $R^4 \to \mathrm{K3}\#\overline{CP^2}$ explains the tiny necessarily nonzero value of the cosmological constant (CC). Thus the value of CC is a topological invariant corresponding to the two topology changes as in the previous section, $S^3 \to \Sigma(2,5,7) \to P\#P$, and is given by the formula

$$\Omega_\Lambda = \frac{c^5}{24\pi^2 hGH_0^2} \exp\left(-\frac{3}{CS(\Sigma(2,5,7))} - \frac{3}{CS(P\#P)} - \frac{\chi(A)}{4}\right) \simeq 0.7029 \tag{9}$$

where quantum corrections are included (represented by $1/4$th part of the Euler characteristic of the Akbulut cork A [1] with $\partial A = \Sigma(2,5,7)$). $CS(\Sigma(2,5,7))$ and $CS(P\#P)$ are the Chern–Simons invariants of $\Sigma(2,5,7)$ and $P\#P$, respectively. Thus we have a topological scenario explaining the realistic value of CC avoiding the zero-point energies excessive contributions. The topological invariance does the job: Such a CC value is not an additive quantity since otherwise the topological invariance would be spoiled. We can understand this also by making use of smallness of exotic R^4s as follows.

The defining property of any small R^4 is its embedding into the standard $\mathbb{R}^4$. The invariant topological part of the cosmological constant in this case reads [9]

$$(\text{the CC of the embedding } R^4 \hookrightarrow \mathbb{R}^4) = \frac{1}{\sqrt[3]{\mathrm{Vol}^2(Y_\infty)}} \exp\left(-\frac{3}{CS(Y_\infty)}\right) \tag{10}$$

where Y_∞ is a 3-sphere widely embedded in R^4 with the volume $\mathrm{Vol}(Y_\infty)$ and $CS(Y_\infty)$ is its Chern–Simons invariant. As noted in [9] the Chern–Simons invariant of such a sphere vanishes and so the value of CC vanishes as well by (10). This is a quite remarkable result by itself. Every small exotic R^4 is embeddable in $\mathbb{R}^4$ and the curvature of R^4 depends on the embedding. However, whatever values the Riemann curvature takes the invariant parts for the embedding are always zero. Thus the CC value vanishes for every small R^4 embedded in $\mathbb{R}^4$.

Consider a quantum field theory defined on the Minkowski spacetime M^4 and allow for the (quantum) fluctuations of curvature which lead to a Lorentzian spacetime manifold $\tilde{M}^4$. Even though we do not know the precise quantum description of gravitational fluctuations we still accept the point of view that in the semiclassical limit the zero-point energies of quantum fields give nonvanishing contributions to the vacuum energy density in spacetime. Is it possible that the curvature of $\tilde{M}^4$ be generated by smoothness structure on $\mathbb{R}^4$? Let us consider a certain exotic R^4. Since it is open we can always find a nonvanishing smooth vector field $X(x)$ on R^4 and define a curved Lorentzian manifold M^4_X (e.g., [3]). This construction depends on $X(x)$ but since the embedding $R^4 \hookrightarrow \mathbb{R}^4$ varies the Riemann curvature of R^4, the curvature of the corresponding Lorentz manifold M^4_X varies as well. Thus for such a class of Lorentzian manifolds which are of the form M^4_X for some R^4 and a nonvanishing vector field X on it, the corresponding invariant value of CC vanishes. This can serve as a topological mechanism explaining the vanishing of CC on certain Lorentzian spacetime manifolds. However, the mechanism works under a supposition that the CC contributions on flat Minkowski spacetime generate the curvature which comes from the exotic R^4 as described above. This means that the vanishing of CC can be achieved via changing the smoothness from the standard $\mathbb{R}^4$ to the small exotic R^4 and subsequently considering embedding of the latter into the standard $\mathbb{R}^4$.

Thus we need a two-step extension of spacetime to understand the observed value of CC by topological means (any exotic R^4 is locally the standard $\mathbb{R}^4$): $\mathbb{R}^4 \to R^4$ and $R^4 \to K3$. In fact this kind of a topological approach is quite universal and a couple of other cosmological parameters can be similarly derived as topological invariants. The following examples show the scope of the approach [7,8].

1. The α parameter in the Starobinsky model (in the units of the Planck mass squared)

$$\alpha \cdot M_P^{-2} = \frac{1}{\left(1+\vartheta+\frac{\vartheta^2}{2}+\frac{\vartheta^3}{6}\right)} \approx 10^{-5} \text{ where } \vartheta = \frac{3}{2\cdot CS(\Sigma(2,5,7))} = \frac{140}{3} .$$

2. The number of e-folds during the inflation

$$N = \frac{3}{2\cdot CS(\Sigma(2,5,7))} + \ln(8\pi) \approx 51 .$$

3. The scalar/tensor ratio $r = \frac{12}{(\vartheta+ln(8\pi^2))^2} \approx 0.0046$.

4. The spectral tilt $n_s = 1 - \frac{2}{\vartheta+ln(8\pi^2)} \approx 0.961$.

5. The GUT energy scale $\big($the energy of the first topology change $S^3 \to \Sigma(2,5,7)\big)$

$$\Delta E_1 = \frac{E_{\text{Planck}}}{1+\vartheta+\frac{\vartheta^2}{2}+\frac{\vartheta^3}{6}} \approx 10^{15}\,\text{GeV} .$$

6. The electroweak energy scale (the energy assigned to the second topological transition $\Sigma(2,5,7) \to P\#P$)

$$E_2 = \frac{E_{\text{Planck}} \cdot \exp\left(-\frac{1}{2\cdot CS(P\#P)}\right)}{1+\vartheta+\frac{\vartheta^2}{2}+\frac{\vartheta^3}{6}} \approx 63\,\text{GeV} .$$

7. The topological bound on the sum of the three neutrino masses $< 0.018\,\text{eV}$.

Together with the value of CC the above list strongly suggests that the topology underlying exotic smooth 4-manifolds, like R^4 and K3, might indeed shed some light on the important domains of physics where certain crucial physical parameters remain free or theoretically undetermined. This property of being topological invariant with respect to physical quantities indicates a fundamental character of the approach.

Is there any fundamental symmetry leading to topologically supported physical parameters? One indication follows from the constructions presented in this paper. Firstly, as presented in Section 2 the 4-cobordism between S^3 and $\Sigma(2,5,7)$ is a driving force for the smooth evolution of the cosmos and it yields the cosmological inflation with the realistic e-fold number and the value of the α parameter. The smoothness of such an evolution is restored as soon as one refers to the modified (exotic) smoothness on $\mathbb{R}^4$. The entire modification is caused by the Akbulut cork with the boundary $S^3 \sqcup \Sigma(2,5,7)$ and its embedding into R^4. This suggests that diffeomorphisms invariance in dimension 4 is somehow replaced by broader cobordisms invariance. Secondly, in order to understand the role of cobordisms between 4-manifolds let us start with recalling the following h-cobordism theorem in dimensions greater or equal to 6.

> Let W be a simply connected compact manifold with a boundary ∂M that has two components, M_1 and M_2 such that the inclusions $i_{1,2} : M_{1,2} \hookrightarrow M$ are homotopy equivalences. Then W is diffeomorphic to the product $M_1 \times [0,1] = M_2 \times [0,1]$, where dimensions of $M_{1,2} \geq 5$. This means that if M_1 and M_2 are two simply connected manifolds of dimension ≥ 5 and there exists an h-cobordism W between them, then W is a product $M_1 \times [0,1]$ and M_1 is diffeomorphic to M_2.

In dimension 5, however, the following holds.

> There exist simply connected compact cobordisms W of dimension 5 with the inclusions of their boundary components $M_{1,2} \overset{i_{1,2}}{\hookrightarrow} W$ being homotopy equivalences such that W is not diffeomorphic to the product $M_1 \times [0,1]$ (or $M_2 \times [0,1]$) hence M_1 is not diffeomorphic to M_2 being h-cobordant to it.

Thus there exists a five-dimensional smooth cobordism between nondiffeomorphic 4-manifolds which is topologically trivial. This phenomenon indicates that something unusual is happening in dimension 4 and in fact there follows the existence of small exotic R^4s. In the particular case $M_1 = 3CP^2\#20\overline{CP^2}$ and $M_2 = \text{K3}\#\overline{CP^2}$, which are homeomorphic (and certainly homotopy equivalent) but not diffeomorphic, the 5-cobordism $W^5 = M_1 \times [0,1]$ is topologically trivial but smoothly nontrivial. Moreover, there exists the same Akbulut cork as considered previously: $\tilde{A} \subset M_1$ and $A \subset M_2$ ($\tilde{A}$ and A differ by a certain involution of the boundary ∂A) such that the neighbourhoods $N(\tilde{A})$ of $\tilde{A}$ in M_1 and $N(A)$ of A in M_2 are both (different) exotic R^4s. $N(A)$ in $\text{K3}\#\overline{CP^2}$ is precisely the exotic R^4 leading to the realistic value of the cosmological constant and which has been referred to in this paper. Consequently, the Akbulut cork A realises the 4-cobordism between S^3 and $\Sigma(2,5,7)$ as described in Section 2.

Thus, four-dimensional nondiffeomorphic smooth manifolds R^4s in M_1 and M_2, and the possibility to attain one of them from the other via a nontrivial 5-cobordism, appears as the fundamental 'symmetry' of a physical theory extending GR. However, here we do not investigate this interesting point any further. It will be addressed in the future work.

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