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Geometric Properties of the Pentablock

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Abstract

In this paper, we give a positive answer to the question raised in Kosiński (Complex Anal Oper Theory 9(6):1349–1359, 2015) and Zapałowski (J Math Anal Appl 430(1):126–143, 2015), i.e., we show that the *pentablock* $\mathcal{P}$ is a $\mathbb{C}$-convex domain.

Contents

1 Introduction

Recently, many authors showed great interest in two domains: the *symmetrized bidisc* and the *tetrablock*, arising from the μ-synthesis, from the aspect of geometric function theory. Actually, both domains are $\mathbb{C}$-convex but non-convex, and they cannot be exhausted by domains biholomorphic to convex ones, with the Lempert's theorem (see Lempert [13,14]) holding on these two domains, i.e., the Lempert function and the Carathéodory distance coincide on them (see [2,6–8,20]). So from the point of view of the Lempert's theorem holding, these two domains play an important role in

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the study of a long-standing open problem whether Lempert's theorem still holds for $\mathbb{C}$-convex domain. However, as far as we know, the answer is positive for $\mathbb{C}$-convex domain with $\mathcal{C}^2$ boundary (see [10]).

In 2015, Agler, Lykova and Young [1] introduced a new bounded domain $\mathcal{P}$ by

$$\mathcal{P} := \left\{(a_{21}, \operatorname{tr} A, \det A) : A = [a_{ij}]_{i,j=1}^2 \in \mathbb{B}\right\},$$

where

$$\mathbb{B} := \left\{A \in \mathbb{C}^{2\times 2} : \ ||A|| < 1\right\}$$

denotes the open unit ball in the space $\mathbb{C}^{2\times 2}$ with the usual operator norm. They called this domain the *pentablock* as $\mathcal{P} \cap \mathbb{R}^3$ is a convex body bounded by five faces, three of which are flat and two are curved (see [1]).

The *pentablock* $\mathcal{P}$ is polynomially convex and starlike about the origin, but neither circled nor convex. Moreover, it does not have a $\mathcal{C}^1$ boundary (see [1]). This new domain is also arising from the μ-synthesis, just like the *symmetrized bidisc* and the *tetrablock*. So it is naturally to consider analogous properties of the *pentablock*, such as the question about $\mathbb{C}$-convexity of $\mathcal{P}$, and Lemperts theorem on the equality of holomorphically invariant functions and metrics for the pentablock (see [1,12,19]). In this paper, we give a positive answer to the $\mathbb{C}$-convexity of $\mathcal{P}$. More precisely, we obtain the following theorem.

Theorem 1.1 *The pntablock $\mathcal{P}$ is a $\mathbb{C}$-convex domain.*

Throughout this paper, $\mathbb{D}$ denotes the open unit disc in the complex plane ,while $\mathbb{T}$ denotes the unit circle. And other basic notions, definitions, and properties from the theory of invariant functions, linearly convex and $\mathbb{C}$-convex domains that we shall use in the paper may be found in [3,9,11].

2 Preliminary Results

2.1 Pentablock

We first recall the definition of the pentablock $\mathcal{P}$.

Theorem 2.1 [1, Theorem 1.1 and Theorem 5.2] *Let*

$$(s, p) = (\lambda_1 + \lambda_2, \lambda_1\lambda_2),$$

where $\lambda_1, \lambda_2 \in \mathbb{D}$. Let $a \in \mathbb{C}$ and

$$\beta = \frac{s - \bar{s}p}{1 - |p|^2}.$$

The following statements are equivalent:

(1) $(a, s, p) \in \mathcal{P}$,
(2) $|a| < \left|1 - \frac{\frac{1}{2}s\bar{\beta}}{1+\sqrt{1-|\beta|^2}}\right|$,
(3) $|a| < \frac{1}{2}|1 - \bar{\lambda}_1\lambda_2| + \frac{1}{2}(1 - |\lambda_1|^2)^{\frac{1}{2}}(1 - |\lambda_2|^2)^{\frac{1}{2}}$,
(4) $\sup_{z\in\mathbb{D}} |\Psi_z(a, s, p)| < 1$, *where* Ψ_z *is the linear fractional map*

$$\Psi_z(a, s, p) = \frac{a(1 - |z|^2)}{1 - sz + pz^2}.$$

2.2 Pentablock as a Hartogs Domain

Following the description of the pentablock $\mathcal{P}$, we can learn that the pentablock $\mathcal{P}$ is closely related to the symmetrized bidisc $\mathbb{G}_2$, which is defined by

$$\mathbb{G}_2 = \left\{(s, p) \in \mathbb{C}^2 : |s - \bar{s}p| + |p|^2 < 1\right\}.$$

In fact, the pentablock $\mathcal{P}$ can be seen as a Hartogs domain in $\mathbb{C}^3$ over the symmetrized bidisc $\mathbb{G}_2$ (see [1]), that is,

$$\mathcal{P} = \left\{(a, s, p) \in \mathbb{D} \times \mathbb{G}_2 : |a|^2 < e^{-\varphi(s,p)}\right\},$$

where

$$\varphi(s, p) = -2\log\left|1 - \frac{\frac{1}{2}s\bar{\beta}}{1 + \sqrt{1 - |\beta|^2}}\right|,$$

$(s, p) \in \mathbb{G}_2$ and $\beta = \frac{s-\bar{s}p}{1-|p|^2}$.

Hartogs domain is a one of important research object in several complex variable. For the studies on Hartogs domain, please refer to [4,5,16–18]. So considering the pentablock $\mathcal{P}$ as a Hartogs domain will be great helpful for us to study the convexity of the pentablock $\mathcal{P}$.

2.3 Some Useful Results

In this subsection, we will give some useful results on the symmetrized bidisc $\mathbb{G}_2$ and the pentablock $\mathcal{P}$. In order to study the pentablock $\mathcal{P}$, it is sufficient to learn the $\mathbb{C}$-convexity of $\mathbb{G}_2$.

Theorem 2.2 [15] *The symmetrized bidisc* $\mathbb{G}_2$ *is* $\mathbb{C}$*-convex.*

Through the study of the boundary of $\mathcal{P}$, we can learn that there are two main part of the boundary, i.e., the smooth part and the non-smooth part. So it is necessary to study some basic convexity property of $\mathcal{P}$ to simplify the problem.

Theorem 2.3 [12, Proposition 9] *The pentablock* $\mathcal{P}$ *is linearly convex.*

In order to study the $\mathbb{C}$-convexity of $\mathcal{P}$ in some simple way, we give the whole holomorphic automorphism group $\mathrm{Aut}(\mathcal{P})$ as follows.

Theorem 2.4 [12, Theorem 15] *All mappings of the form*

$$f_{\omega,\nu}(a, \lambda_1 + \lambda_2, \lambda_1\lambda_2) = \left(\frac{\omega(1-|\alpha|^2)a}{1-\bar{\alpha}(\lambda_1+\lambda_2) + \bar{\alpha}^2\lambda_1\lambda_2}, \nu(\lambda_1) + \nu(\lambda_2), \nu(\lambda_1)\nu(\lambda_2) \right),$$

where $(a, \lambda_1 + \lambda_2, \lambda_1\lambda_2) \in \mathcal{P}$, $\lambda_1, \lambda_2 \in \mathbb{D}$, ν *is a Möbius function of the form* $\nu(\lambda) = \eta\frac{\lambda-\alpha}{1-\bar{\alpha}\lambda}$, *and where* $\omega, \eta \in \mathbb{T}$, $\alpha \in \mathbb{D}$, *form the whole group* $\mathrm{Aut}(\mathcal{P})$ *of holomorphic automorphisms of the pentablock* $\mathcal{P}$.

2.4 $\mathbb{C}$-convex Domain

A domain $D \subset \mathbb{C}^n$ is called $\mathbb{C}$-convex if for any affine complex line ℓ such that $\ell \cap D \neq \emptyset$, and the set $\ell \cap D$ is connected and simply connected. For a domain $D \subset \mathbb{C}^n$ and a point $a \in \mathbb{C}^n$, we denote by $\Gamma_D(a)$ the set of all complex hyperplanes L such that $(a + L) \cap D = \emptyset$. Then we have the basic criterion on $\mathbb{C}$-convexity.

Theorem 2.5 [3, Theorem 2.5.2] *The bounded domain* $D \subset \mathbb{C}^n$, $n > 1$, *is* $\mathbb{C}$*-convex iff for any boundary point* $x \in \partial D$, *the set* $\Gamma_D(x)$ *is non-empty and connected.*

Remark 2.6 By Theorem 2.5, we only need to give a full description of the tangent hyperplanes to the pentablock $\mathcal{P}$. And together with Theorem 2.3, we can only need to consider the non-smooth part of the boundary. Furthermore, through Theorem 2.4 we can simplify the situation into just four different types, i.e., (1) $(a, 1, 0)$ with $|a| \leq \frac{1}{2}$; (2) $(a, 0, -1)$ with $|a| < 1$; (3) $(1, 0, -1)$; (4) $(0, 2, 1)$.

3 The Set of All Tangent Hyperplanes to $\mathcal{P}$ at the Non-smooth Part

In this section, we will give a full description of the tangent hyperplanes to the pentablock $\mathcal{P}$ at the non-smooth boundary part. Set $P_0 = (a_0, s_0, p_0)$ be a non-smooth boundary point of the pentablock $\mathcal{P}$, and let $\Gamma_{\mathcal{P}}(P_0)$ denote the set of all tangent hyperplanes to $\mathcal{P}$ at the boundary point P_0. Now assume the hyperplane in $\mathbb{C}^3$ that

$$\mathcal{L} := \{(a, s, p) \in \mathbb{C}^3 : k_1 a + k_2 s + k_3 p = 0\}.$$

Then

$$\mathcal{L} \in \Gamma_{\mathcal{P}}(P_0) \Longleftrightarrow P_0 \in \mathcal{L} \quad \text{and} \quad \mathcal{L} \cap \mathcal{P} = \emptyset.$$

Together with automorphisms of $\mathcal{P}$ by Theorem 2.4, we can only need to consider some special boundary points P_0:

(1) If $(s_0, p_0) \in \partial\mathbb{G}_2 \backslash \partial_s\mathbb{G}_2$ and $|a_0|^2 \leq e^{-\varphi(s_0,p_0)}$.

Actually we can assume that $(s_0, p_0) = (1, 0)$, and then we have $|a_0| \leq \frac{1}{2}$. Now we consider the hyperplane in $\mathbb{C}^3$ passing through the boundary point P_0,

$$\mathcal{L} := \{(a, s, p) \in \mathbb{C}^3 : k_1(a - a_0) + k_2(s - s_0) + k_3(p - p_0) = 0\}. \tag{3.1}$$

If $k_1 \neq 0$, then $\mathcal{L} = \{(a, s, p) \in \mathbb{C}^3 : a = a_0 + k_2(1 - s) - k_3 p\}$.

Next suppose that $\mathcal{L} \in \Gamma_{\mathcal{P}}(P_0)$, so we get $\mathcal{L} \cap \mathcal{P} = \emptyset$. This means that for any $(s, p) \in \mathbb{G}_2$, we have

$$|a_0 + k_2(1 - s) - k_3 p|^2 \geq e^{-\varphi(s,p)}.$$

Now set $p = 0$, we can learn that for any $s \in \mathbb{D}$,

$$\frac{1}{2} + \frac{1}{2}(1 - |s|^2)^{\frac{1}{2}} \leq |a_0 + k_2(1 - s)| \leq |a_0| + |k_2(1 - s)|.$$

Together with $|a_0| \leq \frac{1}{2}$, we obtain that the following inequality

$$|k_2| \geq \frac{\frac{1}{2}(1 - |s|^2)^{\frac{1}{2}}}{|1 - s|}$$

holds for all $s \in \mathbb{D}$.

Hence, taking s tends to 1, we can conclude that such k_2 does not exist. It follows that if the hyperplane $\mathcal{L}$ in (3.1) belongs to $\Gamma_{\mathcal{P}}(P_0)$, then $k_1 = 0$.

Moreover, if we have a hyperplane $\mathcal{L} \in \Gamma_{\mathcal{P}}(P_0)$, then consider the following hyperplane in $\mathbb{C}^2$:

$$\mathcal{L}' := \{(s, p) \in \mathbb{C}^2 : (a, s, p) \in \mathcal{L}\}.$$

Easily, we can see that $(s_0, p_0) = (1, 0) \in \mathcal{L}' \cap \partial\mathbb{G}_2$ and $\mathcal{L}' \cap \mathbb{G}_2 = \emptyset$. So

$$\mathcal{L}' \in \Gamma_{\mathbb{G}_2}(s_0, p_0).$$

This implies that

$$\Gamma_{\mathcal{P}}(P_0) \subseteq \mathbb{C} \times \Gamma_{\mathbb{G}_2}(s_0, p_0).$$

Clearly the other inclusion holds. Therefore, we have

$$\Gamma_{\mathcal{P}}(a_0, 1, 0) = \mathbb{C} \times \Gamma_{\mathbb{G}_2}(1, 0) \quad \left(|a_0| \leq \frac{1}{2}\right).$$

(2) If $(s_0, p_0) \in \partial_s\mathbb{G}_2$ and $|a_0|^2 < e^{-\varphi(s_0,p_0)}$.

From the assumption, we can learn that $s_0^2 \neq 4p_0$. Otherwise, $e^{-\varphi(s_0,p_0)} = 0$. Hence, we can assume that $(s_0, p_0) = (0, -1)$ and $|a_0| < 1$.

By the same way, consider the hyperplane passing through the boundary point P_0 with the assumption $k_1 \neq 0$, namely,

$$\mathcal{L} = \{(a, s, p) \in \mathbb{C}^3 : a = a_0 + k_2 s + k_3(1 + p)\}.$$

Then suppose that $\mathcal{L} \in \Gamma_{\mathcal{P}}(P_0)$, so this leads to $\mathcal{L} \cap \mathcal{P} = \emptyset$. Hence, for any $(s, p) \in \mathbb{G}_2$, we have

$$|a_0 + k_2 s + k_3(1 + p)|^2 \geq e^{-\varphi(s,p)}.$$

Now set $s = 0$, we obtain that for any $p \in \mathbb{D}$,

$$1 \leq |a_0 + k_3(1+p)| \leq |a_0| + |k_3(1+p)|.$$

This means that the following inequality

$$|k_3(1+p)| \geq 1 - |a_0| > 0$$

holds for any $p \in \mathbb{D}$. Thus such k_3 does not exist. And it follows that k_1 must be zero. Therefore, following by the same procedure, we obtain

$$\Gamma_{\mathcal{P}}(a_0, 0, -1) = \mathbb{C} \times \Gamma_{\mathbb{G}_2}(0, -1) \quad (|a_0| < 1).$$

(3) $(s_0, p_0) \in \partial_s \mathbb{G}_2 \backslash \Sigma$ and $|a_0|^2 = e^{-\varphi(s_0, p_0)}$.

By the assumption, we can also see that $s_0^2 \neq 4p_0$. So we can assume $(a_0, s_0, p_0) = (1, 0, -1)$. Then consider the hyperplane

$$\mathcal{L} := \{(a, s, p) \in \mathbb{C}^3 : a = 1 + k_2 s + k_3(1+p)\},$$

and suppose that $\mathcal{L} \in \Gamma_{\mathcal{P}}(P_0)$. This implies that $\mathcal{L} \cap \mathcal{P} = \emptyset$. Thus for any $(s, p) \in \mathbb{G}_2$, we have

$$|1 + k_2 s + k_3(1+p)|^2 \geq e^{-\varphi(s,p)}. \tag{3.2}$$

Let $s = 0$, then for any $p \in \mathbb{D}$, we have

$$|1 + k_3 + k_3 p| \geq 1.$$

Now we give the following lemma to help us.

Lemma 3.1 *For any $z \in \mathbb{D}$, the inequality $|1 + k + kz| \geq 1$ holds iff $k \geq 0$.*

Proof Set $f(z) = 1 + k + kz$, and then we directly learn that $f(z)$ is a holomorphic function on $\mathbb{D}$. Since $|f(z)| \geq 1$, it means that $f(z)$ has no zero in $\mathbb{D}$. Hence, by the maximal principle, $|f(z)| \geq 1$ holds for all $z \in \overline{\mathbb{D}}$. Now let $|z| = 1$, we can obtain

$$\begin{aligned}
&|1 + k + kz| \geq 1, \\
&\quad \Leftrightarrow |1+k|^2 + |k|^2 - 1 \geq -2\mathrm{Re}\left((|k|^2 + k)z\right), \quad (|z| = 1) \\
&\quad \Leftrightarrow 2|k|^2 + 2\mathrm{Re}k \geq 2\left|k + |k|^2\right|, \quad (\text{the inequality holds for all } z \in \partial\mathbb{D}) \\
&\quad \Rightarrow (\mathrm{Re}k)^2 \geq |k|^2.
\end{aligned}$$

This means that k is real. Now if $k < 0$, then there exists $M > 0$ such that

$$M|k| < 1,$$

and we can choose z_0 with $-1 < z_0 < 0$ such that

$$1 + z_0 < M.$$

Hence, we have

$$0 < 1 + Mk < 1 + k(1 + z_0) < 1.$$

This leads to a contradiction. On the other hand, for $k \geq 0$, the inequality is evidently valid. Therefore, we conclude that $k \geq 0$. $\square$

By Lemma 3.1, we have $k_3 \geq 0$. Now we want to prove the inequality (3.2) on the whole $\overline{\mathbb{G}}_2$.

If there exists $(s_1, p_1) \in \partial\mathbb{G}_2$ such that the inequality (3.2) does not hold. Then set $a_1 = 1 + k_2 s_1 + k_3(1 + p_1)$, and by the assumption, we have

$$|a_1|^2 < e^{-\varphi(s_1, p_1)}. \tag{3.3}$$

Since $(a_1, s_1, p_1) \in \mathcal{L}$ and $\mathcal{L} \in \Gamma_{\mathcal{P}}(a_0, s_0, p_0)$, we can see that

$$\mathcal{L} \in \Gamma_{\mathcal{P}}(a_1, s_1, p_1).$$

So together with (3.3), by the case (1) and case (2), we know that such $\mathcal{L}$ does not exist. Hence, the inequality (3.2) also holds for all $(s, p) \in \overline{\mathbb{G}}_2$.

Back to the inequality (3.2), since $(s, p) \in \overline{\mathbb{G}}_2$, we can set $s = \lambda_1 + \lambda_2$ and $p = \lambda_1\lambda_2$. So for any $(\lambda_1, \lambda_2) \in \partial(\mathbb{D} \times \mathbb{D})$, we have

$$|1 + k_2\lambda_1 + k_3 + \lambda_2(k_2 + k_3\lambda_1)| \geq \frac{1}{2}|1 - \bar{\lambda}_1\lambda_2|.$$

Now let $\lambda_1 = 1$, then for any $\lambda_2 \in \partial\mathbb{D}$, we can obtain

$$\begin{aligned}
&|1 + k_2 + k_3 + \lambda_2(k_2 + k_3)| \geq \frac{1}{2}|1 - \lambda_2|, \\
&\quad \Leftrightarrow |1 + k|^2 + |k|^2 - \frac{1}{2} \geq -\mathrm{Re}\left(\left(2k + 2|k|^2 + \frac{1}{2}\right)\lambda_2\right), \quad (k = k_2 + k_3) \\
&\quad \Leftrightarrow 2|k|^2 + 2\mathrm{Re}k + \frac{1}{2} \geq \left|2|k|^2 + 2k + \frac{1}{2}\right|, \quad (\text{the inequality holds for all } \lambda_2 \in \partial\mathbb{D}) \\
&\quad \Rightarrow (\mathrm{Re}k)^2 \geq |k|^2.
\end{aligned}$$

This implies that k is real . On the other hand, we can learn that if k is real , then we have

$$\begin{aligned}
2|k|^2 + 2\mathrm{Re}k + \frac{1}{2} &= 2k^2 + 2k + \frac{1}{2} \\
&= \frac{1}{2}(2k + 1)^2 \geq 0.
\end{aligned}$$

Thus we can obtain that for any $\lambda_2 \in \partial\mathbb{D}$,

$$|1 + k_2 + k_3 + \lambda_2(k_2 + k_3)| \geq \frac{1}{2}|1 - \lambda_2| \Longleftrightarrow k = k_2 + k_3 \text{ is real .}$$

Since $k_3 \geq 0$, k_2 is real. Following this result, we want to omit the assumption $\lambda_1 = 1$. Thus we give the following lemma.

Lemma 3.2 *For any* $(\lambda_1, \lambda_2) \in \partial\mathbb{D} \times \partial\mathbb{D}$, *the inequality*

$$|1 + k_2\lambda_1 + k_3 + \lambda_2(k_2 + k_3\lambda_1)| \geq \frac{1}{2}|\lambda_1 - \lambda_2|$$

holding is equivalent to $k_3^2 + k_3 \geq k_2^2$.

Proof By direct calculation, we have

$$\begin{aligned}
&|1 + k_2\lambda_1 + k_3 + \lambda_2(k_2 + k_3\lambda_1)| \geq \frac{1}{2}|\lambda_1 - \lambda_2|, \\
&\quad \Leftrightarrow |1 + k_2\lambda_1 + k_3|^2 + |k_2 + k_3\lambda_1|^2 - \frac{1}{2} \\
&\quad \geq -\mathrm{Re}\left(\left(2(k_2 + k_3\lambda_1)(1 + k_2\bar{\lambda}_1 + k_3) + \frac{1}{2}\bar{\lambda}_1\right)\lambda_2\right), \\
&\quad \Leftrightarrow \left(\frac{1}{2} + 2k_2^2\right) + (2k_3 + 2k_3^2) + (2k_2 + 4k_2k_3)\mathrm{Re}\lambda_1 \\
&\quad \geq \left|\left(\frac{1}{2} + 2k_2^2\right)\bar{\lambda}_1 + (2k_3 + 2k_3^2)\lambda_1 + (2k_2 + 4k_2k_3)\right|.
\end{aligned}$$

Now set $a = \frac{1}{2} + 2k_2^2$, $b = 2k_3 + 2k_3^2$ and $c = 2k_2 + 4k_2k_3$. Thus we see that if

$$a + b + c\mathrm{Re}\lambda_1 \geq \left|a\bar{\lambda}_1 + b\lambda_1 + c\right|, \tag{3.4}$$

then we have

$$(1 - (\mathrm{Re}\lambda_1)^2)(4ab - c^2) \geq 0.$$

It follows

$$k_3^2 + k_3 \geq k_2^2.$$

Notice that $a > 0$ and $b \geq 0$. So if we want to get the equivalence condition for the inequality (3.4), we only need to consider the following inequality holding for all $\lambda_1 \in \partial\mathbb{D}$,

$$a + b + c\mathrm{Re}\lambda_1 \geq 0.$$

However, it is not hard to see

$$a + b \geq |c|. \tag{3.5}$$

So together with $a + b \geq 0$, we obtain that the inequality (3.5) is equivalent to

$$(a + b)^2 \geq c^2.$$

In fact, $(a+b)^2 - c^2 \geq (a+b)^2 - 4ab \geq 0$. Hence, this must be an equivalence condition. $\square$

Therefore, we have

$$\Gamma_{\mathcal{P}}(1,0,-1) \subseteq (\mathbb{C} \times \Gamma_{\mathbb{G}_2}(0,-1)) \cup \{(a,s,p) \in \mathbb{C}^3 : a = 1 + k_2 s + k_3(1+p)\},$$

where k_2 is real, $k_3 \geq 0$ and $k_3^2 + k_3 \geq k_2^2$.

Now we want to show the other inclusion. Let k_2 be a real number, and $k_3 \geq 0$ with $k_3^2 + k_3 \geq k_2^2$. Then consider the hyperpalne

$$\mathcal{L} := \{(a,s,p) \in \mathbb{C}^3 : a = 1 + k_2 s + k_3(1+p)\}.$$

In order to prove $\mathcal{L} \in \Gamma_{\mathcal{P}}(1,0,-1)$, we only need to show the following inequality

$$|1 + k_2 s + k_3(1+p)|^2 \geq e^{-\varphi(s,p)} \tag{3.6}$$

holding for any $(s,p) \in \mathbb{G}_2$.

Define

$$h(s,p) = 1 + k_2 s + k_3(1+p).$$

If $k_2 \neq 0$ and $k_3 \neq 0$, then $h(s,p)$ is a well-defined holomorphic function on $\mathbb{C}^2$. By Lemma 3.2, for any $(s,p) \in \partial_s \mathbb{G}_2$, we have

$$|h(s,p)|^2 \geq e^{-\varphi(s,p)}.$$

Now set $s = \lambda_1 + \lambda_2$ and $p = \lambda_1\lambda_2$, and then for any $(\lambda_1,\lambda_2) \in \partial\mathbb{D} \times \partial\mathbb{D}$, we have

$$|1 + k_2(\lambda_1+\lambda_2) + k_3(1+\lambda_1\lambda_2)| \geq \frac{1}{2}|\lambda_1 - \lambda_2|. \tag{3.7}$$

Thus, if there exists $(s_1,p_1) \in \partial_s\mathbb{G}_2$ such that $h(s_1,p_1) = 0$, then we must have $s_1^2 = 4p_1$. Hence, we can assume that $s_1 = 2\lambda_0$ and $p_1 = \lambda_0^2$ for some $\lambda_0 \in \partial\mathbb{D}$. So for $h(s_1,p_1) = 0$, we obtain

$$1 + 2k_2\lambda_0 + k_3(1+\lambda_0^2) = 0. \tag{3.8}$$

Notice that $|\lambda_0| = 1$, and then assume $\lambda_0 = x_0 + y_0 i$. Thus we have $x_0^2 + y_0^2 = 1$. From (3.8), we see that

$$1 + 2k_2(x_0 + y_0 i) + k_3(2x_0^2 + 2x_0 y_0 i) = 0.$$

Thus we get

$$\begin{cases} k_2 y_0 + k_3 x_0 y_0 = 0, \\ 1 + 2k_2 x_0 + 2k_3 x_0^2 = 0. \end{cases}$$

If $y_0 = 0$, then $2|k_2| = 2k_3 + 1$, which contradicts to $k_3^2 + k_3 \geq k_2^2$; if $y_0 \neq 0$, then $k_2 = -k_3 x_0$. It follows that $1 + 2k_2 x_0 + 2k_3 x_0^2 = 1 \neq 0$. Hence, such λ_0 does not exist. This means that for any $(s, p) \in \partial_s \mathbb{G}_2$, we have

$$h(s, p) \neq 0. \tag{3.9}$$

Next we want to prove that the inequality (3.7) still holds for any $(\lambda_1, \lambda_2) \in \partial\mathbb{D} \times \mathbb{D}$, i.e.,

$$|1 + k_2(\lambda_1 + \lambda_2) + k_3(1 + \lambda_1\lambda_2)| \geq \frac{1}{2}|\lambda_1 - \lambda_2|, \quad (\lambda_1, \lambda_2) \in \partial\mathbb{D} \times \mathbb{D}.$$

If there exists $(\lambda_1^0, \lambda_2^0) \in \partial\mathbb{D} \times \mathbb{D}$ such that

$$1 + k_2(\lambda_1^0 + \lambda_2^0) + k_3(1 + \lambda_1^0\lambda_2^0) = 0,$$

then we have

$$1 + k_2\lambda_1^0 + k_3 = -(k_2 + k_3\lambda_1^0)\lambda_2^0.$$

Thus, if $k_2 + k_3\lambda_1^0 = 0$, then λ_1^0 is real. So we have

$$0 = 1 + k_2\lambda_1^0 + k_3 = 1 - k_3(\lambda_1^0)^2 + k_3 = 1.$$

This leads to a contradiction. Hence, since $|\lambda_2^0| < 1$, we can see

$$\begin{aligned} &|1 + k_2\lambda_1^0 + k_3| < |k_2 + k_3\lambda_1^0|, \\ &\quad \Leftrightarrow 1 + 2k_3 < -2k_2\mathrm{Re}\lambda_1^0, \qquad (|\lambda_1^0| = 1) \\ &\quad \Leftrightarrow 1 + 4k_3 + 4k_3^2 < 4k_2^2(\mathrm{Re}\lambda_1^0)^2 \leq 4k_2^2 \leq 4k_3 + 4k_3^2. \end{aligned}$$

This leads to another contradiction. Thus, define

$$g_{\lambda_1}(\lambda_2) := \frac{\frac{1}{2}(\lambda_1 - \lambda_2)}{1 + k_2(\lambda_1 + \lambda_2) + k_3(1 + \lambda_1\lambda_2)},$$

and then we get that for any fixed $\lambda_1 \in \partial\mathbb{D}$, $g_{\lambda_1}(\lambda_2)$ is a well-defined holomorphic function on $\overline{\mathbb{D}}$.Thus, by the maximal principle, together with (3.7), we have

$$|g_{\lambda_1}(\lambda_2)| \leq 1, \qquad \forall\lambda_2 \in \overline{\mathbb{D}}.$$

So, we can obtain that the inequality

$$|h(\lambda_1, \lambda_2)| \geq \frac{1}{2}|\lambda_1 - \lambda_2|$$

holds for all $(\lambda_1, \lambda_2) \in \partial\mathbb{D} \times \mathbb{D}$, and also for all $(\lambda_1, \lambda_2) \in \mathbb{D} \times \partial\mathbb{D}$. Thus, together with (3.9), we can conclude that

$$h(s, p) \neq 0 \qquad (s, p) \in \partial\mathbb{G}_2.$$

Notice that $k_2 \neq 0$ and $k_3 \neq 0$, so by the Hartogs theorem we have

$$h(s, p) \neq 0 \qquad (s, p) \in \overline{\mathbb{G}}_2.$$

Now set $s = \beta + \bar{\beta} p$, then if we want to show (3.6), we only need to prove the following inequality

$$|1 + k_2\beta + k_3 + (k_2\bar{\beta} + k_3)p| \geq \left| 1 - \frac{\frac{1}{2}(|\beta|^2 + \bar{\beta}^2 p)}{1 + \sqrt{1 - |\beta|^2}} \right| \tag{3.10}$$

holds for any $(\beta, p) \in \mathbb{D} \times \mathbb{D}$.

Fixed any $\beta \in \mathbb{D}$, define

$$f_\beta(p) = \frac{1 - \frac{\frac{1}{2}(|\beta|^2 + \bar{\beta}^2 p)}{1 + \sqrt{1 - |\beta|^2}}}{1 + k_2\beta + k_3 + (k_2\bar{\beta} + k_3)p},$$

and then we can see that $f_\beta(p)$ is a well-defined holomorphic function on $\overline{\mathbb{D}}$. So if we want to show (3.10), we only need to prove that for any fixed $\beta \in \mathbb{D}$,

$$|f_\beta(p)| \leq 1, \qquad \forall p \in \mathbb{D}.$$

With the maximal principle, we just need to show

$$|f_\beta(p)| \leq 1, \quad \forall p \in \partial\mathbb{D} \text{ with any fixed } \beta \in \mathbb{D}.$$

However, for any fixed $\beta \in \mathbb{D}$ and $p \in \partial\mathbb{D}$, there exist λ_1, λ_2 such that

$$|\lambda_1| = |\lambda_2| = 1, \ \beta = \frac{\lambda_1 + \lambda_2}{2} \text{ and } p = \lambda_1\lambda_2.$$

So it suffices to show the following inequality

$$|1 + k_2(\lambda_1 + \lambda_2) + k_3(1 + \lambda_1\lambda_2)| \geq \frac{1}{2}|\lambda_1 - \lambda_2|$$

holding for all $(\lambda_1, \lambda_2) \in \partial\mathbb{D} \times \partial\mathbb{D}$. By Lemma 3.2, we can conclude that

$$\mathcal{L} \in \Gamma_{\mathcal{P}}(1, 0, -1).$$

Now if $k_3 = 0$, then $k_2 = 0$. Easily, we can see that

$$\mathcal{L} = \{a = 1\} \in \Gamma_{\mathcal{P}}(1, 0, -1);$$

and if $k_2 = 0$, then $\mathcal{L} = \{a = 1 + k_3(1 + p)\}$. Notice that

$$|1 + k_3(1 + p)| \geq 1 + k_3 - k_3|p| \geq 1.$$

Hence, we can also see that $\mathcal{L} \in \Gamma_{\mathcal{P}}(1, 0, -1)$.

Therefore, we have

$$\Gamma_{\mathcal{P}}(1, 0, -1) = (\mathbb{C} \times \Gamma_{\mathbb{G}_2}(0, -1)) \cup \{(a, s, p) \in \mathbb{C}^3 : a = 1 + k_2 s + k_3(1 + p)\},$$

where k_2 is real, $k_3 \geq 0$ and $k_3^2 + k_3 \geq k_2^2$.

(4) $(s_0, p_0) \in \partial\mathbb{G}_2 \cap \Sigma$ and then $a_0 = 0$.

Now suppose that $(a_0, s_0, p_0) = (0, 2, 1)$, and then by the same way, consider the hyperplane

$$\mathcal{L} := \{(a, s, p) \in \mathbb{C}^3 : a = k_2(2 - s) + k_3(1 - p)\} \in \Gamma_{\mathcal{P}}(0, 2, 1).$$

Using the same argument, we obtain that for any $(s, p) \in \overline{\mathbb{G}}_2$,

$$|k_2(2 - s) + k_3(1 - p)|^2 \geq e^{-\varphi(s,p)}.$$

Now set $p = 1$, then we have $-2 \leq s \leq 2$. So we obtain

$$|k_2(2 - s)| \geq \sqrt{1 - \frac{1}{4}s^2}.$$

Thus, such k_2 does not exist as $s \to 2^-$. This leads that k_1 must be zero. Therefore, we obtain

$$\Gamma_{\mathcal{P}}(0, 2, 1) = \mathbb{C} \times \Gamma_{\mathbb{G}_2}(2, 1).$$

In summary, we can give a full description of $\Gamma_{\mathcal{P}}(P_0)$ as follows.

Theorem 3.3 *We can give the description of the tangent hyperplanes to the pentablock $\mathcal{P}$ at four different boundary points.*

(1) $\Gamma_{\mathcal{P}}(a_0, 1, 0) = \mathbb{C} \times \Gamma_{\mathbb{G}_2}(1, 0)$, $(|a_0| \leq \frac{1}{2})$;
(2) $\Gamma_{\mathcal{P}}(a_0, 0, -1) = \mathbb{C} \times \Gamma_{\mathbb{G}_2}(0, -1)$, $(|a_0| < 1)$;
(3) $\Gamma_{\mathcal{P}}(1, 0, -1) = (\mathbb{C} \times \Gamma_{\mathbb{G}_2}(0, -1)) \cup \{(a, s, p) \in \mathbb{C}^3 : a = 1 + k_2 s + k_3(1 + p)\}$, *where k_2 is real, $k_3 \geq 0$ and $k_3^2 + k_3 \geq k_2^2$;*
(4) $\Gamma_{\mathcal{P}}(0, 2, 1) = \mathbb{C} \times \Gamma_{\mathbb{G}_2}(2, 1)$.

4 Proof of Theorem 1.1

Proof Linear convexity of $\mathcal{P}$ implies that in the case of a smooth boundary point $P_0 \in \partial\mathcal{P}$, the set $\Gamma_{\mathcal{P}}(P_0)$ is a singleton. Consider then the non-smooth point $P_0 \in \partial\mathcal{P}$. By Theorem 2.4, it is sufficient to consider the only four different cases

$$
\begin{aligned}
&P_0 = (a_0, 1, 0), |a_0| \le \tfrac{1}{2};\\
&P_0 = (a_0, 0, -1), |a_0| < 1;\\
&P_0 = (1, 0, -1);\\
&P_0 = (0, 2, 1).
\end{aligned}
$$

Then Theorems 3.3 and 2.2 imply that $\Gamma_{\mathcal{P}}(P_0)$ is the union of connected sets whose intersection is non-empty for the non-smooth boundary point P_0, so it is connected. Thus, by Theorem 2.5, we can conclude that the pentablock $\mathcal{P}$ is $\mathbb{C}$-convex. This finishes the proof. □

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