Journal of Optimization Theory and Applications (2020) 187:391–407
https://doi.org/10.1007/s10957-020-01752-4

Existence of Solutions for Implicit Obstacle Problems of Fractional Laplacian Type Involving Set-Valued Operators

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Received: 19 August 2019 / Accepted: 9 September 2020 / Published online: 25 September 2020

Abstract
The paper is devoted to a new kind of implicit obstacle problem given by a fractional Laplacian-type operator and a set-valued term, which is described by a generalized gradient. An existence theorem for the considered implicit obstacle problem is established, using a surjectivity theorem for set-valued mappings, Kluge's fixed point principle and nonsmooth analysis.

Keywords Implicit obstacle problem · Surjectivity theorem · Generalized fractional Laplacian · Generalized gradient · Fixed point theorem

Mathematics Subject Classification 35R11 · 35J50 · 35J60 · 26E25 · 47J22

1 Introduction

Partial differential equations, involving nonlocal operators, have recently received much attention since the nonlocal operators, which are infinitesimal generators of Lévy-type stochastic processes, describe precisely various phenomena in such fields

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as population dynamics, game theory, finance, image processing (see [1–5] and the references therein). On the other hand, in many physical processes and engineering applications, the mathematical models are formulated as inequalities instead of equations, extensively appearing in the form of variational inequalities and hemivariational inequalities. Roughly speaking, the variational inequalities arise in a convex framework, whereas the hemivariational inequalities address systems with nonconvex and nonsmooth structure (see [6–17]).

Recent works focus on systems governed by nonlocal operators and exhibiting set-valued terms in the form of generalized gradient of a locally Lipschitz function. Frassu et al. [18] proved the existence of three nontrivial solutions for a pseudo-differential inclusion driven by a nonlocal anisotropic operator and with generalized gradient of a locally Lipschitz potential. Teng [19] and Xi et al. [20] applied the nonsmooth critical point theory to obtain multiplicity results for nonlocal elliptic hemivariational inequalities. Liu and Tan [21] employed a surjectivity theorem for pseudomonotone and coercive operators to explore a nonlocal hemivariational inequality. For related works, we refer to [13,14].

In relevant situations encountered in engineering and economic models, such as Nash equilibrium with shared constraints and transport optimization feedback control, the constraint sets depend on the unknown state variable. For this reason, the theory of quasi variational/hemivariational inequalities was prompted to become one of the most promising research domains in applied mathematics. Yet, as far as we know, there is no publication considering differential inclusion problems with nonlocal operators and implicit obstacle effect (i.e., the constraints depend on the unknown function). It is the goal of the present work to study an implicit obstacle problem containing a generalized fractional Laplace operator and a generalized gradient term. Specifically, we establish a general existence theorem for this new type of problem.

The rest of the paper is organized as follows. In Sect. 2, we formulate the problem and survey some preliminary material needed in the study of Problem 2.1. Section 3 provides the variational formulation of Problem 2.1, whereas Sect. 4 comprises the study of variational selection associated to our problem. Section 5 presents our existence theorem with its proof employing Kluge's fixed point principle, a surjectivity theorem for multivalued operators and nonsmooth analysis. Section 6 sets forth our existence result without the relaxed monotonicity condition. Section 7 is devoted to conclusions.

2 Problem Formulation and Mathematical Background

Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary, let $s \in]0, 1[$ with $N > 2s$ and let $f \in L^2(\Omega)$. Denoting $\Omega^c := \mathbb{R}^N \setminus \Omega$, we formulate the following implicit obstacle problem:

Problem 2.1 Find $u\colon \mathbb{R}^N \to \mathbb{R}$ such that

$$\begin{aligned} &(\mathcal{L}_K u)(x) + \partial j(x, u(x)) + \zeta(x) \ni f(x) && \text{in } \Omega,\\ &u = 0 && \text{in } \Omega^c,\\ &\left(\int_{\mathbb{R}^{2N}} \big(u(x) - u(y)\big)^2 K(x-y)\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}} \le U(u), \end{aligned} \tag{1}$$

where $\mathcal{L}_K$ stands for the generalized nonlocal fractional Laplace operator given by $\mathcal{L}_K u(x) := -\int_{\mathbb{R}^N} \big[u(x+y) + u(x-y) - 2u(x)\big] K(y)\,\mathrm{d}y$ for a.e. $x \in \mathbb{R}^N$, $U\colon L^2(\Omega) \to \mathbb{R}$ is a given function, the multivalued term $\partial j(x,\cdot)$ denotes the generalized gradient (see Definition 2.2 below) of a locally Lipschitz function $r \mapsto j(x,r)$, and $\zeta \in L^2(\Omega)$ belongs to the (exterior) normal cone $N_{C(u)}(u)$ in $L^2(\Omega)$ of the set $C(u) = \big\{v \in L^2(\mathbb{R}^N) : v = 0 \text{ in } \Omega^c,\ \mathcal{R}(v) \le U(u)\big\}$ at the point $u \in C(u)$, that is, $\int_\Omega \zeta(x)(v(x) - u(x))\,\mathrm{d}x \le 0$ for all $v \in C(u)$, where $\mathcal{R}(v) := (\int_{\mathbb{R}^{2N}} \big(v(x) - v(y)\big)^2 K(x-y)\,\mathrm{d}x\,\mathrm{d}y)^{\frac{1}{2}}$.

Throughout the paper, K is assumed to fulfill the conditions:
$\underline{H(K)}$: $K\colon \mathbb{R}^N\backslash\{0\} \to]0, +\infty[$ is such that

(i) The function $x \mapsto \min\{|x|^2, 1\}K(x)$ belongs to $L^1(\mathbb{R}^N)$;
(ii) There exists a constant $m_K > 0$ such that $K(x) \ge m_K |x|^{-(N+2s)}$ for all $x \in \mathbb{R}^N\backslash\{0\}$;
(iii) $K(x) = K(-x)$ for all $x \in \mathbb{R}^N\backslash\{0\}$.

If the constraint $(\int_{\mathbb{R}^{2N}} \big(u(x) - u(y)\big)^2 K(x-y)\,\mathrm{d}x\,\mathrm{d}y)^{\frac{1}{2}} \le U(u)$ is dropped in (1), then Problem 2.1 reduces to the nonlocal inclusion problem

$$\begin{aligned} &(\mathcal{L}_K u)(x) + \partial j(x, u(x)) \ni f(x) && \text{in } \Omega,\\ &u(x) = 0 && \text{in } \Omega^c, \end{aligned}$$

which has been studied by Migórski et al. [22] for the particular case of kernel $K(x) := |x|^{-(N+2s)}$ for all $x \in \mathbb{R}^N\backslash\{0\}$, i.e., $\mathcal{L}_K$ is the fractional Laplace operator $(-\Delta)^s u(x) := -\int_{\mathbb{R}^N} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}}\,\mathrm{d}y$ for a.e. $x \in \mathbb{R}^N$.

Then, we briefly review basic notation and results which are needed in the sequel. For more details, we refer to monographs [23–26].

Let us begin with some definitions and properties for set-valued mappings.

Definition 2.1 Let X and Y be topological spaces and let $F\colon X \rightrightarrows Y$ be a set-valued mapping.

(i) We say that F is upper semicontinuous (u.s.c., for short) at $x \in X$, if for every open set $O \subset Y$ with $F(x) \subset O$ there exists a neighborhood $N(x)$ of x such that $F(N(x)) := \bigcup_{y \in N(x)} F(y) \subset O$. If this holds for every $x \in X$, then F is called upper semicontinuous.
(ii) We say that F is sequentially closed at $x_0 \in X$, if for every sequence $\{(x_n, y_n)\} \subset \mathrm{Gr}(F)$ with $(x_n, y_n) \to (x_0, y_0)$ in $X \times Y$, then it holds $(x_0, y_0) \in \mathrm{Gr}(F)$, where $\mathrm{Gr}(F)$ is the graph of the set-valued mapping F defined by $\mathrm{Gr}(F) := \{(x, y) \in X \times Y : y \in F(x)\}$. We say that F is sequentially closed, if it is sequentially closed at every $x_0 \in X$.

The next proposition characterizes the upper semicontinuity of set-valued maps.

Proposition 2.1 *Let $F\colon X \rightrightarrows Y$, with X and Y topological spaces. Then F is upper semicontinuous, if and only if, for each closed set $C \subset Y$, the set $F^-(C) := \{x \in X : F(x) \cap C \neq \emptyset\}$ is closed in X.*

Let E be a Banach space with its dual E^*. A function $J\colon E \to \mathbb{R}$ is said to be locally Lipschitz at $u \in E$, if there exist a neighborhood $N(u)$ of u and a constant $L_u > 0$ such that $|J(w) - J(v)| \le L_u \|w - v\|_E$ for all $w, v \in N(u)$.

Definition 2.2 Let $u, v \in E$ and $J\colon E \to \mathbb{R}$ be a locally Lipschitz function. The generalized directional derivative $J^0(u; v)$ of J at the point u in the direction v is defined by $J^0(u; v) := \limsup\limits_{w\to u,\, t\downarrow 0} \frac{J(w+tv)-J(w)}{t}$. The generalized gradient $\partial J\colon E \rightrightarrows E^*$ of $J\colon E \to \mathbb{R}$ is defined by

$$\partial J(u) := \{\xi \in E^* : J^0(u; v) \ge \langle \xi, v\rangle_{E^*\times E} \quad \text{for all} \quad v \in E\}, \ \forall u \in E.$$

The next proposition collects some basic results (see, e.g., [25, Proposition 3.23]).

Proposition 2.2 *Let $J\colon E \to \mathbb{R}$ be locally Lipschitz of rank $L_u > 0$ at $u \in E$. Then we have*

(a) *the function $v \mapsto J^0(u; v)$ is positively homogeneous, subadditive, and satisfies $|J^0(u; v)| \le L_u \|v\|_E$ for all $v \in E$;*
(b) *$(u, v) \mapsto J^0(u; v)$ is upper semicontinuous;*
(c) *$\partial J(u)$ is a nonempty, convex, and weak* compact subset of E^*;*
(d) *$J^0(u; v) = \max\left\{\langle \xi, v\rangle_{E^*\times E} : \xi \in \partial J(u)\right\}$ for all $v \in E$.*

In what follows, we denote $\mathcal{S} := (\mathbb{R}^N \backslash \Omega) \times (\mathbb{R}^N \backslash \Omega)$, $\mathcal{P} := \mathbb{R}^{2N} \backslash \mathcal{S}$ and the fractional critical exponent $2_s^* = \frac{2N}{N-2s}$ if $2s < N$ and $2_s^* = +\infty$ otherwise, with $s \in]0, 1[$ and $\Omega \subset \mathbb{R}^N$ as in Sect. 1. We introduce the function space

$$X := \big\{u\colon \mathbb{R}^N \to \mathbb{R} \text{ measurable} : (u(x) - u(y))^2 K(x - y) \in L^2(\mathcal{P}) \\ \text{and } u|_\Omega \in L^2(\Omega)\big\},$$

with $K\colon \mathbb{R}^N \backslash \{0\} \to]0, +\infty[$ verifying assumption $H(K)$, endowed with the norm $\|u\|_X := \|u\|_{L^2(\Omega)} + (\int_{\mathcal{P}} |u(x) - u(y)|^2 K(x - y)\, \mathrm{d}y\, \mathrm{d}x)^{\frac{1}{2}}$ for all $u \in X$ (see, e.g., [27, 28]). In order to fit the generalized Dirichlet boundary condition in Problem 2.1 we consider the subspace X_0 of X given by

$$X_0 := \{u \in X : u = 0 \text{ for a.e. } x \in \Omega^c\}.$$

We list a few useful results (see, e.g., [27]).

Lemma 2.1 *Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary and let $s \in]0, 1[$ with $N > 2s$. Then we have:*

(i) *X_0 is a Hilbert space with the inner product for all $u, v \in X_0$*

$$\langle u, v\rangle_{X_0} := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}[u(x)-u(y)][v(x)-v(y)]K(x-y)\,\mathrm{d}x\,\mathrm{d}y;$$

(ii) *If $p \in [1, 2_s^*]$, there exists $c(p) > 0$ such that $\|u\|_{L^p(\mathbb{R}^N)} \le c(p)\|u\|_{X_0}$ for all $u \in X_0$;*

(iii) *The embedding of X_0 into $L^p(\mathbb{R}^N)$ is compact for $p \in [1, 2_s^*[$.*

From Lemma 2.1 (ii) it is seen that the norm $\|\cdot\|_{X_0}$ on X_0 defined by

$$\|u\|_{X_0} := \left(\int_{\mathcal{P}}|u(x)-u(y)|^2K(x-y)\,\mathrm{d}y\,\mathrm{d}x\right)^{\frac{1}{2}}$$

is equivalent to the norm induced by $\|\cdot\|_X$.

We end the section by recalling the fixed point theorem of Kluge [29] and the surjectivity theorem for set-valued mappings of Le [30, Theorem 2.2], which will be used in the proof of our existence result.

Theorem 2.1 *Let Z be a reflexive Banach space and let $C \subset Z$ be nonempty, closed and convex. Assume that $\Psi : C \rightrightarrows C$ is a set-valued mapping such that for every $u \in C$, the set $\Psi(u)$ is nonempty, closed, and convex, and the graph of Ψ is sequentially weakly closed. If $\Psi(C)$ is bounded, then the map Ψ has at least one fixed point in C.*

Remark 2.1 In the statement of Theorem 2.1, we took advantage of Referee's comment pointing out that the usual formulation "If either C is bounded or $\Psi(C)$ is bounded" of hypothesis in Kluge's fixed point theorem is equivalent to "If $\Psi(C)$ is bounded". This is true because $\Psi(C) \subset C$.

Theorem 2.2 *Let E be a reflexive Banach space with dual E^* and pairing $\langle\cdot,\cdot\rangle_{E^*\times E}$, let $A\colon D(A) \subset E \rightrightarrows E^*$ be a maximal monotone operator, let $B\colon D(B) = E \rightrightarrows E^*$ be a bounded pseudomonotone operator, and let $L \in E^*$. Assume that there exist $u_0 \in E$ and $R \ge \|u_0\|_E$ such that $D(A)\cap O_R(0) \ne \emptyset$ and $\langle \xi+\eta-L, u-u_0\rangle_{E^*\times E} > 0$ for all $u \in D(A)$ with $\|u\|_E = R$, all $\xi \in A(u)$ and all $\eta \in B(u)$, where $O_R(0) := \{v \in E : \|v\|_E < R\}$. Then the inclusion $A(u) + B(u) \ni L$ has a solution.*

3 Hypotheses and Variational Formulation

In this section, we set forth our assumptions and give the variational formulation. In what follows, we assume that the functions $j\colon \Omega\times\mathbb{R}\to\mathbb{R}$ and $U\colon L^2(\Omega)\to\mathbb{R}$ verify the following conditions:

$\underline{H(j)}$: $j\colon \Omega\times\mathbb{R}\to\mathbb{R}$ is such that

(i) $j(\cdot, r)$ is measurable on Ω for all $r \in \mathbb{R}$ and there exists $l \in L^2(\Omega)$ such that $j(\cdot, l(\cdot))$ belongs to $L^1(\Omega)$;

(ii) $j(x,\cdot)$ is locally Lipschitz for a.e. $x \in \Omega$;

(iii) There exists a constant $m_j \geq 0$ such that for all $r_1, r_2 \in \mathbb{R}$ and a.e. $x \in \Omega$ it holds $(\xi_1 - \xi_2)(r_1 - r_2) \geq -m_j|r_1 - r_2|^2$, whenever $\xi_1 \in \partial j(x, r_1)$ and $\xi_2 \in \partial j(x, r_2)$, where $\partial j(x, r)$ stands for the generalized gradient of j with respect to the variable r;
(iv) There exist constants $\alpha_j \geq 0$ and $\beta_j \in]0, 1/c(2)^2[$ such that for all $r \in \mathbb{R}$ and a.e. $x \in \Omega$ there holds $j^0(x, r; -r) \leq \alpha_j + \beta_j r^2$;
(v) There exist $c_j > 0$ and $a \in L^2(\Omega)$ with $a(x) \geq 0$ satisfying $|\xi| \leq a(x) + c_j|r|$ for all $\xi \in \partial j(x, r)$ and a.e. $x \in \Omega$.

Remark 3.1 Assumption $H(j)$(iii) is usually called relaxed monotone condition (see, e.g., [25]) for the locally Lipschitz function $j(x, \cdot)$. It is equivalent to the inequality $j^0(x, s_1; s_2 - s_1) + j^0(x, s_2; s_1 - s_2) \leq m_j|s_1 - s_2|^2$ for all $s_1, s_2 \in \mathbb{R}$ and for a.e. $x \in \Omega$.

Remark 3.2 The relaxed monotone condition $H(j)$(iii) reads as

$$(\xi_1 + m_j r_1 - (\xi_2 + m_j r_2))(r_1 - r_2) \geq 0$$

for all $\xi_1 \in \partial j(x, r_1)$, $\xi_2 \in \partial j(x, r_2)$, and $r_1, r_2 \in \mathbb{R}$. Observing that

$$\xi_k + m_j r_k \in \partial \left(j(x, r_k) + m_j \frac{r_k^2}{2} \right), \quad k = 1, 2,$$

we infer that the function $r \mapsto g(x, r) := j(x, r) + m_j \frac{r^2}{2}$ is convex (see [23, Proposition 2.2.9]) with the subdifferential $\partial_C g(x, r) = \partial j(x, r) + m_j r$. Consequently, the variational–hemivariational inequality in Problem 2.1 can be equivalently rewritten as a linearly perturbed variational inequality by replacing $\partial j(x, u)$ with $\partial_C g(x, u) - m_j u$.

Define the set-valued mapping $C\colon X_0 \rightrightarrows X_0$ by

$$C(u) := \{v \in X_0 : \|v\|_{X_0} \leq U(u)\} \tag{2}$$

(i.e., the set $C(u)$ introduced in the statement of Problem 2.1) for all $u \in X_0$. We note that the set $C(u)$ in (2) is closed and convex in X_0, and $0_{X_0} \in C(u)$.

Remark 3.3 The constraint in Problem 2.1 can be expressed as $\|u\|_{X_0} \leq U(u)$. This is an implicit nonlocal formulation substantially different with respect to classical statements as the pointwise constraints $u(x) \leq f(x)$ in obstacle problem. The motivation is to locate the solution in a nonlocal way relying on the continuous embedding $X_0 \subset L^2(\Omega)$. A natural choice of the continuous map $U\colon L^2(\Omega) \to \mathbb{R}$ (see hypothesis $H(U)$) is $U(u) = a\|u\|_{L^2(\Omega)} + b$, with constants $a > 0$ and $b > 0$ sufficiently large, giving rise to a constraint $\|u\|_{L^2(\Omega)} \leq \|u\|_{X_0} \leq a\|u\|_{L^2(\Omega)} + b$.

To obtain the variational formulation of Problem 2.1, let $u\colon \mathbb{R}^N \to \mathbb{R}$ be a smooth function such that (1) holds. For any $v \in C(u)$, we act on the inclusion $(\mathcal{L}_K u)(x) + \partial j(x, u(x)) \ni f(x)$ in Ω with $v(x) - u(x)$ and then integrate over Ω to get

$$\int_{\Omega}(\mathcal{L}_K u)(x)[v(x)-u(x)]\,\mathrm{d}x+\int_{\Omega}\xi(x)[v(x)-u(x)]\,\mathrm{d}x$$
$$+\int_{\Omega}\zeta(x)[v(x)-u(x)]\,\mathrm{d}x=\int_{\Omega}f(x)[v(x)-u(x)]\,\mathrm{d}x,$$

where the function $\xi\colon \Omega\to\mathbb{R}$ is such that $\xi(x)\in\partial j(x,u(x))$ for a.e. $x\in\Omega$ and $\zeta\in N_{C(u)}(u)$, thus $\int_{\Omega}\zeta(x)[v(x)-u(x)]\,\mathrm{d}x\le 0$. By virtue of the definitions of X_0 and generalized gradient we find

$$\int_{\Omega}(\mathcal{L}_K u)(x)[v(x)-u(x)]\,\mathrm{d}x=\int_{\mathbb{R}^N}(\mathcal{L}_K u)(x)[v(x)-u(x)]\,\mathrm{d}x,$$
$$\int_{\Omega}\xi(x)[v(x)-u(x)]\,\mathrm{d}x\le\int_{\Omega}j^0(x,u(x);v(x)-u(x))\,\mathrm{d}x.$$

Taking into account the preceding discussion, the variational formulation of Problem 2.1 reads as follows.

Problem 3.1 Find $u\in X_0$ such that $u\in C(u)$ and

$$\int_{\mathbb{R}^N}(\mathcal{L}_K u)(x)[v(x)-u(x)]\,\mathrm{d}x+\int_{\Omega}j^0(x,u(x);v(x)-u(x))\,\mathrm{d}x$$
$$\ge\int_{\Omega}f(x)[v(x)-u(x)]\,\mathrm{d}x\quad\forall v\in C(u)\ (\text{with } C(u) \text{ in (2)}).$$

Further, let us introduce the function $J\colon L^2(\Omega)\to\mathbb{R}$ defined by

$$J(u):=\int_{\Omega}j(x,u(x))\,\mathrm{d}x \tag{3}$$

for all $u\in L^2(\Omega)$. On account of hypothesis $H(j)$ and the definition of J in (3), the next lemma is a direct consequence of [25, Theorem 3.47].

Lemma 3.1 *If $H(j)$ holds , then J defined in (3) has the properties:*

(i) *$J\colon L^2(\Omega)\to\mathbb{R}$ is locally Lipschitz;*
(ii) *For all $u,v\in L^2(\Omega)$, there hold the inequalities*

$$J^0(u;v)\le\int_{\Omega}j^0(x,u(x);v(x))\,\mathrm{d}x,$$
$$J^0(u;-u)\le\alpha_j|\Omega|+\beta_j\int_{\Omega}|u(x)|^2\,\mathrm{d}x,$$
$$J^0(u;v-u)+J^0(v;u-v)\le m_j\|u-v\|^2_{L^2(\Omega)};$$

(iii) *For each $u\in L^2(\Omega)$, one has $\partial J(u)\subset\int_{\Omega}\partial j(x,u(x))\,\mathrm{d}x$ and*

$$\|\xi\|_{L^2(\Omega)}\le c_J(1+\|u\|_{L^2(\Omega)})\ \text{ for all } \xi\in\partial J(u) \text{ and all } u\in L^2(\Omega),$$

with some $c_J>0$.

4 Variational Selection

The section is concerned with the existence of solutions to Problem 3.1. To this end, we pass through a related inequality problem.

Problem 4.1 Find $u \in X_0$ such that $u \in C(u)$ and

$$\int_{\mathbb{R}^N} (\mathcal{L}_K u)(x)[v(x) - u(x)]\,\mathrm{d}x + J^0(u; v - u) \geq \int_\Omega f(x)[v(x) - u(x)]\,\mathrm{d}x$$

for all $v \in C(u)$.

Lemma 3.1(ii) ensures that if $u \in X_0$ is a solution to Problem 4.1, then u solves Problem 3.1 as well. Consequently, it is enough to establish the solvability of Problem 4.1. This will be achieved by means of the auxiliary problem:

Problem 4.2 Given $w \in X_0$, find $u \in C(w)$ such that

$$\int_{\mathbb{R}^N} (\mathcal{L}_K u)(x)[v(x) - u(x)]\,\mathrm{d}x + J^0(u; v - u) \geq \int_\Omega f(x)[v(x) - u(x)]\,\mathrm{d}x$$

for all $v \in C(w)$, with $C(w)$ in (2).

We are going to find a fixed point of the set-valued mapping $S\colon X_0 \rightrightarrows X_0$ that we call variational selection, which is defined by

$$S(w) := \{u \in X_0 : \ u \text{ solves Problem 4.2}\} \quad \text{for all } w \in X_0.$$

Theorem 4.1 *If $H(K)$, $H(j)$ and $H(U)$ are satisfied and $m_j c(2)^2 \leq 1$, where $c(2)$ is the constant in Lemma 2.1(ii), then for each $w \in X_0$, the set $S(w)$ is nonempty, closed, bounded, and convex in X_0.*

Proof Let $A\colon X_0 \to X_0^*$ be such that $\langle Au, v\rangle_{X_0} = \int_{\mathbb{R}^N} (\mathcal{L}_K u)(x)v(x)\,\mathrm{d}x$ for all $u, v \in X_0$, and consider the indicator function of $C(w)$, that is the function $I_{C(w)}\colon X_0 \to \overline{\mathbb{R}} := \mathbb{R} \bigcup \{+\infty\}$ given by

$$I_{C(w)}(v) := \begin{cases} 0, & \text{if } v \in C(w), \\ +\infty, & \text{otherwise.} \end{cases}$$

From the respective definitions and the fact that $f \in L^2(\Omega) \subset X_0^*$ (see Lemma 2.1), Problem 4.2 can be expressed as the variational–hemivariational inequality: find $u \in X_0$ such that

$$\langle Au, v - u\rangle_{X_0} + J^0(u; v - u) + I_{C(w)}(v) - I_{C(w)}(u) \geq \langle f, v - u\rangle_{X_0} \tag{4}$$

for all $v \in X_0$.

Claim 1 *For each $w \in X_0$, $I_{C(w)}\colon X_0 \to \overline{\mathbb{R}}$ is a proper, convex, and lower semicontinuous function with $0_{X_0} \in D(I_{C(w)})$ (the effective domain).*

This follows readily from the expression of the set $C(w)$ (see (2)).

Claim 1 confirms that the inclusion problem: find $u \in X_0$ such that

$$Au + \partial J(u) + \partial_C(I_{C(w)})(u) \ni f \tag{5}$$

is meaningful, where the notation $\partial_C(I_{C(w)})$ stands for the subdifferential of $I_{C(w)}$ in the sense of convex analysis. Furthermore, Lemma 3.1 shows that every solution to inclusion (5) is a solution to problem (4) too.

Next we aim to apply Theorem 2.2 to show that problem (5) is solvable.

Claim 2 $A\colon X_0 \to X_0^*$ *is a linear, continuous and strongly monotone operator.*

Notice that

$$\begin{aligned}\langle A(u), v\rangle_{X_0} &:= \int_{\mathbb{R}^N} (\mathcal{L}_K u)(x) v(x)\,\mathrm{d}x\\ &= -\int_{\mathbb{R}^{2N}} [u(x+y) + u(x-y) - 2u(x)] v(x) K(y)\,\mathrm{d}y\,\mathrm{d}x\\ &= -\int_{\mathbb{R}^{2N}} [u(x+y) - u(x)] v(x) K(y)\,\mathrm{d}y\,\mathrm{d}x\\ &\quad -\int_{\mathbb{R}^{2N}} [u(x-y) - u(x)] v(x) K(y)\,\mathrm{d}y\,\mathrm{d}x\\ &= \int_{\mathbb{R}^{2N}} [u(x) - u(y)][v(x) - v(y)] K(x-y)\,\mathrm{d}y\,\mathrm{d}x = \langle u, v\rangle_{X_0}\end{aligned}$$

for all $u, v \in X_0$, where the change of variable and symmetry requirement $H(j)$(iii) have been used. Thus we infer that

$$\|Au\|_{X_0^*} = \|u\|_{X_0} \text{ and } \langle Au, u\rangle_{X_0} = \|u\|^2_{X_0} \quad \text{for all } u \in X_0. \tag{6}$$

From (6), it follows that the linear operator A is bounded and strongly elliptic.

Claim 3 $A + \partial J\colon X_0 \to 2^{X_0^*}$ *is a bounded and pseudomonotone set-valued operator such that for each $u \in X_0$, the set $A(u) + \partial J(u)$ is closed and convex in X_0^*.*

Indeed, Proposition 2.2 and Lemma 3.1 ensure that for each $u \in X_0$, the set $Au + \partial J(u)$ is nonempty, closed and convex in X_0^*. But, Lemma 3.1(iii) and equality (6) imply $\|A(u) + \xi\|_{X_0^*} \le \|u\|_{X_0} + c_J\big(1 + c(2)\|u\|_{X_0}\big)$ for all $u \in X_0$ and $\xi \in \partial J(u)$. We infer that $A + \partial J$ is a bounded map.

Next we apply Proposition 2.1 to prove that the set-valued mapping $A + \partial J$ is upper semicontinuous from X_0 to X_0^* with weak topology. It is sufficient to check that for each weakly closed subset D in X_0^*, the set $(A + \partial J)^-(D)$ is closed in X_0. Let a sequence $\{u_n\} \subset (A + \partial J)^-(D)$ be such that

$$u_n \to u \quad \text{in } X_0 \text{ as } n \to \infty, \text{ for some } u \in X_0. \tag{7}$$

Then, $u_n^* := Au_n + \xi_n \in \big(A(u_n) + \partial J(u_n)\big) \cap D$ for some $\xi_n \in \partial J(u_n)$. The continuity of A (as shown in Claim 2) implies that $A(u_n) \to A(u)$ in X_0^* as $n \to \infty$. According to Lemma 3.1(iii) and (7), $\{\xi_n\}$ is bounded in $L^2(\Omega)$, so we can assume that $\xi_n \to \xi$ weakly in $L^2(\Omega)$ with some $\xi \in L^2(\Omega)$. Since ∂J is upper semicontinuous from $L^2(\Omega)$ to w-$L^2(\Omega)$ (i.e., $L^2(\Omega)$ with weak topology) and has bounded, convex, closed values, it has a closed graph in $X_0 \times \mathrm{w} - X_0^*$ (see [31, Theorem 1.1.4]). Hence, owing to the weak closedness of D and the continuity of the embeddings $X_0 \subset L^2(\Omega) \subset X_0^*$, we derive that $A(u) + \xi \in D$ and $\xi \in \partial J(u)$, which provides that $u \in \big(A + \partial J\big)^-(D)$. Consequently, $A + \partial J$ is upper semicontinuous from X_0 to X_0^* with weak topology.

To prove that $A + \partial J$ is pseudomonotone, let sequences $\{u_n\}$ and $\{u_n^*\}$ be such that

$$u_n \to u \quad \text{weakly in } X_0, \tag{8}$$

$$u_n^* \in A(u_n) + \partial J(u_n) \quad \text{with} \quad \limsup_{n\to\infty} \langle u_n^*, u_n - u\rangle_{X_0} \le 0. \tag{9}$$

Our goal is to produce for each $v \in X_0$ an element $u^*(v) \in A(u) + \partial J(u)$ with

$$\liminf_{n\to\infty} \langle u_n^*, u_n - v\rangle_{X_0} \ge \langle u^*(v), u - v\rangle_{X_0}. \tag{10}$$

Indeed, by (9) we can find a sequence $\{\xi_n\} \subset X_0^*$ such that for each $n \in \mathbb{N}$, $\xi_n \in \partial J(u_n)$ and $u_n^* = A(u_n) + \xi_n$. From (9) and the above equality it follows

$$\limsup_{n\to\infty} \langle Au_n, u_n - u\rangle_{X_0} + \liminf_{n\to\infty} \langle \xi_n, u_n - u\rangle_{X_0} \le 0. \tag{11}$$

Using (8) and the compact embedding of X_0 into $L^2(\Omega)$ (see Lemma 2.1), it holds $u_n \to u$ in $L^2(\Omega)$ as $n \to \infty$. Moreover, in view of [32, Thoerem 2.2], it turns out $\partial(J|_{X_0})(u) \subset \partial(J|_{L^2(\Omega)})(u)$ for all $u \in X_0$, which leads to

$$\langle \xi_n, u_n - u\rangle_{X_0} = \langle \xi_n, u_n - u\rangle_{L^2(\Omega)}. \tag{12}$$

Lemma 3.1 and the boundedness of $\{u_n\}$ in X_0 entail that the sequence $\{\xi_n\}$ is bounded both in $L^2(\Omega)$ and X_0^*. Then, through (12), we pass to the limit as $n \to \infty$ to get $\lim_{n\to\infty} \langle \xi_n, u_n - u\rangle_{X_0} = \lim_{n\to\infty} \langle \xi_n, u_n - u\rangle_{L^2(\Omega)} = 0$. The latter, in conjunction with (11) and (6), yields

$$\begin{aligned}
&\limsup_{n\to\infty} \|u_n - u\|_{X_0}^2 \\
&= \limsup_{n\to\infty} \langle A(u_n) - A(u), u_n - u\rangle_{X_0} + \lim_{n\to\infty} \langle A(u), u_n - u\rangle_{X_0} \le 0.
\end{aligned}$$

This means that $u_n \to u$ in X_0. On the other hand, the reflexivity of X_0^* and boundedness of $\{\xi_n\} \subset X_0^*$ permit us to suppose that $\xi_n \to \xi$ weakly in X_0^* for some $\xi \in X_0^*$. Now we can assert that $\xi \in \partial J(u)$ (see, e.g., [31, Theorem 1.1.4]). Since

$$\liminf_{n\to\infty} \langle u_n^*, u_n - v\rangle_{X_0} = \liminf_{n\to\infty} \langle A(u_n) + \xi_n, u_n - v\rangle_{X_0} = \langle A(u) + \xi, u - v\rangle_{X_0},$$

it is clear that (10) is verified with $u^* = A(u) + \xi \in A(u) + \partial J(u)$. We conclude that $A + \partial J$ is pseudomonotone.

Claim 4 *There exists* $R > 0$ *such that* $\langle Au + \xi + \eta - f, u\rangle_{X_0} > 0$ *for all* $u \in C(w)$ *with* $\|u\|_{X_0} = R$, *all* $\xi \in \partial J(u)$ *and all* $\eta \in \partial_C(I_{C(w)})$.

For any $u \in D(\partial_C(I_{C(w)}))$, $\xi \in \partial J(u)$ and $\eta \in \partial_C(I_{C(w)})$, on the basis of previous considerations and Lemma 3.1(ii), it holds

$$\begin{aligned}
&\langle Au + \xi + \eta - f, u\rangle_{X_0}\\
&\quad \geq \|u\|^2_{X_0} - \|f\|_{X_0^*}\|u\|_{X_0} - J^0(u; -u)\\
&\quad \geq \|u\|^2_{X_0} - \|f\|_{X_0^*}\|u\|_{X_0} - \alpha_j|\Omega| - \beta_j \int_\Omega |u(x)|^2\,\mathrm{d}x\\
&\quad \geq \|u\|^2_{X_0} - \|f\|_{X_0^*}\|u\|_{X_0} - \alpha_j|\Omega| - \beta_j c(2)^2\|u\|^2_{X_0}.
\end{aligned} \tag{13}$$

Let $R > 0$ be such that $R((1 - \beta_j c(2)^2)R - \|f\|_{X_0^*}) - \alpha_j|\Omega| > 0$, which is possible since $\beta_j \in]0, 1/c(2)^2[$. Then estimate (13) proves the validity of Claim 4.

By Claims 1–4 and Theorem 2.2, there exists $u_w \in X_0$ resolving inclusion (5). Thus $S(w) \neq \emptyset$ holds true for each $w \in X_0$.

Now we claim that the solution set $S(w)$ of Problem 4.2 is closed. Let $\{u_n\} \subset S(w)$ be such that $u_n \to u$ in X_0. For each $n \in \mathbb{N}$, by (4), there holds

$$\langle Au_n, v - u_n\rangle_{X_0} + J^0(u_n; v - u_n) + I_{C(w)}(v) - I_{C(w)}(u_n) \geq \langle f, v - u_n\rangle_{X_0}$$

for all $v \in X_0$, or $\langle Au_n, v - u_n\rangle_{X_0} + J^0(u_n; v - u_n) \geq \langle f, v - u_n\rangle_{X_0}$ for all $v \in C(w)$ because $u_n \in C(w)$. Passing to the upper limit as $n \to \infty$ yields $u \in S(w)$, hence the claim that $S(w)$ is closed in X_0 is proved.

Moreover, we show that for each $w \in X_0$, the set $S(w)$ is convex. Toward this, we note that assumption $m_j c(2)^2 \leq 1$ turns $A + \partial J\colon X_0 \to 2^{X_0^*}$ be monotone because for all $u, v \in X_0$, $\xi_u \in \partial J(u)$ and $\xi_v \in \partial J(v)$ we obtain

$$\begin{aligned}
&\langle A(u) + \xi_u - A(v) - \xi_v, u - v\rangle_{X_0}\\
&\quad \geq \|u - v\|^2_{X_0} - m_j\|u - v\|^2_{L^2(\Omega)} \geq \|u - v\|^2_{X_0} - m_j c(2)^2\|u - v\|^2_{X_0} \geq 0.
\end{aligned}$$

Let $u_1, u_2 \in S(w)$, $t \in]0, 1[$, and denote $u_t = tu_1 + (1 - t)u_2$. The monotonicity of $A + \partial J$ implies for $i = 1, 2$ that $\langle Av + \xi_v, v - u_i\rangle_{X_0} \geq \langle f, v - u_i\rangle_{X_0}$, whenever $\xi_v \in \partial J(v)$ and $v \in X_0$. The latter in conjunction with Proposition 2.2 (d) results in

$$\begin{aligned}
&\langle Av, v - u_t\rangle_{X_0} + J^0(v; v - u_t) \geq \langle Av + \xi_v, v - u_t\rangle_{X_0}\\
&\quad = t\langle Av + \xi_v, v - u_1\rangle_{X_0} + (1 - t)\langle Av + \xi_v, v - u_2\rangle_{X_0} \text{ (for all } \xi_v \in \partial J(v))\\
&\quad \geq t\langle f, v - u_1\rangle_{X_0} + (1 - t)\langle f, v - u_2\rangle_{X_0} = \langle f, v - u_t\rangle_{X_0}
\end{aligned} \tag{14}$$

for all $v \in C(w)$. Let $z \in X_0$ and $\lambda \in]0, 1[$. Inserting $v = z_\lambda := \lambda z + (1 - \lambda)u_t$ in (14) gives $\langle Az_\lambda, z - u_t\rangle_{X_0} + J^0(z_\lambda; z - u_t) \geq \langle f, z - u_t\rangle_{X_0}$. We pass to the upper limit as $\lambda \to 0^+$ finding

$$\langle Au_t, z-u_t\rangle_{X_0} + J^0(u_t; z-u_t) \geq \limsup_{\lambda\to 0^+}\left[\langle Az_\lambda, z-u_t\rangle_{X_0} + J^0(z_\lambda; z-u_t)\right]$$
$$\geq \langle f, z-u_t\rangle_{X_0}.$$

Recall that $z \in C(w)$ was arbitrary, so this renders $u_t \in S(w)$. Therefore the set $S(w)$ is convex.

Next we demonstrate that the set $S(w)$ is bounded in X_0 for each $w \in X_0$. Fix $w \in X_0$. Arguing by contradiction, suppose that $S(w)$ is unbounded, whence there exists a sequence $\{u_n\} \subset S(w)$ such that

$$\|u_n\|_{X_0} \to \infty \quad \text{as } n \to \infty. \tag{15}$$

Since $0_{X_0} \in C(w)$, we have $\langle Au_n, u_n\rangle_{X_0} - J^0(u_n; -u_n) \leq \langle f, u_n\rangle_{X_0}$. Reasoning as in (13) enables us to find $\|f\|_{X_0^*}\|u_n\|_{X_0} \geq \|u_n\|^2_{X_0} - \alpha_j|\Omega| - \beta_j c(2)^2\|u_n\|^2_{X_0}$. As it is known from assumption $H(j)$(iv) that $\beta_j \in]0, 1/c(2)^2[$, then (15) and the above estimate generate a contradiction, which completes the proof. □

5 Existence Result

We are in a position to state our main result.

Theorem 5.1 *Assume that $H(K)$, $H(j)$, $H(U)$, and $m_j c(2)^2 \leq 1$ are fulfilled. Then Problem 3.1 possesses at least a solution.*

Proof As already remarked, every fixed point of S solves Problem 4.1 as well. Besides, Lemma 3.1 reveals that the set of solutions for Problem 4.1 is a subset of the set of solutions for Problem 3.1. Consequently, it suffices to show that the set of fixed points of S is nonempty.

Claim 5 *The graph of S is sequentially weakly closed.*

Let $\{w_n\} \subset X_0$ and $\{u_n\} \subset X_0$ be sequences such that $u_n \in S(w_n)$, $w_n \to w$ weakly in X_0 and $u_n \to u$ weakly in X_0. Hence, for each $n \in \mathbb{N}$, it holds $u_n \in C(w_n)$, i.e., $\|u_n\|_{X_0} \leq U(w_n)$. The compactness of the embedding of X_0 in $L^2(\Omega)$ and the continuity of U postulated in condition $H(U)$ provide

$$\|u\|_{X_0} \leq \liminf_{n\to\infty}\|u_n\|_{X_0} \leq \liminf_{n\to\infty} U(w_n) = U(w),$$

so $u \in C(w)$.

The fact that $u_n \in S(w_n)$ reads as

$$\langle Au_n, v-u_n\rangle_{X_0} + J^0(u_n; v-u_n) \geq \langle f, v-u_n\rangle_{X_0} \tag{16}$$

for all $v \in C(w_n)$, whereas the monotonicity of $u \mapsto Au + \partial J(u)$ reveals

$$\langle Aw_n, v-u_n\rangle_{X_0} + J^0(w_n; v-u_n) \geq \langle f, v-u_n\rangle_{X_0}. \tag{17}$$

For each $v \in C(w)$, consider the sequence $\{v_n\}$ defined by $v_n = \frac{U(w_n)}{U(w)} v$ for each $n \in \mathbb{N}$. Clearly, $\|v_n\|_{X_0} = \frac{U(w_n)}{U(w)} \|v\|_{X_0} \leq U(w_n)$ and

$$\lim_{n\to\infty} \|v_n - v\|_{X_0} = \lim_{n\to\infty} \big|U(w) - U(w_n)\big| \frac{\|v\|_{X_0}}{U(w)} = 0.$$

We deduce that $\{v_n\}$ converges strongly to v in X_0 and $v_n \in C(w_n)$ for every $n \in \mathbb{N}$. It is thus permitted to insert $v = v_n$ in (17). Passing to the upper limit as $n \to \infty$ produces

$$\begin{aligned}
&\langle Av, v-u\rangle_{X_0} + J^0(v; v-u) \\
&\quad \geq \limsup_{n\to\infty} \langle Av_n, v_n - u_n\rangle_{X_0} + \limsup_{n\to\infty} J^0(v_n; v_n - u_n) \\
&\quad \geq \limsup_{n\to\infty} \langle f, v_n - u_n\rangle_{X_0} = \langle f, v-u\rangle_{X_0},
\end{aligned}$$

where we have used (6), the compact embedding of X_0 in $L^2(\Omega)$ and that $L^2(\Omega) \times L^2(\Omega) \ni (v,u) \to J^0(u; v) \in \mathbb{R}$ is upper semicontinuous (see Lemma 3.1 and Proposition 2.2). The arbitrariness of $v \in C(w)$ and Minty approach guarantee that $u \in S(w)$. Therefore, Claim 5 is proved.

Claim 6 *The set* $S(X_0)$ *is bounded in* X_0.

If the claim were not true, then there would exist sequences $\{u_n\}$ and $\{w_n\}$ with $u_n \in S(w_n)$ such that

$$\|u_n\|_{X_0} \to \infty \quad \text{as } n \to \infty. \tag{18}$$

For every $n \in \mathbb{N}$, one has (16) for all $v \in C(w_n)$. Bearing in mind that $0_{X_0} \in C(w)$ for each $w \in X_0$, we take $v = 0_{X_0}$ as test function in (16) obtaining $\langle Au_n, u_n\rangle_{X_0} - J^0(u_n; -u_n) \leq \|f\|_{X_0^*} \|u_n\|_{X_0}$. The latter combined with (6) and Lemma 3.1(ii) shows $\|u_n\|_{X_0} - \frac{\alpha_j |\Omega|}{\|u_n\|_{X_0}} - \beta_j c(2)^2 \|u_n\|_{X_0} \leq \|f\|_{X_0^*}$. This triggers a contradiction with (18) owing to $\beta_j \in]0, 1/c(2)^2[$. We conclude that Claim 6 holds true.

On account of Claims 5 and 6, the required conditions to apply Theorem 2.1 are verified for the set-valued mapping S. Hence it has a fixed point in X_0, which from Lemma 3.1 is a solution to Problem 3.1. □

6 Dropping Assumption *H*(*j*)(iii)

The aim of this section is to point out that assumption $H(j)$(iii) can be dropped in the statement of Theorem 5.1. This important fact has been pointed out in one of Referee's reports, where it was also outlined the proof. We have preferred to keep the statement of Theorem 5.1 because our original approach was completely different relying on Theorem 2.2. Here is the improved statement.

Theorem 6.1 *Theorem 5.1 holds true without assuming condition* $H(j)$(iii).

Proof Let us introduce the set-valued mapping $\mathcal{T} : X_0 \times L^2(\Omega) \rightrightarrows X_0 \times L^2(\Omega)$ by $\mathcal{T}(v, w) = (u, F(u))$, where u is the unique solution of the classical variational inequality in the Hilbert space X_0: find $u \in X_0$ such that $u \in C(v)$ and

$$\int_{\mathbb{R}^N} (\mathcal{L}_K u)(x)[z(x) - u(x)]\,\mathrm{d}x + \int_\Omega w(x)[z(x) - u(x)]\,\mathrm{d}x \\ \geq \int_\Omega f(x)[z(x) - u(x)]\,\mathrm{d}x \tag{19}$$

for all $z \in C(v)$ and, for a constant $R > 0$,

$$F(u) = \begin{cases} \partial J(u), & \text{if } \|u\|_{L^2(\Omega)} \leq R, \\ \partial J(Ru/\|u\|_{L^2(\Omega)}), & \text{otherwise.} \end{cases}$$

The set of constraints $C(v)$ is given in (2), and it was already noted that $C(v)$ is closed and convex in X_0, and $0_{X_0} \in C(v)$.

Setting $z = 0_{X_0}$ in (19) yields

$$\|u\|_{X_0}^2 \leq (\|w\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)})c(2)\|u\|_{X_0}. \tag{20}$$

Combining with the continuous embedding $X_0 \subset L^2(\Omega)$, there exists a constant $R_0 > 0$ (independent of R) for which $\|u\|_{L^2(\Omega)} \leq R_0$. Now we fix $R > R_0$. Then the definition of the set-valued mapping F entails $F(u) = \partial J(u)$, so by (19) with the fixed R, if $(u, w) \in X_0 \times L^2(\Omega)$ is a fixed point of $\mathcal{T}$, i.e., $(u, w) \in \mathcal{T}(u, w)$, one has that u is a solution of Problem 4.1 and we are done.

We prove that the set-valued mapping $\mathcal{T}$ possesses a fixed point by applying Theorem 2.1. To this end we check that the graph of $\mathcal{T}$ is sequentially weakly closed. Let the sequences $\{v_n\} \subset X_0$ and $\{w_n\} \subset L^2(\Omega)$ satisfy $v_n \to v$ weakly in X_0, $w_n \to w$ weakly in $L^2(\Omega)$, and let $(u_n, \sigma_n) \in \mathcal{T}(v_n, w_n)$ be such that $(u_n, \sigma_n) \to (u, \sigma)$ weakly in $X_0 \times L^2(\Omega)$, with $\sigma_n \in \partial J(u_n)$. By the definition of $\mathcal{T}$ and knowing that $(u_n, \sigma_n) \in \mathcal{T}(v_n, w_n)$, it turns out $u_n \in C(v_n)$ and

$$\int_{\mathbb{R}^N} (\mathcal{L}_K u_n)(x)[z(x) - u_n(x)]\,\mathrm{d}x + \int_\Omega w_n(x)[z(x) - u_n(x)]\,\mathrm{d}x \\ \geq \int_\Omega f(x)[z(x) - u_n(x)]\,\mathrm{d}x \tag{21}$$

for all $z \in C(v_n)$. Due to $u_n \to u$ weakly in X_0 and the compact embedding $X_0 \subset L^2(\Omega)$ we have along a relabeled subsequence that $u_n \to u$ strongly in $L^2(\Omega)$, thereby $\|u\|_{X_0} \leq R_0$. Exploiting $\sigma_n \to \sigma$ weakly in $L^2(\Omega)$ in conjunction with $\sigma_n \in \partial J(u_n)$ enables us to deduce that $\sigma \in \partial J(u)$, thus $\sigma \in F(u)$. Since $\|u_n\|_{X_0} \leq U(v_n)$, by the compact embedding of X_0 in $L^2(\Omega)$ and the continuity of U on $L^2(\Omega)$ (note hypotheesis $H(U)$), we infer that $\|u\|_{X_0} \leq U(v)$ or equivalently $u \in C(v)$.

In order to conclude that the graph of $\mathcal{T}$ is sequentially weakly closed it remains to show that $u \in X_0$ is a solution of

$$\int_{\mathbb{R}^N} (\mathcal{L}_K u)(x)[z(x) - u(x)]\,\mathrm{d}x + \int_{\Omega} w(x)[z(x) - u_(x)]\,\mathrm{d}x \\ \geq \int_{\Omega} f(x)[z(x) - u(x)]\,\mathrm{d}x \tag{22}$$

for all $z \in C(v)$. We proceed on the pattern in the proof of Theorem 5.1. Let $z \in C(v)$. For each $n \in \mathbb{N}$, we construct $z_n = \frac{U(v_n)}{U(v)} z$. From the compact embedding $X_0 \subset L^2(\Omega)$ and hypothesis $H(U)$, it follows that $z_n \to z$ in X_0 in view of $\lim_{n\to\infty} \|z_n - z\|_{X_0} = \lim_{n\to\infty} \left|U(v) - U(v_n)\right| \frac{\|z\|_{X_0}}{U(v)} = 0$. Furthermore, we have $\|z_n\|_{X_0} = \frac{U(v_n)}{U(v)} \|z\|_{X_0} \leq U(v_n)$ ensuring that $z_n \in C(v_n)$ for every $n \in \mathbb{N}$, which allows us to insert $z = z_n$ in (21). We arrive at

$$\int_{\mathbb{R}^N} (\mathcal{L}_K u_n)(x) z_n(x)\,\mathrm{d}x + \int_{\Omega} w_n(x)[z_n(x) - u_n(x)]\,\mathrm{d}x \\ \geq \int_{\Omega} f(x)[z_n(x) - u_n(x)]\,\mathrm{d}x + \|u_n\|_{X_0}^2.$$

Through the weak lower semicontinuity of the norm, in the limit as $n \to \infty$ we obtain (22).

Next we prove that the range $\mathcal{T}(X_0 \times L^2(\Omega))$ of the set-valued mapping $\mathcal{T}$ is bounded in $X_0 \times L^2(\Omega)$. From the definition of $\mathcal{T}$, we see that it suffices to show that the first component u of $\mathcal{T}(v, w)$ is bounded. It is so since the function J in (3) is Lipschitz continuous on every bounded set in $L^2(\Omega)$, which reflects in the fact that the generalized gradient $\partial J(u)$ is bounded whenever u is bounded. For $z = 0_{X_0}$ in (19) we are led to estimate (20). Hence, admitting that w is bounded in $L^2(\Omega)$ we get that u is bounded in X_0, whence $\mathcal{T}(X_0 \times L^2(\Omega))$ is bounded in $X_0 \times L^2(\Omega)$.

We have shown that the hypotheses of Theorem 2.1 are fulfilled in the case of the set-valued mapping $\mathcal{T} \colon X_0 \times L^2(\Omega) \to 2^{X_0 \times L^2(\Omega)}$. Then Theorem 2.1 provides the existence of a fixed point of $\mathcal{T}$. As remarked before, this completes the proof. □

7 Conclusions

In this paper, we consider an implicit obstacle problem driven by a fractional Laplace operator and a set-valued mapping which is described by a generalized gradient. Under quite general assumptions on the data, and employing Kluge's fixed point principle for multivalued operators, and a surjectivity theorem, we prove that the set of weak solutions for the implicit obstacle problem is nonempty. Finally, implementing an idea suggested by one of the Referees, we improve our existence result. Problems of this type are encountered in transport optimization, Nash equilibrium theory and related fields. In the future we plan to apply the theoretical results established in the current paper to Nash equilibrium problems and population dynamics models.

Acknowledgements The authors would like to express their warm thanks to Reviewers for their helpful comments which substantially improved the paper. This project has received funding from the NNSF of China Grant No. 12001478, and European Union's Horizon 2020 Research and Innovation Programme under the Marie Skłodowska-Curie grant agreement No. 823731 CONMECH. It is supported by the National Science Center of Poland under Preludium Project No. 2017/25/N/ST1/00611, the Startup Project of Doctor Scientific Research of Yulin Normal University No. G2020ZK07, and International Project co-financed by the Ministry of Science and Higher Education of Republic of Poland under Grant No. 3792/GGPJ/H2020/2017/0.

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