



Generalized Multipliers for Left-Invertible Operators and Applications

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Abstract. Generalized multipliers for a left-invertible operator T , whose formal Laurent series $U_x(z) = \sum_{n=1}^{\infty} (P_E T^n x) \frac{1}{z^n} + \sum_{n=0}^{\infty} (P_E T'^{*n} x) z^n$, $x \in \mathcal{H}$ actually represent analytic functions on an annulus or a disc are investigated. We show that they are coefficients of analytic functions and characterize the commutant of some left-invertible operators, which satisfies certain conditions in its terms. In addition, we prove that the set of multiplication operators associated with a weighted shift on a rootless directed tree lies in the closure of polynomials in z and $\frac{1}{z}$ of the weighted shift in the topologies of strong and weak operator convergence.

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1. Introduction

One of the key ideas in operator theory is that of viewing an operator as multiplication by z on a Hilbert space consisting of (vector-valued) holomorphic functions. The point is that this multiplication operator is much easier to analyze than is the case in the original setting because of the richer structure of a space of holomorphic functions. This is its great advantage and one of the reasons why it attracts the attention of researchers.

As was mentioned by A.L. Shields in the paper [24] the fact that weighted shift can be viewed as multiplication by z on a Hilbert space of formal power series (in the unilateral case) or formal Laurent series (in the bilateral case) has been long folklore and this point of view was taken by R. Gellar (see [11, 12]). Given a standard orthonormal basis $\{e_n\}_{n \in \mathbb{Z}}$ of $\ell^2(\mathbb{Z})$, let $\{\lambda_n\}_{n \in \mathbb{Z}}$ be a sequence of nonzero scalars such that the bilateral weighted shift $S_\lambda : \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$ given by

$$S_\lambda e_n = \lambda_{n+1} e_{n+1}, \quad n \in \mathbb{Z}$$

is bounded. To each vector $x = \sum_{n=-\infty}^{\infty} x_n e_n$ in $\ell^2(\mathbb{Z})$ Gellar associate the series

$$U_x(z) = \sum_{n=1}^{\infty} \left(\prod_{i=-n+1}^0 \lambda_i \right) x_{-n} \frac{1}{z^n} + \sum_{n=0}^{\infty} \left(\prod_{i=1}^n \lambda_i \right)^{-1} x_n z^n.$$

This endow $\ell^2(\mathbb{Z})$ with the structure of a Hilbert space whose elements are formal Laurent series. Moreover operator S_λ is unitarily equivalent to the operator \mathcal{M}_z of multiplication by z on $\mathcal{H} := \{U_x : x \in \ell^2(\mathbb{Z})\}$. He considered those formal Laurent series $\varphi = \sum_{n=-\infty}^{\infty} \widehat{\varphi}(n)z^n$, such that multiplication by φ in \mathcal{H} is a bounded operator. Those series are called *multipliers*. He proved that the commutant of bilateral weighted shift of multiplicity one may be identified with the algebra of its multipliers.

In [25] S. Shimorin obtain a weak analog of the Wold decomposition theorem, representing operator close to isometry in some sense as a direct sum of a unitary operator and a shift operator acting in some reproducing kernel Hilbert space of vector-valued holomorphic functions defined on a disc. In particular he constructed an analytic model for a left-invertible analytic operator $T \in \mathcal{B}(\mathcal{H})$. The construction of the Shimorin’s model is as follows. Let $E := \mathcal{N}(T^*)$ and define a vector-valued holomorphic functions U_x as

$$U_x(z) = \sum_{n=0}^{\infty} (P_E T'^{*n} x) z^n, \quad z \in \mathbb{D}(r(T')^{-1}),$$

where T' is the Cauchy dual of T . Then we equip the obtained space of analytic functions $\mathcal{H} := \{U_x : x \in \mathcal{H}\}$ with the inner product induced by \mathcal{H} . The operator $U : \mathcal{H} \ni x \rightarrow U_x \in \mathcal{H}$ becomes a unitary isomorphism. Shimorin proved that \mathcal{H} is a reproducing kernel Hilbert space and the operator T is unitary equivalent to the operator \mathcal{M}_z of multiplication by z on \mathcal{H} and T'^* is unitary equivalent to the operator \mathcal{L} given by the

$$(\mathcal{L}f)(z) = \frac{f(z) - f(0)}{z}, \quad f \in \mathcal{H}.$$

In [9] P. Dymek, A. Płaneta and M. Ptak extended the notion of multipliers to left-invertible analytic operators using Shimorin’s analytic function theory approach.

Namely, they defined generalized multipliers for left-invertible analytic operator T , whose coefficients are bounded operators on $\mathcal{N}(T^*)$ and characterized the commutant of such operators in its terms (see [9, Theorem 4]).

In the recent paper [22, Section 3], the author provided a new analytic model on an annulus for left-invertible operators, which are not necessarily analytic operators. The construction of the analytic model on an annulus for a left-invertible operator $T \in \mathcal{B}(\mathcal{H})$ is based on the following unitary isomorphism:

$$U : \mathcal{H} \ni x \rightarrow \sum_{n=1}^{\infty} (P_E T^n x) \frac{1}{z^n} + \sum_{n=0}^{\infty} (P_E T'^{*n} x) z^n \in \mathcal{H},$$

where E is a closed subspace of \mathcal{H} satisfying some certain condition (see (3.1)). The model extends both Shimorin’s analytic model for left-invertible

analytic operators (see [22, Theorem 3.3]) and Gellar’s model for a bilateral weighted shift (see [22, Example 5.2]). As shown in [22, Theorem 3.2 and 3.8] a left-invertible operator T , which satisfies certain conditions can be modelled as a multiplication operator \mathcal{M}_z on a reproducing kernel Hilbert space of vector-valued analytic functions on an annulus or a disc.

In this paper, we extend the notion of generalized multipliers for left-invertible analytic operator introduced in [9] to the case of left-invertible operator T , whose model constructed in [22, Section 3] actually represent analytic functions on an annulus or a disc. We show that they are coefficients of analytic functions and characterize the commutant of some left-invertible operators, which satisfies certain conditions in its terms. In addition, we prove that the set of multiplication operators associated with a weighted shift on a rootless directed tree lies in the closure of polynomials in z and $\frac{1}{z}$ of the weighted shift in the topologies of strong and weak operator convergence.

2. Preliminaries

In this paper, we use the following notation. The fields of rational, real and complex numbers are denoted by \mathbb{Q} , \mathbb{R} and \mathbb{C} , respectively. The symbols \mathbb{Z} , \mathbb{Z}_+ , \mathbb{N} and \mathbb{R}_+ stand for the sets of integers, positive integers, non-negative integers, and nonnegative real numbers, respectively. Set $\mathbb{D}(r) = \{z \in \mathbb{C} : |z| < r\}$ and $\mathbb{A}(r^-, r^+) = \{z \in \mathbb{C} : r^- < |z| < r^+\}$ for $r, r^-, r^+ \in \mathbb{R}_+$. The characteristic function of a subset A of \mathbb{Z} is denoted by χ_A .

All Hilbert spaces considered in this paper are assumed to be complex. Let T be a linear operator in a complex Hilbert space \mathcal{H} . Denote by T^* the adjoint of T . We write $\mathbf{B}(\mathcal{H})$ for the C^* -algebra of all bounded operators and the cone of all positive operators in \mathcal{H} , respectively. The spectrum and spectral radius of $T \in \mathbf{B}(\mathcal{H})$ is denoted by $\sigma(T)$ and $r(T)$ respectively. Let W be a subset of \mathcal{H} . Then $\text{lin } W$, $\bigvee W$ stands for the smallest linear subspace, closed subspace generated by W , respectively. We use the notation $\langle x \rangle$ in place of $\text{lin}\{x\}$, for $x \in \mathcal{H}$. Let $T \in \mathbf{B}(\mathcal{H})$. We say that T is left-invertible if there exists $S \in \mathbf{B}(\mathcal{H})$ such that $ST = I$. The *Cauchy dual operator* T' of a left-invertible operator $T \in \mathbf{B}(\mathcal{H})$ is defined by

$$T' = T(T^*T)^{-1}.$$

The notion of the Cauchy dual operator has been introduced and studied by Shimorin in the context of the wandering subspace problem for Bergman-type operators [25]. We call T *analytic* if $\mathcal{H}_\infty = \bigcap_{i=1}^\infty T^i \mathcal{H} = \{0\}$.

Let $\mathcal{T} = (V, \mathcal{E})$ be a directed tree (V and \mathcal{E} are the sets of vertices and edges of \mathcal{T} , respectively). For any vertex $u \in V$ we put $\text{Chi}(u) = \{v \in V : (u, v) \in \mathcal{E}\}$. Denote by par the partial function from V to V which assigns to a vertex u a unique $v \in V$ such that $(v, u) \in \mathcal{E}$. A vertex $u \in V$ is called a root of \mathcal{T} if u has no parent. If \mathcal{T} has a root, we denote it by root . Put $V^\circ = V \setminus \{\text{root}\}$ if \mathcal{T} has a root and $V^\circ = V$ otherwise. The Hilbert space of square summable complex functions on V equipped with the standard inner product is denoted by $\ell^2(V)$. For $u \in V$, we define $e_u \in \ell^2(V)$ to be the characteristic function of the set $\{u\}$. It turns out that the set $\{e_v\}_{v \in V}$

is an orthonormal basis of $\ell^2(V)$. We put $V_{\prec} := \{v \in V : \text{card}(\text{Chi}(v)) \geq 2\}$ and call the a member of this set a *branching vertex* of \mathcal{T} . A subgraph of a directed tree \mathcal{T} which itself is a directed tree will be called a subtree of \mathcal{T} .

Given a system $\lambda = \{\lambda_v\}_{v \in V^\circ}$ of complex numbers, we define the operator S_λ in $\ell^2(V)$, which is called a *weighted shift* on \mathcal{T} with weights λ , as follows

$$\mathcal{D}(S_\lambda) = \{f \in \ell^2(V) : A_{\mathcal{T}} f \in \ell^2(V)\} \quad \text{and} \quad S_\lambda f = A_{\mathcal{T}} f \quad \text{for} \quad f \in \mathcal{D}(S_\lambda),$$

where

$$(A_{\mathcal{T}} f)(v) = \begin{cases} \lambda_v f(\text{par}(v)) & \text{if } v \in V^\circ, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.1. [13] *If S_λ is a densely defined weighted shift on a directed tree \mathcal{T} with weights $\lambda = \{\lambda_v\}_{v \in V^\circ}$, then*

$$\mathcal{N}(S_\lambda^*) = \begin{cases} \langle e_{\text{root}} \rangle \oplus \bigoplus_{u \in V_{\prec}} (\ell^2(\text{Chi}(u)) \ominus \langle \lambda^u \rangle) & \text{if } \mathcal{T} \text{ has a root,} \\ \bigoplus_{u \in V_{\prec}} (\ell^2(\text{Chi}(u)) \ominus \langle \lambda^u \rangle) & \text{otherwise,} \end{cases}$$

where $\lambda^u \in \ell^2(\text{Chi}(u))$ is given by $\lambda^u : \ell^2(\text{Chi}(u)) \ni v \rightarrow \lambda_v \in \mathbb{C}$

Following [6], we say that \mathcal{T} has *finite branching index* if there exist $m \in \mathbb{N}$ such that

$$\text{Chi}^k(V_{\prec}) \cap V_{\prec} = \emptyset, \quad k \geq m, \quad k \in \mathbb{N}.$$

The next lemma shows that in the case of rootless directed tree with finite branching index there exist some special vertex.

Lemma 2.2. [6] *Let $\mathcal{T} = (V, \mathcal{E})$ be a rootless directed tree with finite branching index m . Then there exist a vertex $\omega \in V_{\prec}$ such that*

$$\text{card}(\text{Chi}(\text{par}^{(n)}(\omega))) = 1, \quad n \in \mathbb{Z}_+. \tag{2.1}$$

Moreover, if V_{\prec} is non-empty, then there exists a unique $\omega \in V_{\prec}$ satisfying (2.1).

The vertex $\omega \in V_{\prec}$ appearing in the statement of Lemma 2.2 is called *generalized root*.

We now describe Cauchy dual of a left-invertible weighted shift on a directed tree

Lemma 2.3. [6] *Let $\mathcal{T} = (V, \mathcal{E})$ be a directed tree and S_λ be a left-invertible weighted shift on a directed tree \mathcal{T} with weights $\lambda = \{\lambda_v\}_{v \in V^\circ}$. Then the Cauchy dual S'_λ of S_λ is also a weighted shift on a directed tree \mathcal{T} given by:*

$$S'_\lambda e_u = \sum_{v \in \text{Chi}(u)} \frac{\lambda_v}{\|S_\lambda e_{\text{par}(v)}\|^2} e_v, \quad u \in V.$$

Note that S'_λ is a weighted shift with weights $\{\lambda'_v\}_{v \in V^\circ}$, where

$$\lambda'_v := \frac{\lambda_v}{\|S_\lambda e_{\text{par}(v)}\|^2}, \quad v \in V.$$

Throughout this text, we find it convenient to use the notation S_λ in place of S'_λ . We refer the reader to [13] for more details on weighted shifts on directed trees.

Assume that $\mathcal{T} = (V, \mathcal{E})$ is a countably infinite rooted and leafless directed tree, and $\lambda = \{\lambda_v\}_{v \in V^\circ} \subset (0, \infty)$. For $u \in V$ and $v \in \text{Des}(u)$ we set

$$\lambda_{u|v} = \begin{cases} 1 & \text{if } u = v, \\ \prod_{n=0}^{k-1} \lambda_{\text{par}^n(v)} & \text{if } \text{par}^k(v) = u. \end{cases}$$

Let $\widehat{\varphi} : \mathbb{N} \rightarrow \mathbb{C}$. Define the mapping $\Gamma_{\widehat{\varphi}}^\lambda : \mathbb{C}^V \rightarrow \mathbb{C}^V$ by

$$(\Gamma_{\widehat{\varphi}}^\lambda)(v) = \sum_{k=0}^{|v|} \lambda_{\text{par}^k(v)|v} \widehat{\varphi}(k) f(\text{par}^k(v)), \quad v \in V.$$

The *multiplication operator* $M_{\widehat{\varphi}}^\lambda : \ell^2(V) \supseteq \mathcal{D}(M_{\widehat{\varphi}}^\lambda) \rightarrow \ell^2(V)$ is given by

$$\begin{aligned} \mathcal{D}(M_{\widehat{\varphi}}^\lambda) &:= \{f \in \ell^2(V) : \Gamma_{\widehat{\varphi}}^\lambda f \in \ell^2(V)\}, \\ M_{\widehat{\varphi}}^\lambda f &:= \Gamma_{\widehat{\varphi}}^\lambda f, \quad f \in \mathcal{D}(M_{\widehat{\varphi}}^\lambda). \end{aligned}$$

By $\mathcal{M}(\lambda)$ we denote the *multiplier algebra* induced by S_λ , i.e., the commutative Banach algebra consisting of all $\widehat{\varphi} : \mathbb{N} \rightarrow \mathbb{C}$ such that $\mathcal{D}(M_{\widehat{\varphi}}^\lambda) = \ell^2(V)$ with the norm

$$\|\widehat{\varphi}\| := \|M_{\widehat{\varphi}}^\lambda\|, \quad \widehat{\varphi} \in \mathcal{M}(\lambda).$$

For the basic concepts of the theory of multipliers, we refer the reader to [3].

3. Generalized Multipliers

In this section, we provide a characterization of the commutant of left-invertible operator, which is not necessarily analytic in terms of generalized multipliers. Since in the case of such operators the first sum in (3.2) does not have to vanish (cf. [22, Theorem 3.3] this requires considering two-sided sequence of operators. Hence, we have to extend the notion of generalized multipliers for left-invertible and analytic operators introduced in [9].

Since the analytic model for left-invertible operator introduced in the recent paper [22] by the author plays a major role in this paper, we outline it in the following discussion. Let $T \in \mathbf{B}(\mathcal{H})$ be a left-invertible operator and E be a closed subspace of \mathcal{H} denote by $[E]_{T^*, T'}$ the following subspace of \mathcal{H} :

$$[E]_{T^*, T'} := \bigvee (\{T^{*n}x : x \in E, n \in \mathbb{N}\} \cup \{T'^n x : x \in E, n \in \mathbb{N}\}),$$

where T' is the Cauchy dual of T .

To avoid the repetition, we state the following assumption which will be used frequently in this paper.

$$\begin{aligned} \text{The operator } T \in \mathbf{B}(\mathcal{H}) \text{ is left-invertible and } E \\ \text{is a closed subspace of } \mathcal{H} \text{ such that } [E]_{T^*, T'} = \mathcal{H}. \end{aligned} \tag{3.1}$$

Suppose (3.1) holds. In this case we may construct a Hilbert \mathcal{H} associated with T , of formal Laurent series with vector coefficients. We proceed

as follows. For each $x \in \mathcal{H}$, define a formal Laurent series U_x with vector coefficients as

$$U_x(z) = \sum_{n=1}^{\infty} (P_E T^n x) \frac{1}{z^n} + \sum_{n=0}^{\infty} (P_E T'^{*n} x) z^n. \tag{3.2}$$

Let \mathcal{H} denote the vector space of formal Laurent series with vector coefficients of the form U_x , $x \in \mathcal{H}$. Consider the map $U : \mathcal{H} \rightarrow \mathcal{H}$ defined by $Ux := U_x$. As shown in [22, Lemma 3.1] U is injective. In particular, we may equip the space \mathcal{H} with the norm induced from \mathcal{H} , so that U is unitary.

By [22, Theorem 3.2] the operator T is unitary equivalent to the operator $\mathcal{M}_z : \mathcal{H} \rightarrow \mathcal{H}$ of multiplication by z on \mathcal{H} given by

$$(\mathcal{M}_z f)(z) = z f(z), \quad f \in \mathcal{H}$$

and operator T'^{*} is unitary equivalent to the operator $\mathcal{L} : \mathcal{H} \rightarrow \mathcal{H}$ given by

$$(\mathcal{L} f)(z) = \frac{f(z) - (P_{N(\mathcal{M}_z)} f)(z)}{z}, \quad f \in \mathcal{H}.$$

Following [25], the reproducing kernel for \mathcal{H} is an $\mathbf{B}(E)$ -valued function of two variables $\kappa_{\mathcal{H}} : \Omega \times \Omega \rightarrow \mathbf{B}(E)$ such that

(i) for any $e \in E$ and $\lambda \in \Omega$

$$\kappa_{\mathcal{H}}(\cdot, \lambda) e \in \mathcal{H},$$

(ii) for any $e \in E$, $f \in \mathcal{H}$ and $\lambda \in \Omega$

$$\langle f(\lambda), e \rangle_E = \langle f, \kappa_{\mathcal{H}}(\cdot, \lambda) e \rangle_{\mathcal{H}}.$$

It turns out that if the series (3.2) is convergent in E on $\Omega \subset \mathbb{C}$ for every $x \in \mathcal{H}$, then \mathcal{H} is a reproducing kernel Hilbert space of vector-valued holomorphic functions on Ω (see [22, Theorem 3.8]).

For left-invertible operator $T \in \mathbf{B}(\mathcal{H})$, among all subspaces satisfying condition (3.1) we will distinguish those subspaces E which satisfy the following condition

$$E \perp T^n E \quad \text{and} \quad E \perp T'^{*n} E, \quad n \in \mathbb{Z}_+. \tag{3.3}$$

Observe that every $f \in \mathcal{H}$ can be represented as follows

$$f = \sum_{n=-\infty}^{\infty} \hat{f}(n) z^n,$$

where

$$\hat{f}(n) = \begin{cases} P_E T'^{*n} U^* f & \text{if } n \in \mathbb{N}, \\ P_E T^{-n} U^* f & \text{if } n \in \mathbb{Z} \setminus \mathbb{N}. \end{cases} \tag{3.4}$$

As an immediate consequence of the above representation, we obtain the following lemma.

Lemma 3.1. *Let $\{f_n\}_{n=0}^{\infty} \subset \mathcal{H}$ and $f \in \mathcal{H}$ be such that $\lim_{n \rightarrow \infty} f_n = f$. Then*

$$\lim_{n \rightarrow \infty} \hat{f}_n(k) = \hat{f}(k) \quad \text{for } k \in \mathbb{Z}.$$

Proof. It follows directly from (3.4). □

We extend the notion of generalized multipliers for left-invertible analytic operators introduced in [9] to the case of left-invertible operators which are not necessarily analytic. If $\widehat{\varphi} \in \mathbf{B}(E)^{\mathbb{Z}}$ and $\widehat{f} \in E^{\mathbb{Z}}$, their Cauchy-type multiplication $\widehat{\varphi} * \widehat{f}$ is defined by

$$(\widehat{\varphi} * \widehat{f})(n) := \sum_{k=-\infty}^{\infty} \widehat{\varphi}(k)\widehat{f}(n - k), \quad n \in \mathbb{Z}, \tag{3.5}$$

provided that the series is convergent for all $n \in \mathbb{Z}$. We define the multiplication operator $M_{\widehat{\varphi}} : \mathcal{H} \supseteq \mathcal{D}(M_{\widehat{\varphi}}) \rightarrow \mathcal{H}$ given by

$$\begin{aligned} \mathcal{D}(M_{\widehat{\varphi}}) &= \{f \in \mathcal{H} : \text{there is } g \in \mathcal{H} \text{ such that } \widehat{\varphi} * \widehat{f} = \widehat{g}\}, \\ M_{\widehat{\varphi}}f &= g \text{ if } \widehat{\varphi} * \widehat{f} = \widehat{g}. \end{aligned}$$

Below, we collect some properties of generalized multipliers.

Lemma 3.2. *Suppose (3.1) holds. Let $\widehat{\varphi} : \mathbb{Z} \rightarrow \mathbf{B}(E)$. Then the following assertions are satisfied:*

(i) *if additionally (3.3) holds then for every $e \in E$ and $n \in \mathbb{Z}$,*

$$(\widehat{\varphi} * (\widehat{Ue}))(n) = \widehat{\varphi}(n)e,$$

(ii) *for every $f \in \mathcal{D}(M_{\widehat{\varphi}})$, $n \in \mathbb{Z}$,*

$$\begin{aligned} \widehat{M_{\widehat{\varphi}}f}(n) &= \sum_{k=-\infty}^{\infty} \widehat{\varphi}(n - k)\widehat{f}(k) = \sum_{k=1}^{\infty} \widehat{\varphi}(n + k)P_E T^k U^* f \\ &\quad + \sum_{k=0}^{\infty} \widehat{\varphi}(n - k)P_E T'^{*k} U^* f, \end{aligned}$$

(iii) *for every $f \in \mathcal{D}(M_{\widehat{\varphi}})$*

$$(M_{\widehat{\varphi}}f)(z) = \sum_{n=-\infty}^{\infty} \left(\sum_{k=1}^{\infty} \widehat{\varphi}(n + k)P_E T^k U^* f + \sum_{k=0}^{\infty} \widehat{\varphi}(n - k)P_E T'^{*k} U^* f \right) z^n.$$

Proof. (i) Fix $e \in E$. Combining (3.3) with (3.4) and (3.5), we get

$$\begin{aligned} (\widehat{\varphi} * (\widehat{Ue}))(n) &= \sum_{k=1}^{\infty} \widehat{\varphi}(n + k)P_E T^k e + \sum_{k=0}^{\infty} \widehat{\varphi}(n - k)P_E T'^{*k} e \\ &= \widehat{\varphi}(n)e, \quad n \in \mathbb{Z}. \end{aligned}$$

(ii) It follows from (3.2), (3.4) and (3.5).

(iii) is a direct consequence of (ii).

We call $\widehat{\varphi}$ a *generalized multiplier* of T and $M_{\widehat{\varphi}}$ a *generalized multiplication operator* if $M_{\widehat{\varphi}} \in \mathbf{B}(\mathcal{H})$. The set of all generalized multipliers of the operator T is denoted by $\mathcal{GM}(T)$. One can easily verify that the set $\mathcal{GM}(T)$ is a linear subspace of $\mathbf{B}(E)^{\mathbb{Z}}$. Consider the map $V : \mathcal{GM}(T) \ni \widehat{\varphi} \rightarrow M_{\widehat{\varphi}} \in \mathbf{B}(\mathcal{H})$. By Lemma 3.2, the kernel of V is trivial. In particular, we may equip the space $\mathcal{GM}(T)$ with the norm $\|\cdot\| : \mathcal{GM}(T) \rightarrow [0, \infty)$ induced from $\mathbf{B}(\mathcal{H})$, so that V is isometry:

$$\|\widehat{\varphi}\| := \|M_{\widehat{\varphi}}\|, \quad \widehat{\varphi} \in \mathcal{GM}(T).$$

For an operator $A \in \mathbf{B}(\mathcal{H})$ let $\widehat{\varphi}_A : \mathbb{Z} \rightarrow \mathbf{B}(E)$ be a function defined by

$$\widehat{\varphi}_A(m) = \begin{cases} P_E T'^{*m} A|_E & \text{if } m \in \mathbb{N}, \\ P_E T^{-m} A|_E & \text{if } m \in \mathbb{Z} \setminus \mathbb{N}. \end{cases}$$

Since the operator T is unitary equivalent to the operator \mathcal{M}_z of multiplication by z it is obvious that if $f \in \mathcal{H}$ then also $z^n f \in \mathcal{H}$, $n \in \mathbb{N}$. In contrast to this, the question whether $\frac{1}{z^n} f \in \mathcal{H}$ for $n \in \mathbb{N}$ provided that $f \in \mathcal{H}$ is much more delicate. The following result provides a complete answer to this problem. In particular, it gives a characterization of the range of the n -th power of \mathcal{M}_z , $n \in \mathbb{N}$.

Theorem 3.3. *Suppose (3.1) holds. Let $f \in \mathcal{H}$. Then $\frac{1}{z^n} f \in \mathcal{H}$ if and only if $f \in \mathcal{R}(\mathcal{M}_z^n)$, $n \in \mathbb{N}$.*

Proof. Fix $n \in \mathbb{N}$. Suppose $\frac{1}{z^n} f \in \mathcal{H}$ then there exist $g \in \mathcal{H}$ such that $\frac{1}{z^n} f = g$ hence $f = z^n g$. This implies that $f \in \mathcal{R}(\mathcal{M}_z^n)$.

On the other hand if $f \in \mathcal{R}(\mathcal{M}_z^n)$, then there exist $g \in \mathcal{H}$ such that $z^n g = f$. Hence, $\frac{1}{z^n} f \in \mathcal{H}$. □

It turns out that some special generalized multiplication operators can be described with the help of operators \mathcal{M}_z and \mathcal{L} .

Theorem 3.4. *Suppose (3.1) holds. The following assertions are satisfied:*

- (i) for $n \in \mathbb{N}$ the sequence $\chi_{\{n\}} I_E$ is a generalized multiplier and $M_{\chi_{\{n\}} I_E} = \mathcal{M}_z^n$,
- (ii) if $n \in \mathbb{Z} \setminus \mathbb{N}$, then $\mathcal{D}(M_{\chi_{\{n\}} I_E}) = \mathcal{R}(\mathcal{M}_z^n)$ and

$$M_{\chi_{\{n\}} I_E} f = \mathcal{L}^{-n} f, \quad f \in \mathcal{D}(M_{\chi_{\{n\}} I_E}),$$

- (iii) \mathcal{M}_z commutes with $M_{\widehat{\varphi}}$, for $\widehat{\varphi} \in \mathcal{GM}(T)$.

Proof. (i) Fix $f \in \mathcal{H}$ and $n \in \mathbb{N}$. Set $g = \mathcal{M}_z^n f$ and $\widehat{\varphi} = \chi_{\{n\}} I_E$. Then $\widehat{\varphi} * \hat{f} = \hat{g}$. Hence, $M_{\chi_{\{n\}} I_E} f = \mathcal{M}_z^n f$, $f \in \mathcal{H}$.

- (ii) Fix $f \in \mathcal{H}$ and $n \in \mathbb{Z} \setminus \mathbb{N}$. Set $\widehat{\varphi} = \chi_{\{n\}} I_E$. One can observe that $\widehat{\varphi} * \hat{f} = \widehat{z^n f}$. This combined with Theorem 3.3 gives $\mathcal{D}(M_{\chi_{\{n\}} I_E}) = \mathcal{R}(\mathcal{M}_z^n)$ and

$$M_{\chi_{\{n\}} I_E} f = \mathcal{L}^{-n} f, \quad f \in \mathcal{D}(M_{\chi_{\{n\}} I_E}).$$

- (iii) Since the operator T is unitary equivalent to the operator \mathcal{M}_z of multiplication by z on \mathcal{H} , we infer from Lemma 3.2

$$\begin{aligned} & (M_{\widehat{\varphi}} \mathcal{M}_z f)(z) \\ &= \sum_{n=-\infty}^{\infty} \left(\sum_{k=1}^{\infty} \widehat{\varphi}(n+k) P_E T^k U^* \mathcal{M}_z f + \sum_{k=0}^{\infty} \widehat{\varphi}(n-k) P_E T'^{*k} U^* \mathcal{M}_z f \right) z^n \\ &= \sum_{n=-\infty}^{\infty} \left(\sum_{k=0}^{\infty} \widehat{\varphi}(n+k) P_E T^{k+1} U^* f + \sum_{k=1}^{\infty} \widehat{\varphi}(n-k) P_E T'^{*k-1} U^* f \right) z^n \\ &= \sum_{n=-\infty}^{\infty} \left(\sum_{k=1}^{\infty} \widehat{\varphi}(n-1+k) P_E T^k U^* f + \sum_{k=0}^{\infty} \widehat{\varphi}(n-1+k) P_E T'^{*k} U^* f \right) z^n \end{aligned}$$

$$\begin{aligned}
 &= \mathcal{M}_z \sum_{n=-\infty}^{\infty} \left(\sum_{k=1}^{\infty} \widehat{\varphi}(n+k) P_E T^k U^* f + \sum_{k=0}^{\infty} \widehat{\varphi}(n-k) P_E T'^{*k} U^* f \right) z^n \\
 &= (\mathcal{M}_z M_{\widehat{\varphi}} f)(z) \\
 &\text{for } f \in \mathcal{H}. \quad \square
 \end{aligned}$$

In the next theorem, we show that $\mathcal{GM}(T)$ consists of coefficients of analytic functions.

Theorem 3.5. *Suppose (3.1), (3.3) hold and the series (3.2) is convergent in E on an annulus $\mathbb{A}(r^-, r^+)$ with $r^- < r^+$ and $r^-, r^+ \in [0, \infty)$ for every $x \in \mathcal{H}$. Let $\widehat{\varphi} \in \mathcal{GM}(T)$. Then the following assertions hold:*

- (i) *the series $\sum_{n=-\infty}^{\infty} \widehat{\varphi}(n)\lambda^n$ converges absolutely and uniformly in the norm of $\mathbf{B}(E)$ on any compact set contained in $\mathbb{A}(r^-, r^+)$,*
- (ii) *for every $f \in \mathcal{H}$*

$$(M_{\widehat{\varphi}} f)(\lambda) = \left(\sum_{n=-\infty}^{\infty} \widehat{\varphi}(n)\lambda^n \right) f(\lambda), \quad \lambda \in \mathbb{A}(r^-, r^+),$$

- (iii) *$\widehat{\varphi} * \widehat{\psi} \in \mathcal{GM}(T)$ for every $\widehat{\varphi}, \widehat{\psi} \in \mathcal{GM}(T)$ and*

$$M_{\widehat{\varphi}} M_{\widehat{\psi}} = M_{\widehat{\varphi} * \widehat{\psi}}.$$

Proof. (i) Fix $r < r^+$. We infer from assertion (i) of Lemma 3.2 that

$$M_{\widehat{\varphi}} U e = \sum_{n=-\infty}^{\infty} \widehat{\varphi}(n) e z^n \in \mathcal{H}, \quad e \in E.$$

It follows from our assumptions on the series in (3.2) that

$$\sum_{n=0}^{\infty} \widehat{\varphi}(n) e r^n,$$

converges for every $e \in E$. Thus, there exists a constant $C(r, e) > 0$ such that

$$\|\widehat{\varphi}(n) e r^n\| < C(r, e), \quad n \in \mathbb{N}.$$

By uniform boundedness principle (see [23, Theorem 2.6]) we obtain that there exists a constant $M(r) > 0$ such that

$$\|\widehat{\varphi}(n) r^n\| < M(r), \quad n \in \mathbb{N}.$$

If $|\lambda| < r$, then applying the above, we see that

$$\begin{aligned}
 \left\| \sum_{n=0}^{\infty} \widehat{\varphi}(n)\lambda^n \right\| &\leq \sum_{n=0}^{\infty} \|\widehat{\varphi}(n)\lambda^n\| \leq \sum_{n=0}^{\infty} \|\widehat{\varphi}(n)r^n\| \left[\frac{|\lambda|}{r} \right]^n \\
 &\leq M(r) \sum_{n=0}^{\infty} \left[\frac{|\lambda|}{r} \right]^n.
 \end{aligned}$$

This proves our claim. Following steps analogous to those above, we obtain that

$$\sum_{n=1}^{\infty} \widehat{\varphi}(-n) \frac{1}{\lambda^n}$$

also converges absolutely and uniformly in the norm of $\mathbf{B}(E)$ on any compact set contained in $\mathbb{A}(r^-, r^+)$.

(ii) Observe that the series $\sum_{n=0}^{\infty} \widehat{\varphi}(n)\lambda^n$ and $\sum_{n=-\infty}^{\infty} \widehat{f}(n)\lambda^n$ are convergent absolutely for every $\lambda \in \mathbb{A}(r^-, r^+)$, $e \in E$ (the first one by assertion (i), the second one by assertion (i) of [22, Theorem 3.8]). This implies that the Cauchy product of these series is convergent absolutely for every $\lambda \in \mathbb{A}(r^-, r^+)$, $e \in E$, when combined with assertion (iii) of Lemma 3.2, yields

$$(M_{\widehat{\varphi}}f)(\lambda) = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \widehat{\varphi}(k)\widehat{f}(n-k)\lambda^n = \left(\sum_{n=-\infty}^{\infty} \widehat{\varphi}(n)\lambda^n \right) f(\lambda)$$

for $\lambda \in \mathbb{A}(r^-, r^+)$.

(iii) By assertion (i) series $\sum_{n=-\infty}^{\infty} \widehat{\varphi}(n)\lambda^n$ and $\sum_{n=-\infty}^{\infty} \widehat{\psi}(n)\lambda^n$ converges absolutely and uniformly in the norm of $\mathbf{B}(E)$ on any compact set contained in $\mathbb{A}(r^-, r^+)$. Thus the same is true for Cauchy products of these series and

$$\left(\sum_{n=-\infty}^{\infty} \widehat{\varphi}(n)\lambda^n \right) \left(\sum_{n=-\infty}^{\infty} \widehat{\psi}(n)\lambda^n \right) = \sum_{n=-\infty}^{\infty} (\widehat{\varphi} * \widehat{\psi})(n)\lambda^n.$$

This combined with assertion (iii) of Lemma 3.2 gives

$$\begin{aligned} (M_{\widehat{\varphi}}M_{\widehat{\psi}}f)(\lambda) &= \left(\sum_{n=-\infty}^{\infty} \widehat{\varphi}(n)\lambda^n \right) ((M_{\widehat{\psi}}f)(\lambda)) \\ &= \left(\sum_{n=-\infty}^{\infty} \widehat{\varphi}(n)\lambda^n \right) \left(\sum_{n=-\infty}^{\infty} \widehat{\psi}(n)\lambda^n \right) f(\lambda) \\ &= \left(\sum_{n=-\infty}^{\infty} (\widehat{\varphi} * \widehat{\psi})(n)\lambda^n \right) f(\lambda) = (M_{\widehat{\varphi} * \widehat{\psi}}f)(\lambda). \end{aligned}$$

This implies that $(\widehat{\varphi} * \widehat{\psi}) * \widehat{f} = \widehat{M_{\widehat{\varphi}}M_{\widehat{\psi}}f}$ for every $f \in \mathcal{H}$. Hence $\mathcal{D}(M_{\widehat{\varphi} * \widehat{\psi}}) = \mathcal{H}$ and $M_{\widehat{\varphi}}M_{\widehat{\psi}} = M_{\widehat{\varphi} * \widehat{\psi}}$. This in turn implies that $\widehat{\varphi} * \widehat{\psi} \in \mathcal{GM}(T)$, because $M_{\widehat{\varphi}}, M_{\widehat{\psi}} \in \mathbf{B}(\mathcal{H})$.

In [9, Theorem 4] P. Dymek, A. Płaneta and M. Ptak used Shimorin’s analytic model to characterize the commutant of left-invertible analytic operators in terms of generalized multipliers. A close inspection of their proof reveals that the equality $I - TT^* = P_{\mathcal{N}(T^*)}$ play a pivotal role in the proof. Since in the model for a left-invertible operator considered in [22] (see (3.2)), which is not analytic the subspace $E \neq \mathcal{N}(T^*)$ a new approach is needed. Our proof is completely different from the proof given by the above mentioned authors. The next theorem provides a characterization of the commutant of left-invertible operator, which is not necessarily analytic in terms of generalized multipliers.

Theorem 3.6. *Let $T \in \mathbf{B}(\mathcal{H})$ be left-invertible and $E \subset \mathcal{H}$ be a closed subspace such that,*

- (i) $[E]_{T^*, T'} = \mathcal{H}$ and $[E]_{T'^*, T} = \mathcal{H}$,
(ii)
$$T^n T'^{*n} E \subset E, \quad n \in \mathbb{N}, \quad (3.6)$$

(iii) $E \perp T^n E$ and $E \perp T'^n E$, $n \in \mathbb{Z}_+$,
(iv) the series (3.2) is convergent in E on an annulus $\mathbb{A}(r^-, r^+)$ with $r^- < r^+$ and $r^-, r^+ \in [0, \infty)$ for every $x \in \mathcal{H}$.

If $A \in \mathcal{B}(\mathcal{H})$ commutes with T , then $\widehat{\varphi}_A \in \mathcal{GM}(T)$ and $A = U^* M_{\widehat{\varphi}_A} U$.

Proof. Let $\mathcal{A} = UAU^*$. All we need to prove is the following equality

$$(\widehat{\varphi}_A * \widehat{f})(n) = \widehat{\mathcal{A}f}(n), \quad f \in \mathcal{H}, \quad n \in \mathbb{Z}. \quad (3.7)$$

Fix $n \in \mathbb{Z}$. Consider first the case when $f = UT^m e$, $e \in E$, $m \in \mathbb{N}$. By (iii)

$$\widehat{\varphi}_A(n+k) P_E T^k U^* f = \widehat{\varphi}_A(n+k) P_E T^{k+m} e = \chi_{\{0\}}(k+m) \widehat{\varphi}_A(n) e,$$

for $k \in \mathbb{N}$. Arguing as above, we see that

$$\widehat{\varphi}_A(n-k) P_E T'^{*k} U^* f = \begin{cases} \widehat{\varphi}_A(n-k) P_E T'^{*k-m} e = 0 & \text{if } k > m, \\ \widehat{\varphi}_A(n-k) P_E T'^{m-k} e = 0 & \text{if } k < m, \\ \widehat{\varphi}_A(n-m) e & \text{if } k = m. \end{cases}$$

Since A and T commute, we have

$$\widehat{\varphi}_A(n-m) e = \begin{cases} P_E T'^{*n-m} A e = P_E T'^{*n} A T^m e & \text{if } n \geq m, \\ P_E T^{m-n} A e = P_E T'^{*n} A T^m e & \text{if } n < m \text{ and } n \geq 0, \\ P_E T^{m-n} A e = P_E T^{-n} A T^m e & \text{if } n < 0. \end{cases}$$

This altogether implies that

$$\begin{aligned} (\widehat{\varphi}_A * \widehat{f})(n) &= \sum_{k=1}^{\infty} \widehat{\varphi}_A(n+k) P_E T^k U^* f + \sum_{k=0}^{\infty} \widehat{\varphi}_A(n-k) P_E T'^{*k} U^* f \\ &= \chi_{\mathbb{N}}(n) P_E T'^{*n} A T^m e + \chi_{\mathbb{Z} \setminus \mathbb{N}}(n) P_E T^{-n} A T^m e \\ &= \widehat{\mathcal{A}f}(n), \end{aligned}$$

where $f = UT^m e$ for $e \in E$, $m \in \mathbb{N}$ and $n \in \mathbb{Z}$.

In turn, if $f = UT'^{*m} e$, $m \in \mathbb{Z}_+$, then

$$\widehat{\varphi}_A(n-k) P_E T'^{*k} U^* f = \widehat{\varphi}_A(n-k) P_E T'^{*k+m} e = 0,$$

$k \in \mathbb{N}$. It follows from (3.3) and inclusion (3.6) that

$$\widehat{\varphi}_A(n+k) P_E T^k U^* f = \begin{cases} \widehat{\varphi}_A(n+k) P_E T^{k-m} (T^m T'^{*m} e) = 0 & \text{if } k > m, \\ \widehat{\varphi}_A(n+k) P_E T'^{*m-k} (T^m T'^{*m} e) = 0 & \text{if } k < m, \\ \widehat{\varphi}_A(n+m) (T^m T'^{*m} e) & \text{if } k = m. \end{cases}$$

Let $g_e = T^m T'^{*m} e$. Then

$$\widehat{\varphi}_A(n+m) g_e = \begin{cases} P_E T'^{*n+m} A g_e = P_E T'^{*n} A T'^{*m} e & \text{if } n+m \geq 0, n \geq 0, \\ P_E T'^{*n+m} A g_e = P_E T^{-n} A T'^{*m} e & \text{if } n+m \geq 0, n < 0, \\ P_E T^{-(n+m)} A g_e = P_E T^{-n} A T'^{*m} e & \text{if } n+m < 0. \end{cases}$$

As a consequence, we have

$$(\widehat{\varphi}_A * \widehat{f})(n) = \sum_{k=1}^{\infty} \widehat{\varphi}_A(n+k) P_E T^k U^* f + \sum_{k=0}^{\infty} \widehat{\varphi}_A(n-k) P_E T'^{*k} U^* f$$

$$\begin{aligned} &= \chi_{\mathbb{N}}(n)P_E T'^{*n} A T'^{*m} e + \chi_{\mathbb{Z} \setminus \mathbb{N}}(n)P_E T^{-n} A T'^{*m} e \\ &= \widehat{\mathcal{A}}f(n), \end{aligned}$$

where $f = UT'^{*m}e$, for $e \in E$, $m \in \mathbb{N}$ and $n \in \mathbb{Z}$. We extend the previous equality by linearity to the following space

$$\text{lin}(\{UT^n x : x \in E, n \in \mathbb{N}\} \cup \{UT'^{*n} x : x \in E, n \in \mathbb{N}\}). \tag{3.8}$$

We now show that the operator $\mathcal{H} \ni f \rightarrow (\widehat{\varphi}_A * \widehat{f})(n) \in E, n \in \mathbb{Z}$ is bounded. Indeed, it follows from our assumptions on the series in (3.2) that series

$$\sum_{n=0}^{\infty} (P_E T'^{*n} x) \lambda^n,$$

converges for every $x \in \mathcal{H}$ on annulus $\mathbb{A}(r^-, r^+)$. Fix $\lambda \in \mathbb{A}(r^-, r^+)$. There exists a constant $C(x) > 0, x \in \mathcal{H}$, such that

$$\|(P_E T^n x) \frac{1}{\lambda^n}\| < C(x), \quad n \in \mathbb{N}.$$

By uniform boundedness principle (see [23, Theorem 2.6]) we obtain that there exists a constant $M > 0$ such that

$$\|(P_E T^n) \frac{1}{\lambda^n}\| < M, \quad n \in \mathbb{N}.$$

We show that series

$$\sum_{n=0}^{\infty} (P_E T^{n+p} A P_E T'^{*n+q}), \quad p, q \in \mathbb{N} \tag{3.9}$$

converges absolutely in the norm of $\mathbf{B}(\mathcal{H}, E)$. We infer from assertion (i) [22, Theorem 3.8] that the series $\sum_{n=0}^{\infty} (P_E T'^{*n}) \lambda^n$ converges absolutely in E . As a consequence, we have

$$\begin{aligned} \sum_{n=0}^k \|P_E T^{n+p} A P_E T'^{*n+q}\| &\leq |\lambda|^{p-q} \sum_{n=0}^k \|(P_E T^{n+p}) \frac{1}{\lambda^{n+p}} A (P_E T'^{*n+q}) \lambda^{n+q}\| \\ &\leq |\lambda|^{p-q} \sum_{n=0}^k \|(P_E T^{n+p}) \frac{1}{\lambda^{n+p}}\| \|A\| \|(P_E T'^{*n+q}) \lambda^{n+q}\| \\ &\leq |\lambda|^{p-q} M \|A\| \sum_{n=0}^{\infty} \|(P_E T'^{*n+q}) \lambda^{n+q}\| < \infty. \end{aligned}$$

By the same kind of reasoning one can show that series

$$\sum_{n=0}^{\infty} (P_E T'^{*n+p} A P_E T^{n+q}), \quad p, q \in \mathbb{N} \tag{3.10}$$

also converges absolutely in the norm of $\mathbf{B}(\mathcal{H}, E)$. Note that

$$\begin{aligned}
 (\widehat{\varphi}_A * \widehat{f})(n) &= \sum_{k=-\infty}^{\infty} \widehat{\varphi}_A(n-k)\widehat{f}(k) = \left[\sum_{k=1}^{\infty} P_E T^{*!n-k} A P_E T^k U^* \right. \\
 &\quad \left. + \sum_{k=0}^{n-1} P_E T^{*!n-k} A P_E T^{*!k} U^* + \sum_{k=n}^{\infty} P_E T^{k-n} A P_E T^{*!k} U^* \right] f
 \end{aligned}
 \tag{3.11}$$

for $f \in \mathcal{H}$, $n \in \mathbb{N}$ and

$$\begin{aligned}
 (\widehat{\varphi}_A * \widehat{f})(n) &= \sum_{k=-\infty}^{\infty} \widehat{\varphi}_A(n-k)\widehat{f}(k) = \left[\sum_{k=n}^{\infty} P_E T^{*!n-k} A P_E T^k U^* \right. \\
 &\quad \left. + \sum_{k=0}^{n-1} P_E T^{k-n} A P_E T^k U^* + \sum_{k=1}^{\infty} P_E T^{k-n} A P_E T^{*!k} U^* \right] f
 \end{aligned}
 \tag{3.12}$$

for $f \in \mathcal{H}$, $n \in \mathbb{Z} \setminus \mathbb{N}$. Combining (3.11) and (3.12) with the fact that series (3.9) and (3.10) converge absolutely in the norm of $\mathbf{B}(\mathcal{H}, E)$, one can convince himself that the operator $\mathcal{H} \ni f \rightarrow (\widehat{\varphi}_A * \widehat{f})(n) \in E$, $n \in \mathbb{Z}$ is bounded. This combined with the fact that equality (3.7) holds on a dense subspace (3.8) completes the proof. \square

Following [25, Definition 2.4], we say that $T \in \mathbf{B}(\mathcal{H})$ possesses the wandering subspace property, if

$$\mathcal{N}(T^*)|_T = \bigvee \{T^n \mathcal{N}(T^*) : n \in \mathbb{N}\} = \mathcal{H}.$$

It turns out that for a left-invertible operator T , T is analytic if and only if the Cauchy dual T' of T possesses wandering subspace property (see [25, Proposition 2.7]). It is interesting to observe that the class of left-invertible analytic operators with the wandering subspace property and the class of weighted shift on leafless (both rooted and rootless) directed trees satisfy the assumptions of the previous theorem.

Example 3.7. Let $T \in \mathbf{B}(\mathcal{H})$ be a left-invertible analytic operator with the wandering subspace property and $E := \mathcal{N}(T^*)$. By [25, Proposition 2.7], $\mathcal{H}_{\infty}^{\perp} = [E]_{T'}$. Since T is analytic, $\mathcal{H}_{\infty} = \{0\}$. Thus $[E]_{T'^*, T} \supset [E]_T = \mathcal{H}$ and $[E]_{T^*, T'} \supset [E]_{T'} = \mathcal{H}$, which yields $[E]_{T^*, T'} = \mathcal{H}$ and $[E]_{T'^*, T} = \mathcal{H}$. Since $E = \mathcal{N}(T^*)$, one can show that (3.3) holds and $T^n T'^{*n} E = \{0\} \subset E$.

It turns out that any weighted shift on a rooted directed tree is analytic (see [6, Lemma 3.3]). Thus, it remains to consider weighted shift on a rootless directed tree.

Example 3.8. Let \mathcal{T} be a rootless and leafless directed tree and $\lambda = \{\lambda_v\}_{v \in V}$ be a system of weights. Let S_{λ} be the weighted shift on \mathcal{T} and $E := \langle e_{\omega} \rangle \oplus \mathcal{N}(S_{\lambda}^*)$. Assume that $S_{\lambda} \in \mathbf{B}(\mathcal{H})$ and formal Laurent series (3.2) converges absolutely in E on Ω such that $\text{int } \Omega \neq \emptyset$. By [22, Lemma 4.2] and [14, Lemma 4.3.1] we have $[E]_{T^*, T'} = \mathcal{H}$ and $[E]_{T'^*, T} = \mathcal{H}$, where $T = S_{\lambda}$. It is a matter of routine to verify that $T^n T'^{*n} E \subset \langle e_{\omega} \rangle \subset E$.

4. Weighted Shifts on Rootless Directed Trees

The implementation of methods of graph theory into operator theory gives rise to a new class of operators known as weighted shifts on directed trees. This class was introduced in [13] and intensively studied since then (see e.g., [1–4, 6, 7, 10, 14, 18]). Z.J. Jabłoński, I. B. Jung and J. Stochel realized the importance of this class as a vehicle to collect a number of interesting examples and counterexamples (see e.g., [5, 9, 14–17, 20, 21, 26]).

In this section as an application of generalized multipliers for left-invertible operators which are not necessarily analytic, we show a way to extend the notion of multipliers for weighted shifts on rooted directed trees which was introduced in [3] to the setting of weighted shifts on rootless directed trees.

To avoid the repetition, we state the following assumption which will be used frequently in this section.

Let \mathcal{T} be a rootless and leafless directed tree with finite branching index and $\lambda = \{\lambda_v\}_{v \in V}$ be a system of weights. Let $S_\lambda \in \mathbf{B}(\ell^2(V))$ be a left-invertible weighted shift on \mathcal{T} , $E := \langle e_\omega \rangle \oplus \mathcal{N}(S_\lambda^*)$ and $\tilde{E} := \langle e_\omega \rangle$.

(4.1)

Since the analytic structure of weighted shifts on rootless directed trees plays a major role in this section, we outline it in the following discussion. Suppose (4.1) holds and the series (3.2) with S_λ in place of T is convergent in E on an annulus $\mathbb{A}(r^-, r^+)$ with $r^- < r^+$ and $r^-, r^+ \in [0, \infty)$ for every $x \in \mathcal{H}$. In [22, equations (4.7) and (4.8)] the inner and outer radius of convergence for weighted shifts on rootless directed trees was described only in terms of its weights. In this case (see [22, Theorem 4.3]), there exist a z -invariant reproducing kernel Hilbert space \mathcal{H} of E -valued holomorphic functions defined on the annulus $\mathbb{A}(r^-, r^+)$ and a unitary mapping $U : \ell^2(V) \rightarrow \mathcal{H}$ such that $\mathcal{M}_z U = U S_\lambda$, where \mathcal{M}_z denotes the operator of multiplication by z on \mathcal{H} . Moreover, the linear subspace generated by E -valued polynomials in z and \tilde{E} -valued polynomials involving only negative powers of z is dense in \mathcal{H} , that is

$$\bigvee (\{z^n E : n \in \mathbb{N}\} \cup \{\frac{1}{z^n} \tilde{E} : n \in \mathbb{Z}_+\}) = \mathcal{H}. \tag{4.2}$$

Looking more closely at the equation (3.2) and using the kernel-range decomposition one can observe that every $f \in \mathcal{H}$ can be represented as follows

$$f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n) z^n,$$

where $\hat{f}(n) \in \tilde{E}$ for $n \in \mathbb{Z} \setminus \mathbb{N}$. One of the major differences between the case of weighted shift on rootless directed tree and weighted shift on rooted directed tree is that ranges of coefficients of generalized multipliers with negative indexes may be nonzero and are always contained in a smaller subspace \tilde{E} of E .

Recently, the analytic structure of weighted shifts on directed trees was studied by several authors (see [1, 3, 6–8, 22]).

Lemma 4.1. *Suppose (4.1) holds. Let $\hat{\varphi} \in \mathcal{GM}(S_\lambda)$. Then $\mathcal{R}(\hat{\varphi}(n)) \subset \tilde{E}$ for $n \in \mathbb{Z} \setminus \mathbb{N}$.*

Proof. Fix $e \in E$ and $n \in \mathbb{Z} \setminus \mathbb{N}$. Let $f(z) := e$ be a constant polynomial. Then there exist $g \in \mathcal{H}$ such that $\hat{\varphi} * \hat{f} = \hat{g}$. Combining Lemmas 3.2 and [22, Lemma 4.2], we get $\hat{g}(n) = \hat{\varphi}(n)e$. It is plain that $\hat{g}(n) \in \tilde{E}$ and consequently that $\mathcal{R}(\hat{\varphi}(n)) \subset \tilde{E}$. □

Now, we describe the rational functions which correspond to the elements of the form $S_\lambda^n e_1$ and $S_{\lambda'}^{*n} e_2$ for $e_1 \in E$ and $e_2 \in \tilde{E}$. As we recalled above elements of this form span \mathcal{H} and therefore the following lemma turns out to be useful. It will be used several times in this paper.

Lemma 4.2. *Suppose (4.1) holds. Then for every $n \in \mathbb{N}$, $e_1 \in E$, and $e_2 \in \tilde{E}$ we have*

$$US_\lambda^n e_1 = e_1 z^n \quad \text{and} \quad US_{\lambda'}^{*n} e_2 = e_2 \frac{1}{z^n}.$$

Proof. Let $f, g \in \mathcal{H}$ be such that $US_\lambda^n e_1 = f$ and $US_{\lambda'}^{*n} e_2 = g$. Applying (3.4) to vector $S_\lambda^n e_1$, we see that

$$\hat{f}(m) = \begin{cases} P_E S_{\lambda'}^{*m-n} e_1 & \text{if } m \geq n \text{ and } m \geq 0, \\ P_E S_\lambda^{n-m} e_1 & \text{if } m < n \text{ and } m \geq 0, \\ P_E S_\lambda^{-m+n} e_1 & \text{if } m < 0. \end{cases}$$

By (3.3) we obtain that $f(z) = e_1 z^n$.

Now, applying (3.4) to vector $S_{\lambda'}^{*n} e_2$, we get

$$\hat{g}(m) = \begin{cases} P_E S_{\lambda'}^{*n+m} e_2 & m \geq 0, \\ P_E S_\lambda^{-m-n} S_\lambda^n S_{\lambda'}^{*n} e_2 & \text{if } -m > n \text{ and } m < 0, \\ P_E S_{\lambda'}^{*n-m} S_\lambda^n S_{\lambda'}^{*n} e_2 & \text{if } -m \leq n \text{ and } m < 0. \end{cases}$$

By (3.3) again and the fact that (see Example 3.8)

$$S_\lambda^n S_{\lambda'}^{*n} e = e, \quad n \in \mathbb{N}, e \in \tilde{E},$$

we deduce that $g(z) = e_2 \frac{1}{z^n}$.

Now we introduce some operator which will play an essential role in this section. We define the operator \mathcal{R} in \mathcal{H} by

$$\mathcal{D}(\mathcal{R}) := \left\{ \sum_{n=-\infty}^{\infty} a_n z^n \in \mathcal{H} : \sum_{n=-\infty}^{\infty} P_{\tilde{E}} a_n z^n \in \mathcal{H} \right\}$$

$$\mathcal{R} \left(\sum_{n=-\infty}^{\infty} a_n z^n \right) := \sum_{n=-\infty}^{\infty} P_{\tilde{E}} a_n z^n, \quad \sum_{n=-\infty}^{\infty} a_n z^n \in \mathcal{D}(\mathcal{R}).$$

It is obvious that this operator is an idempotent ($\mathcal{R}^2 = \mathcal{R}$). However, it may happen that \mathcal{R} is not an orthogonal projection (see Example 4.5). It turns out that \mathcal{R} is closed.

Lemma 4.3. *Operator \mathcal{R} is closed.*

Proof. Let $\{f_n\}_{n=0}^\infty \subset \mathcal{D}(\mathcal{R})$ and $f, g \in \mathcal{H}$ be such that $\lim_{n \rightarrow \infty} f_n = f$ and $\lim_{n \rightarrow \infty} \mathcal{R}f_n = g$. By Lemma 3.1 $\lim_{n \rightarrow \infty} \hat{f}_n(k) = \hat{f}(k)$ and $\lim_{n \rightarrow \infty} \mathcal{R}f_n(k) = \lim_{n \rightarrow \infty} P_{\hat{E}} \hat{f}_n(k) = \hat{g}(k)$ for $k \in \mathbb{Z}$. This implies that $\lim_{n \rightarrow \infty} P_{\hat{E}} \hat{f}_n(k) = P_{\hat{E}} \hat{f}(k)$. As a consequence $f \in \mathcal{D}(\mathcal{R})$ and $\mathcal{R}f = g$. \square

The next theorem provides boundedness conditions for operator \mathcal{R} .

Theorem 4.4. *Suppose that (4.1) holds and the sequence $\{S_\lambda^n S_{\lambda'}^{*n}\}_{n=1}^\infty$ of operators is uniformly bounded. Then the following assertions hold:*

- (i) *the sequence $\{S_\lambda^n S_{\lambda'}^{*n}\}_{n=1}^\infty$ is convergent in the strong operator topology,*
- (ii) *\mathcal{R} is bounded and $\mathcal{R} = (\text{SOT}) \lim_{n \rightarrow \infty} \mathcal{M}_z^n \mathcal{L}^n,$*
- (iii)

$$\text{if } \sum_{n=-\infty}^\infty a_n z^n \in \mathcal{H} \text{ then also } \sum_{n=-\infty}^\infty P_{\hat{E}} a_n z^n \in \mathcal{H},$$

- (iv) *operator \mathcal{R} commutes with \mathcal{M}_z and \mathcal{L} .*

Proof. (i) We begin by showing that $\{S_\lambda^n S_{\lambda'}^{*n} x\}_{n=1}^\infty$ is a Cauchy sequence for every $x \in X$, where $X := \{S_\lambda^n E : n \in \mathbb{N}\} \cup \{S_{\lambda'}^{*n} E : n \in \mathbb{N}\}$. Indeed, if $e \in E$ and $m \in \mathbb{N}$, then

$$\begin{aligned} \lim_{n \rightarrow \infty} S_\lambda^n S_{\lambda'}^{*n} (S_\lambda^m e) &= \lim_{n \rightarrow \infty} S_\lambda^m (S_\lambda^{n-m} S_{\lambda'}^{*n-m} e) = \lim_{n \rightarrow \infty} S_\lambda^m (S_\lambda^{n-m} S_{\lambda'}^{*n-m} P_{\hat{E}} e) \\ &= \lim_{n \rightarrow \infty} S_\lambda^m P_{\hat{E}} e = S_\lambda^m P_{\hat{E}} e, \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} S_\lambda^n S_{\lambda'}^{*n} (S_{\lambda'}^{*m} e) &= \lim_{n \rightarrow \infty} S_{\lambda'}^{*m} (S_\lambda^{n+m} S_{\lambda'}^{*n+m} e) \\ &= \lim_{n \rightarrow \infty} S_{\lambda'}^{*m} (S_\lambda^{n+m} S_{\lambda'}^{*n+m} P_{\hat{E}} e) = \lim_{n \rightarrow \infty} S_{\lambda'}^{*m} P_{\hat{E}} e \\ &= S_{\lambda'}^{*m} P_{\hat{E}} e. \end{aligned} \tag{4.4}$$

Due to the linearity of the operators $S_\lambda^n S_{\lambda'}^{*n}$, $n \in \mathbb{N}$ also $\{S_\lambda^n S_{\lambda'}^{*n} x\}_{n=1}^\infty$ is a Cauchy sequence for every $x \in \text{lin } X$. Since by Lemma [22, Lemma 4.2] subset X is linearly dense in \mathcal{H} , it follows from [27, Theorem 4.23] that uniformly bounded sequence of operators $\{S_\lambda^n S_{\lambda'}^{*n}\}_{n=1}^\infty$ is convergent in the strong operator topology.

(ii) Fix $x \in \mathcal{H}$. Since subset X is linearly dense in \mathcal{H} there exist $\{x_k\}_{k=1}^\infty \subset \mathcal{H}$ such that

$$\begin{aligned} x &= \lim_{n \rightarrow \infty} x_n, \\ x_k &= \sum_{n=0}^\infty S_\lambda^n a_n^k + \sum_{n=1}^\infty S_{\lambda'}^{*n} b_n^k, \end{aligned} \tag{4.5}$$

and there are only finitely many nonzero terms in these sums, where $a_n^k, b_n^k \in E$ for $k, n \in \mathbb{N}$. Combining Lemma 3.1 and 4.2, we deduce that

$$P_E S_{\lambda'}^{*m} x = \lim_{k \rightarrow \infty} a_m^k$$

and

$$P_E S_\lambda^m x = \lim_{k \rightarrow \infty} b_m^k.$$

Since, by assertion (i) the sequence $\{S_\lambda^n S_{\lambda'}^{*n}\}_{n=1}^\infty$ is convergent in the strong operator topology, there exist operator $A \in \mathbf{B}(\mathcal{H})$ such that

$$A := (\text{SOT}) \lim_{n \rightarrow \infty} S_\lambda^n S_{\lambda'}^{*n}.$$

It follows from (4.3) and (4.4) that

$$\begin{aligned} Ax_k &= A \left(\sum_{n=0}^{\infty} S_\lambda^n a_n^k + \sum_{n=1}^{\infty} S_{\lambda'}^{*n} b_n^k \right) \\ &= \sum_{n=0}^{\infty} S_\lambda^n P_{\bar{E}} a_n^k + \sum_{n=1}^{\infty} S_{\lambda'}^{*n} P_{\bar{E}} b_n^k. \end{aligned}$$

Combining this with Lemma 3.1, we deduce that

$$\begin{aligned} (\widehat{UAx})(m) &= P_E S_{\lambda'}^{*m} Ax = \lim_{k \rightarrow \infty} P_E S_{\lambda'}^{*m} Ax_k \\ &= \lim_{k \rightarrow \infty} P_{\bar{E}} a_m^k = P_{\bar{E}} P_E S_{\lambda'}^{*m} x = P_{\bar{E}} (\widehat{Ux})(m) \end{aligned}$$

and

$$\begin{aligned} (\widehat{UAx})(-m) &= P_E S_\lambda^m Ax = \lim_{k \rightarrow \infty} P_E S_\lambda^m Ax_k \\ &= \lim_{k \rightarrow \infty} P_{\bar{E}} b_m^k = P_{\bar{E}} P_E S_\lambda^m x = P_{\bar{E}} (\widehat{Ux})(m), \end{aligned}$$

for $m \in \mathbb{N}$. Therefore, the assertions (ii) is justified.

(iii) is a direct consequence of (ii).

(iv) As a consequence of (ii), we get

$$\begin{aligned} AS_\lambda &= (\text{SOT}) \lim_{n \rightarrow \infty} S_\lambda^n S_{\lambda'}^{*n} S_\lambda = (\text{SOT}) \lim_{n \rightarrow \infty} S_\lambda S_\lambda^{n-1} S_{\lambda'}^{*n-1} \\ &= S_\lambda (\text{SOT}) \lim_{n \rightarrow \infty} S_\lambda^{n-1} S_{\lambda'}^{*n-1} = S_\lambda A \end{aligned}$$

and

$$\begin{aligned} AS_{\lambda'}^* &= (\text{SOT}) \lim_{n \rightarrow \infty} S_\lambda^n S_{\lambda'}^{*n} S_{\lambda'}^* = (\text{SOT}) \lim_{n \rightarrow \infty} S_{\lambda'}^* S_\lambda^{n+1} S_{\lambda'}^{*n+1} \\ &= S_{\lambda'}^* (\text{SOT}) \lim_{n \rightarrow \infty} S_\lambda^{n+1} S_{\lambda'}^{*n+1} = S_{\lambda'}^* A. \quad \square \end{aligned}$$

The following example illustrate that if the sequence $\{S_\lambda^n S_{\lambda'}^{*n}\}_{n=1}^\infty$ is not convergent in the strong operator topology then operator \mathcal{R} may not be bounded and it may happen that

$$\sum_{n=-\infty}^{\infty} a_n z^n \in \mathcal{H} \quad \text{but} \quad \sum_{n=-\infty}^{\infty} P_{\bar{E}} a_n z^n \notin \mathcal{H}. \quad (4.6)$$

Moreover, idempotent \mathcal{R} in this example is not an orthogonal projection.

Example 4.5. Let $\mathcal{T} = (V, \mathcal{E})$ be the directed tree (see Figure 1) with

$$\begin{aligned} V &:= \mathbb{N} \sqcup \{(i, j) : i \in J_2, j \in \mathbb{N}\}, \\ \mathcal{E} &:= \{(i+1, i) : i \in \mathbb{N}\} \sqcup \{(0, (i, 0)) : i \in J_2\} \\ &\quad \sqcup \{((i, j), (i, j+1)) : i \in J_2, j \in \mathbb{N}\}, \end{aligned}$$

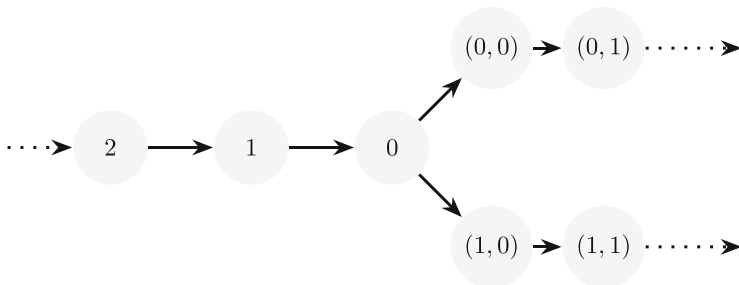


FIGURE 1.

where $J_2 = \{0, 1\}$ and $\lambda = \{\lambda_v\}_{v \in V}$ be the system of positive weights (The symbol “ \sqcup ” denotes disjoint union of sets). It is now easily seen that

$$S_\lambda e_x = \begin{cases} \lambda_{(i,j+1)} e_{(i,j+1)} & \text{for } x = (i, j), i \in \{0, 1\}, j \in \mathbb{N}, \\ \lambda_{i-1} e_{i-1} & \text{for } x = i \text{ and } i \in \mathbb{Z}_+, \\ \lambda_{(0,0)} e_{(0,0)} + \lambda_{(1,0)} e_{(1,0)} & \text{for } x = 0. \end{cases} \tag{4.7}$$

It is a matter of routine to verify that the adjoint of the Cauchy dual $S_{\lambda'}^*$ of S_λ has the following form

$$S_{\lambda'}^* e_x = \begin{cases} \frac{1}{\lambda_{(i,j)}} e_{(i,j-1)} & \text{for } x = (i, j), i \in \{0, 1\}, j \in \mathbb{Z}_+, \\ \frac{1}{\lambda_i} e_{i+1} & \text{for } x = i \text{ and } i \in \mathbb{Z}_+, \\ \frac{\lambda_{(i,0)}}{\lambda_{(1,0)}^2 + \lambda_{(0,0)}^2} e_0 & \text{for } x = (i, 0) \text{ and } i \in \{0, 1\}. \end{cases} \tag{4.8}$$

This implies that

$$S_\lambda^{n+k} S_{\lambda'}^{*n+k} e_{(0,n)} = \frac{\lambda_{(0,0)}}{\lambda_{(1,0)}^2 + \lambda_{(0,0)}^2} (\lambda_{(0,0)} e_{(0,n)} + \prod_{j=1}^n \frac{\lambda_{(1,j)}}{\lambda_{(0,j)}} e_{(1,n)}), \tag{4.9}$$

for $k \in \mathbb{Z}_+$. By (4.7) and (4.8), we have $P_{\tilde{E}} S_\lambda^m e_{(0,n)} = 0$, $m \in \mathbb{N}$ and $P_{\tilde{E}} S_{\lambda'}^{*m} e_{(0,n)} = 0$, $m > n + 1$. Combining this with Lemma 4.2, we see that there exists $a_0, \dots, a_n \in E$ and $a_{n+1} \in \tilde{E}$ such that $e_{(0,n)} = a_0 + S_\lambda a_1 + \dots + S_\lambda^{n+1} a_{n+1}$. Using (4.3), (4.9) and applying Lemma 4.2, we obtain

$$U e_{(0,n)} \in \mathcal{D}(\mathcal{R}) \quad \text{and} \quad \mathcal{R} U e_{(0,n)} = U S_\lambda^{n+1} S_{\lambda'}^{*n+1} e_{(0,n)}, \quad n \in \mathbb{N}.$$

Now, taking

$$\lambda_{(0,j)} = 1, \quad \lambda_{(1,j)} = 2, \quad \lambda_j = 1, \quad j \in \mathbb{N},$$

we see that

$$\|\mathcal{R} U e_{(0,n)}\| \geq \|S_\lambda^{n+1} S_{\lambda'}^{*n+1} e_{(0,n)}\| = \frac{1}{5} \sqrt{1 + 4^{n+1}}.$$

This implies that \mathcal{R} is not bounded. Since by Lemma 4.3 operator \mathcal{R} is closed, we infer from [27, Theorem 5.6] that there exist $x \in \mathcal{H}$ such that $x \notin \mathcal{D}(\mathcal{R})$, which means that there exist series with the property (4.6).

Let $e := \lambda_{(2,1)} e_{(1,1)} - \lambda_{(1,1)} e_{(2,1)}$ and $x := S_\lambda^2 e_0 + S_\lambda e$. It is a matter of routine to verify that $e \in \mathcal{N}(S_\lambda^*)$. Then, we have

$$U(S_\lambda e + S_\lambda^2 e_0) = ez + e_0 z^2 \quad \text{and} \quad \mathcal{R} U x = e_0 z^2.$$

This leads to

$$\langle \mathcal{R}Ux, Ux \rangle_{\mathcal{H}} = \langle e_0z^2, ez + e_0z^2 \rangle_{\mathcal{H}} = \langle S_\lambda^2e_0, S_\lambda^2e_0 + S_\lambda e \rangle_{\mathcal{H}}.$$

On the other hand,

$$\langle \mathcal{R}Ux, \mathcal{R}Ux \rangle_{\mathcal{H}} = \langle e_0z^2, e_0z^2 \rangle_{\mathcal{H}} = \langle S_\lambda^2e_0, S_\lambda^2e_0 \rangle_{\mathcal{H}}.$$

We claim that

$$\langle \mathcal{R}Ux, Ux \rangle_{\mathcal{H}} - \langle \mathcal{R}Ux, \mathcal{R}Ux \rangle_{\mathcal{H}} = \langle S_\lambda^2e_0, S_\lambda e \rangle_{\mathcal{H}} \neq 0$$

and therefore \mathcal{R} is not an orthogonal projection. Indeed,

$$\begin{aligned} \langle S_\lambda^2e_0, S_\lambda e \rangle_{\mathcal{H}} &= \langle \lambda_{(1,1)}\lambda_{(1,2)}e_{(1,2)} + \lambda_{(2,1)}\lambda_{(2,2)}e_{(2,2)}, \\ &\quad \lambda_{(2,1)}\lambda_{(1,2)}e_{(1,2)} - \lambda_{(1,1)}\lambda_{(2,2)}e_{(2,1)} \rangle \\ &= \lambda_{(1,1)}\lambda_{(2,1)}(\lambda_{(1,2)}^2 - \lambda_{(2,2)}^2). \end{aligned}$$

It turns out that the operator $M_{\chi_{\{n\}}P_{\tilde{E}}}$ can be described with the help of the operators \mathcal{R} and \mathcal{L} in the case of weighted shifts on rootless directed trees.

Theorem 4.6. *Suppose (4.1) holds and the sequence $\{S_\lambda^n S_{\lambda'}^{*n}\}_{n=1}^\infty$ of operators is uniformly bounded. Then the following assertions hold:*

(i)

$$\mathcal{L}\left(\sum_{n=-\infty}^\infty \hat{f}(n)z^n\right) = -P_{\mathcal{N}(\mathcal{M}_z^*)}\hat{f}(0)\frac{1}{z} + \sum_{n=-\infty}^\infty \hat{f}(n)z^{n-1}, \quad f \in \mathcal{H},$$

(ii) for every $n \in \mathbb{N}$ the sequence $\chi_{\{-n\}}P_{\tilde{E}}$ is a generalized multiplier and

$$M_{\chi_{\{-n\}}P_{\tilde{E}}} = \mathcal{R}\mathcal{L}^n,$$

(iii) if $\{a_n\}_{n=1}^\infty \subset \tilde{E}$ and

$$\sum_{n=-\infty}^\infty a_n z^n \in \mathcal{H} \quad \text{then also} \quad \sum_{n=-\infty}^\infty a_n z^n \in \bigcap_{n=1}^\infty \mathcal{R}(\mathcal{M}_z^n).$$

Proof. (i) and (ii) Fix $f \in \mathcal{H}$. There exist $x \in \mathcal{H}$ such that $Ux = f := \sum_{n=-\infty}^\infty a_n z^n$. Let $\{x_k\}_{k=1}^\infty \subset \mathcal{H}$ be such that conditions (4.5) holds. Then

$$S_{\lambda'}^* x_k = \sum_{n=1}^\infty S_\lambda^{n-1} a_n^k + S_{\lambda'}^* P_{\tilde{E}} a_0^k + \sum_{n=1}^\infty S_{\lambda'}^{*n+1} b_n^k.$$

Applying Lemma 4.2, we get

$$US_{\lambda'}^* x_k = \sum_{n=1}^\infty a_n^k z^{n-1} + P_{\tilde{E}} a_0^k \frac{1}{z} + \sum_{n=1}^\infty b_n^k \frac{1}{z^{n+1}}.$$

Employing Lemma 3.1 completes the proof of assertion (i). Note that for $m, p \in \mathbb{N}$

$$S_\lambda^m S_{\lambda'}^{*m+p} x_k = \sum_{n=0}^\infty S_\lambda^{m+n} a_{m+n+p}^k + \sum_{n=0}^{m-1} S_\lambda^n S_\lambda^{m-n} S_{\lambda'}^{*(m-n)} P_{\tilde{E}} a_{p+n}^k$$

$$\begin{aligned}
 & + \sum_{n=0}^{p-1} S_{\lambda'}^{*(p-n)} S_{\lambda}^{m+p-n} S_{\lambda'}^{*(m+p-n)} P_{\tilde{E}} a_n^k \\
 & + \sum_{n=1}^{\infty} S_{\lambda}^m S_{\lambda'}^{*(n+m+p)} P_{\tilde{E}} b_n^k.
 \end{aligned}$$

Since $S_{\lambda}^l S_{\lambda'}^{*l} e = e$ for $e \in \tilde{E}$ and $l \in \mathbb{Z}_+$, applying Lemma 4.2 again, we get

$$\begin{aligned}
 U S_{\lambda}^m S_{\lambda'}^{*m+p} x_k & = \sum_{n=0}^{\infty} a_{m+n+p}^k z^{m+n} + \sum_{n=0}^{m-1} P_{\tilde{E}} a_{p+n}^k z^n \\
 & + \sum_{n=0}^{p-1} P_{\tilde{E}} a_n^k \frac{1}{z^{p-n}} + \sum_{n=1}^{\infty} P_{\tilde{E}} b_n^k \frac{1}{z^{p+n}},
 \end{aligned} \tag{4.10}$$

$m, p \in \mathbb{N}$. Since by Lemma 4.4, $U^* \mathcal{R}U = (\text{SOT}) \lim_{n \rightarrow \infty} S_{\lambda}^n S_{\lambda'}^{*n}$, we deduce from (4.10) and Lemma 4.2 assertions (ii).

(iii) Fix $n \in \mathbb{N}$. Let $\{a_n\}_{n=1}^{\infty} \subset \tilde{E}$ be such that $f := \sum_{n=-\infty}^{\infty} a_n z^n \in \mathcal{H}$. It follows from (ii) that $g := \frac{1}{z^n} f = \mathcal{L}^n f \in \mathcal{H}$. Observe that $f = z^n g$, which implies that $f \in \mathcal{R}(S_{\lambda}^n)$. \square

In the case of left-invertible analytic operator T the linear subspace $\mathcal{M}(T)$ of $\mathcal{GM}(T)$ consisting of all generalized multipliers whose all coefficients are scalar multiples of the identity operator is a Banach algebra (see [9, Theorem 2]). It turns out that $\mathcal{M}(S_{\lambda})$ and classical multipliers $\mathcal{M}(\lambda)$ for a weighted shifts on directed tree S_{λ} are in fact the same object (see [9, Proposition 16]). By $\mathcal{RM}(T)$ we denote the linear subspace of $\mathcal{GM}(T)$ consisting of all generalized multipliers whose non-negative coefficients are scalar multiples of the identity operator and negative coefficients are scalar multiples of the operator $P_{\tilde{E}}$. If $\hat{\varphi} \in \mathcal{RM}(T)$, then there exists a sequence $\{a_n\}_{n=-\infty}^{\infty} \subset \mathbb{C}$ such that

$$\hat{\varphi}(n) = \begin{cases} a_n I_E & \text{if } n \in \mathbb{N}, \\ a_n P_{\tilde{E}} & \text{if } n \in \mathbb{Z} \setminus \mathbb{N}. \end{cases} \tag{4.11}$$

It turns out that the operator $M_{\hat{\varphi}}$ with $\hat{\varphi} \in \mathcal{RM}(T)$ can be described with the help of \mathcal{M}_z , \mathcal{L} and \mathcal{R} .

Theorem 4.7. *Suppose (4.1) holds, the sequence $\{S_{\lambda}^n S_{\lambda'}^{*n}\}_{n=1}^{\infty}$ is uniformly bounded and the series (3.2) is convergent in E on an annulus $\mathbb{A}(r^-, r^+)$ with $r^- < r^+$ and $r^-, r^+ \in [0, \infty)$ for every $x \in \mathcal{H}$. Let $\hat{\varphi} \in \mathcal{RM}(S_{\lambda})$. Then the following equality holds*

$$M_{\hat{\varphi}} f = \sum_{k=1}^{\infty} \hat{\varphi}(-k) \mathcal{R} \mathcal{L}^k f + \sum_{k=0}^{\infty} \hat{\varphi}(k) \mathcal{M}_z^k f, \quad f \in \mathcal{H}.$$

Proof. First we note that

$$\sum_{n=-\infty}^{\infty} \hat{f}(n-k) z^n = \mathcal{M}_z^k f, \quad f \in \mathcal{H}, k \in \mathbb{N}. \tag{4.12}$$

Since the sequence of operators $\{S_\lambda^n S_{\lambda'}^{*n}\}_{n=1}^\infty$ is uniformly bounded the assertion (ii) of Theorem 4.6, shows that

$$\sum_{n=-\infty}^\infty P_{\bar{E}} \hat{f}(n+k) z^n = \mathcal{R} \mathcal{L}^k f, \quad f \in \mathcal{H}, k \in \mathbb{N}. \tag{4.13}$$

Combining (4.12) and (4.13) with Theorem 3.5, we deduce that

$$\begin{aligned} (M_{\hat{\varphi}} f)(\lambda) &= \left(\sum_{n=-\infty}^{-1} a_n P_{\bar{E}} \lambda^n + \sum_{n=0}^\infty a_n I_E \lambda^n \right) f(\lambda) \\ &= \sum_{n=-\infty}^{-1} a_n P_{\bar{E}} \lambda^n f(\lambda) + \sum_{n=0}^\infty a_n I_E \lambda^n f(\lambda) \\ &= \sum_{n=1}^\infty a_{-n} (\mathcal{R} \mathcal{L}^n f)(\lambda) + \sum_{n=0}^\infty a_n (\mathcal{M}_z^n f)(\lambda) \end{aligned}$$

for $\lambda \in \mathbb{A}(r^-, r^+)$. □

Let $\hat{\varphi} \in \mathcal{M}(\lambda)$. As shown in [3, Proposition 4.8] if the series $\sum_{k=0}^\infty \hat{\varphi}(k) z^k$ converges for every $z \in \mathbb{D}(r)$, where $r \in (r(S_\lambda), \infty)$ then

$$M_{\hat{\varphi}}^\lambda f = \sum_{k=0}^\infty \hat{\varphi}(k) S_\lambda^k f, \quad f \in \ell^2(V). \tag{4.14}$$

In view of Theorem 4.7 and (4.14) it seems natural to extend the definition of multipliers to the case of weighted shifts on rootless directed trees as follows. Assume \mathcal{T} is a rootless and leafless directed tree and $\lambda = \{\lambda_v\}_{v \in V}$ is a system of weights. Let $S_\lambda \in \mathcal{B}(\ell^2(V))$ be a left-invertible weighted shift on \mathcal{T} and $\hat{\varphi} : \mathbb{Z} \rightarrow \mathbb{C}$. If the sequence $\{S_\lambda^n S_{\lambda'}^{*n}\}_{n=1}^\infty$ is uniformly bounded and the series

$$\sum_{k=1}^\infty \hat{\varphi}(-k) R S_{\lambda'}^{*k} f + \sum_{k=0}^\infty \hat{\varphi}(k) S_\lambda^k f,$$

converges for every $f \in \ell^2(V)$, where $R := (\text{SOT}) \lim_{n \rightarrow \infty} S_\lambda^n S_{\lambda'}^{*n}$, $n \in \mathbb{N}$ then we define multiplication operator by

$$M_{\hat{\varphi}}^\lambda f := \sum_{k=1}^\infty \hat{\varphi}(-k) R S_{\lambda'}^{*k} f + \sum_{k=0}^\infty \hat{\varphi}(k) S_\lambda^k f, \quad f \in \ell^2(V).$$

5. Polynomial Approximation

In this section we prove that the set of multiplication operators associated with a weighted shift on a rootless directed tree lies in the closure of polynomials in z and $\frac{1}{z}$ of the weighted shift in the topologies of strong and weak operator convergence.

If $v \in \bigcup_{n=0}^\infty \text{Chi}^n(\omega)$, then $|v|$ denotes the unique $k \in \mathbb{N}$ such that $\text{par}^k(v) = \omega$ and if $v \in V \setminus \bigcup_{n=0}^\infty \text{Chi}^n(\omega)$, then $|v|$ denotes $-k$ where $k \in \mathbb{N}$ is the unique number such that $\text{par}^k(\omega) = v$. Recall that the k -th generation of vertices is the set $V_k := \{v \in V : |v| = k\}$, for $k \in \mathbb{Z}$. We regard the

Hilbert space $\ell^2(V_k)$ as a closed linear subspace of $\ell^2(V)$ by identifying each $f \in \ell^2(V_k)$ with the function $\tilde{f} \in \ell^2(V)$ which extends f and vanishes on the set $V \setminus V_k$, $k \in \mathbb{Z}$. That is

$$\ell^2(V_k) := \{f \in \ell^2(V) : f(u) = 0 \text{ if } |u| \neq k\}, \quad k \in \mathbb{Z}.$$

By P_k we denote the orthogonal projection from the space $\ell^2(V)$ onto $\ell^2(V_k)$, $k \in \mathbb{Z}$. The following lemma is an immediate consequence of [9, Lemma 9].

Lemma 5.1. *Suppose that \mathcal{T} is a countably infinite rootless and leafless directed tree with finite branching index. Let $S_\lambda \in \mathbf{B}(\ell^2(V))$ be a left-invertible weighted shift. Then there exists an orthonormal basis $\{e'_j\}_{j \in J}$ of $E := \langle \omega \rangle \oplus \mathcal{N}(S_\lambda^*)$ such that for every $j \in J$ vector e'_j belongs to the space $\ell^2(V_{k_j})$ for some $k_j \in \mathbb{Z}$.*

Proof. See [9, Lemma 9]. □

An orthonormal basis satisfying condition from the above lemma will be called *separated basis*. Let $\mathcal{T} = (V, \mathcal{E})$ be a rootless directed tree. For $\theta \in \mathbb{T}$, $f \in \ell^2(V)$, we define $f_\theta : V \rightarrow \mathbb{C}$

$$f_\theta(v) = \theta^{|v|} f(v), \quad v \in V.$$

The diagonal operator $D_\theta : E \rightarrow E$ is given by

$$D_\theta e'_j = \theta^{k_j} e'_j, \quad j \in J.$$

For $\theta \in \mathbb{T}$, $\widehat{\varphi} : \mathbb{Z} \rightarrow \mathbf{B}(E)$ by $\widehat{\varphi}_\theta$, we denote the sequence $\{\widehat{\varphi}_\theta(n)\}_{n=-\infty}^\infty$, where

$$\widehat{\varphi}_\theta(n) := \theta^n D_\theta \widehat{\varphi}(n) D_{\bar{\theta}}. \tag{5.1}$$

We denote by f_θ the analytic function $U(U^* f)_\theta \in \mathcal{H}$ for $f \in \mathcal{H}$ and $\theta \in \mathbb{T}$.

The following lemma is an analogue of [9, Proposition 13].

Lemma 5.2. *Suppose (4.1) holds. Let $S_\lambda \in \mathbf{B}(\ell^2(V))$ be a left-invertible weighted shift and $\{e'_j\}_{j \in J}$ be a separated basis of $E := \langle e_\omega \rangle \oplus \mathcal{N}(S_\lambda^*)$ such that $e'_j \in \ell^2(V_{k_j})$, for $j \in J$, where $k_j \in \mathbb{Z}$. Then the following assertions hold:*

- (i) $\widehat{f}_\theta(n) = \theta^n D_\theta \widehat{f}(n)$, $f \in \mathcal{H}$, $\theta \in \mathbb{T}$ and $n \in \mathbb{Z}$,
- (ii) if $\theta \in \mathbb{T}$ and $f \in \mathcal{H}$, then $f_\theta \in \mathcal{H}$ and $\|f\| = \|f_\theta\|$,
- (iii) if $f \in \mathcal{H}$, then $\mathbb{T} \ni \theta \rightarrow f_\theta \in \mathcal{H}$ is continuous,
- (iv) if $\theta \in \mathbb{T}$ and $\widehat{\varphi} \in \mathcal{RM}(S_\lambda)$, then $\widehat{\varphi}_\theta \in \mathcal{RM}(S_\lambda)$ and

$$M_{\widehat{\varphi}_\theta} f = (M_{\widehat{\varphi}} f_{\bar{\theta}})_\theta,$$

- (v) if $\widehat{\varphi} \in \mathcal{RM}(S_\lambda)$, then $\mathbb{T} \ni \theta \rightarrow M_{\widehat{\varphi}_\theta} \in \mathcal{RM}(S_\lambda)$ is continuous in the strong operator topology.

Proof. (ii), (iii) and (v) the proof of these assertions is essentially the same as that of assertions (ii), (iii) and (v) of [9, Proposition 13].

(i) Let $x := U^* f$ and $x_\theta := U^* f_\theta$. Since $S_\lambda^n e'_j \in \ell^2(V_{n+k_j})$ for $j \in J$ and $n \in \mathbb{N}$, we see that

$$\begin{aligned} \langle \widehat{f}_\theta(n), e'_j \rangle &= \langle P_E S_\lambda^{*n} U^* f_\theta, e'_j \rangle = \langle S_\lambda^{*n} x_\theta, e'_j \rangle = \langle x_\theta, S_\lambda^n e'_j \rangle \\ &= \langle x_\theta, P_{n+k_j} S_\lambda^n e'_j \rangle = \langle P_{n+k_j} x_\theta, S_\lambda^n e'_j \rangle = \langle \theta^{n+k_j} P_{n+k_j} x, S_\lambda^n e'_j \rangle \\ &= \theta^{n+k_j} \langle P_E S_\lambda^n x, e'_j \rangle = \theta^{n+k_j} \langle \widehat{f}(n), e'_j \rangle = \langle \theta^n D_\theta \widehat{f}(n), e'_j \rangle. \end{aligned}$$

Note that $S_\lambda^{*n} e'_j \in \ell^2(V_{-n+k_j})$. Similar reasoning leads to

$$\begin{aligned} \langle \widehat{f}_\theta(-n), e'_j \rangle &= \langle P_E S_\lambda^n U^* f_\theta, e'_j \rangle = \langle S_\lambda^n x_\theta, e'_j \rangle = \langle x_\theta, S_\lambda^{*n} e'_j \rangle \\ &= \langle x_\theta, P_{-n+k_j} S_\lambda^{*n} e'_j \rangle = \langle P_{-n+k_j} x_\theta, S_\lambda^{*n} e'_j \rangle \\ &= \langle \theta^{-n+k_j} P_{-n+k_j} x, S_\lambda^{*n} e'_j \rangle = \theta^{-n+k_j} \langle P_E S_\lambda^n x, e'_j \rangle \\ &= \theta^{-n+k_j} \langle \widehat{f}(-n), e'_j \rangle = \langle \theta^{-n} D_\theta \widehat{f}(-n), e'_j \rangle, \end{aligned}$$

for $j \in J$. As a consequence, we have

$$\widehat{f}_\theta(n) = \sum_{j \in J} \langle \widehat{f}_\theta(n), e'_j \rangle e'_j = \sum_{j \in J} \theta^{n+k_j} \langle \widehat{f}(n), e'_j \rangle e'_j = \theta^n D_\theta \widehat{f}(n), \quad n \in \mathbb{Z}.$$

(iv) Let $\widehat{\varphi} \in \mathcal{RM}(S_\lambda)$ and $\theta \in \mathbb{T}$. Then we have

$$\begin{aligned} (\widehat{\varphi}_\theta * \widehat{f})(n) &= \sum_{k=-\infty}^{\infty} \widehat{\varphi}_\theta(k) \widehat{f}(n-k) = \sum_{k=-\infty}^{\infty} \theta^k D_\theta \widehat{\varphi}(k) D_{\bar{\theta}} \widehat{f}(n-k) \\ &= \theta^n D_\theta \sum_{k=-\infty}^{\infty} \widehat{\varphi}(k) \bar{\theta}^{n-k} D_{\bar{\theta}} \widehat{f}(n-k) = \theta^n D_\theta \sum_{k=-\infty}^{\infty} \widehat{\varphi}(k) \widehat{f}_{\bar{\theta}}(n-k) \\ &= \theta^n D_\theta ((\widehat{\varphi} * \widehat{f}_{\bar{\theta}})(n)) = \theta^n D_\theta ((\widehat{M_{\widehat{\varphi}} \widehat{f}_{\bar{\theta}}})(n)) = (\widehat{M_{\widehat{\varphi}} \widehat{f}_{\bar{\theta}}})_\theta(n), \end{aligned}$$

for $n \in \mathbb{Z}$. □

In view of Lemma 5.2 for any $\widehat{\varphi} \in \mathcal{RM}(S_\lambda)$, $f \in \ell^2(V)$ and continuous function $q : \mathbb{T} \rightarrow \mathbb{C}$, the mapping

$$\mathbb{T} \ni \theta \rightarrow q(\theta) M_{\widehat{\varphi}_\theta} f \in \ell^2(V)$$

is continuous, which implies that the mapping

$$\mathbb{T} \ni \theta \rightarrow q(\theta) M_{\widehat{\varphi}_\theta} \in \mathbf{B}(\ell^2(V))$$

is SOT-integrable with respect to the normalized Lebesgue measure on \mathbb{T} . This enables us to consider a bounded linear operator on $\ell^2(V)$ given by the integral

$$\int_{\mathbb{T}} q(\theta) M_{\widehat{\varphi}_\theta} d\theta,$$

where

$$\left(\int_{\mathbb{T}} q(\theta) M_{\widehat{\varphi}_\theta} d\theta \right) f = \int_{\mathbb{T}} q(\theta) M_{\widehat{\varphi}_\theta} f d\theta, \quad f \in \ell^2(V).$$

We refer the reader to [1, Section 3] and [19, Section 3.1.2] for more details on SOT-integrability with respect to the Lebesgue measure.

In the following lemma we construct for every $\widehat{\varphi} \in \mathcal{RM}(S_\lambda)$ a sequence $\widehat{\varphi}_n \in \mathcal{RM}(S_\lambda)$ of multipliers with finite support such that $M_{\widehat{\varphi}_n} \rightarrow M_{\widehat{\varphi}}$

in the strong operator topology. Let $\mathbb{C}_{\mathbb{T}}[z]$ denotes the set of *trigonometric polynomials* on \mathbb{T} i.e. polynomials of the form

$$p_n(\theta) = \sum_{k=-n}^n p_k \theta^k, \quad n \in \mathbb{N},$$

where $\{p_k\}_{k=-n}^n \subset \mathbb{C}$, $n \in \mathbb{N}$. For $p \in \mathbb{C}_{\mathbb{T}}[z]$, we define $\hat{p} : \mathbb{Z} \rightarrow \mathbb{C}$ by

$$\hat{p}(k) = \begin{cases} p_k, & \text{if } -n \leq k \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 5.3. *Suppose (4.1) holds, $\{S_\lambda^n S_\lambda^{*n}\}_{n=1}^\infty$ is uniformly bounded and the series (3.2) is convergent in E on an annulus $\mathbb{A}(r^-, r^+)$ with $r^- < r^+$ and $r^-, r^+ \in [0, \infty)$ for every $x \in \mathcal{H}$. Let $S_\lambda \in \mathcal{B}(\ell^2(V))$ be a left-invertible operator and $\hat{\varphi} \in \mathcal{RM}(S_\lambda)$. If $p \in \mathbb{C}_{\mathbb{T}}[z]$ then $\hat{p}\hat{\varphi} \in \mathcal{RM}(S_\lambda)$ and*

$$\int_{\mathbb{T}} p(\bar{\theta}) M_{\hat{\varphi}_\theta} d\theta = M_{\hat{p}\hat{\varphi}}.$$

Proof. Fix $\hat{\varphi} \in \mathcal{RM}(S_\lambda)$. There exists the sequence $\{a_n\}_{n=-\infty}^\infty \subset \mathbb{C}$ such that (4.11) holds. We claim that

$$\langle \hat{\varphi}(n) z^n f(z), e \rangle = \begin{cases} \langle (M_{a_n \chi_{\{n\}}} f)(z), e \rangle & \text{if } n \in \mathbb{N}, \\ \langle (M_{a_n P_{\bar{E}} \chi_{\{n\}}} f)(z), e \rangle & \text{if } n \in \mathbb{Z} \setminus \mathbb{N}, \end{cases} \tag{5.2}$$

for $e \in E$. First, we consider the case when $n \in \mathbb{N}$. Indeed, we have

$$\langle \hat{\varphi}(n) z^n f(z), e \rangle = a_n \langle z^n f(z), e \rangle = \langle (M_{a_n \chi_{\{n\}}} f)(z), e \rangle$$

for $z \in \mathbb{A}(r^-, r^+)$. It remains to consider the other case when $n \in \mathbb{Z} \setminus \mathbb{N}$. Applying Theorem 4.6, we obtain

$$\begin{aligned} \langle \hat{\varphi}(n) z^n f(z), e \rangle &= a_n \langle P_{\bar{E}} z^n f(z), e \rangle = a_n \langle P_{\bar{E}} z^n \sum_{k=-\infty}^\infty \hat{f}(k) z^k, e \rangle \\ &= a_n \langle \sum_{k=-\infty}^\infty P_{\bar{E}} \hat{f}(k) z^{n+k}, e \rangle = a_n \langle (\mathcal{RL}^{-n} f)(z), e \rangle \\ &= \langle (M_{a_n \chi_{\{n\}} P_{\bar{E}}} f)(z), e \rangle, \end{aligned}$$

for $z \in \mathbb{A}(r^-, r^+)$. By the linearity of the integral, it is enough to prove the case where $p(\theta) = \theta^k$, $k \in \mathbb{Z}$. Using assertion (ii) of Theorem 3.5, we verify that

$$\begin{aligned} \langle (\int_{\mathbb{T}} \bar{\theta}^k M_{\hat{\varphi}_\theta} d\theta f)(z), e \rangle &= \int_{\mathbb{T}} \langle \bar{\theta}^k (M_{\hat{\varphi}_\theta} f)(z), e \rangle d\theta \\ &= \int_{\mathbb{T}} \bar{\theta}^k \langle (\sum_{n=-\infty}^\infty \hat{\varphi}_\theta(n) z^n) f(z), e \rangle d\theta \end{aligned}$$

$$\begin{aligned}
 &= \int_{\mathbb{T}} \bar{\theta}^k \langle (\sum_{n=-\infty}^{\infty} \theta^n \widehat{\varphi}(n) z^n) f(z), e \rangle d\theta \\
 &= \sum_{n=-\infty}^{\infty} \int_{\mathbb{T}} \theta^{n-k} \langle \widehat{\varphi}(n) z^n f(z), e \rangle d\theta = \langle \widehat{\varphi}(k) z^k f(z), e \rangle,
 \end{aligned}$$

for $z \in \mathbb{A}(r^-, r^+)$. This combined with (5.2) completes the proof. □

We show now that under certain condition $\mathcal{RM}(S_\lambda)$ is closed in the strong operator topology.

Theorem 5.4. *Suppose (4.1) holds and the sequence $\{S_\lambda^n S_\lambda^{*n}\}_{n=1}^\infty$ is uniformly bounded and the series (3.2) is convergent in E on an annulus $\mathbb{A}(r^-, r^+)$ with $r^- < r^+$ and $r^-, r^+ \in [0, \infty)$ for every $x \in \mathcal{H}$. Then the space $\mathcal{RM}(S_\lambda)$ is closed in the strong operator topology.*

Proof. Let $A \in \mathcal{B}(\ell^2(V))$ and $\{\widehat{\varphi}_\sigma\}_{\sigma \in \Sigma} \subset \mathcal{RM}(S_\lambda)$ be a net such such that (SOT) $\lim_\sigma M_{\widehat{\varphi}_\sigma} = A$. Then

$$\lim_\sigma M_{\widehat{\varphi}_\sigma} e = \lim_\sigma \sum_{n=-\infty}^{-1} a_n^\sigma P_{\tilde{E}} e z^n + \sum_{n=0}^\infty a_n^\sigma e z^n = Ae, \quad e \in E.$$

Applying Lemma 3.1, we get

$$\widehat{Ae}(n) = \begin{cases} \lim_\sigma a_n^\sigma e, & \text{if } n \in \mathbb{N}, \\ \lim_\sigma a_n^\sigma P_{\tilde{E}} e & \text{if } n \in \mathbb{Z} \setminus \mathbb{N}. \end{cases}$$

This implies that there exist a sequence $\{a_n\}_{n=-\infty}^\infty \subset \mathbb{C}$ such that for every $e \in E$, we have

$$\widehat{Ae}(n) = \begin{cases} a_n e & \text{if } n \in \mathbb{N}, \\ a_n P_{\tilde{E}} e & \text{if } n \in \mathbb{Z} \setminus \mathbb{N}. \end{cases}$$

Since $Ae \in \mathcal{H}$, we infer from assertion (i) of [22, Theorem 3.8] that

$$\sum_{n=-\infty}^{-1} a_n P_{\tilde{E}} e \lambda^n + \sum_{n=0}^\infty a_n e \lambda^n$$

converges absolutely for every $e \in E$ and $\lambda \in \mathbb{A}(r^-, r^+)$. In particular for every $e \in \tilde{E}$, we get

$$\sum_{n=-\infty}^{-1} a_n P_{\tilde{E}} e \lambda^n + \sum_{n=0}^\infty a_n e \lambda^n = (\sum_{n=-\infty}^\infty a_n \lambda^n) e.$$

This implies that $\sum_{n=-\infty}^\infty a_n \lambda^n$ converges absolutely for every $\lambda \in \mathbb{A}(r^-, r^+)$. Since $\|a_n P_{\tilde{E}} \lambda^n\| \leq |a_n \lambda^n|$, $n \in \mathbb{Z} \setminus \mathbb{N}$ and $\|a_n I_E \lambda^n\| \leq |a_n \lambda^n|$, $n \in \mathbb{N}$, we deduce that

$$\sum_{n=-\infty}^{-1} a_n P_{\tilde{E}} \lambda^n + \sum_{n=0}^\infty a_n I_E \lambda^n \tag{5.3}$$

converges absolutely in the norm of $\mathcal{B}(E)$ for every $\lambda \in \mathbb{A}(r^-, r^+)$ and consequently

$$(Ae)(\lambda) = (\sum_{n=-\infty}^{-1} a_n P_{\tilde{E}} \lambda^n + \sum_{n=0}^\infty a_n I_E \lambda^n) e, \quad e \in E, \lambda \in \mathbb{A}(r^-, r^+). \tag{5.4}$$

Let $\widehat{\psi} : \mathbb{Z} \rightarrow \mathbf{B}(E)$ be a function defined by

$$\widehat{\psi}(n) = \begin{cases} a_n I_E & \text{if } n \in \mathbb{N}, \\ a_n P_{\tilde{E}} & \text{if } n \in \mathbb{Z} \setminus \mathbb{N}. \end{cases}$$

By Theorem 3.4 operator \mathcal{M}_z commutes with $M_{\widehat{\psi}}$ for every $\widehat{\psi} \in \mathcal{GM}(S_\lambda)$. This leads to

$$\begin{aligned} A(ez^n) &= \lim_{\sigma} M_{\widehat{\psi}_\sigma}(ez^n) = \lim_{\sigma} M_{\widehat{\psi}_\sigma}(\mathcal{M}_z^n e) = \lim_{\sigma} \mathcal{M}_z^n M_{\widehat{\psi}_\sigma} e \\ &= \mathcal{M}_z^n \lim_{\sigma} M_{\widehat{\psi}_\sigma} e = \mathcal{M}_z^n A e, \quad e \in E. \end{aligned} \tag{5.5}$$

Using (5.4), we deduce that

$$\begin{aligned} (\mathcal{M}_z^n A e)(\lambda) &= \lambda^n \left(\sum_{k=-\infty}^{-1} a_k P_{\tilde{E}} \lambda^k + \sum_{k=0}^{\infty} a_k I_E \lambda^k \right) e \\ &= \left(\sum_{k=-\infty}^{-1} a_k P_{\tilde{E}} \lambda^k + \sum_{k=0}^{\infty} a_k I_E \lambda^k \right) (e \lambda^n) \end{aligned}$$

for $\lambda \in \mathbb{A}(r^-, r^+)$ and $e \in E$. Combining this fact with (5.5), we get

$$A(ez^n) = M_{\widehat{\psi}}(ez^n), \quad e \in E, n \in \mathbb{N}.$$

Similarly, we see that

$$\begin{aligned} A\left(e \frac{1}{z^n}\right) &= \lim_{\sigma} M_{\widehat{\psi}_\sigma}\left(e \frac{1}{z^n}\right) = \mathcal{L}^n \mathcal{M}_z^n \lim_{\sigma} M_{\widehat{\psi}_\sigma}\left(e \frac{1}{z^n}\right) \\ &= \mathcal{L}^n \lim_{\sigma} M_{\widehat{\psi}_\sigma} \mathcal{M}_z^n \left(e \frac{1}{z^n}\right) = \mathcal{L}^n \lim_{\sigma} M_{\widehat{\psi}_\sigma} e \end{aligned} \tag{5.6}$$

for $e \in \tilde{E}$. Similar reasoning as above leads to

$$\begin{aligned} (\lim_{\sigma} M_{\widehat{\psi}_\sigma} e)(\lambda) &= \left(\sum_{k=-\infty}^{-1} a_k P_{\tilde{E}} \lambda^k + \sum_{k=0}^{\infty} a_k I_E \lambda^k \right) e \\ &= \left(\sum_{k=-\infty}^{-1} a_k P_{\tilde{E}} \lambda^k + \sum_{n=0}^{\infty} a_n I_E \lambda^n \right) \lambda^n \left(e \frac{1}{\lambda^n} \right) \\ &= \lambda^n \left(\sum_{k=-\infty}^{-1} a_k P_{\tilde{E}} \lambda^k + \sum_{k=0}^{\infty} a_k I_E \lambda^k \right) \left(e \frac{1}{\lambda^n} \right) \end{aligned}$$

for $\lambda \in \mathbb{A}(r^-, r^+)$ and $e \in \tilde{E}$. This combined with (5.6) yields

$$A\left(e \frac{1}{z^n}\right) = M_{\widehat{\psi}}\left(e \frac{1}{z^n}\right), \quad e \in \tilde{E}, n \in \mathbb{N}.$$

Fix $f \in \mathcal{H}$. By (4.2) there exist sequence $\{f_n\}_{n=0}^{\infty}$ such that $\lim_{n \rightarrow \infty} f_n = f$ and

$$f_n \in \text{lin}\{z^n E : n \in \mathbb{N}\} \cup \left\{ \frac{1}{z^n} \tilde{E} : n \in \mathbb{Z}_+ \right\}, \quad n \in \mathbb{N}.$$

Since the series (5.3) converges absolutely in the norm of $\mathbf{B}(E)$ for every $\lambda \in \mathbb{A}(r^-, r^+)$, we have

$$\begin{aligned}
 \langle Af(\lambda), e \rangle_E &= \langle Af, \kappa_{\mathcal{H}}(\cdot, \lambda)e \rangle_{\mathcal{H}} = \lim_{k \rightarrow \infty} \langle Af_k, \kappa_{\mathcal{H}}(\cdot, \lambda)e \rangle_{\mathcal{H}} \\
 &= \lim_{k \rightarrow \infty} \langle (\sum_{n=-\infty}^{-1} a_n P_{\tilde{E}} \lambda^n + \sum_{n=0}^{\infty} a_n I_E \lambda^n) f_k(\lambda), e \rangle_E \\
 &= \lim_{k \rightarrow \infty} \langle f_k(\lambda), (\sum_{n=-\infty}^{-1} a_n P_{\tilde{E}} \lambda^n + \sum_{n=0}^{\infty} a_n I_E \lambda^n)^* e \rangle_E \\
 &= \langle f(\lambda), (\sum_{n=-\infty}^{-1} a_n P_{\tilde{E}} \lambda^n + \sum_{n=0}^{\infty} a_n I_E \lambda^n)^* e \rangle_E \\
 &= \langle (\sum_{n=-\infty}^{-1} a_n P_{\tilde{E}} \lambda^n + \sum_{n=0}^{\infty} a_n I_E \lambda^n) f(\lambda), e \rangle_E
 \end{aligned}$$

for $\lambda \in \mathbb{A}(r^-, r^+)$ and $e \in E$. Thus, we obtain

$$(Af)(\lambda) = (\sum_{n=-\infty}^{-1} a_n P_{\tilde{E}} \lambda^n + \sum_{n=0}^{\infty} a_n I_E \lambda^n) f(\lambda)$$

for $\lambda \in \mathbb{A}(r^-, r^+)$. □

Now, we prove that truncated multipliers tends to the original multiplier as the size of support grows to infinity.

Lemma 5.5. *Suppose (4.1) holds and the sequence of operators $\{S_{\lambda}^n S_{\lambda'}^{*n}\}_{n=1}^{\infty}$ is uniformly bounded. Let $\widehat{\varphi} \in \mathcal{RM}(S_{\lambda})$ and let $\{p_n\}_{n=0}^{\infty}$ be a Fejer kernels i.e.*

$$p_n(\theta) = \sum_{k=-n}^n (1 - \frac{|k|}{n+1}) \theta^k, \quad n \in \mathbb{N}.$$

Then the following assertions hold:

- (i) $\widehat{p}_n \widehat{\varphi} \in \mathcal{RM}(S_{\lambda})$,
- (ii) $\|M_{\widehat{p}_n \widehat{\varphi}}\| \leq M_{\widehat{\varphi}}$,
- (iii) $M_{\widehat{p}_n \widehat{\varphi}} \rightarrow M_{\widehat{\varphi}}$ in the strong operator topology.

Proof. (i) is a direct consequence of assertion (i) of Theorem 3.4 and assertion (ii) of Theorem 4.6.

(ii) See the proof of [1, Lemma 3.3(ii)] and Lemmas 5.2 and 5.3.

(iii) First we show that

$$M_{\widehat{p}_n \widehat{\varphi}} Ue \rightarrow M_{\widehat{\varphi}} Ue, \quad e \in E. \tag{5.7}$$

Let $\{e'_j\}_{j \in J}$ be a separated basis for E . Clearly, it is enough to prove that

$$M_{\widehat{p}_n \widehat{\varphi}} Ue'_j \rightarrow M_{\widehat{\varphi}} Ue'_j, \quad j \in J. \tag{5.8}$$

Fix $j \in J$. Since $M_{\widehat{\varphi}} Ue'_j \in \mathcal{H}$ there exists $g \in \ell^2(V)$ such that $M_{\widehat{\varphi}} Ue'_j = Ug$. The assertion (iii) of Lemma 3.2, shows that

$$\widehat{\varphi}(m)e'_j = \widehat{M_{\widehat{\varphi}} Ue'_j}(m) = \widehat{Ug}(m) = \begin{cases} P_E S_{\lambda'}^{*m} g & \text{if } m \in \mathbb{N}, \\ P_E S_{\lambda}^{-m} g, & \text{if } m \in \mathbb{Z} \setminus \mathbb{N}. \end{cases}$$

Let $g_n : V \rightarrow \mathbb{C}$ be a function defined by

$$g_n(u) = \hat{p}_n(|u| - k_j)g(u), \quad u \in V.$$

We claim that

$$M_{\hat{p}_n \widehat{\varphi}} \widehat{U} e'_j(m) = \widehat{U} g_n(m), \quad m \in \mathbb{Z}. \tag{5.9}$$

First, we consider the case when $m \in \mathbb{N}$. Using the properties of inner product and definition of g_n , one gets the following

$$\begin{aligned} \langle P_E S_{\lambda'}^{*m} g_n, e'_l \rangle &= \langle g_n, S_{\lambda'}^m e'_l \rangle = \langle P_{m+k_l} g_n, S_{\lambda'}^m e'_l \rangle \\ &= \hat{p}_n(m + k_l - k_j) \langle P_{m+k_l} g, S_{\lambda'}^m e'_l \rangle \\ &= \hat{p}_n(m + k_l - k_j) \langle P_E S_{\lambda'}^{*m} g, e'_l \rangle \\ &= \hat{p}_n(m + k_l - k_j) \langle \widehat{\varphi}(m) e'_j, e'_l \rangle \\ &= \hat{p}_n(m + k_l - k_j) a_m \langle e'_j, e'_l \rangle. \end{aligned}$$

As a consequence, we see that

$$\begin{aligned} \widehat{U} g_n(m) &= P_E S_{\lambda'}^{*m} g_n = \sum_{l \in J} \langle P_E S_{\lambda'}^{*m} g_n, e'_l \rangle e'_l \\ &= \sum_{l \in J} \hat{p}_n(m + k_l - k_j) a_m \langle e'_j, e'_l \rangle e'_l \\ &= \hat{p}_n(m) a_m e'_j = \hat{p}_n(m) \widehat{\varphi}(m) e'_j = M_{\hat{p}_n \widehat{\varphi}} \widehat{U} e'_j(m). \end{aligned}$$

It remains to consider the other case when $m \in \mathbb{Z} \setminus \mathbb{N}$. It is easily seen that

$$\langle P_E S_{\lambda}^{-m} g_n, e'_l \rangle = \langle g_n, S_{\lambda}^{*(-m)} e'_l \rangle = 0, \quad e'_l \in \mathcal{N}(S_{\lambda}^*). \tag{5.10}$$

Arguing as above, we deduce that

$$\begin{aligned} \langle P_E S_{\lambda}^{-m} g_n, e'_l \rangle &= \langle g_n, S_{\lambda}^{*(-m)} e'_l \rangle = \langle P_{m+k_l} g_n, S_{\lambda}^{*(-m)} e'_l \rangle \\ &= \hat{p}_n(m + k_l - k_j) \langle P_{m+k_l} g, S_{\lambda}^{*(-m)} e'_l \rangle \\ &= \hat{p}_n(m + k_l - k_j) \langle P_E S_{\lambda}^{-m} g, e'_l \rangle \\ &= \hat{p}_n(m + k_l - k_j) \langle \widehat{\varphi}(m) e'_j, e'_l \rangle \\ &= \hat{p}_n(m + k_l - k_j) a_m \langle e'_j, e'_l \rangle, \quad e'_l \notin \mathcal{N}(S_{\lambda}^*). \end{aligned}$$

This and (5.10) imply that

$$\langle P_E S_{\lambda}^m g_n, e'_l \rangle = \hat{p}_n(m + k_l - k_j) a_m \langle P_{\bar{E}} e'_j, e'_l \rangle.$$

Following steps analogous to those above, we obtain that

$$\begin{aligned} \widehat{U} g_n(m) &= P_E S_{\lambda}^{-m} g_n = \sum_{l \in J} \langle P_E S_{\lambda}^{-m} g_n, e'_l \rangle e'_l \\ &= \sum_{l \in J} \hat{p}_n(m + k_l - k_j) a_m \langle P_{\bar{E}} e'_j, e'_l \rangle e'_l \\ &= \hat{p}_n(m) a_m P_{\bar{E}} e'_j = \hat{p}_n(m) \widehat{\varphi}(m) e'_j = M_{\hat{p}_n \widehat{\varphi}} \widehat{U} e'_j(m). \end{aligned}$$

Since $g_n(u) \rightarrow g(u)$, we infer from (5.9) that

$$\langle M_{\hat{p}_n \widehat{\varphi}} \widehat{U} e'_j, e_u \rangle = \langle g_n, e_u \rangle = g_n(u) \rightarrow g(u) = \langle g, e_u \rangle = \langle M_{\widehat{\varphi}} U e'_j, e_u \rangle.$$

Therefore $\{M_{\hat{p}_n \hat{\varphi}} U e'_j\}_{n=1}^\infty$ is weakly convergent to $M_{\hat{\varphi}} U e'_j$ in \mathcal{H} . By inequality $|g_n(u)| \leq |g(u)|$, we have

$$\|M_{\hat{p}_n \hat{\varphi}} U e'_j\| = \|g_n\| \leq \|g\| = \|M_{\hat{\varphi}} U e'_j\|.$$

Combining this with the fact that $\{M_{\hat{p}_n \hat{\varphi}} U e'_j\}_{n=1}^\infty$ is weakly convergent, we deduce (5.8). This completes the proof of (5.7).

By Theorem 3.4 operator S_λ commutes with $M_{\hat{\varphi}}$ for every $\hat{\varphi} \in \mathcal{GM}(S_\lambda)$. Combining this fact with (5.7), we deduce that

$$M_{\hat{p}_n \hat{\varphi}} U S_\lambda^m e \rightarrow M_{\hat{\varphi}} U S_\lambda^m e. \tag{5.11}$$

Using (3.6) and (5.7), one can verify that

$$M_{\hat{p}_n \hat{\varphi}} U S_\lambda^m S_{\lambda'}^{*m} e \rightarrow M_{\hat{\varphi}} U S_\lambda^m S_{\lambda'}^{*m} e, \quad e \in E.$$

Hence, by the fact that S_λ commutes with $M_{\hat{\varphi}}$ for every $\hat{\varphi} \in \mathcal{GM}(S_\lambda)$ again, we have

$$S_\lambda^m M_{\hat{p}_n \hat{\varphi}} U S_{\lambda'}^{*m} e \rightarrow S_\lambda^m M_{\hat{\varphi}} U S_{\lambda'}^{*m} e, \quad e \in E.$$

Multiplying the above equation left by $S_{\lambda'}^{*m}$

$$M_{\hat{p}_n \hat{\varphi}} U S_{\lambda'}^{*m} e \rightarrow M_{\hat{\varphi}} U S_{\lambda'}^{*m} e, \quad e \in E. \tag{5.12}$$

It follows from assertion (iii) of [22, Lemma 4.2] that

$$\bigvee \{S_\lambda^n E : n \in \mathbb{N}\} \oplus \bigvee \{S_{\lambda'}^{*n} E : n \in \mathbb{N}\} = \mathcal{H}.$$

This, combined with (5.11), (5.12) and assertion (ii) completes the proof. \square

Now we are in a position to prove the main result of this section.

Theorem 5.6. *Suppose (4.1) holds, the sequence of operators $\{S_\lambda^n S_{\lambda'}^{*n}\}_{n=1}^\infty$ is uniformly bounded and the series (3.2) is convergent in E on an annulus $\mathbb{A}(r^-, r^+)$ with $r^- < r^+$ and $r^-, r^+ \in [0, \infty)$ for every $x \in \mathcal{H}$. Then*

$$\mathcal{RM}(S_\lambda) = \overline{\{p[S_\lambda] : p \in \mathbb{C}_\mathbb{T}[z]\}}^{\text{SOT}} = \overline{\{p[S_\lambda] : p \in \mathbb{C}_\mathbb{T}[z]\}}^{\text{WOT}},$$

where $p[T] := \sum_{k=-n}^{-1} a_k R T^k + \sum_{k=0}^n a_k T^k$ for $p(\theta) = \sum_{k=-n}^n a_k \theta^k$, $n \in \mathbb{N}$, $R := (\text{SOT}) \lim_{n \rightarrow \infty} T^n T'^{*n}$ and T is a left-invertible operator.

Proof. By Theorem 3.4, we get $M_{\chi_{\{n\}} I_E} = S_\lambda^n$, $n \in \mathbb{N}$ and by Theorem 4.6 $M_{\chi_{\{-n\}} P_E} = \mathcal{R} S_{\lambda'}^{*n}$, $n \in \mathbb{N}$. Combining this with Lemma 5.5, we deduce that $\mathcal{RM}(S_\lambda)$ lies in the set $\overline{\{p[S_\lambda] : p \in \mathbb{C}_\mathbb{T}[z]\}}^{\text{SOT}}$. Since, by Theorem 5.4 the space $\mathcal{RM}(S_\lambda)$ is closed in the strong operator topology, we get

$$\mathcal{RM}(S_\lambda) = \overline{\{p[S_\lambda] : p \in \mathbb{C}_\mathbb{T}[z]\}}^{\text{SOT}}.$$

The fact that closures of convex sets in the strong and weak operator topologies coincide completes the proof.

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