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**A STUDY OF CHAOS FOR PROCESSES  
UNDER SMALL PERTURBATIONS II:  
RIGOROUS PROOF OF CHAOS**

**Abstract.** In the present paper we prove distributional chaos for the Poincaré map in the perturbed equation

$$\dot{z} = \left(1 + e^{i\kappa t} |z|^2\right) \bar{z}^2 - Ne^{-i\frac{\pi}{3}}.$$

Heteroclinic and homoclinic connections between two periodic solutions bifurcating from the stationary solution 0 present in the system when  $N = 0$  are also discussed.

**Keywords:** distributional chaos, isolating segments, fixed point index, bifurcation.

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## 1. INTRODUCTION

Recently in [10], we introduced topological tools to deal with distributional chaos and homoclinic/heteroclinic solutions in discrete dynamical systems, when a semi-conjugacy with symbolic dynamics, covering a periodic itinerary of at most two to one is available. As a testing ground for our method we used the following perturbed time-periodic nonautonomous planar differential equation

$$\dot{z} = \left(1 + e^{i\kappa t} |z|^2\right) \bar{z}^2 - Ne^{-i\frac{\pi}{3}} \tag{1.1}$$

when the parameter value  $N$  is between 0 and 0.01 and  $\kappa \in (0, 0.18]$ . In [10] we presented the following two theorems (providing only a sketch of the proofs).

**Theorem 1.1.** *Let the inequalities*

$$0 < \kappa \leq 0.18, \tag{1.2}$$

$$0 \leq N \leq 0.01 \tag{1.3}$$

*be satisfied. Then the equation (1.1) is uniform distributionally chaotic.*

**Theorem 1.2.** *Let the inequalities (1.2) and*

$$0 < N \leq 0.01 \tag{1.4}$$

*be satisfied. In that case, the set of solutions of (1.1) which are:*

- 1.5. *homoclinic to  $\psi_1$ ,*
- 1.6. *homoclinic to  $\psi_2$ ,*
- 1.7. *heteroclinic from  $\psi_1$  to  $\psi_2$ ,*
- 1.8. *heteroclinic from  $\psi_2$  to  $\psi_1$*

*is infinite (for each of the listed possibilities) where  $\psi_1, \psi_2$  are the periodic solutions which bifurcate from the trivial one for  $N = 0$  such that  $\Re[\psi_1] > 0$  and  $\Re[\psi_2] < 0$ .*

In the present paper we fill this gap and provide rigorous proofs of the above facts. They are the main results of the present work.

Theorem 1.2 is especially important, because it shows how the method of isolating segments can be used to prove existence of homoclinic and heteroclinic solutions. We strongly believe that this kind of analysis can be repeated in applications, by use of other topological tools (i.e. other than isolating segments), when existence of semiconjugacy with symbolic dynamics can be combined with additional knowledge about the vector field.

The article is organized as follows. In section 2 we recall all definitions and basic facts used in the further parts of the article. Next, in section 3 we present detailed proofs of Theorems 1.1 and 1.2.

## 2. BASIC NOTIONS

### 2.1. TOPOLOGICAL DYNAMICS

Let  $(X, f)$  be a dynamical system on a compact metric space. By *positive orbit* of  $x$  we mean the set

$$\text{Orb}^+(x, f) = \{x, f(x), f^2(x), \dots\}.$$

If, additionally,  $f$  is a homeomorphism, we may define its *negative orbit* and (*full orbit*) by, respectively

$$\text{Orb}^-(x, f) = \{x, f^{-1}(x), f^{-2}(x), \dots\}, \quad \text{Orb}(x, f) = \text{Orb}^-(x, f) \cup \text{Orb}^+(x, f).$$

A point  $y \in X$  is an  $\omega$ -*limit point* ( $\alpha$ -*limit point*) of a point  $x$  if it is an accumulation point of the sequence  $x, f(x), f^2(x), \dots$  (resp.  $x, f^{-1}(x), f^{-2}(x), \dots$ ). The set of all  $\omega$ -limit points ( $\alpha$ -limit points) of  $x$  is called a  $\omega$ -*limit set* (resp.  $\alpha$ -*limit set*) of  $x$  and denoted  $\omega_f(x)$  (resp.  $\alpha_f(x)$ ). A point  $p \in X$  is said to be *periodic* if  $f^n(p) = p$  for some  $n \geq 1$ . The set of all periodic points for  $f$  is denoted by  $\text{Per}(f)$ .

Let  $(X, f), (Y, g)$  be dynamical systems on compact metric spaces. A continuous map  $\Phi : X \rightarrow Y$  is called a *semiconjugacy* (or a *factor map*) between  $f$  and  $g$  if  $\Phi$  is surjective and  $\Phi \circ f = g \circ \Phi$ .

**Definition 2.1.** Let  $(X, \rho)$  be a compact metric space. We say that  $f \in C(X)$  has the *weak specification property* (briefly *WSP*) if, for any  $\delta > 0$ , there is a positive integer  $N_\delta$  such that for any points  $y_1, y_2 \in X$  and any sequence  $0 = j_1 \leq k_1 < j_2 \leq k_2$  satisfying  $j_2 - k_1 \geq N_\delta$  there is a point  $x$  in  $X$  such that, for  $m = 1, 2$  and all integers  $i$  with  $j_m \leq i \leq k_m$ , the following condition holds:

$$\rho(f^i(x), f^i(y_m)) < \delta. \tag{2.1}$$

**Proposition 2.2** ([10, Proposition 12]). *Let  $(X, d), (Y, \rho)$  be compact metric spaces and  $f \in C(X), g \in C(Y)$ . Let  $\Phi : X \rightarrow Y$  be a semi-conjugacy such that  $\Phi^{-1}(y) = \{p_1, \dots, p_k\} \subset \text{Per}(f)$  for some  $y \in \text{Per}(g)$  and  $k \in \mathbb{N}$ . Then, for any  $z \in Y$  with the property  $\omega_g(z) = \alpha_g(z) = \text{Orb}(y, g)$  and for any  $q \in \Phi^{-1}(z)$  there exist  $u, v \in \Phi^{-1}(y)$  such that  $\alpha_f(q) = \text{Orb}(u, f)$  and  $\omega_f(q) = \text{Orb}(v, f)$ .*

## 2.2. SHIFT SPACES

Let  $\mathcal{A} = \{0, 1, \dots, n - 1\}$ . We denote

$$\Sigma_n = \mathcal{A}^{\mathbb{Z}}.$$

By a *word*, we mean any element of a free monoid  $\mathcal{A}^*$  with the set of generators equal to  $\mathcal{A}$ . If  $x \in \Sigma_n$  and  $i < j$  then by  $x_{[i,j]}$  we mean a sequence  $x_i, x_{i+1}, \dots, x_j$ . We may naturally identify  $x_{[i,j]}$  with the word  $x_{[i,j]} = x_i x_{i+1} \dots x_j \in \mathcal{A}^*$ . It is also convenient to denote  $x_{[i,j]} = x_{[i,j-1]}$ .

We introduce a metric  $\rho$  in  $\Sigma_n$  by

$$\rho(x, y) = 2^{-k}, \text{ where } k = \min \{m \geq 0 : x_{[-m,m]} \neq y_{[-m,m]}\}.$$

By the  $0^\infty$  we denote the element  $x \in \Sigma_n$  such that  $x_i = 0$  for all  $i \in \mathbb{Z}$ . The usual map on  $\Sigma_n$  is the shift map  $\sigma$  defined by  $\sigma(x)_i = x_{i+1}$  for all  $i \in \mathbb{Z}$ . The dynamical system  $(\Sigma_n, \sigma)$  is called *full two-sided shift* over  $n$  symbols. If  $X \subset \Sigma_n$  is closed and invariant (i.e.  $\sigma(X) = X$ ) then we say that  $X$  is a shift. There are many equivalent ways to define shifts, e.g.  $X$  is a shift iff there exists a set (of forbidden words)  $\mathcal{F} \subset \mathcal{A}^*$  such that  $X = X_{\mathcal{F}}$  where

$$X_{\mathcal{F}} = \{x \in \Sigma_n : x_{[i,j]} \notin \mathcal{F} \text{ for every } i \leq j\}.$$

One of the most important classes of shifts is the class of shifts of finite type. It contains all shifts which can be defined by finite sets of forbidden words. Equivalently,  $X \subset \Sigma_n$  is a shift of finite type if there is an integer  $m > 0$  and  $M \subset \mathcal{A}^m$  such that

$$x \in X \iff x_{[i,i+m]} \in M \text{ for all } i \in \mathbb{Z}.$$

A shift which may be obtained as a factor of a shift of finite type is called a *sofic shift*.

Another way to define shifts of finite type and sofic shifts is to use directed graphs and labeled directed graphs respectively, called their presentations (elements of shift are identified with bi-infinite paths on graph). The reader not familiar with this approach is once again referred to [7] or [9].

### 2.3. DYNAMICAL SYSTEMS AND WAŻEWSKI METHOD

Let  $X$  be a topological space and  $W$  be its subset. Denote by  $\text{cl}W$  the closure of  $W$ . The following definitions come from [14].

Let  $D$  be an open subset of  $\mathbb{R} \times X$ . By a *local flow* on  $X$  we mean a continuous map  $\phi : D \rightarrow X$ , such that three conditions are satisfied:

- (i)  $I_x = \{t \in \mathbb{R} : (t, x) \in D\}$  is an open interval  $(\alpha_x, \omega_x)$  containing 0, for every  $x \in X$ ,
- (ii)  $\phi(0, x) = x$ , for every  $x \in X$ ,
- (iii)  $\phi(s + t, x) = \phi(t, \phi(s, x))$ , for every  $x \in X$  and  $s, t \in \mathbb{R}$  such that  $s \in I_x$  and  $t \in I_{\phi(s, x)}$ .

In the sequel we write  $\phi_t(x)$  instead of  $\phi(t, x)$ . We distinguish three subsets of  $W$  given by

$$\begin{aligned} W^- &= \{x \in W : \phi([0, t] \times \{x\}) \not\subset W, \text{ for every } t > 0\}, \\ W^+ &= \{x \in W : \phi([-t, 0] \times \{x\}) \not\subset W, \text{ for every } t > 0\}, \\ W^* &= \{x \in W : \phi(t, x) \notin W, \text{ for some } t > 0\}. \end{aligned}$$

It is easy to see that  $W^- \subset W^*$ . We call  $W^-$  the *exit set* of  $W$ , and  $W^+$  the *entrance set* of  $W$ . We call  $W$  a *Ważewski set* provided

- 2.2. if  $x \in W$ ,  $t > 0$ , and  $\phi([0, t] \times \{x\}) \subset \text{cl}W$  then  $\phi([0, t] \times \{x\}) \subset W$ ,
- 2.3.  $W^-$  is closed relative to  $W^*$ .

**Proposition 2.3.** *If both  $W$  and  $W^-$  are closed subsets of  $X$  then  $W$  is a Ważewski set.*

The function  $\sigma^* : W^* \rightarrow [0, \infty)$

$$\sigma^*(x) = \sup\{t \in [0, \infty) : \phi([0, t] \times \{x\}) \subset W\}$$

is called the *escape-time function* of  $W$ . The following lemma is called the Ważewski lemma.

**Lemma 2.4** ([14, Lemma 2.1 (iii)]). *Let  $W$  be a Ważewski set and  $\sigma^*$  be its escape-time function. Then  $\sigma^*$  is continuous.*

### 2.4. PROCESSES

Let  $X$  be a topological space and  $\Omega \subset \mathbb{R} \times \mathbb{R} \times X$  be an open set.

By a *local process* on  $X$  we mean a continuous map  $\varphi : \Omega \rightarrow X$ , such that the following three conditions are satisfied:

- (i)  $\forall \sigma \in \mathbb{R}, x \in X, \{t \in \mathbb{R} : (\sigma, t, x) \in \Omega\}$  is an open interval containing 0,
- (ii)  $\forall \sigma \in \mathbb{R}, \varphi(\sigma, 0, \cdot) = \text{id}_X$ ,
- (iii)  $\forall x \in X, \sigma, s \in \mathbb{R}, t \in \mathbb{R}$  if  $(\sigma, s, x) \in \Omega, (\sigma + s, t, \varphi(\sigma, s, x)) \in \Omega$ , then  $(\sigma, s + t, x) \in \Omega$  and  $\varphi(\sigma, s + t, x) = \varphi(\sigma + s, t, \varphi(\sigma, s, x))$ .

For abbreviation, we write  $\varphi_{(\sigma,t)}(x)$  instead of  $\varphi(\sigma, t, x)$ .

Given a local process  $\varphi$  on  $X$  one can define a local flow  $\phi$  on  $\mathbb{R} \times X$  by

$$\phi(t, (\sigma, x)) = (t + \sigma, \varphi(\sigma, t, x)).$$

Let  $M$  be a smooth manifold and let  $v : \mathbb{R} \times M \rightarrow TM$  be a time-dependent vector field. We assume that  $v$  is regular enough to guarantee that for every  $(t_0, x_0) \in \mathbb{R} \times M$  the Cauchy problem

$$\dot{x} = v(t, x), \tag{2.4}$$

$$x(t_0) = x_0 \tag{2.5}$$

has a unique solution. Then the equation (2.4) generates a local process  $\varphi$  on  $M$  by  $\varphi_{(t_0,t)}(x_0) = x(t_0, x_0, t + t_0)$ , where  $x(t_0, x_0, \cdot)$  is the solution of the Cauchy problem (2.4), (2.5).

Let  $T$  be a positive number. We assume that  $v$  is  $T$ -periodic in  $t$ . It follows that the local process  $\varphi$  is  $T$ -periodic, i.e.,

$$\forall \sigma, t \in \mathbb{R} \quad \varphi_{(\sigma+T,t)} = \varphi_{(\sigma,t)},$$

hence there is a one-to-one correspondence between  $T$ -periodic solutions of (2.4) and fixed points of the Poincaré map  $P_T = \varphi_{(0,T)}$ .

## 2.5. PERIODIC ISOLATING SEGMENTS

Let  $X$  be a topological space and  $T$  be a positive number. We assume that  $\varphi$  is a  $T$ -periodic local process on  $X$ .

For any set  $Z \subset \mathbb{R} \times X$  and  $a, b, t \in \mathbb{R}$ ,  $a < b$  we define

$$Z_t = \{x \in X : (t, x) \in Z\},$$

$$Z_{[a,b]} = \{(t, x) \in Z : t \in [a, b]\}.$$

Let  $\pi_1 : \mathbb{R} \times X \rightarrow \mathbb{R}$  and  $\pi_2 : \mathbb{R} \times X \rightarrow X$  be projections on the time and space variable, respectively.

A compact set  $W \subset [a, b] \times X$  is called an *isolating segment over  $[a, b]$*  for  $\varphi$  if it is ENR (Euclidean neighborhood retract – cf. [5]) and there are  $W^{--}, W^{++} \subset W$  compact ENR's (called, respectively, the *proper exit set* and *proper entrance set*) such that:

2.6.  $\partial W = W^- \cup W^+$ ,

2.7.  $W^- = W^{--} \cup (\{b\} \times W_b)$ ,  $W^+ = W^{++} \cup (\{a\} \times W_a)$ ,

2.8. there exists homeomorphism  $h : [a, b] \times W_a \rightarrow W$  such that  $\pi_1 \circ h = \pi_1$  and  $h([a, b] \times W_a^{--}) = W^{--}$ ,  $h([a, b] \times W_a^{++}) = W^{++}$ .

Every isolating segment is also a Ważewski set (for the local flow associated to a process  $\varphi$ ). We say that an isolating segment  $W$  over  $[a, b]$  is  $(b - a)$ -periodic

(or simply *periodic*) if  $W_a = W_b$ ,  $W_a^{--} = W_b^{--}$  and  $W_a^{++} = W_b^{++}$ . Let  $T > 0$ . Given the set  $Z \subset [0, T] \times X$  such that  $Z_0 = Z_T$  we define its infinite catenation by

$$Z^\infty = \{(t, z) \in \mathbb{R} \times X : z \in Z_{t \bmod T}\}.$$

Let  $W$  be a periodic isolating segment over  $[a, b]$ . The homeomorphism  $h$  induces  $m : (W_a, W_a^{--}) \rightarrow (W_b, W_b^{--}) = (W_a, W_a^{--})$  a *monodromy homeomorphism* given by

$$m(x) = \pi_2 h(b, \pi_2 h^{-1}(a, x)).$$

A different choice of the homeomorphism  $h$  leads to a map which is homotopic to  $m$ . It follows that the automorphism in singular homology

$$\mu_W = H(m) : H(W_a, W_a^{--}) \rightarrow H(W_a, W_a^{--})$$

is an invariant of segment  $W$ .

The following theorem, proved by R. Szrednicki in [13], plays a crucial role in the method of isolating segments.

**Theorem 2.5** ([15, Theorem 5.1]). *Let  $\varphi$  be a local process on  $X$  and let  $W$  be a periodic isolating segment over  $[a, b]$ . Then the set*

$$U = U_W = \{x \in W_a : \varphi_{(a, t-a)}(x) \in W_t \setminus W_t^{--} \forall t \in [a, b]\}$$

*is open in  $W_a$  and the set of fixed points of the restriction*

$$\varphi_{(a, b-a)}|_U : U \rightarrow W_a$$

*is compact. Moreover*

$$\text{ind}(\varphi_{(a, b-a)}|_U) = \text{Lef}(\mu_W),$$

*where  $\text{Lef}(\mu_W)$  denotes the Lefschetz number of  $\mu_W$ .*

*In particular, if*

$$\text{Lef}(\mu_W) \neq 0$$

*then  $\varphi_{(a, b-a)}$  has a fixed point in  $W_a$ .*

(For the definition of the fixed point index and Lefschetz number see e.g. [5].)

## 2.6. CONTINUATION METHOD

Let  $X$  be a metric space. We denote by  $\rho$  the corresponding distance on  $\mathbb{R} \times X$ . Let  $\varphi$  be a local process on  $X$ ,  $T > 0$  and  $W, U$  be two subsets of  $\mathbb{R} \times X$ . We consider the following conditions (see [15, 17]):

(G1)  $W$  and  $U$  are  $T$ -periodic segments for  $\varphi$  which satisfy

$$U \subset W, \quad (U_0, U_0^{--}) = (W_0, W_0^{--}),$$

(G2) there exists  $\eta > 0$  such that for every  $(t, w) \in W^{--}$  and  $(t, z) \in U^{--}$  there exists  $\tau_0 > 0$  such that for  $0 < \tau < \tau_0$  it holds that  $(t + \tau, \varphi(t, \tau, w)) \notin W$ ,  $\rho((t + \tau_0, \varphi(t, \tau_0)(w)), W) > \eta$  and  $(t + \tau, \varphi(t, \tau, z)) \notin U$ ,  $\rho((t + \tau_0, \varphi(t, \tau_0)(z)), U) > \eta$ .

Let  $K$  be a positive integer and let  $E[1], \dots, E[K]$  be disjoint closed subsets of the essential exit set  $U^{--}$  which are  $T$ -periodic, i.e.  $E[l]_0 = E[l]_T$ , and such that

$$U^{--} = \bigcup_{l=1}^K E[l].$$

(In applications we will use the decomposition of  $U^{--}$  into connected components).

Before we can recall the method of continuation, we need one more definition related to the set  $W \subset \mathbb{R} \times X$ . For  $n \in \mathbb{N}$ ,  $D \subset W_0$  and every finite sequence  $c = (c_0, \dots, c_{n-1}) \in \{0, 1, \dots, K\}^{\{0, 1, \dots, n-1\}}$  we define  $D_c$  as a set of points satisfying the following conditions:

- (H1)  $\varphi_{(0, lT)}(x) \in D$  for  $l \in \{0, 1, \dots, n\}$ ,
- (H2)  $\varphi_{(0, lT+t)}(x) \in W_t \setminus W_t^{--}$  for  $t \in [0, T]$  and  $l \in \{0, 1, \dots, n-1\}$ ,
- (H3) for each  $l = 0, 1, \dots, n-1$ , if  $c_l = 0$ , then  $\varphi_{(0, lT+t)}(x) \in U_t \setminus U_t^{--}$  for  $t \in (0, T)$ ,
- (H4) for each  $l = 0, 1, \dots, n-1$ , if  $c_l > 0$ , then  $\varphi_{(0, lT)}(x)$  leaves  $U$  in time less than  $T$  through  $E[c_l]$ .

Let  $\Omega \subset \mathbb{R} \times \mathbb{R} \times X$  be open and

$$[0, 1] \times \Omega \ni (\lambda, \sigma, t, x) \mapsto \varphi_{(\sigma, t)}^\lambda(x) \in X$$

be a continuous family of  $T$ -periodic local processes on  $X$ . We say that the conditions (G1) and (G2) are satisfied *uniformly* (with respect to  $\lambda$ ) if they are satisfied with  $\varphi$  replaced by  $\varphi^\lambda$  and the same  $\eta$  in (G2) is valid for all  $\lambda \in [0, 1]$ .

We write  $D_c^\lambda$  for the set defined by the conditions (H1)–(H4) for the local process  $\varphi^\lambda$ .

The following theorem plays a crucial role in the method of continuation.

**Theorem 2.6** (see [15, 16]). *Let  $\varphi^\lambda$  be a continuous family of  $T$ -periodic local processes such that (G1) and (G2) hold uniformly. Then for every  $n > 0$  and every finite sequence  $c = (c_0, \dots, c_{n-1}) \in \{0, 1, \dots, K\}^{\{0, 1, \dots, n-1\}}$  the fixed point indices  $\text{ind}(\varphi_{(0, nT)}^\lambda |_{(W_0 \setminus W_0^{--})_c^\lambda})$  are correctly defined and equal to each other (i.e. do not depend on  $\lambda \in [0, 1]$ ).*

## 2.7. DISTRIBUTIONAL CHAOS

Let  $\mathbb{N}$  denote the set of positive integers and let  $f$  be a continuous self map of a compact metric space  $(X, \rho)$ . We define a function  $\xi_f : X \times X \times \mathbb{R} \times \mathbb{N} \rightarrow \mathbb{N}$  by:

$$\xi_f(x, y, t, n) = \#\{i : \rho(f^i(x), f^i(y)) < t, \quad 0 \leq i < n\},$$

where  $\#A$  denotes the cardinality of the set  $A$ . Using  $\xi_f$ , we define the following two functions:

$$F_{xy}(f, t) = \liminf_{n \rightarrow \infty} \frac{1}{n} \xi_f(x, y, t, n), \quad F_{xy}^*(f, t) = \limsup_{n \rightarrow \infty} \frac{1}{n} \xi_f(x, y, t, n).$$

For brevity, we often write  $\xi$ ,  $F_{xy}(t)$ ,  $F_{xy}^*(t)$  instead of  $\xi_f$ ,  $F_{xy}(f, t)$ ,  $F_{xy}^*(f, t)$  respectively.

Both functions  $F_{xy}$  and  $F_{xy}^*$  are nondecreasing,  $F_{xy}(t) = F_{xy}^*(t) = 0$  for  $t < 0$  and  $F_{xy}(t) = F_{xy}^*(t) = 1$  for  $t > \text{diam } X$ . Functions  $F_{xy}$  and  $F_{xy}^*$  are called *lower* and *upper distribution* functions, respectively.

The following definition is a stronger version of definition stated first by Schweizer and Smítal in [11].

**Definition 2.7.** A pair of points  $(x, y) \in X \times X$  is called *distributionally chaotic (of type 1)* if:

- 2.9.  $F_{xy}(s) = 0$  for some  $s > 0$ ,
- 2.10.  $F_{xy}^*(t) = 1$  for all  $t > 0$ .

A set containing at least two points is called a *distributionally scrambled set of type 1* (or a *d-scrambled set* for short) if any pair of its distinct points is distributionally chaotic.

A map  $f$  is *distributionally chaotic* (DC1) if it has an uncountable *d-scrambled set*. Distributional chaos is said to be *uniform* if the constant  $s$  from condition (2.9.) may be chosen the same for all the pairs of distinct points of a *d-scrambled set*.

We are working in the context of periodic local processes, so it is more natural to us the following definition in this context.

**Definition 2.8.** We say that a  $T$ -periodic local process  $\varphi$  on  $M$  is *(uniform) distributionally chaotic* if there exists a compact set  $\Lambda \subset M$  invariant for the Poincaré map  $P_T = \varphi_{(0, T)}$  such that  $P_T|_{\Lambda}$  is *(uniform) distributionally chaotic*.

We say that the equation (2.4) is *(uniform) distributionally chaotic* if it generates a local process which is *(uniform) distributionally chaotic*.

Now, it is possible to recall the following fact which will be used in the sequel.

**Theorem 2.9** ([10, Theorem 11]). *Let  $(X, d)$ ,  $(Y, \rho)$  be compact metric spaces and let  $f \in C(X)$ ,  $g \in C(Y)$ . Let  $\Phi : X \rightarrow Y$  be a semi-conjugacy such that  $\#\Phi^{-1}(y) \leq 2$  for some  $y \in \text{Per}(g)$ . If  $g$  is surjective and has WSP then  $f$  is distributionally chaotic and distributional chaos is uniform.*

### 3. RIGOROUS PROOF OF THE MAIN THEOREMS

#### 3.1. PROOF OF THEOREM 1.1

Before we start the proof of Theorem 1.1 we need a few preliminary lemmas and we also have to define some important regions in the extended phase space of our equation.



Let  $T = \frac{2\pi}{\kappa}$  be the minimal period of equation (1.1). We define sets

$$\begin{aligned}\widehat{B} &= \left\{ z \in \mathbb{C} : |\operatorname{Arg} z| \leq \frac{\pi}{6}, \Re z \leq \cos \frac{\pi}{6} \right\} \cup \{0\}, \\ \widehat{B}^b &= \left\{ z \in \widehat{B} : \Re z = \cos \frac{\pi}{6} \right\}\end{aligned}$$

and a rotation  $h : \mathbb{C} \ni z \mapsto e^{i\frac{\pi}{3}} z \in \mathbb{C}$ . We write

$$B = \bigcup_{k=0}^5 h^k(\widehat{B}), \quad B^b = \bigcup_{k=0}^2 h^{2k}(\widehat{B}^b), \quad B^\sharp = h(B^b)$$

and for any  $s \in \mathbb{R}^+$  we define

$$B(s) = \{sz : z \in B\}, \quad B(s)^b = \{sz : z \in B^b\}, \quad B(s)^\sharp = \{sz : z \in B^\sharp\}.$$

Note that  $B(s)$  is a regular hexagon centered at the origin with diameter  $2s$ . Next, we define sets

$$W(R, \kappa) = \left\{ (t, z) \in [0, T] \times \mathbb{C} : e^{-i\frac{t\kappa}{3}} z \in B(R) \right\}, \quad (3.1)$$

$$\widetilde{W}(R, \kappa) = \left\{ (t, z) \in [0, T] \times \mathbb{C} : e^{-i\frac{t\kappa}{3}} z \in B(R)^b \right\}, \quad (3.2)$$

$$\widetilde{\widetilde{W}}(R, \kappa) = \left\{ (t, z) \in [0, T] \times \mathbb{C} : e^{-i\frac{t\kappa}{3}} z \in B(R)^\sharp \right\}. \quad (3.3)$$

Let  $0 < r < R$ ,  $0 < \Delta < \frac{T}{2}$ ,  $\omega = \frac{R-r}{\Delta}$ . We define function  $s : [0, T] \rightarrow [r, R]$  given by

$$s(t) = \begin{cases} R - \omega t, & \text{for } t \in [0, \Delta], \\ r, & \text{for } t \in [\Delta, T - \Delta], \\ R - \omega(T - t), & \text{for } t \in [T - \Delta, T] \end{cases} \quad (3.4)$$

and sets

$$U(R, r, \kappa, \Delta) = \{(t, z) \in [0, T] \times \mathbb{C} : z \in B(s(t))\}, \quad (3.5)$$

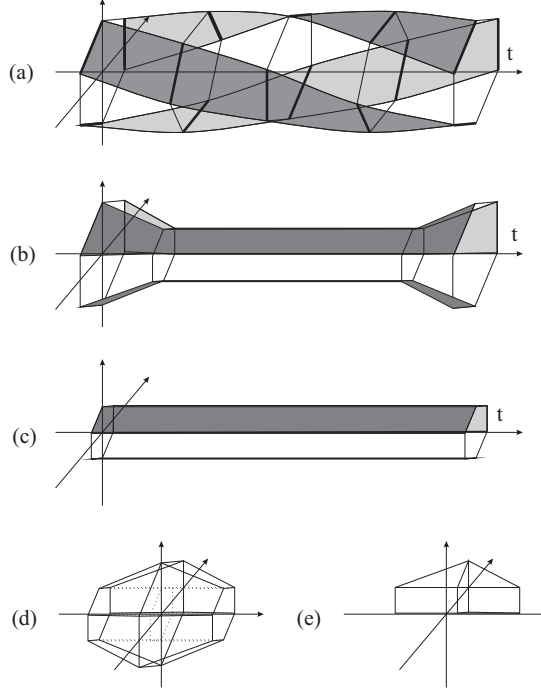
$$\widetilde{U}(R, r, \kappa, \Delta) = \{(t, z) \in [0, T] \times \mathbb{C} : z \in B(s(t))^b\}, \quad (3.6)$$

$$\widetilde{\widetilde{U}}(R, r, \kappa, \Delta) = \{(t, z) \in [0, T] \times \mathbb{C} : z \in B(s(t))^\sharp\}. \quad (3.7)$$

Finally, for any  $\xi > 0$  we define sets

$$V(\xi) = [0, T] \times B(\xi), \quad \widetilde{V}(\xi) = [0, T] \times B(\xi)^b, \quad \widetilde{\widetilde{V}}(\xi) = [0, T] \times B(\xi)^\sharp. \quad (3.8)$$

When the numbers  $R, r, \kappa, \Delta$  are fixed, and it causes no confusion, we simply write  $W, U$  instead of  $W(R, \kappa), U(R, r, \kappa, \Delta)$ , respectively, see Figure 1(a), 1(b).



**Fig. 1.** Isolating segments: (a)  $W$ ; (b)  $U$ ; (c)  $V(\xi)$  and sets: (d)  $Z$  and (e)  $\widehat{Z}$ .  
Sets  $W^{--}$ ,  $U^{--}$ ,  $V(\xi)^{--}$  are marked in grey

**Lemma 3.1.** Let  $\varphi$  be the local process generated by (1.1),  $\kappa > 0$ ,  $r = 0.69$ ,  $T = \frac{2\pi}{\kappa}$  and  $V(\xi)$  be given by (3.8). Then:

3.9.  $V(\xi)$  is a  $T$ -periodic isolating segment for  $\varphi$  for every  $\xi \in [2\sqrt{N}, r]$  such that the following equalities hold

$$V(\xi)^{--} = \widetilde{V}(\xi), \quad V(\xi)^{++} = \widetilde{V}(\xi), \quad (3.10)$$

provided the condition (1.3) is satisfied,

3.11.  $V(\xi)$  is a  $T$ -periodic isolating segment for  $\varphi$  for every  $\xi \in (0, r]$  such that (3.10) holds, provided that  $N = 0$ .

*Proof.* Similarly to the case of  $W(R, \kappa)$  and  $U(R, r, \kappa, \Delta)$  (see the proof of Lemma 3.6), it is enough to show that the vector field  $(1, v)$  points outward  $V(\xi)$  on the set

$$K(3, \xi) = V(\xi) \cap \left\{ (t, z) \in [0, T] \times \mathbb{C} : \Re(z) = \xi \cos \frac{\pi}{6} \right\}.$$

We parametrize the set  $K(3, \xi)$  by

$$s_3 : [0, T] \times [-1, 1] \ni (t, o) \mapsto \left( t, \xi \left( \cos \frac{\pi}{6} + io \sin \frac{\pi}{6} \right) \right) \in \mathbb{R} \times \mathbb{C},$$

An outward orthogonal vector  $n_3(t, o)$  in the point  $s_3(t, o)$  has the form

$$n_3(t, o) = [0, 1]^T \in \mathbb{R} \times \mathbb{C},$$

Thus, by Lemma 3.3, the inner product of the vector field  $(1, v)$  and an outward normal vector in the point  $s_3(t, o)$  can be estimated by

$$\begin{aligned} \langle (1, v(s_3(t, o))), n_3(t, o) \rangle &= \Re \left[ \xi^2 \left( \cos \frac{\pi}{6} - io \sin \frac{\pi}{6} \right)^2 \times \right. \\ &\quad \times \left. \left( 1 + e^{ikt} \xi^2 \left[ \cos^2 \frac{\pi}{6} + o^2 \sin^2 \frac{\pi}{6} \right] \right) - N e^{\frac{-i\pi}{3}} \right] \geq \\ &\geq \frac{1}{2} \xi^2 - \xi^4 - N = (\star). \end{aligned}$$

It is easy to see, that  $(\star) > 0$  provided that  $N = 0$  and  $\xi \in (0, r]$  or conditions (1.3) and  $\xi \in [2\sqrt{N}, r]$  are satisfied.  $\square$

**Lemma 3.2.** *Let  $\kappa$  and  $N$  satisfy (1.2), (1.3) and  $r = 0.69$ ,  $R = 1.45$ ,  $\Delta = 2.9$ ,  $T = \frac{2\pi}{\kappa}$ . Let  $\varphi$  be the local process generated by (1.1) and  $U, V(r)$  be given by (3.5) and (3.8). Then every solution  $\eta$  of (1.1) which enters  $(U \setminus V(r))^\infty$  has to leave  $U^\infty$  i.e. if there exists  $t \in \mathbb{R}$  such that  $(t, \eta(t)) \in (U \setminus V(r))^\infty$  then there exists  $t_0 \in \mathbb{R}$  such that  $(t_0, \eta(t_0)) \notin U^\infty$ .*

*Proof.* Let us define

$$Z = ((U \setminus V(r))^\infty)_{[-\Delta, \Delta]}. \quad (3.12)$$

By the symmetries of the vector field  $\bar{z}^2$ , it is enough to prove that every solution of (1.1) which enters

$$\widehat{Z} = Z \cap \left\{ (t, z) \in \mathbb{R} \times \mathbb{C} : |\text{Arg } z| \leq \frac{\pi}{6} \right\} \quad (3.13)$$

leaves  $Z$  in time interval  $[-\Delta - \varepsilon, \Delta + \varepsilon]$  for any  $\varepsilon > 0$ .

By Lemma 3.1, the set

$$\widehat{Z} \cap \left\{ (t, z) \in \mathbb{R} \times \mathbb{C} : \Re(z) = r \cos \frac{\pi}{6} \right\}$$

is a subset of  $((V(r))^\infty)_{[-T, T]}^-$ , so it is also a subset of  $\widehat{Z}^+$ . To finish the proof, it is enough to show that

$$\widehat{Z} \cap \left\{ (t, z) \in \mathbb{R} \times \mathbb{C} : |\text{Arg } z| = \frac{\pi}{6} \right\} \subset \widehat{Z}^+$$

holds. By the symmetries of  $\bar{z}^2$ , it is enough to prove that

$$\widetilde{Z} = \widehat{Z} \cap \left\{ (t, z) \in \mathbb{R} \times \mathbb{C} : \text{Arg } z = \frac{\pi}{6} \right\} \subset \widehat{Z}^+.$$

We show that the vector field  $(1, v)$  points inward  $Z$  on the set  $\widetilde{Z}$ .

We claim that for every point of the set  $\widetilde{Z}$  the inner product of an outward normal vector of  $Z$  and the vector field  $(1, v)$  is negative.

Let

$$s_4 : [-\Delta, \Delta] \times [0, 1] \ni (t, o) \mapsto \left( t, e^{i\frac{\pi}{6}} \left[ r + o(R-r) \left( 1 - \frac{|t|}{\Delta} \right) \right] \right) \in \mathbb{R} \times \mathbb{C}$$

be a parametrization of  $\tilde{Z}$ . An outward normal vector in every point of the set  $\tilde{Z}$  has the form  $n_4 = \left[ 0, e^{i\frac{2}{3}\pi} \right]^T$ . Thus

$$\begin{aligned} \langle (1, v(s_4(t, o))), n_4(t, o) \rangle &= \Re \left[ e^{-i\pi} \left[ r + o(R-r) \left( 1 - \frac{|t|}{\Delta} \right) \right]^2 + \right. \\ &\quad \left. + e^{i\kappa t - i\pi} \left[ r + o(R-r) \left( 1 - \frac{|t|}{\Delta} \right) \right]^4 - N e^{-i\pi} \right] \leq \\ &\leq -r^2 - \cos(\kappa t) \left[ r + o(R-r) \left( 1 - \frac{|t|}{\Delta} \right) \right]^4 + N \leq \\ &\leq -r^2 + N \end{aligned}$$

holds, so it is enough if the following inequality is satisfied

$$r^2 > N.$$

But it is guaranteed by the assumptions.  $\square$

**Lemma 3.3.** *Let  $|\alpha| \leq \frac{\pi}{6}$ . Then the equality*

$$\min_{\tau \in [-1, 1]} \Re \left[ e^{i\alpha} \left( \cos \frac{\pi}{6} - i\tau \sin \frac{\pi}{6} \right)^2 \left[ \cos^2 \frac{\pi}{6} + \tau^2 \sin^2 \frac{\pi}{6} \right] \right] = \sin \left( \frac{\pi}{6} - |\alpha| \right)$$

holds. Moreover,

$$\min_{\tau \in [-1, 1]} \Re \left[ \left( \cos \frac{\pi}{6} - i\tau \sin \frac{\pi}{6} \right)^2 \right] = \frac{1}{2}$$

holds.

*Proof.* The proof is a consequence of direct calculations. It is left to the reader.  $\square$

**Lemma 3.4.** *Let  $n$  be a positive integer,  $T > 0$ ,  $0 < r_1 < r_2$  and  $\varphi$  be a  $T$ -periodic local process on a Banach space  $X$ . Let  $\{V(\xi)\}_{\xi \in [r_1, r_2]}$  be a family of  $T$ -periodic isolating segments for  $\varphi$  such that:*

- 3.14.  $V(\xi) = \left\{ (t, x) \in [0, T] \times X : \left( t, \frac{r_2}{\xi} x \right) \in V(r_2) \right\}$  for every  $\xi \in [r_1, r_2]$ ,
- 3.15.  $V(\xi_1) \subset V(\xi_2)$  for every  $r_1 \leq \xi_1 < \xi_2 \leq r_2$ ,
- 3.16.  $\text{dist}(\partial[V(\xi_1)^\infty], \partial[V(\xi_2)^\infty]) > 0$  for every  $r_1 \leq \xi_1 < \xi_2 \leq r_2$ .

Then:

- 3.17. for every  $t \in [0, T]$  the trajectory of every point  $x$  from  $V(r_2)_t \setminus \text{int}(V(r_1)_t)$  leaves  $V(r_2)^\infty$  i.e. for every  $x \in V(r_2)_t \setminus \text{int}(V(r_1)_t)$  there exists a time  $\tau \in \mathbb{R}$  such that  $(t + \tau, \varphi_{(t, \tau)}(x)) \notin V(r_2)^\infty$ ,
- 3.18. for every  $\xi \in [r_1, r_2)$  there exist  $\tau_0 > 0$  and  $0 < \varepsilon \leq r_2 - \xi$  such that for every  $(t, x) \in V(\xi)^{-}$  the condition  $(t + \tau, \varphi_{(t, \tau)}(x)) \notin V(\xi)^\infty$  holds for every  $0 < \tau < \tau_0$  and additionally  $(t + \tau_0, \varphi_{(t, \tau_0)}(x)) \notin V(\xi + \varepsilon)^\infty$  is satisfied.

**Remark 3.5.**

- 3.19. The context makes it clear whether  $\partial$  denotes a boundary with respect to the space  $X$  or to  $\mathbb{R} \times X$ .
- 3.20. If  $\text{int}(V(\xi)_t) = \emptyset$  for some  $t \in [0, T]$  and  $\xi \in [r_1, r_2]$ , then  $\text{int}(V(r_1)_t) = \text{int}(V(r_2)_t) = \emptyset$  and  $\partial(V(r_1)_t) \subset \partial(V(r_2)_t)$  which contradicts the assumption (3.16.).

*Proof of Lemma 3.4.* Let:

$$D^b = \{(t, x) \in \mathbb{R} \times X : \text{there exists } \xi \in [r_1, r_2] \text{ such that } (t, x) \in V(\xi)^{-}\},$$

$$D^\sharp = \{(t, x) \in \mathbb{R} \times X : \text{there exists } \xi \in [r_1, r_2] \text{ such that } (t, x) \in V(\xi)^{++}\}.$$

We write

$$D = D^b \cup D^\sharp = \bigcup_{t \in [0, T]} \{t\} \times [V(r_2)_t \setminus \text{int}(V(r_1)_t)].$$

Let  $(t, x) \in D$ . We define numbers  $\alpha_{(t, x)}$ ,  $\omega_{(t, x)}$  by

$$\alpha_{(t, x)} = \inf \{\tau_0 \in \mathbb{R} : (t + \tau, \varphi_{(t, \tau)}(x)) \in D^\infty \text{ for every } \tau_0 \leq \tau \leq 0\},$$

$$\omega_{(t, x)} = \sup \{\tau_0 \in \mathbb{R} : (t + \tau, \varphi_{(t, \tau)}(x)) \in D^\infty \text{ for every } 0 \leq \tau \leq \tau_0\}$$

and mapping

$$L_{(t, x)} : (\alpha_{(t, x)}, \omega_{(t, x)}) \ni \tau \mapsto L_{(t, x)}(\tau) \in [r_1, r_2]$$

where  $L_{(t, x)}(\tau) = \xi$ , provided

$$\varphi_{(t, \tau)}(x) \in \partial [[V(\xi)^\infty]_{t+\tau}].$$

Observe that by assumption (3.16.),  $L_{(t, x)}$  is well defined. To see that  $L_{(t, x)}$  is continuous let us consider  $\{\tau_n\}_{n \in \mathbb{N}} \subset (\alpha_{(t, x)}, \omega_{(t, x)})$  such that  $\lim_{n \rightarrow \infty} \tau_n = \tau$ . We denote  $\xi_n = L_{(t, x)}(\tau_n)$  and  $\xi = L_{(t, x)}(\tau)$ . We claim that  $\lim_{n \rightarrow \infty} \xi_n = \xi$ . To obtain a contradiction, let us assume that there exists a subsequence  $\{\xi_{n_k}\}_{k \in \mathbb{N}} \subset \{\xi_n\}_{n \in \mathbb{N}}$  such that its limit  $\xi^*$  satisfies  $\xi^* \neq \xi$ .

We have to consider the two possible cases. First, let us assume that  $\xi^* < \xi$  and define  $\mu = \frac{\xi^* + \xi}{2}$ . Then  $\xi_{n_k} < \mu$  for almost every  $n_k$ . Thus  $\varphi_{(t, \tau_{n_k})}(x) \in V(\mu)^\infty$  and so  $\varphi_{(t, \tau)}(x) \in V(\mu)^\infty$ . But by the assumptions of the lemma, we get  $\text{dist}(V(\mu)^\infty, \partial[V(\xi)^\infty]) > 0$  which is the desired contradiction.

For the second case, let  $\xi^* > \xi$ . Once again we denote  $\mu = \frac{\xi^* + \xi}{2}$ . Then  $\xi_{n_k} > \mu$  for almost every  $n_k$ . Thus  $\varphi_{(t, \tau_{n_k})}(x) \in V(r_2)^\infty \setminus V(\mu)^\infty$  and so  $\varphi_{(t, \tau)}(x) \in V(r_2)^\infty \setminus \text{int}[V(\mu)^\infty]$ . But by the assumptions of the lemma,  $\text{dist}(V(\xi)^\infty, \partial[V(\mu)^\infty]) > 0$  and, as previously, we have a contradiction. The proof of the claim is finished.

Let  $\tau > 0$  and  $\varphi(\{t\} \times [0, \tau] \times \{x\}) \subset D^b$ . Then by the definition of the exit set,  $L_{(t, x)}|_{[0, \tau]}$  is strictly increasing. Similarly, if  $\varphi(\{t\} \times [-\tau, 0] \times \{x\}) \subset D^\sharp$ , then  $L_{(t, x)}|_{[-\tau, 0]}$  is strictly decreasing.

Fix any  $(t, x) \in D^b$  and  $\tau > 0$  such that  $\varphi(\{t\} \times [0, \tau] \times \{x\}) \subset D^b$ . We claim that  $\varphi_{(t, \tau)}(x) \in D^b \setminus D^\sharp$ . Indeed, if  $(t + \tau, y) = \varphi_{(t, \tau)}(x) \in D^b \cap D^\sharp$ , then, by the monotonicity of  $L_{(t, x)}|_{[0, \tau]}$  we get

$$\varphi(\{t + \tau\} \times [-\tau, 0] \times \{y\}) = \varphi(\{t\} \times [0, \tau] \times \{x\}) \subset V(L_{(t+\tau, y)}(0)).$$

But  $(t + \tau, y) \in V(L_{(t+\tau, y)}(0))^{++}$ , so  $\varphi(\{t + \tau\} \times [-\tau, 0] \times \{y\}) \not\subset V(L_{(t+\tau, y)}(0))$ , which is the desired contrary.

We have just proved that if  $(t, x) \in D^b$  then  $L_{(t, x)}([0, \omega_{(t, x)}]) \subset D^b$ . Since points from  $V(r_1)^{--}$  cannot leave  $D$  at positive time through  $V(r_1)^{--}$ , every point from  $D^b$  leaves  $V(r_2)^\infty$  at positive time, i.e. for every  $(t, x) \in D$  there exists  $\tau > \omega_{(t, x)} > 0$  such that  $\varphi(t, \tau, x) \notin V(r_2)^\infty$  and  $\varphi(t, \omega_{(t, x)}, x) \in (V(r_2)^{--})^\infty$ .

Analogously, if  $(t, x) \in D^\sharp$ , then there exists  $\tau < \alpha_{(t, x)} < 0$  such that  $\varphi(t, \tau, x) \notin V(r_2)^\infty$  and  $\varphi(t, \alpha_{(t, x)}, x) \in (V(r_2)^{++})^\infty$ . This ends the proof of (3.17).

We fix  $\xi \in [r_1, r_2)$ . Let  $\sigma^*$  be the Ważewski escape-time function for  $V(r_2)^\infty$ . Observe that  $\tau_0 = \min\{\sigma^*(w) : w \in V(\xi)^{--}\}$  is well defined by compactness of  $V(\xi)^{--}$  and continuity of  $\sigma^*$  (see Lemma 2.4). Then for  $\tau_0 > 0$ , the set

$$L = \{(t + \tau_0, \varphi(t, \tau_0, x)) : (t, x) \in V(\xi)^{--}\} \subset V(r_2)$$

is compact and  $L \cap V(\xi)^\infty = \emptyset$ , so there exists  $\varepsilon > 0$  such that  $V(\xi + \varepsilon)^\infty \cap L = \emptyset$ . The proof of (3.18.) is finished.  $\square$

**Lemma 3.6.** *Let  $\kappa$  and  $N$  satisfy (1.2), (1.3) and  $r = 0.69$ ,  $R = 1.45$ ,  $\Delta = 2.9$ ,  $T = \frac{2\pi}{\kappa}$ . Let  $\varphi^\lambda$  be the local process generated by (1.1) where  $\lambda = N$ . Then:*

- 3.21.  $\{\varphi^\lambda\}_{\lambda \in [0, N]}$  is a continuous family of local processes,
- 3.22. sets  $W(R, \kappa)$  and  $U(R, r, \kappa, \Delta)$  given by (3.1) and (3.5), respectively, satisfy conditions (G1) and (G2) uniformly with respect to  $\lambda$ ,
- 3.23. the equalities

$$W(R, \kappa)^{--} = \widetilde{W}(R, \kappa), \tag{3.24}$$

$$W(R, \kappa)^{++} = \widetilde{\widetilde{W}}(R, \kappa), \tag{3.25}$$

$$U(R, r, \kappa, \Delta)^{--} = \widetilde{U}(R, r, \kappa, \Delta), \tag{3.26}$$

$$U(R, r, \kappa, \Delta)^{++} = \widetilde{\widetilde{U}}(R, r, \kappa, \Delta) \tag{3.27}$$

hold for  $\widetilde{W}(R, \kappa)$ ,  $\widetilde{\widetilde{W}}(R, \kappa)$ ,  $\widetilde{U}(R, r, \kappa, \Delta)$  and  $\widetilde{\widetilde{U}}(R, r, \kappa, \Delta)$  given by (3.2), (3.3), (3.6) and (3.7), respectively,

3.28. *there is a decomposition into disjointed connected components*

$$U(R, r, \kappa, \Delta)^{-} = E[1] \cup E[2] \cup E[3]$$

where  $E[1]$ ,  $E[2]$  and  $E[3]$  are given by

$$E[1] = \left\{ (t, z) \in \tilde{U}(R, r, \kappa, \Delta) : |\operatorname{Arg} z| \leq \frac{\pi}{6} \right\}, \quad (3.29)$$

$$E[2] = \left\{ (t, z) \in \tilde{U}(R, r, \kappa, \Delta) : (t, h^{-2}(z)) \in E[1] \right\}, \quad (3.30)$$

$$E[3] = \left\{ (t, z) \in \tilde{U}(R, r, \kappa, \Delta) : (t, h^{-4}(z)) \in E[1] \right\}. \quad (3.31)$$

*Proof.* First, let us notice that  $\{\varphi^\lambda\}_{\lambda \in [0, N]}$  is a continuous family of local processes by the continuous dependence of the vector field.

Now we prove that  $W(R, \kappa)$  and  $U(R, r, \kappa, \Delta)$  satisfy condition (G1) and (G2) uniformly with respect to  $\lambda$ . We start with showing that  $W(R, \kappa)$  is an isolating segment for every  $\varphi^\lambda$ , where  $\lambda \in [0, N]$ .

By the symmetries of the vector field  $u(z) = \bar{z}^2$  and estimation

$$\left| \Re \left[ e^{i\kappa t} N e^{-\frac{i\pi}{3}} \right] \right| \leq N,$$

it is enough to show that the vector field  $(1, v)$  points outward  $W(R, \kappa)$  on the set

$$K(1, R, \kappa) = W(R, \kappa) \cap \left\{ (t, z) \in [0, T] \times \mathbb{C} : \Re \left[ z e^{-i\frac{\kappa t}{3}} \right] = R \cos \frac{\pi}{6} \right\}.$$

We parametrize the set  $K(1, R, \kappa)$  by

$$s_1 : [0, T] \times [-1, 1] \ni (t, o) \mapsto \left( t, R e^{\frac{it\kappa}{3}} \left( \cos \frac{\pi}{6} + io \sin \frac{\pi}{6} \right) \right) \in [0, T] \times \mathbb{C}.$$

An outward orthogonal vector  $n_1(t, o)$  in the point  $s_1(t, o)$  has the form

$$n_1(t, o) = \left[ R \frac{\kappa}{3} o \sin \frac{\pi}{6}, e^{\frac{it\kappa}{3}} \right]^T \in \mathbb{R} \times \mathbb{C}.$$

Thus, by Lemma 3.3, the inner product of the vector field  $(1, v)$  and an outward orthogonal vector in the point  $s_1(t, o)$  can be estimated by

$$\begin{aligned} \langle (1, v(s_1(t, o))), n_1(t, o) \rangle &= R \frac{\kappa}{6} o + \Re \left[ e^{-\frac{it\kappa}{3}} R^2 e^{-\frac{2it\kappa}{3}} \left( \cos \frac{\pi}{6} - io \sin \frac{\pi}{6} \right)^2 \times \right. \\ &\quad \left. \times \left( 1 + e^{it\kappa} R^2 \left[ \cos^2 \frac{\pi}{6} + o^2 \sin^2 \frac{\pi}{6} \right] \right) - e^{-\frac{it\kappa}{3}} N e^{-\frac{i\pi}{3}} \right] \geq \\ &\geq -R \frac{\kappa}{6} - R^2 + R^4 \sin \frac{\pi}{6} - N. \end{aligned}$$

Then the vector field  $(1, v)$  points outward  $W(R, \kappa)$  in every point of  $K(1, R, \kappa)$ , provided that

$$\frac{1}{2} R^4 > R \frac{\kappa}{6} + R^2 + N \quad (3.32)$$

holds.

Now we show that  $U(R, r, \kappa, \Delta)$  is an isolating segment for every  $\varphi^\lambda$ . Let  $\omega = \frac{R-r}{\Delta}$ . Analogous to the case of  $W(R, \kappa)$ , it is enough to show that the vector field  $(1, v)$  points outward  $U(R, r, \kappa, \Delta)$  on the set

$$K(2, R, r, \kappa, \Delta) = U(R, r, \kappa, \Delta) \cap \left\{ (t, z) \in [0, T] \times \mathbb{C} : \Re z = s(t) \cos \frac{\pi}{6} \right\}$$

where  $s$  is given by (3.4). We parametrize the set  $K(2, R, r, \kappa, \Delta) \in \mathbb{R} \times \mathbb{C}$  by

$$\begin{aligned} s_{2,1} : [0, \Delta] \times [-1, 1] \ni (t, o) &\mapsto \left( t, (R - \omega t) \left( \cos \frac{\pi}{6} + io \sin \frac{\pi}{6} \right) \right), \\ s_{2,2} : [\Delta, T - \Delta] \times [-1, 1] \ni (t, o) &\mapsto \left( t, r \left( \cos \frac{\pi}{6} + io \sin \frac{\pi}{6} \right) \right), \\ s_{2,3} : [T - \Delta, T] \times [-1, 1] \ni (t, o) &\mapsto \left( t, [R - \omega(T - t)] \left( \cos \frac{\pi}{6} + io \sin \frac{\pi}{6} \right) \right). \end{aligned}$$

An outward orthogonal vector  $n_{2,j}(t, o)$  in the point  $s_{2,j}(t, o)$  for  $j \in \{1, 2, 3\}$  has the form

$$\begin{aligned} n_{2,1}(t, o) &= \left[ \omega \cos \frac{\pi}{6}, 1 \right]^T \in \mathbb{R} \times \mathbb{C}, \\ n_{2,2}(t, o) &= [0, 1]^T \in \mathbb{R} \times \mathbb{C}, \\ n_{2,3}(t, o) &= \left[ -\omega \cos \frac{\pi}{6}, 1 \right]^T \in \mathbb{R} \times \mathbb{C}. \end{aligned}$$

Let  $\kappa\Delta \leq \frac{\pi}{6}$ . Thus, by Lemma 3.3, the inner product of the vector field  $(1, v)$  and an outward orthogonal vector in the point  $s_{2,j}(t, o)$  for  $j \in \{1, 2, 3\}$  can be estimated by

$$\begin{aligned} \langle (1, v(s_{2,1}(t, o))), n_{2,1}(t, o) \rangle &= \frac{\sqrt{3}}{2}\omega + \Re \left[ N e^{\frac{2i\pi}{3}} + (R - \omega t)^2 \left( \cos \frac{\pi}{6} - io \sin \frac{\pi}{6} \right)^2 \times \right. \\ &\quad \left. \times \left( 1 + e^{i\kappa t} (R - \omega t)^2 \left[ \cos^2 \frac{\pi}{6} + o^2 \sin^2 \frac{\pi}{6} \right] \right) \right] \geq \\ &\geq \frac{\sqrt{3}}{2}\omega + \frac{1}{2}(R - \omega t)^2 + (R - \omega t)^4 \sin \left( \frac{\pi}{6} - \kappa\Delta \right) - N \geq \\ &\geq \frac{1}{2}r^2 + r^4 \sin \left( \frac{\pi}{6} - \kappa\Delta \right) + \frac{\sqrt{3}}{2}\omega - N, \end{aligned}$$

$$\begin{aligned} \langle (1, v(s_{2,2}(t, o))), n_{2,2}(t, o) \rangle &= \Re \left[ r^2 \left( \cos \frac{\pi}{6} - io \sin \frac{\pi}{6} \right)^2 \times \right. \\ &\quad \left. \times \left( 1 + e^{i\kappa t} r^2 \left[ \cos^2 \frac{\pi}{6} + o^2 \sin^2 \frac{\pi}{6} \right] \right) - N e^{-\frac{i\pi}{3}} \right] \geq \\ &\geq \frac{1}{2}r^2 - r^4 - N, \end{aligned}$$

$$\begin{aligned} \langle (1, v(s_{2,3}(t, o))), n_{2,3}(t, o) \rangle &= -\frac{\sqrt{3}}{2}\omega + \Re \left[ [R - \omega(T - t)]^2 \left( \cos \frac{\pi}{6} - io \sin \frac{\pi}{6} \right)^2 \times \right. \\ &\quad \left. \times \left( 1 + e^{i\kappa t} [R - \omega(T - t)]^2 \left[ \cos^2 \frac{\pi}{6} + o^2 \sin^2 \frac{\pi}{6} \right] \right) - N e^{-\frac{i\pi}{3}} \right] \geq \\ &\geq \frac{1}{2}r^2 + r^4 \sin \left( \frac{\pi}{6} - \kappa\Delta \right) - \frac{\sqrt{3}}{2}\omega - N. \end{aligned}$$



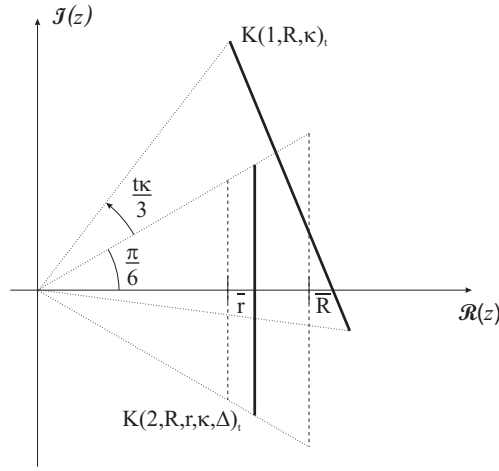
Thus the vector field  $(1, v)$  points outward  $U(R, r, \kappa, \Delta)$  in every point of the set  $K(2, R, r, \kappa, \Delta)$ , provided that  $R > r$  and

$$\kappa\Delta \leq \frac{\pi}{6}, \tag{3.33}$$

$$\frac{1}{2}r^2 + r^4 \sin\left(\frac{\pi}{6} - \kappa\Delta\right) > \frac{\sqrt{3}}{2} \frac{R-r}{\Delta} + N, \tag{3.34}$$

$$\frac{1}{2}r^2 > r^4 + N \tag{3.35}$$

hold.



**Fig. 2.** Sets  $K(1, R, \kappa)_t$  and  $K(2, R, r, \kappa, \Delta)_t$  (here  $\bar{R} = R \cos(\frac{\pi}{6})$ ,  $\bar{r} = r \cos(\frac{\pi}{6})$ )

Now we show that  $U(R, r, \kappa, \Delta) \subset W(R, \kappa)$  holds. Let  $t \in [\Delta, T - \Delta]$ . Then  $U(R, r, \kappa, \Delta)_t \subset W(R, \kappa)_t$  provided that

$$r < R \cos \frac{\pi}{6}. \tag{3.36}$$

Now let  $t \in [0, T] \cup [T - \Delta, T]$ . It is enough to consider the set  $K(1, R, \kappa)$  and  $K(2, R, r, \kappa, \Delta)$  (see Fig. 2). We parametrize the set  $K(1, R, \kappa)$  by

$$\tilde{s}_1 : [0, T] \times \left[-\frac{\pi}{6}, \frac{\pi}{6}\right] \ni (t, \beta) \mapsto (t, \tilde{s}_{(1,t)}(\beta)),$$

where

$$\tilde{s}_{(1,t)}(\beta) = Re^{\frac{it\kappa}{3}} (1 + i \tan \beta) \cos \frac{\pi}{6}.$$

Let  $t \in (0, \Delta]$ . If  $U(R, r, \kappa, \Delta)_t \subset W(R, \kappa)_t$ , then the point  $\tilde{s}_{(1,t)}\left(\frac{\pi}{6} - \frac{t\kappa}{3}\right)$  is a point of  $K(1, R, \kappa)_t$  closest to  $K(2, R, r, \kappa, \Delta)$ . But  $\text{Arg} \left[\tilde{s}_{(1,t)}\left(\frac{\pi}{6} - \frac{t\kappa}{3}\right)\right] = \frac{\pi}{6}$  and

$$\left\{z \in K(2, R, r, \kappa, \Delta)_t : \text{Arg } z = \frac{\pi}{6}\right\} = \{(R - \omega t)e^{i\frac{\pi}{6}}\}.$$

Thus  $U(R, r, \kappa, \Delta)_t \subset W(R, \kappa)_t$ , provided the inequality

$$\begin{aligned} R - \omega t &\leq \left| Re^{\frac{it\kappa}{3}} \left( 1 + i \tan \left( \frac{\pi}{6} - \frac{t\kappa}{3} \right) \right) \cos \frac{\pi}{6} \right| = \\ &= R \frac{\cos \frac{\pi}{6}}{\cos \left( \frac{\pi}{6} - \frac{t\kappa}{3} \right)} \end{aligned}$$

holds for every  $t \in [0, \Delta]$ . Since there is equality for  $t = 0$ , it is enough to compare the derivatives. We get

$$\omega \geq \frac{\kappa}{3} R \frac{\sin \left( \frac{\pi}{6} - \frac{t\kappa}{3} \right)}{\cos^2 \left( \frac{\pi}{6} - \frac{t\kappa}{3} \right)} \cos \frac{\pi}{6}.$$

The right side attains its maximum for  $t = 0$  thus it is enough if the following condition is satisfied

$$\omega \geq \frac{\kappa}{3} R \tan \frac{\pi}{6}.$$

Finally, we get

$$3\sqrt{3} \left( 1 - \frac{r}{R} \right) \geq \kappa \Delta. \quad (3.37)$$

By the fact that

$$K(2, R, r, \kappa, \Delta)_{T-t} = K(2, R, r, \kappa, \Delta)_t,$$

the inclusion  $U(R, r, \kappa, \Delta)_t \subset W(R, \kappa)_t$  holds for  $t \in [T - \Delta, T]$ , provided that (3.37) is satisfied.

We have just proved that  $U(R, r, \kappa, \Delta)$  and  $W(R, \kappa)$  satisfy condition (G1) uniformly with respect to  $\lambda$ , provided that (3.32), (3.33), (3.34), (3.35), (3.36) and (3.37) hold. But these inequalities hold for the values of  $R, r, \kappa, \Delta$  and  $N$  fixed in the assumptions of the lemma.

Note that there exists  $0 < \delta_0$  such that the inequalities (3.32), (3.34) and (3.35) hold for every  $R + \delta, r + \delta$  (i.e.  $U(R + \delta, r + \delta, \kappa, \Delta)$  and  $W(R + \delta, \kappa)$  are isolating segments for every  $\varphi^\lambda$ ) where  $0 < \delta < \delta_0$  and  $R, r$  are such as in the assumptions of the lemma. Since  $\text{dist}(U(R + \delta_1, r + \delta_1, \kappa, \Delta)^\infty, \partial U(R + \delta_2, r + \delta_2, \kappa, \Delta)^\infty) > 0$  and  $\text{dist}(W(R + \delta_1, \kappa)^\infty, \partial W(R + \delta_2, \kappa)^\infty) > 0$  for  $0 \leq \delta_1 < \delta_2 \leq \delta_0$ , point (2) of Lemma 3.4 implies that condition (G2) is satisfied uniformly with respect to  $\lambda$  for  $\lambda \in [0, N]$ .  $\square$

The following lemma refers to some objects defined later in the proof of Theorem 1.1. We decided to state these calculations outside of the main proof, to make the presentation more transparent.

**Lemma 3.7.** *Let  $\kappa > 0$ ,  $T = \frac{2\pi}{\kappa}$  and let (1.4) be satisfied. We denote by  $\varphi$  the local process generated by (1.1). Then the sets  $E\left(\frac{\sqrt{N}}{20}\right)$ ,  $\tilde{E}\left(\frac{\sqrt{N}}{20}\right)$  given by (3.41) and (3.42), respectively, are  $T$ -periodic isolating segments for  $\varphi$  satisfying conditions (3.43), (3.44), (3.45) and (3.46).*

*Proof.* By the symmetry (3.52), it is enough to prove that  $E\left(\frac{\sqrt{N}}{20}\right)$  is a  $T$ -periodic isolating segment for  $\varphi$  such that (3.43) and (3.44) hold.

Let  $M = \sqrt{N}$ . Then one gets

$$v(t, z) = \overline{(z - Me^{i\frac{\pi}{6}})(z + Me^{i\frac{\pi}{6}})} + e^{i\kappa t}|z|^2\bar{z}^2.$$

Now we parametrize one side of  $E\left(\frac{M}{20}\right)$  by

$$s_5 : [0, T] \times [-1, 1] \ni (t, o) \mapsto \left( t, Me^{i\frac{\pi}{6}} + \frac{M}{20}e^{-i\frac{\pi}{12}} \left( \frac{\sqrt{2}}{2} + io\frac{\sqrt{2}}{2} \right) \right).$$

An outward normal vector in every point of this side has the form  $n_5 = [0, e^{-i\frac{\pi}{12}}]^T$ . Thus

$$\begin{aligned} \langle (1, v(s_5(t, o))), n_5(t, o) \rangle &= \Re \left[ \frac{M}{20}e^{i\frac{\pi}{6}} \left( \frac{\sqrt{2}}{2} - io\frac{\sqrt{2}}{2} \right) 2Me^{-i\frac{\pi}{6}} \right] + \\ &+ \Re \left[ \frac{M^2}{20^2}e^{i\frac{\pi}{3}} \left( \frac{\sqrt{2}}{2} - io\frac{\sqrt{2}}{2} \right)^2 \right] + \\ &+ \Re \left[ e^{i\kappa t + i\frac{\pi}{12}} \left( Me^{-i\frac{\pi}{6}} + \frac{M}{20}e^{i\frac{\pi}{12}} \left( \frac{\sqrt{2}}{2} - io\frac{\sqrt{2}}{2} \right) \right)^2 \right] \times \\ &\times \left| Me^{i\frac{\pi}{6}} + \frac{M}{20}e^{-i\frac{\pi}{12}} \left( \frac{\sqrt{2}}{2} + io\frac{\sqrt{2}}{2} \right) \right|^2 \geq \\ &\geq \sqrt{2}\frac{M^2}{20} - \frac{M^2}{20^2} - \left( \frac{21}{20}M \right)^4 > \\ &> 0 \end{aligned}$$

holds, so the vector field  $(1, v)$  points outward  $E\left(\frac{M}{20}\right)$  on the side. Calculations for other sides of  $E\left(\frac{M}{20}\right)$  are similar. This proves (3.43) and (3.44).  $\square$

*Proof of Theorem 1.1.* The proof consists of the following steps. Firstly we construct two periodic isolating segments. Then we apply Theorem 2.6, [17, Theorem 7] and Theorem 2.9.

We fix  $\kappa$  and  $N$  satisfying (1.2), (1.3) and set  $r = 0.69$ ,  $R = 1.45$ ,  $\Delta = 2.9$ . Let  $\varphi^\lambda$  be the local process generated by (1.1) where  $\lambda = N$ . Then  $\{\varphi^\lambda\}_{\lambda \in [0, N]}$  is a continuous family of local processes. By Lemma 3.6, sets  $W(R, \kappa)$  and  $U(R, r, \kappa, \Delta)$  satisfy the

conditions (G1) and (G2) uniformly with respect to  $\lambda$ . Furthermore, the conditions (3.24)–(3.27) hold, that is

$$\begin{aligned} W(R, \kappa)^{-} &= \widetilde{W}(R, \kappa), \\ W(R, \kappa)^{++} &= \widetilde{\widetilde{W}}(R, \kappa), \\ U(R, r, \kappa, \Delta)^{-} &= \widetilde{U}(R, r, \kappa, \Delta), \\ U(R, r, \kappa, \Delta)^{++} &= \widetilde{\widetilde{U}}(R, r, \kappa, \Delta) \end{aligned}$$

and  $U(R, r, \kappa, \Delta)^{-} = E[1] \cup E[2] \cup E[3]$  is a decomposition into disjointed connected components given by (3.29)–(3.31), that is

$$\begin{aligned} E[1] &= \left\{ (t, z) \in \widetilde{U}(R, r, \kappa, \Delta) : |\operatorname{Arg} z| \leq \frac{\pi}{6} \right\}, \\ E[2] &= \left\{ (t, z) \in \widetilde{U}(R, r, \kappa, \Delta) : (t, h^{-2}(z)) \in E[1] \right\}, \\ E[3] &= \left\{ (t, z) \in \widetilde{U}(R, r, \kappa, \Delta) : (t, h^{-4}(z)) \in E[1] \right\}. \end{aligned}$$

Let a set  $\Pi \subset \Sigma_4$  be such that a sequence  $c$  belongs to  $\Pi$  if the following conditions hold:

- (M1) if  $c_i = k$  for some  $k \in \{1, 2, 3\}$ , then  $c_{i+1} \in \{0, k, (k \bmod 3) + 1\}$ ,
- (M2) if  $c_p = 0$  for all  $p \leq i$ , then  $c_{i+1} \in \{0, 1, 2, 3\}$ ,
- (M3) if  $c_j = 0$  for all  $j \in \{p+1, \dots, i\}$  and  $c_p = k \neq 0$ , then  $c_{i+1} \in \{0, k, (k \bmod 3) + 1\}$ .

In other words,  $\Pi$  is a sofic shift whose presentation is given in Figure 3.

A set of forbidden words for  $\Pi$  is presented below

$$\{13, 21, 32, 20^k 1, 30^k 2, 10^k 3 : k \in \mathbb{N}\}.$$

Observe that this shift is strictly sofic (i.e. it is not a shift of finite type). Indeed, if  $\Pi$  is of finite type then there is  $n > 0$  such that if  $uv$  and  $vw$  are allowed words then  $uvw$  is also allowed, provided that  $|v| \geq n$ , i.e.  $v$  has at least  $n$  symbols (see [6] for more comments). In our case, in particular,  $10^n$  and  $0^n 3$  are allowed but  $10^n 3$  is not.

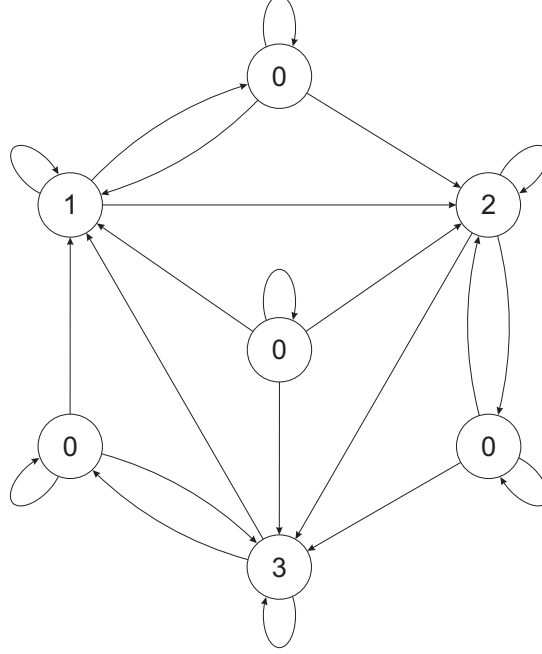
Let  $c \in \Pi$  be an  $n$ -periodic,  $\varphi = \varphi^0$  be the local process generated by (1.1) and  $\varphi^1$  be the local process generated by (1.1) with  $N = 0$ . Then, by Theorem 2.6, the equality

$$\operatorname{ind} \left( \varphi_{(0, nT)}^0 \big|_{(W_0 \setminus W_0^{-})_c^0} \right) = \operatorname{ind} \left( \varphi_{(0, nT)}^1 \big|_{(W_0 \setminus W_0^{-})_c^1} \right)$$

holds. But, by the proof of Theorem 7 from [17],  $\operatorname{ind} \left( \varphi_{(0, nT)}^1 \big|_{(W_0 \setminus W_0^{-})_c^1} \right) \neq 0$ . Finally

$$\operatorname{ind} \left( \varphi_{(0, nT)}^0 \big|_{(W_0 \setminus W_0^{-})_c^0} \right) \neq 0, \quad (3.38)$$

so there exists a  $z \in (W_0 \setminus W_0^{-})_c^0$  such that  $\varphi_{(0, nT)}(z) = z$ .



**Fig. 3.** Presentation of the sofic shift  $\Pi$

Let

$$\tilde{I} = \{z \in W_0 : \varphi_{(0,t+kT)}(z) \in W_t \text{ for } t \in [0, T], k \in \mathbb{Z}\}. \quad (3.39)$$

We define a continuous map  $g : \tilde{I} \rightarrow \Sigma_4$  by

$$g(z)_l = \begin{cases} 0, & \text{if } \varphi_{(0,t+lT)}(z) \in U_t \text{ for all } t \in (0, T), \\ k, & \text{if } \varphi_{(0,lT)}(z) \text{ leaves } U \text{ in a time less than } T \text{ through } E[k]. \end{cases}$$

It is easy to see that  $\sigma \circ g = g \circ \varphi_{(0,T)}$ . Moreover, by (3.38), if  $c \in \Pi$  is  $n$ -periodic, then  $g^{-1}(c)$  contains an  $n$ -periodic point for  $\varphi_{(0,T)}$ . Thus  $\Pi \subset g(\tilde{I})$ . Let  $I = g^{-1}(\Pi)$ . Then  $g|_I : I \rightarrow \Pi$  is a semiconjugacy between  $\varphi_{(0,T)}|_I$  and  $\sigma|_\Pi$ .

We claim that  $\Pi$  has the WSP. First note that  $\sigma|_\Pi$  has an irreducible presentation (i.e. there is a path between any two vertices in that presentation), e.g. that presented in Figure 3 (let us call this graph  $\mathcal{G}$ ). Because the graph  $\mathcal{G}$  has self-loops it is easy to deduce that there is an integer  $N$  such that for any two vertices of  $\mathcal{G}$  there is a path between them consisting of exactly  $N$  edges. This immediately implies that  $\sigma|_\Pi$  has the WSP (see [9, Prop. 4.5.10.] for the case of mixing). Obviously  $\sigma|_\Pi : \Pi \rightarrow \Pi$  is onto and in fact a homeomorphism.

Thus to apply [10, Theorem 5] and finish the proof we only need to show that

$$\#g^{-1}(\{0^\infty\}) \leq 2. \quad (3.40)$$

Condition (3.40) says that there are at most two solutions  $z : \mathbb{R} \rightarrow \mathbb{C}$  of (1.1) which stays in the set  $U^\infty$ , which means that  $z(t + nT) \in U_t$  for every  $t \in [0, T]$  and  $n \in \mathbb{Z}$ .

(I) Let  $N = 0$ . Then  $\psi \equiv 0$  is a solution of (1.1). We show that it is the only solution in  $U^\infty$ .

By Lemma 3.1, the set  $V(\xi)$  is an isolating segment for every  $\xi \in (0, r]$  (see Fig. 1(c)), and, moreover, (3.10) holds, that is

$$V(\xi)^{--} = \tilde{V}(\xi), \quad V(\xi)^{++} = \tilde{\tilde{V}}(\xi).$$

Thus, by Lemma 3.4,  $\psi$  is the only solution which stays in  $V(r)^\infty$ . By Lemma 3.2, every solution which enters  $(U \setminus V(r))^\infty$  has to leave  $U^\infty$ . Thus condition (3.40) is satisfied.

(II) Now, let (1.4) be satisfied (i.e.  $N > 0$ ). For  $\eta > 0$  we define sets

$$D(\eta) = \left\{ z \in \mathbb{C} : \left| \Re \left[ e^{i\frac{\pi}{12}} \left( z - \sqrt{N} e^{i\frac{\pi}{6}} \right) \right] \right| \leq \frac{\sqrt{2}}{2} \eta, \right. \\ \left. \left| \Im \left[ e^{i\frac{\pi}{12}} \left( z - \sqrt{N} e^{i\frac{\pi}{6}} \right) \right] \right| \leq \frac{\sqrt{2}}{2} \eta \right\},$$

and

$$E(\eta) = [0, T] \times D(\eta), \quad (3.41)$$

$$\tilde{E}(\eta) = [0, T] \times (-D(\eta)). \quad (3.42)$$

Then by Lemma 3.7,  $E\left(\frac{\sqrt{N}}{20}\right)$  and  $\tilde{E}\left(\frac{\sqrt{N}}{20}\right)$  are  $T$ -periodic isolating segments such that

$$E\left(\frac{\sqrt{N}}{20}\right)^{--} = [0, T] \times \left\{ z \in D : \left| \Re \left[ e^{i\frac{\pi}{12}} \left( z - \sqrt{N} e^{i\frac{\pi}{6}} \right) \right] \right| = \frac{\sqrt{2N}}{40} \right\}, \quad (3.43)$$

$$E\left(\frac{\sqrt{N}}{20}\right)^{++} = [0, T] \times \left\{ z \in D : \left| \Im \left[ e^{i\frac{\pi}{12}} \left( z - \sqrt{N} e^{i\frac{\pi}{6}} \right) \right] \right| = \frac{\sqrt{2N}}{40} \right\}, \quad (3.44)$$

$$\tilde{E}\left(\frac{\sqrt{N}}{20}\right)^{--} = [0, T] \times \left\{ z \in D : \left| \Im \left[ e^{i\frac{\pi}{12}} \left( z + \sqrt{N} e^{i\frac{\pi}{6}} \right) \right] \right| = \frac{\sqrt{2N}}{40} \right\}, \quad (3.45)$$

$$\tilde{E}\left(\frac{\sqrt{N}}{20}\right)^{++} = [0, T] \times \left\{ z \in D : \left| \Re \left[ e^{i\frac{\pi}{12}} \left( z + \sqrt{N} e^{i\frac{\pi}{6}} \right) \right] \right| = \frac{\sqrt{2N}}{40} \right\} \quad (3.46)$$

hold.

Let  $\chi$  denote the Euler characteristic. Then

$$\begin{aligned} \text{Lef} \left( \mu_{E\left(\frac{\sqrt{N}}{20}\right)} \right) &= \text{Lef} \left( \mu_{\tilde{E}\left(\frac{\sqrt{N}}{20}\right)} \right) = \\ &= \chi \left( E \left( \frac{\sqrt{N}}{20} \right)_0 \right) - \chi \left( E \left( \frac{\sqrt{N}}{20} \right)_0^{--} \right) = 1 - 2 = -1 \neq 0. \end{aligned}$$

By [15, Theorem 5.1], there exist  $T$ -periodic solutions  $\psi_1$  and  $\psi_2$  of the equation (1.1) inside the sets  $E\left(\frac{\sqrt{N}}{20}\right)^\infty$  and  $\tilde{E}\left(\frac{\sqrt{N}}{20}\right)^\infty$ , respectively. Moreover  $\psi_1(0), \psi_2(0) \in g^{-1}(\{0^\infty\})$ . It easy to see that the following inequality holds

$$|\psi_1| \leq \frac{21}{20} \sqrt{N}. \quad (3.47)$$

To finish the proof it is enough to show that  $\psi_1$  and  $\psi_2$  are the only solutions in  $U^\infty$ . We do it in four steps. We prove that:

3.48.  $\psi_1$  and  $\psi_2$  are the only solutions in  $E\left(\sqrt{N}\right)^\infty$  and  $\tilde{E}\left(\sqrt{N}\right)^\infty$ , respectively,

3.49.  $\psi_1$  and  $\psi_2$  are the only solutions in  $V\left(2\sqrt{N}\right)^\infty$ ,

3.50.  $\psi_1$  and  $\psi_2$  are the only solutions in  $V(r)^\infty$ ,

3.51. every solution which enters  $(U \setminus V(r))^\infty$  has to leave  $U^\infty$ .

*Step 1.* By the symmetry

$$v(t, z) = v(t, -z) \text{ for all } (t, z) \in \mathbb{R} \times \mathbb{C}, \quad (3.52)$$

it is enough to prove that  $\psi_1$  is the only solution in  $E\left(\sqrt{N}\right)^\infty$ . The idea and calculations in the case of  $\psi_2$  and  $\tilde{E}\left(\sqrt{N}\right)^\infty$  are similar.

We make the following change of variables

$$w = z - \psi_1 \quad (3.53)$$

and denote

$$\xi_1 = \psi_1 - \sqrt{N} e^{i\frac{\pi}{6}}.$$

Thus, by (3.47), we obtain that

$$|\xi_1| \leq \frac{\sqrt{N}}{20}. \quad (3.54)$$

The equation (1.1) has the form (in new variables)

$$\dot{w} = \tilde{v}(t, w) = 2\sqrt{N} e^{-i\frac{\pi}{6}} \bar{w} + \bar{w}^2 + 2\bar{\xi}_1 \bar{w} + e^{i\kappa t} G(w), \quad (3.55)$$

where

$$\begin{aligned} G(w) &= |w + \psi_1|^2 \overline{w + \psi_1}^2 - |\psi_1|^2 \overline{\psi_1}^2 = \\ &= \bar{w} |w + \psi_1|^2 \overline{w + 2\psi_1} + \left( |w|^2 + 2\Re[\bar{w}\psi_1] \right) \overline{\psi_1}^2. \end{aligned} \quad (3.56)$$

Let

$$F(\eta) = [0, T] \times \left\{ z \in \mathbb{C} : |\Re(e^{i\frac{\pi}{12}} z)| \leq \frac{\sqrt{2}}{2}\eta, |\Im(e^{i\frac{\pi}{12}} z)| \leq \frac{\sqrt{2}}{2}\eta \right\}. \quad (3.57)$$

Then by Lemma 3.8, for  $\eta \in \left(0, \frac{21}{20}\sqrt{N}\right]$  the set  $F(\eta)$  is a  $T$ -periodic segment for the local process generated by (3.55) and

$$F(\eta)^{-} = [0, T] \times \left\{ z \in F(\eta)_0 : |\Re(e^{i\frac{\pi}{12}} z)| = \frac{\sqrt{2}}{2}\eta \right\}, \quad (3.58)$$

$$F(\eta)^{+} = [0, T] \times \left\{ z \in F(\eta)_0 : |\Im(e^{i\frac{\pi}{12}} z)| = \frac{\sqrt{2}}{2}\eta \right\}. \quad (3.59)$$

Thus by Lemma 3.4,  $w \equiv 0$  is the only solution which stays in  $F\left(\frac{21}{20}\sqrt{N}\right)^\infty$ . The set  $F\left(\frac{21}{20}\sqrt{N}\right)$  in  $(t, z)$  coordinates, centered at  $\psi_1$  and  $\psi_1$ , is contained in  $E\left(\frac{\sqrt{N}}{20}\right)^\infty$ . Thus  $E(\sqrt{N})$  is contained in  $F\left(\frac{21}{20}\sqrt{N}\right)$  in  $(t, z)$  coordinates. Finally,  $\psi_1$  is the only solution which never leaves the set  $E(\sqrt{N})^\infty$ .

*Step 2.* By the symmetry (3.52), it is enough to prove that every solution which enters

$$K = \left( V\left(2\sqrt{N}\right) \cap \{(t, z) \in [0, T] \times \mathbb{C} : \Re(z) \geq 0\} \right)^\infty \quad (3.60)$$

except  $\psi_1$  has to leave  $V\left(2\sqrt{N}\right)^\infty$ . But it is a direct consequence of Lemma 3.10.

*Step 3.* By Lemma 3.1, the set  $V(\xi)$  is an isolating segment for every  $\xi \in [2\sqrt{N}, r]$  such that (3.10) holds. Thus, by Lemma 3.4, every solution which enters  $\left(V(r) \setminus \text{int } V\left(2\sqrt{N}\right)\right)^\infty$  has to leave  $V(r)^\infty$ .

*Step 4.* It is a consequence of Lemma 3.2.  $\square$

### 3.2. PROOF OF THEOREM 1.2

Before we can start the proof of Theorem 1.2, a further analysis of the vector field is necessary.

**Lemma 3.8.** *Let  $\kappa > 0$ ,  $T = \frac{2\pi}{\kappa}$ ,  $\eta \in \left(0, \frac{21}{20}\sqrt{N}\right]$  and condition (1.4) be satisfied and let  $F(\eta)$  be given by (3.57). Then  $F(\eta)$  is a  $T$ -periodic isolating segment for the local process generated by (3.55) such that (3.58) and (3.59) hold.*

*Proof.* Let  $M = \sqrt{N}$ . We parametrize one side of  $F(\eta)$  by

$$s_6 : [0, T] \times [-1, 1] \ni (t, o) \mapsto \left( t, \eta e^{-i\frac{\pi}{12}} \left( \frac{\sqrt{2}}{2} + io \frac{\sqrt{2}}{2} \right) \right) \in \mathbb{R} \times \mathbb{C},$$



An outward normal vector  $n_6$  in every point of the side equals  $n_6 = [0, e^{-i\frac{\pi}{12}}]^T$ . By (3.47), the expression  $|G(s_6(t, o))|$ , where  $G$  is given by (3.56), can be estimated by

$$|G(s_6(t, o))| \leq \eta \left( \eta + \frac{21}{20}M \right)^2 \left( \eta + 2M\frac{21}{20} \right) + \left( \eta^2 + 2\eta\frac{21}{20}M \right) \left( \frac{21}{20}M \right)^2.$$

By (3.54), the inner product of the vector field  $(1, v)$  and an outward orthogonal vector in the point  $s_6(t, o)$  can be estimated by

$$\begin{aligned} \langle (1, v(s_6(t, o))), n_6(t, o) \rangle &= \Re \left[ e^{i\frac{\pi}{12}} 2Me^{-i\frac{\pi}{6}} \eta e^{i\frac{\pi}{12}} \left( \frac{\sqrt{2}}{2} - io\frac{\sqrt{2}}{2} \right) \right] + \\ &+ \Re \left[ e^{i\frac{\pi}{12}} \eta^2 e^{i\frac{\pi}{6}} \left( \frac{\sqrt{2}}{2} - io\frac{\sqrt{2}}{2} \right)^2 \right] + \\ &+ \Re \left[ e^{i\frac{\pi}{12}} 2\xi_1 \eta e^{i\frac{\pi}{12}} \left( \frac{\sqrt{2}}{2} - io\frac{\sqrt{2}}{2} \right) \right] + \\ &+ \Re [e^{i\frac{\pi}{12} + i\kappa t} G(s_6(t, o))] \geq \\ &\geq \sqrt{2}M\eta - \eta^2 - \frac{M}{10}\eta - \\ &\quad - \eta \left( \eta + 2M\frac{21}{20} \right) \left[ \left( \eta + \frac{21}{20}M \right)^2 + \left( \frac{21}{20}M \right)^2 \right] \geq \\ &\geq \sqrt{2}\eta M - \frac{21}{20}M\eta - \frac{M}{10}\eta - \eta 15M^3 \left( \frac{21}{20} \right)^3 = \\ &= \eta M \left[ \sqrt{2} - \frac{23}{20} - 15M^2 \left( \frac{21}{20} \right)^3 \right] > \\ &> 0, \end{aligned}$$

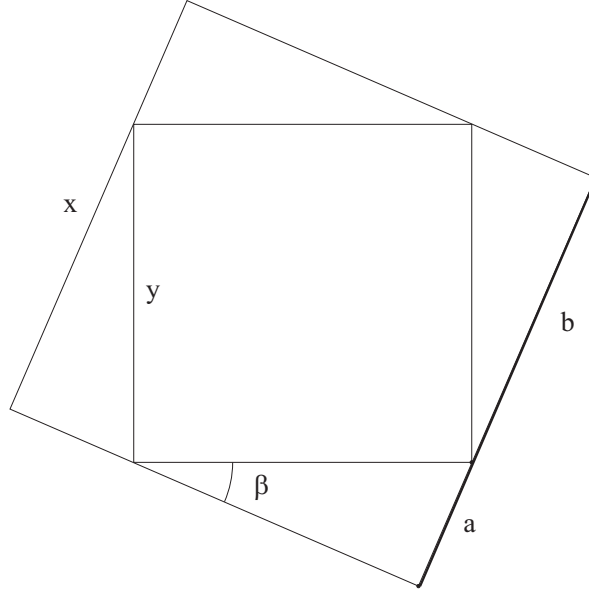
so the vector field  $(1, v)$  points outward  $F(\eta)$  on the side. Calculations for other sides of  $F(\eta)$  are similar. This proves (3.58) and (3.59).  $\square$

**Lemma 3.9.** *Let  $\gamma, \zeta > 0$  and*

$$\begin{aligned} \Lambda(\gamma) &= \left\{ z \in \mathbb{C} : |\Re [e^{i\frac{\pi}{12}} z]| \leq \frac{\sqrt{2}}{2}\gamma, |\Im [e^{i\frac{\pi}{12}} z]| \leq \frac{\sqrt{2}}{2}\gamma \right\}, \\ \Upsilon(\zeta) &= \left\{ z \in \mathbb{C} : |\Re [z]| \leq \frac{\sqrt{2}}{2}\zeta, |\Im [z]| \leq \frac{\sqrt{2}}{2}\zeta \right\}. \end{aligned}$$

*Then  $\Upsilon(\zeta) \subset \Lambda(\gamma)$  iff  $\zeta \leq \frac{\sqrt{6}}{3}\gamma$  holds.*

*Proof.* Let  $x, y$  be the lengths of the side of the bigger and smaller square, respectively. Let  $\beta = \frac{\pi}{12}$  and  $a, b$  be the lengths of the parts of the side of the bigger square (see Fig. 4).



**Fig. 4.** Sets  $\Upsilon(\zeta)$  and  $\Lambda(\gamma)$

Then  $\sin \beta = \frac{a}{y}$ ,  $\cos \beta = \frac{b}{y}$ . Then

$$\begin{aligned} x &= a + b = y(\sin \beta + \cos \beta) = \\ &= \sqrt{2}y \sin \left( \beta + \frac{\pi}{4} \right) = \\ &= y \frac{\sqrt{6}}{2} \end{aligned}$$

holds, so  $y = \frac{\sqrt{6}}{3}x$ , which finishes the proof.  $\square$

**Lemma 3.10.** *Let the inequalities (1.2), (1.4) be satisfied and let  $V(2\sqrt{N})$ ,  $E(\sqrt{N})$ ,  $K$  be given by (3.8), (3.41), (3.60), respectively. We denote by  $\psi_1$  the only solution of (1.1) which never leaves the set  $E(\sqrt{N})^\infty$ . Then every solution which enters  $K$  except  $\psi_1$  has to leave  $V(2\sqrt{N})^\infty$ .*

*Proof.* Let  $M = \sqrt{N}$ . By Lemma 3.1, the set  $V(2M)$  is an isolating segment such that (3.10) holds.

Let  $L(1) = K \cap \{(t, z) \in \mathbb{R} \times \mathbb{C} : \mathfrak{I}(z) \leq 0\}$  and  $(t, z) \in L(1)$ . Then

$$\begin{aligned} \mathfrak{I}(v(t, z)) &= \mathfrak{I} \left[ \bar{z}^2 + e^{i\kappa t} |z|^2 \bar{z}^2 - N e^{-i\frac{\pi}{3}} \right] \geq \\ &\geq -2\mathfrak{I}(z)\Re(z) - |z|^4 + \frac{\sqrt{3}}{2}N \geq \\ &\geq N \left( \frac{\sqrt{3}}{2} - 16N \right) > \\ &> 0 \end{aligned} \tag{3.61}$$

holds. But for  $(t, z) \in K \cap \{(t, z) \in \mathbb{R} \times \mathbb{C} : \Re(z) = 0\}$  we get

$$\begin{aligned} \Re(v(t, z)) &= -\mathfrak{I}(z)^2 + \Re \left( e^{i\kappa t} |z|^2 \bar{z}^2 \right) - \frac{N}{2} \leq \\ &\leq 16N^2 - \frac{N}{2} < \\ &< 0. \end{aligned}$$

Thus if  $(\tau, z) \in L(1)$ , then there exists  $t < 0$  such that  $(\tau + t, \varphi_{(\tau, t)}(z)) \notin V(2M)^\infty$ .

Let

$$H := \left\{ (t, z) \in [0, T] \times \mathbb{C} : |\Re [z - M e^{i\frac{\pi}{6}}]| \leq \frac{\sqrt{3}}{3}M, |\mathfrak{I} [z - M e^{i\frac{\pi}{6}}]| \leq \frac{\sqrt{3}}{3}M \right\}.$$

Then, by Lemma 3.9, the inclusion  $H_0 \subset E(M)_0$  holds and so  $H \subset E(M)$ . Moreover,  $\psi_1$  is the only solution which never leaves  $H^\infty$ , so it is enough to show that every solution which enters  $K \setminus (L(1) \cup H)$  has to leave  $V(2M)^\infty$ . We define three subsets  $L(2)$ ,  $L(3)$ ,  $L(4)$  of  $K$  by the formula (see Fig. 5):

$$\begin{aligned} L(2) &:= K \cap \left\{ (t, z) \in \mathbb{R} \times \mathbb{C} : 0 \leq \Re[z] \leq \frac{\sqrt{3}}{3}M, 0 \leq \mathfrak{I}[z] \right\}, \\ L(3) &:= K \cap \left\{ (t, z) \in \mathbb{R} \times \mathbb{C} : \frac{\sqrt{3}}{3}M \leq \Re[z], M \leq \mathfrak{I}[z] \right\}, \\ L(4) &:= K \cap \left\{ (t, z) \in \mathbb{R} \times \mathbb{C} : \frac{5}{6}\sqrt{3}M \leq \Re[z], 0 \leq \mathfrak{I}[z] \leq M \right\}. \end{aligned}$$

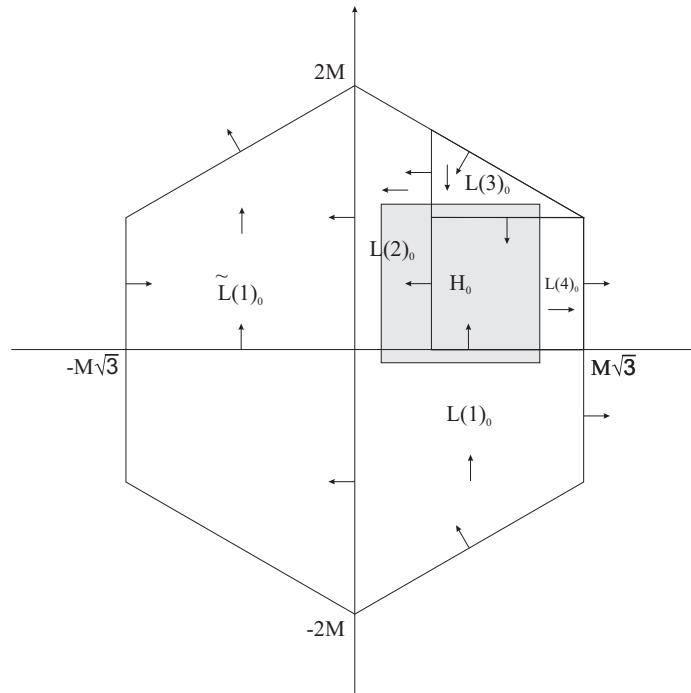
Observe that  $K \setminus (L(1) \cup H) \subset L(2) \cup L(3) \cup L(4)$ , so it is enough to analyze the case of each of these sets (i.e. what happens if a trajectory has a nonempty intersection with one of them).

Let  $(t, z) \in L(2)$ . Then

$$\begin{aligned} \Re[v(t, z)] &\leq \Re[z]^2 - \Im[z]^2 + \Re[e^{i\kappa t}|z|^2\bar{z}^2] - \frac{N}{2} \leq \\ &\leq \frac{N}{3} + 16N^2 - \frac{N}{2} < \\ &< 0, \end{aligned}$$

so the solution passing through  $(t, z)$  has to enter

$$\tilde{L}(1) = \{(t, z) \in [0, T] \times \mathbb{C} : (t, -z) \in L(1)\}.$$



**Fig. 5.** Sets  $L(1)_0$ ,  $\tilde{L}(1)_0$ ,  $L(2)_0$ ,  $L(3)_0$ ,  $L(4)_0$  and  $H_0$ . Arrows show direction of the vector field  $v(t, z)$  inside sets and on its boundaries

Owing to (3.52) and (3.61), every solution entering  $\tilde{L}(1)$  has to leave  $V(2M)^\infty$  (see Fig. 5).

Let now  $(t, z) \in L(3)$ . Then

$$\begin{aligned} \Im[v(t, z)] &\leq -2\Im(z)\Re(z) + \Im[e^{i\kappa t}|z|^2\bar{z}^2] + \frac{\sqrt{3}}{2}N \leq \\ &\leq -\frac{2\sqrt{3}}{3}N + 16N^2 + \frac{\sqrt{3}}{2}N < \\ &< 0, \end{aligned}$$

so the solution passing through  $(t, z)$  has to leave  $V(2M)^\infty$  (for a negative value of the time parameter).

Finally, let  $(t, z) \in L(4)$ . Then

$$\begin{aligned} \Re[v(t, z)] &= \Re[z]^2 - \Im[z]^2 + \Re[e^{i\kappa t}|z|^2\bar{z}^2] - \frac{N}{2} \geq \\ &\geq \frac{75}{36}N - N - 16N^2 - \frac{N}{2} > \\ &> 0, \end{aligned}$$

and once again we obtain that the solution passing through  $(t, z)$  has to leave the set  $V(2M)^\infty$ .  $\square$

*Proof of Theorem 1.2.* We use all the notation introduced in the proof of Theorem 1.1. We start with the following observations:

**Fact 1.** Let  $\xi$  be a solution such that  $\xi(0)$  is sufficiently close to  $\psi_1(0)$  and there exists  $t_0 > 0$  such that  $(t, \xi(t)) \in U^\infty$  for  $t \in [0, t_0)$  and  $(t_0, \xi(t_0)) \in (U^{--})^\infty$ . It is easy to see (cf. Fig. 5 and the proof of Lemma 3.10) that one of the following conditions hold:  $(t_0, \xi(t_0)) \in (E[1])^\infty$  or  $(t_0, \xi(t_0)) \in (E[2])^\infty$ . Similarly, if  $\xi$  is a solution such that  $\xi(0)$  is sufficiently close to  $\psi_2(0)$  and there exists  $t_0 > 0$  such that  $(t, \xi(t)) \in U^\infty$  for  $t \in [0, t_0)$  and  $(t_0, \xi(t_0)) \in (U^{--})^\infty$  then either  $(t_0, \xi(t_0)) \in (E[2])^\infty$  or  $(t_0, \xi(t_0)) \in (E[3])^\infty$ .

**Fact 2.** By the continuity of  $g$ , for every  $\varepsilon > 0$  there exists  $\delta = \delta(\varepsilon)$  such that for every solution  $\xi$  of (1.1) and  $x \in \Pi$  satisfying  $g(\xi(0)) = x$  the inequality  $\rho(x, 0^\infty) < \varepsilon$  implies that  $\text{dist}(\xi(0), g^{-1}(0^\infty)) < \delta$ . It is obvious that

$$\lim_{\varepsilon \rightarrow 0} \delta(\varepsilon) = 0 \tag{3.62}$$

holds. So if  $\varepsilon$  is small enough then exactly one of the following inequalities holds  $|\xi(0) - \psi_1(0)| < \delta$  or  $|\xi(0) - \psi_2(0)| < \delta$ .

Let  $x \in \Pi$  and  $k, l \in \mathbb{Z}$ ,  $k < l$  be such that  $x_j = 0$  for all integers  $j \notin [k, l]$  and  $x_k, x_l \in \{1, 2, 3\}$ . Let  $\xi$  be a solution of (1.1) such that  $g(\xi(0)) = x$ . Then by Proposition 2.2,  $\xi$  is heteroclinic from  $\psi_1$  to  $\psi_2$  or from  $\psi_2$  to  $\psi_1$  or is homoclinic to  $\psi_1$  or  $\psi_2$ . We claim that it is possible to construct a solution  $\xi$  with the properties described below (depending only on  $x$ ):

- (i) if  $x_k = 1$  then  $\lim_{t \rightarrow -\infty} |\xi(t) - \psi_1(t)| = 0$ ,
- (ii) if  $x_k = 3$  then  $\lim_{t \rightarrow -\infty} |\xi(t) - \psi_2(t)| = 0$ ,

and:

- (iii) if  $x_l = 1$  then  $\lim_{t \rightarrow +\infty} |\xi(t) - \psi_1(t)| = 0$ ,
- (iv) if  $x_l = 2$  then  $\lim_{t \rightarrow +\infty} |\xi(t) - \psi_2(t)| = 0$ ,
- (v) if  $x_l = 3$  then  $\lim_{t \rightarrow +\infty} |\xi(t) - \psi_1(t)| = 0$  or  $\lim_{t \rightarrow +\infty} |\xi(t) - \psi_2(t)| = 0$ .

Let  $x_k = 1$ . Then for every  $n \in \mathbb{N}$  and  $j < k - n - 1$  the equality  $(\sigma^j(x))_{[-n, n]} = 0^{2n+1}$  holds, so  $\rho(\sigma^j(x), 0^\infty) < 2^{-n-1}$  and, by Fact 2, the inequality

$$\text{dist}(\xi(0), \{\psi_1(0), \psi_2(0)\}) < \delta(2^{-n-1})$$

holds. But  $x_k = 1$  then, by Fact 1,  $|\xi(0) - \psi_1(0)| < \delta(2^{-n-1})$ . By (3.62), we get (i).

The proof for the case (ii) is similar.

Now, let  $x_l = 1$  and  $x_k = 1$ . It is enough to set  $x = \lim_{n \rightarrow \infty} x^{(n)}$  where  $(x^{(n)})_{[k-n, l+2n+2]} = 0^n x_{[k, l]} 0^{2n+1} 1$  and  $x^{(n)}$  is an  $(l - k + 3n + 3)$ -periodic point from  $\Pi$ . By (3.38), there exists a periodic solution  $\xi_n$  such that  $g(\xi_n(0)) = x^{(n)}$ . As before,

$$\left| \varphi_{(0, T)}^{l+m+1}(\xi_n(0)) - \psi_1(0) \right| < \delta(2^{-m-1})$$

holds for every  $0 < m < n$ . Let  $\xi$  be a solution such that  $\xi(0)$  is an accumulation point of  $\{\xi_n(0)\}_{n \in \mathbb{N}}$ . Then  $g(\xi(0)) = x$  and

$$\left| \varphi_{(0, T)}^{l+m+1}(\xi(0)) - \psi_1(0) \right| \leq \delta(2^{-m-1})$$

holds for every  $m > 0$ . This proves (iii) for  $x_k = 1$ . If  $x_k = 3$  then we define  $(x^{(n)})_{[k-n, l+2n+3]} = 0^n x_{[k, l]} 0^{2n+1} 12$  and follow the previous idea.

The choice of

$$\begin{aligned} (x^{(n)})_{[k-n, l+2n+2]} &= 0^n x_{[k, l]} 0^{2n+1} 3 \text{ for } x_l = 2, x_k = 1, \\ (x^{(n)})_{[k-n, l+2n+2]} &= 0^n x_{[k, l]} 0^{2n+1} 3 \text{ for } x_l = 2, x_k = 3 \end{aligned}$$

proves (iv).

The choice of

$$\begin{aligned} (x^{(n)})_{[k-n, l+2n+2]} &= 0^n x_{[k, l]} 0^{2n+1} 1 \text{ for } x_l = 3, x_k = 1, \\ (x^{(n)})_{[k-n, l+2n+3]} &= 0^n x_{[k, l]} 0^{2n+1} 12 \text{ for } x_l = 3, x_k = 3 \end{aligned}$$

proves (v) in the case of  $\psi_1$  while

$$\begin{aligned} (x^{(n)})_{[k-n, l+2n+2]} &= 0^n x_{[k, l]} 0^{2n+1} 3 \text{ for } x_l = 3, x_k = 1, \\ (x^{(n)})_{[k-n, l+2n+2]} &= 0^n x_{[k, l]} 0^{2n+1} 3 \text{ for } x_l = 3, x_k = 3 \end{aligned}$$

proves (v) in the case of  $\psi_2$ .

To finish the proof observe that sequences  $0^\infty a 0^n b c 0^\infty \in \Pi$  (with fixed symbols  $a, b, c \in \{1, 2, 3\}$  and  $n = 1, 2, \dots$ ) define an infinite set of (different) solutions of the equation (1.1). Then it is enough to fix symbols  $a, c$  according to conditions (i)–(v). A symbol  $b$  is needed because, e.g. between any occurrence of symbols 1, 3 (in an element of  $\Pi$ ) the symbol 2 must occur (see Fig. 3).  $\square$

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