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A Remark on the Continuous Subsolution Problem for the Complex Monge-Ampère Equation

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Abstract
We prove that if the modulus of continuity of a plurisubharmonic subsolution satisfies a Dini-type condition then the Dirichlet problem for the complex Monge-Ampère equation has the continuous solution. The modulus of continuity of the solution also given if the right hand side is locally dominated by capacity.

Keywords Dirichlet problem · Complex Monge-Ampère equation · Weak solutions · Subsolution problem

Mathematics Subject Classification (2010) 32W20 · 32U40

1 Introduction

In this note, we consider the Dirichlet problem for the complex Monge-Ampère equation in a strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$. Let ψ be a continuous function on the boundary of Ω. We look for the solution to the equation:

$$\begin{aligned} & u \in PSH(\Omega) \cap C^0(\bar{\Omega}), \\ & (dd^c u)^n = d\mu, \\ & u = \psi \quad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

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Here, PSH stands for plurisubharmonic functions, and $d^c = i(\bar{\partial} - \partial)$. It was shown in [9] and [10] that for the measures satisfying certain bound in terms of the Bedford-Taylor capacity [4], the Dirichlet problem has a (unique) solution. The precise statement is as follows.

Let $h : \mathbb{R}_+ \to (0, \infty)$ be an increasing function such that

$$\int_1^\infty \frac{1}{x[h(x)]^{\frac{1}{n}}} dx < +\infty.$$

We call such a function admissible. If h is admissible, then so is Ah for any number $A > 0$. Define

$$F_h(x) = \frac{x}{h(x^{-\frac{1}{n}})}.$$

Suppose that for such a function $F_h(x)$ a Borel measure μ satisfies

$$\int_E d\mu \leq F_h(\mathrm{cap}(E)) \tag{1.2}$$

for any Borel set $E \subset \Omega$. Then, by [9] the Dirichlet problem (1.1) has a solution.

This statement is useful as long as we can verify the condition (1.2). In particular, if μ has density with respect to the Lebesgue measure in L^p, $p > 1$ then this bound is satisfied [9]. By the recent results in [12, 13] if μ is bounded by the Monge-Ampère measure of a Hölder continuous plurisubharmonic function φ

$$\mu \leq (dd^c\varphi)^n \quad \text{in } \Omega,$$

then (1.2) holds for a specific h, and consequently, the Dirichlet problem (1.1) is solvable with Hölder continuous solution. The main result in this paper says that we can considerably weaken the assumption on φ and still get a continuous solution of the equation.

Let $\varpi(t) := \varpi(t; \varphi, \bar{\Omega})$ denote the modulus of continuity of φ on $\bar{\Omega}$, i.e.,

$$\varpi(t) = \sup\left\{|\varphi(z) - \varphi(w)| \ : \ z, w \in \bar{\Omega}, |z - w| \leq t\right\}.$$

Thus $|\varphi(z) - \varphi(w)| \leq \varpi(|z - w|)$ for every $z, w \in \bar{\Omega}$. Let us state the first result.

Theorem 1.1 *Let $\varphi \in PSH(\Omega) \cap C^0(\bar{\Omega})$, $\varphi = 0$ on $\partial\Omega$. Assume that its modulus of continuity satisfies the Dini type condition*

$$\int_0^1 \frac{[\varpi(t)]^{\frac{1}{n}}}{t|\log t|} dt < +\infty. \tag{1.3}$$

If the measure μ satisfies $\mu \leq (dd^c\varphi)^n$ in Ω, then the Dirichlet problem (1.1) admits a unique solution.

Let us mention in this context that it is still an open problem if a continuous subsolution φ implies the solvability of (1.1).

The modulus of continuity of the solution to the Dirichlet problem (1.1) was obtained in [3] for $\mu = f dV_{2n}$ with $f(x)$ being continuous on $\bar{\Omega}$. We also wish to study this problem for the measures which satisfy the inequality (1.2). For simplicity, we restrict ourselves to measures belonging to $\mathcal{H}(\alpha, \Omega)$. In other words, we take the function $h(x) = Cx^{n\alpha}$ for positive constants $C, \alpha > 0$ in the inequality (1.2).

We introduce the following notion, which generalizes the one in [8]. Consider a continuous increasing function $F_0 : [0, \infty) \to [0, \infty)$ with $F(0) = 0$.

Definition 1.2 The measure μ is called uniformly locally dominated by capacity with respect to F_0 if for every cube $I(z,r) =: I \subset B_I := B(z,2r) \subset\subset \Omega$ and for every set $E \subset I$,

$$\mu(E) \le \mu(I) F_0 \left(\operatorname{cap}(E, B_I)\right). \tag{1.4}$$

According to [1], the Lebesgue measure dV_{2n} satisfies this property with $F_0 = C_\alpha \exp(-\alpha/x^{-1/n})$ for every $0 < \alpha < 2n$. The case $F_0(x) = Cx$ was considered in [8]. We refer the reader to [5] for more examples of measures satisfying this property. Here is our second result.

Theorem 1.3 *Assume $\mu \in \mathcal{H}(\alpha, \Omega)$ with compact support and satisfying the condition (1.4) for some F_0. Then, the modulus of continuity of the solution u of the Dirichlet problem (1.1) satisfies for $0 < \delta < R_0$ and $2R_0 = \operatorname{dist}(\operatorname{supp}\mu, \partial\Omega) > 0$,*

$$\varpi(\delta; u, \Omega) \le \varpi(\delta; \psi, \partial\Omega) + C \left[\left(\log \frac{R_0}{\delta} \right)^{-\frac{1}{2}} + F_0 \left(\frac{C_0}{[\log(R_0/\delta)]^{\frac{1}{2}}} \right) \right]^{\alpha_1},$$

where the constants C, α_1 depend only on α, μ, Ω.

2 Preliminaries

Here, we gather some basic facts from pluripotential theory taken from [4], and used in the sequel. Given a compact set K in a domain $\Omega \subset \mathbb{C}^n$, its relative extremal function u_K is given by

$$u_K = \sup\{u \in PSH(\Omega) : u < 0, u \le -1 \text{ on } K\}.$$

Its upper semicontinuous regularization u_K^* is plurisubharmonic. When u_K is continuous, we call K a regular set. It is easy to see that the ϵ-envelope

$$K_\epsilon = \{z : \operatorname{dist}(z, K) \le \epsilon\}$$

of a compact set K is regular, and thus any compact set can be approximated from above by regular compact sets.

The relative capacity of a compact set K with respect to Ω (now usually called the Bedford-Taylor capacity) is defined by the formula

$$\operatorname{cap}(K, \Omega) = \sup \left\{ \int_K (dd^c u)^n : u \in PSH(\Omega), -1 \le u \le 0 \right\},$$

and by [4], can be expressed as

$$\operatorname{cap}(E, \Omega) = \int_K (dd^c u_K^*)^n.$$

We say that a positive Borel measure μ belongs to $\mathcal{H}(\alpha, \Omega)$, $\alpha > 0$, if there exists a uniform constant $C > 0$ such that for every compact set $E \subset \Omega$,

$$\mu(E) \le C \left[\operatorname{cap}(E, \Omega)\right]^{1+\alpha}.$$

3 Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1. We need the following lemma. The proof of this lemma is based on a similar idea as the one in [11, Lemma 3.1] where the complex Hessian equation is considered. The difference is that we have much stronger volume-capacity inequality for the Monge-Ampère equation.

Lemma 3.1 *Assume the measure μ is compactly supported. Fix $0 < \alpha < 2n$ and $\tau = \alpha/(2n+1)$. There exists a uniform constant C such that for every compact set $K \subset \Omega$,*

$$\mu(K) \leq C \left\{ \varpi \left(\exp\left(\frac{-\tau}{2[\mathrm{cap}(K)]^{\frac{1}{n}}} \right) \right) + \exp\left(\frac{2n\tau - \alpha}{2[\mathrm{cap}(K)]^{\frac{1}{n}}} \right) \right\} \cdot \mathrm{cap}(K), \tag{3.1}$$

where $\mathrm{cap}(K) := \mathrm{cap}(K, \Omega)$.

Proof Fix a compact subset $K \subset\subset \Omega$. Without loss of generality, we may assume that K is regular. Denote by φ_ε the standard regularization of φ in the terminology of [10]. We choose $\varepsilon > 0$ so small that

$$\mathrm{supp}\, \mu \subset \Omega'' \subset\subset \Omega' \subset \Omega_\varepsilon \subset \Omega,$$

where $\Omega_\varepsilon = \{z \in \Omega \,:\, \mathrm{dist}(z, \partial\Omega) > \varepsilon\}$. Since for every $K \subset \Omega''$ we have

$$C_0 \mathrm{cap}(K, \Omega) \leq \mathrm{cap}(K, \Omega') \leq C_0^{-1} \mathrm{cap}(K, \Omega)$$

(for a constant C_0 depending only on Ω, Ω') in what follows we shall write $\mathrm{cap}(K)$ for either one of these capacities. We have

$$0 \leq \varphi_\varepsilon - \varphi \leq \varpi(\varepsilon) := \delta \quad \text{on } \Omega'.$$

Let u_K be the relative extremal function of K with respect to Ω'. Consider the set $K' = \{3\delta u_K + \varphi_\varepsilon < \varphi - 2\delta\}$. Then,

$$K \subset K' \subset \left\{ u_K < -\frac{1}{2} \right\} \subset \Omega'. \tag{3.2}$$

Hence, by the comparison principle [4],

$$\mathrm{cap}(K') \leq 2^n \mathrm{cap}(K). \tag{3.3}$$

Note that

$$dd^c \varphi_\varepsilon \leq \frac{C}{\varepsilon^2}\, dd^c |z|^2, \quad \|\varphi_\varepsilon + u_K\|_\infty =: M \leq \|\varphi\|_\infty + 1. \tag{3.4}$$

The comparison principle, the bounds (3.4), and the volume-capacity inequality from [1, Theorem A] (in the last inequality below) give the following:

$$\begin{aligned}
\int_{K'} (dd^c\varphi)^n &\leq \int_{K'} (dd^c(3\delta u_K + \varphi_\varepsilon))^n \\
&\leq 3\delta \int_{K'} \left[dd^c(u_K + \varphi_\varepsilon) \right]^n + \int_{K'} (dd^c \varphi_\varepsilon)^n \\
&\leq 3\delta M^n \mathrm{cap}(K') + C(\alpha)\varepsilon^{-2n} \exp\left(\frac{-\alpha}{[\mathrm{cap}(K')]^{\frac{1}{n}}} \right) \mathrm{cap}(K').
\end{aligned}$$

Choose

$$\varepsilon = \exp\left(\frac{-\tau}{[\mathrm{cap}(K')]^{\frac{1}{n}}} \right)$$

(we assume that ε is so small that it satisfies (3.2), otherwise the inequality (3.1) holds true by increasing the constant) and plug in the formula for δ to get that

$$\begin{aligned}\mu(K) &\le \int_{K'} (dd^c(\varphi))^n \\ &\le 3M^n \varpi\left(\exp\left(\frac{-\tau}{[\operatorname{cap}(K')]^{\frac{1}{n}}}\right)\right)\cdot \operatorname{cap}(K') + C\exp\left(\frac{2n\tau-\alpha}{[\operatorname{cap}(K')]^{\frac{1}{n}}}\right).\end{aligned}$$

This combined with (3.3) gives the desired inequality. □

We are ready to finish the proof of the theorem. It follows from Lemma 3.1 that

$$h(x) = \frac{1}{C\varpi(\exp(-\tau x))}$$

is a function which satisfies (1.2) for the measure μ once we have

$$\int_1^\infty \frac{1}{x[h(x)]^{\frac{1}{n}}}\, dx < +\infty.$$

By changing the variable $s = 1/x$, and then $t = e^{-\tau/s}$, this is equivalent to

$$\int_0^{e^{-\tau}} \frac{[\varpi(t)]^{\frac{1}{n}}}{t|\log t|} dt < +\infty.$$

The last inequality is guaranteed by (1.3). Thus, our assumption on the modulus of continuity $\varpi(t)$ implies that h is admissible in the case of μ with compact support. Then, by [10, Theorem 5.9] the Dirichlet problem (1.1) has a unique solution.

To deal with the general case, consider the exhaustion of Ω by compact sets

$$E_j = \{\varphi \le -1/j\}$$

and define μ_j to be the restriction of μ to E_j. Denote by u_j the solution of (1.1) with μ replaced by μ_j. By the comparison principle

$$u_j + \max(\varphi, -1/j) \le u \le u_j,$$

and so the sequence u_j tends to $u = \lim u_j$ uniformly and the continuity of u follows. The proof is complete.

4 The Modulus of Continuity of Solutions

In this section, we study the modulus of continuity of the solution of the Dirichlet problem with the right hand side in the class $\mathcal{H}(\alpha, \Omega)$ under the additional condition that a given measure is locally dominated by capacity.

In what follows we need [8, Lemma 2] whose proof is based on the lemma due to Alexander and Taylor [2, Lemma 3.3]. For the reader's convenience, we give the proofs. The latter can be simplified by using the Błocki inequality [6].

Lemma 4.1 *Let $B' = \{|z - z_0| < r\} \subset\subset B = \{|z - z_0| < R\}$ be two concentric balls centered at z_0 in $\mathbb{C}^n$. Let $u \in PSH(B) \cap L^\infty(B)$ with $u < 0$. There is a constant $C = C(n, \frac{R}{r})$ independent of u such that*

$$\int_{B'} (dd^c u)^n \le C|u(z_0)| \sup_{z\in B} |u(z)|^{n-1}.$$

In particular, if $R/r = 3$ then the constant C depends only on n.

Proof Without loss of generality, we may assume $z_0 = 0$. Set $\rho := (r + R)/2$ and $B(\rho) = \{|z - z_0| < \rho\}$. We use the Błocki inequality [6] for $v(z) = |z|^2 - \rho^2$ and $\beta := dd^c v = dd^c|z|^2$, to get

$$\int_{B'} (dd^c u)^n \leq \frac{1}{(\rho^2 - r^2)^{n-1}} \int_{B(\rho)} |v|^{n-1} (dd^c u)^n$$
$$\leq \frac{(n-1)!\|u\|_{B_\rho}^{n-1}}{(\rho^2 - r^2)^{n-1}} \int_{B(\rho)} dd^c u \wedge \beta^{n-1}.$$

By Jensen's formula

$$u(0) + N(\rho) = \frac{1}{\sigma_{2n-1}} \int_{\{|\zeta|=1\}} u(\rho\zeta) d\sigma(\zeta),$$

where σ_{2n-1} is the area of the unit sphere,

$$N(\rho) = \int_0^\rho \frac{n(t)}{t^{2n-1}} dt$$

and

$$n(t) = \frac{1}{\sigma_{2n-1}} \int_{\{|z|\leq t\}} \Delta u(z) dV_{2n}(z) = a_n \int_{\{|z|\leq t\}} dd^c u \wedge \beta^{n-1}.$$

Since $n(t)/t^{2n-2}$ is increasing, we have

$$N(R) \geq \int_\rho^R \frac{n(t)}{t^{2n-1}} dt \geq \frac{n(\rho)}{\rho^{2n-2}} \log(R/\rho).$$

From $u < 0$, it follows that $N(R) < -u(0)$. Hence,

$$\int_{B_\rho} dd^c u \wedge \beta^{n-1} \leq \frac{n(\rho)}{a_n} \leq \frac{N(R)\rho^{n-2}}{\log(R/\rho)} \leq \frac{\rho^{2n-2}|u(0)|}{\log(R/\rho)}.$$

Combining the above inequalities, we get the desired estimate with the constant

$$C = \frac{(n-1)!\rho^{2n-2}}{(\rho^2 - r^2)^{n-1} \log(R/\rho)}.$$

If $R = 3r$, then C is also independent of r. □

Lemma 4.2 *Denote for* $\rho \geq 0$, $B_\rho = \{|z - z_0| < e^\rho R_0\}$. *Given* $z_0 \in \Omega$ *and two numbers* $R > 1$, $R_0 > 0$ *such that* $B_M \subset\subset \Omega$, *and given* $v \in PSH(\Omega)$ *such that* $-1 < v < 0$, *denote by E the set*

$$E = E(\delta) = \left\{ z \in B_0 \,:\, (1-\delta)v \leq \sup_{B_0} v \right\},$$

where $\delta \in (0, 1)$. *Then, there exists* C_0 *depending only on n such that*

$$\operatorname{cap}(E, B_2) \leq \frac{C_0}{R\delta}.$$

Proof From the logarithmic convexity of the function $r \mapsto \sup_{|z-z_0|<r} v(z)$ it follows that for $z \in B_R \setminus B_0$ and $a_0 := \sup_{B_0} v$ we have

$$v(z) \leq a_0 \left(1 - \frac{1}{R} \log \frac{|z - z_0|}{R_0}\right).$$

Hence,

$$a := \sup_{B_2} v \leq a_0 \left(1 - \frac{2}{R}\right).$$

Let $u = u_{E,B_2}$ the relative extremal function of E with respect to B_2. One has

$$\frac{v - a}{a - a_0/(1-\delta)} \leq u.$$

So, for some $z_1 \in \mathrm{B}_0$, we have

$$u(z_1) \geq \frac{a_0 - a}{a - a_0/(1-\delta)} \geq \frac{2(\delta - 1)}{(M-2)\delta + 2}.$$

Note that $E \subset \{|z - z_1| < 2R_0\} \subset B_2$. Therefore, Lemma 4.1 gives

$$\mathrm{cap}(E, B_2) = \int_{\{|z-z_1|<6R_0\}} (dd^c u)^n \leq C_0 \|u\|_{B_2}^{n-1} |u(z_1)| \leq \frac{C_0}{R\delta}.$$

This is the desired inequality. □

Let us proceed with the proof of Theorem 1.3. Since $\mu \in \mathcal{H}(\alpha, \Omega)$, according to [9] and [10, Theorem 5.9] we can solve the Dirichlet problem (1.1) to obtain a unique continuous solution u. Define for $\delta > 0$ small

$$\Omega_\delta := \{z \in \Omega \ : \ \mathrm{dist}(z, \partial\Omega) > \delta\};$$

and for $z \in \Omega_\delta$ set

$$u_\delta(z) := \sup_{|\zeta| \leq \delta} u(z + \zeta).$$

Thanks to the arguments in [12, Lemma 2.11] it is easy to see that there exists $\delta_0 > 0$ such that

$$u_\delta(z) \leq u(z) + \varpi(\delta; \psi, \partial\Omega) \tag{4.1}$$

for every $z \in \partial\Omega_\delta$ and $0 < \delta < \delta_0$. Here, we used the result of Bedford and Taylor [3, Theorem 6.2] (with minor modifications) to extend ψ plurisubharmonically onto Ω so that its modulus of continuity on $\bar{\Omega}$ is controlled by the one on the boundary. Therefore, for a suitable extension of u_δ to Ω, using the stability estimate for measure in $\mathcal{H}(\alpha, \Omega)$ as in [7, Theorem 1.1] (see also [12, Proposition 2.10]), we get

Lemma 4.3 *There are uniform constants C, α_1 depending only on Ω, α, μ such that*

$$\sup_{\Omega_\delta}(u_\delta - u) \leq \varpi(\delta; \psi, \partial\Omega) + C \left(\int_{\Omega_\delta} (u_\delta - u) d\mu\right)^{\alpha_1}$$

for every $0 < \delta < \delta_0$.

Thanks to this lemma, we know that the right hand side tends to zero as δ decreases to zero. We shall use the property "locally dominated by capacity" to obtain a quantitative bound via Lemma 4.2.

Let us denote the support of μ by K. Since $\|u\|_\infty$ is controlled by a constant $C = C(\alpha, \Omega, \mu)$, without loss of generality, we may assume that

$$-1 \leq u \leq 0.$$

Then for every $0 < \varepsilon < 1$

$$\int_{\Omega_\delta} (u_\delta - u) d\mu \leq \varepsilon\ \mu(\Omega) + \int_{\{u < u_\delta - \varepsilon\} \cap K} d\mu. \tag{4.2}$$

We shall now estimate the second term on the right hand side. We may assume that $\Omega \subset\subset [0,1]^{2n}$. Let us write $z = (x^1, \dots, x^{2n}) \in \mathbb{R}^{2n}$ and denote the semi-open cube centered at a point z_0, of diameter $2r$ by

$$I(z_0, r) := \{z = (x^1, \dots, x^{2n}) \in \mathbb{C}^n : -r \leq x^i - x_0^i < r, \forall i = 1, \dots, 2n\}.$$

Then, by the assumption, μ satisfies for every cube

$$I(z, r) =: I \subset B_I := B(z, 2r) \subset\subset \Omega$$

and for every set $E \subset I$, the inequality

$$\mu(E) \leq \mu(I(z, r)) F_0 (\mathrm{cap}(E, B_I)), \tag{4.3}$$

where $F_0 : [0, \infty] \to [0, \infty]$ is an increasing continuous function and $F_0(0) = 0$.

Consider the semi-open cube decomposition of $\Omega \subset\subset I_0 := [0, 1)^{2n} \subset \mathbb{R}^{2n}$ into 3^{2ns} congruent cubes of diameter $3^{-s} = 2\delta$, where $s \in \mathbb{N}$. Then

$$\{u < u_\delta - \varepsilon\} \cap I_s \subset \left\{ z \in B_{I_s} \ : \ u < \sup_{B_{I_s}} u - \varepsilon \right\}, \tag{4.4}$$

where $I_s = I(z_s, \delta)$ and $B_{I_s} = B(z_s, 2\delta)$ for some $z_s \in I_0$. Hence,

$$\int_{\{u<u_\delta-\varepsilon\}} d\mu \leq \sum_{I_s \cap K \neq \emptyset} \int_{\{u<u_\delta-\varepsilon\}\cap I_s} d\mu.$$

Using (4.3), (4.4), and then applying Lemma 4.2 for $r = 2\delta$ and $R = 2R_0$, we have for $B_s := B(z_s, 4\delta)$ corresponding to each cube I_s

$$\int_{\{u<u_\delta-\varepsilon\}\cap I_s} d\mu \leq \mu(I_s) F_0(\mathrm{cap}(E(\varepsilon, u, B_{I_s}), B_s)) \leq \mu(I_s) \ F_0 \left(\frac{C_0}{\varepsilon \log(R_0/\delta)} \right), \tag{4.5}$$

where $2R_0 = \mathrm{dist}(K, \partial\Omega)$. Therefore, combining the above inequalities, we get that

$$\int_{\{u<u_\delta-\varepsilon\}} d\mu \leq \mu(\Omega) F_0 \left(\frac{C_0}{\varepsilon \log(R_0/\delta)} \right).$$

We conclude from this and Lemma 4.3 that

$$\omega(\delta; u, \bar{\Omega}) \leq \sup_{\Omega_\delta} (u_\delta - u) \leq \varpi(\delta; \psi, \partial\Omega) + C \left[\varepsilon + F_0 \left(\frac{C_0}{\varepsilon \log(R_0/\delta)} \right) \right]^{\alpha_1}.$$

If we choose $\varepsilon = (\log R_0/\delta)^{-1/2}$, then Theorem 1.3 follows.

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