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Stability analysis of partial differential variational inequalities in Banach spaces

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Abstract. In this paper, we study a class of partial differential variational inequalities. A general stability result for the partial differential variational inequality is provided in the case the perturbed parameters are involved in both the nonlinear mapping and the set of constraints. The main tools are theory of semigroups, theory of monotone operators, and variational inequality techniques.

Keywords: partial differential variational inequality, mild solution, stability, upper semicontinuity, continuity.

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1 Introduction and preliminaries

Let X and Y be two real reflexive and separable Banach spaces, X^* be the dual space of X, and $K \subset X$ be a nonempty, compact, and convex set. We denote by $\langle \cdot, \cdot \rangle$ the duality pair between X^* and X. In this paper, we aim to study the following partial differential variational inequality (PDVI) in infinite dimensional space. Find functions $x\colon [0,T] \to Y$ and $u\colon [0,T] \to X$ such that

$$\begin{aligned}
&\dot{x}(t) = Ax(t) + Bu(t) \quad \text{for a.e. } t \in [0,T],\\
&u(t) \in \mathrm{SOL}\big(K, G\big(t, x(t)\big) + F(\cdot)\big) \quad \text{for a.e. } t \in [0,T],\\
&x(0) = x_0.
\end{aligned} \tag{1}$$

Here $A\colon D(A) \subset Y \to Y$ is the infinitesimal generator of a C_0-semigroup $T(t)$, $B\colon X \to Y$ is a linear and bounded operator, $\dot{x}(t) = \mathrm{d}x(t)/\mathrm{d}t$ stands for the derivative of a function x with respect to time variable t, $x_0 \in D(A)$ is a given element. Moreover, $\mathrm{SOL}(K, G(t, x(t)) + F(\cdot))$ denotes the solution set of the following time-dependent variational inequality governed by a nonlinear mapping $F\colon X \to X^*$, a time-dependent mapping $G\colon [0,T] \times Y \to X^*$, and a constraint set K. Find $u\colon [0,T] \to K$ such that, for a.e. $t \in [0,T]$, it holds

$$\big\langle G\big(t, x(t)\big) + F\big(u(t)\big), v - u(t)\big\rangle \geqslant 0 \quad \text{for all } v \in K.$$

Due to many applications in several disciplines such as economics, engineering operation research, and mechanics, variational inequalities and related optimal control problems have been widely studied, and many important results have been recently obtained in [11, 19, 28–32, 35–37]. In the study of control problems governed by variational inequalities, there is a special kind of problems, which consist of a differential equation and a variational inequality. Since more and more problems existing in economical dynamics, dynamic traffic networks and control systems can be converted to an evolution system with a control set being a solution set of a variational inequality; see, for example [27]. Such kind of control problems have received much attention in recent years. In 2008, Pang and Stewart [24] first introduced a control problem in finite dimensional space consisting with an ordinary differential equation and a variational inequality. We refer to such systems as a differential variational inequality (DVI). In their study, they proved the existence of Carathéodory weak solution to the DVI and established the convergence of the Euler time-stepping procedure for the initial value problem for the DVI. In addition to solvability and convergence of solutions to the DVI, another important research issue is the stability analysis; see, for example, [7] and the references therein. In [25], the authors studied the dependence of a solution to DVIs on the initial conditions. Wang et al. in [33] studied the upper semicontinuity and continuity properties for the set of Carathéodory weak solution mapping for a differential mixed variational inequality when both the mapping and the constraint set are perturbed by different parameters in finite dimensional spaces. Gwinner in [7] studied the stability of the solution set to linear differential variational inequalities and gave a result on the upper convergence with respect to perturbations in the data, including perturbations in the associated linear maps and the constraint set in Hilbert spaces. Various theoretical results, numerical algorithms, and applications of the DVIs in finite dimensional spaces have been explored in [2, 8, 9, 12–15, 23, 34].

Furthermore, in many distributed parameter optimal control problems with differential equation as in (1) (see [4, 5]), more precise models are obtained when the ordinary differential equation is replaced by a partial differential equation, where the operator A represents a partial differential operator with respect to spatial variables. Thus, following Liu et al. [17], such problems are called a partial differential variational inequality (PDVI). For various kinds of PDVIs, Liu and his co-authors have made many contributions as indicated below. In [17], Liu et al. as pioneers first explored a class of evolutionary equations driven by variational inequalities in Banach spaces, and proved the nonemptiness and compactness of the solution set. After that, Liu et al. in [16] established the existence of solution for a class of partial differential variational inequalities involving nonlocal boundary conditions in infinite Banach spaces by using fixed point theorem for condensing set-valued operators, theory of measure of noncompactness, and the Filippov implicit function lemma. More recently, Liu et al. in [18] studied a class of partial differential hemivariational inequalities which consists of a nonlinear evolution equation and a hemivariational inequality of elliptic type. In fact, the PDVIs are very useful in optimization, and mechanical and electrical engineering, see [21, 22, 39]. As far as we know, until now there are very few research results on the stability analysis for the PDVIs. Motivated by the aforementioned works, in this paper, we are devoted to stability analysis for the PDVI (1) in infinite dimensional spaces when both the mapping F and the constraint set K are perturbed by two different parameters.

In the rest of this section, we introduce the perturbed problem for the PDVI (1), which is needed to carry out the stability analysis. We conclude this section with basic definitions and lemmata, which will be used in the following sections.

Let (X_1, d_1) and (X_2, d_2) be metric spaces. In what follows, we assume that the constraint set $K \subset X$ in (1) is perturbed through a parameter p, which varies over (X_1, d_1), and the mapping $F\colon X \to X^*$ in (1) is perturbed by a parameter λ, which varies over (X_2, d_2). The corresponding perturbed problem for the (1), which is referred to as the perturbed partial differential variational inequality (PPDVI), can be specified as the following parametric partial differential variational inequality:

$$\begin{aligned} &\dot{x}(t) = Ax(t) + Bu(t) \quad \text{for a.e. } t \in [0, T],\\ &u(t) \in \mathrm{SOL}\big(K(p), G\big(t, x(t)\big) + F(\cdot, \lambda)\big) \quad \text{for a.e. } t \in [0, T],\\ &x(0) = x_0. \end{aligned} \tag{2}$$

Given a Banach space E, we denote by $C([0, T]; E)$ the space of continuous functions on $[0, T]$ with values in E. We adopt the following definition.

Definition 1. A pair of functions (x, u) is called to be a mild solution to PDVI (1) if $x \in C([0, T]; Y)$ and $u\colon [0, T] \to K$ is measurable with

$$u(t) \in \mathrm{SOL}\big(K, G\big(t, x(t)\big) + F(\cdot)\big) \quad \text{for a.e. } t \in [0, T],$$

and

$$x(t) = T(t)x_0 + \int_0^t T(t-s)Bu(s)\,\mathrm{d}s \quad \text{for all } t \in [0,T].$$

In what follows, to simplify notation, we denote the set of mild solutions to the PPDVI (2) and the set of u in the mild solution by $\mathcal{S}(p,\lambda)$ and $SD_u(p,\lambda)$, respectively. In this paper, when both the mapping F and the constraint set K are perturbed by two different parameters p and λ, respectively, we prove the continuity of the solution set $(p,\lambda) \mapsto \mathcal{S}(p,\lambda)$, which is based on its closedness and the upper semicontinuity property.

The following definitions and results will be useful in the next sections. We refer to [6, 10, 20, 26, 38] and the references therein for details.

Definition 2. Let Y and Z be topological spaces, and let $F\colon Y \to 2^Z$ be a set-valued mapping with nonempty values. The mapping F is said to be

(i) closed if its graph $\operatorname{Gr} F = \{(y,z) \in Y \times Z \mid z \in F(y)\}$ is a closed subset of $Y \times Z$.
(ii) upper semicontinuous at $y_0 \in Y$ if, for any neighborhood $N(F(y_0))$ of $F(y_0)$, there exists a neighborhood $N(y_0)$ of y_0 such that $F(y) \subset N(F(y_0))$ for all $y \in N(y_0)$.
(iii) lower semicontinuous at $y_0 \in Y$ if, for any $z_0 \in F(y_0)$ and any neighborhood $N(z_0)$ of z_0, there exists a neighborhood $N(y_0)$ of y_0 such that $F(y) \bigcap N(z_0) \neq \emptyset$ for all $y \in N(y_0)$.

It is evident that a mapping $F\colon Y \to 2^Z$ is lower semicontinuous at $y_0 \in Y$ if and only if, for any sequence $y_n \to y_0$ and $z_0 \in F(y_0)$, there exists a sequence $\{z_n\} \subset Z$ with $z_n \in F(y_n)$ such that $z_n \to z_0$.

Definition 3. Let X and Y be topological spaces. A set-valued mapping F from X to 2^Y is said to be uniformly compact near a point $x \in X$ if and only if there exists a neighborhood U of x such that the closure of the set $\bigcup\{F(x) \mid x \in U\}$ is compact.

Definition 4. Let K be a subset of a real Banach space X. Let X^* be the dual to X and $\langle\cdot,\cdot\rangle$ denote the duality pairing between X^* and X. We say that $F\colon K \to X^*$ is

(i) monotone on K if

$$\bigl\langle F(v) - F(u),\, v - u \bigr\rangle \geqslant 0 \quad \text{for all } u, v \in K.$$

(ii) strongly monotone on K with constant $m_F > 0$ if

$$\bigl\langle F(v) - F(u),\, v - u \bigr\rangle \geqslant m_F \|v - u\|^2 \quad \text{for all } u, v \in K.$$

Given metric spaces X and Y, we introduce the notation

$$P_f(Y) = \{D \subset Y \mid D \text{ is nonempty, closed}\},$$
$$P_K(Y) = \{D \subset Y \mid D \text{ is nonempty, compact}\}.$$

Lemma 1. *Let X and Y be metric spaces. Then the following statements hold*:

(i) *If a set-valued mapping $F\colon X \to P_f(Y)$ is upper semicontinuous, then F is closed.*

(ii) *If a set-valued mapping $F\colon X \to P_f(Y)$ is closed and locally compact, then F is upper semicontinuous.*

(iii) *The set-valued mapping $F\colon X \to P_K(Y)$ is upper semicontinuous if and only if, for every $x \in X$ and every sequence $\{(x_n, y_n)\} \subset X \times Y$, $y_n \in F(x_n)$ with $x_n \to x$ in X, there exists a converging subsequence of $\{y_n\}$ whose limit belongs to $F(x)$.*

Definition 5. Let U, V be two nonempty and bounded subsets of a metric space X. The Hausdorff metric between U and V, denoted by $\mathcal{H}(U, V)$, is defined by

$$\mathcal{H}(U, V) = \max\Big\{\sup_{a\in U} d(a, V),\ \sup_{b\in V} d(b, U)\Big\},$$

where $d(a, V)$ is the distance from a pint a to a set V.

2 Main results

In this section, we shall focus our attention on the stability of solution set for the PDVI (1) with respect to two parameters p and λ by considering its perturbed problem PPDVI (2). To this end, we first recall the following result on existence of mild solutions, whose proof can be found in [16–18].

Theorem 1. *Let X and Y be real, reflexive, and separable Banach spaces, and $K \subset X$ be a nonempty, compact, and convex set. Let $A\colon D(A) \subset Y \to Y$ be the infinitesimal generator of C_0-semigroup $T(t)$ in Y, and $B\colon X \to Y$ be a bounded and linear operator. If the function $[0, T] \times Y \ni (t, x) \mapsto G(t, x) \in X^*$ is continuous and the operator $F\colon K \to X^*$ is monotone and continuous on K, then the PDVI* (1) *has at least one mild solution in the sense of Definition 1.*

Remark 1. We note that if (x, u) is a mild solution to PDVI, then the mild variational trajectory u is not only a measurable function, but it belongs to $L^2(0, T; Y)$ due to the compactness of the set K.

Now, we shall discuss the upper semicontinuity and continuity of the set of mild solutions to PDVI with respect to two parameters, where the first parameter p is assumed to be a perturbation of the constraint set K, and the second parameter λ is considered to appear in the nonlinear operator F. First, we provided a result on the closedness of the solution map to PDVI with respect to parameters (p, λ).

Theorem 2. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be real, reflexive, and separable Banach spaces, and (X_1, d_1) and (X_2, d_2) be metric spaces. Let $K\colon X_1 \to 2^X$ be a continuous* (*with respect to the Hausdorff metric*) *set-valued mapping with nonempty, compact, and convex* *values. Suppose that* $G\colon (0,T)\times Y \to X^*$ *and* $F\colon X\times X_2 \to X^*$ *are continuous functions such that* $u \mapsto F(u,\lambda)$ *is monotone for all* $\lambda \in X_2$. *Then the set-valued mapping* $(p,\lambda) \mapsto \mathcal{S}(p,\lambda)$ *is closed.*

Proof. Let $(p,\lambda) \in X_1 \times X_2$ be fixed. Recall that the set $K(p)$ is nonempty, compact, and convex in X, and the function $u \mapsto F(u,\lambda)$ is monotone and continuous. It follows from Theorem 1 that the set $\mathcal{S}(p,\lambda)$ is nonempty.

We will show that the set-valued mapping $(p,\lambda) \mapsto \mathcal{S}(p,\lambda)$ is closed. Let sequences $\{(p_n,\lambda_n)\} \in X_1 \times X_2$ and $\{(x_n,u_n)\} \subset C([0,T];Y) \times L^2(0,T;X)$ be such that $(x_n,u_n) \in \mathcal{S}(p_n,\lambda_n)$ with $(p_n,\lambda_n) \to (p^*,\lambda^*)$ in $X_1 \times X_2$, $x_n \to x$ in $C([0,T];Y)$, and $u_n \to u$ in $L^2(0,T;X)$. Hence, one has

$$\begin{aligned}
&x_n(t) = T(t)x_0 + \int_0^t T(t-s)Bu_n(s)\,\mathrm{d}s \quad \text{for all } t \in [0,T], &(3)\\
&u_n(t) \in K(p_n),\\
&\big\langle G\big(t,x_n(t)\big) + F\big(u_n(t),\lambda_n\big), w - u_n(t)\big\rangle \geqslant 0 \quad \text{for all } w \in K(p_n),\\
&\quad \text{a.e. } t \in [0,T].
\end{aligned}$$

From the convergence $u_n \to u$ in $L^2(0,T;X)$, without any loss of generality, we may assume that $u_n(t) \to u(t)$ in X for a.e. $t \in [0,T]$, and there exists a function $h \in L^2_+(0,T)$ such that $\|u_n(t)\|_X \leqslant h(t)$ for a.e. $t \in [0,T]$. Using (3), by the Lebesgue dominated convergence theorem (see [6, Thm. 2.2.9]), we infer that

$$x_n(t) = T(t)x_0 + \int_0^t T(t-s)Bu_n(s)\,\mathrm{d}s \to T(t)x_0 + \int_0^t T(t-s)Bu(s)\,\mathrm{d}s$$

in X for all $t \in [0,T]$. Moreover, the convergence $x_n \to x$ in $C([0,T];Y)$ implies

$$x(t) = T(t)x_0 + \int_0^t T(t-s)Bu(s)\,\mathrm{d}s \tag{4}$$

for all $t \in [0,T]$. On the other hand, since $u_n(t) \to u(t)$ in X for a.e. $t \in [0,T]$, and K is continuous in the sense of Hausdorff with compact and convex values, we have $u(t) \in K(p^*)$ for a.e. $t \in [0,T]$; see Lemma 1. Furthermore, for any $v \in K(p^*)$, the continuity of K ensures that there exits a sequence $\{v_n\} \subset X$ with $v_n \in K(p_n)$ such that $v_n \to v$ in X as $n \to \infty$; see [1, Thm. 2.7]. The latter entails

$$\begin{aligned}
0 &\leqslant \lim_{n\to\infty} \big\langle G\big(t,x_n(t)\big) + F\big(u_n(t),\lambda_n\big),\, v_n - u_n(t)\big\rangle\\
&= \big\langle G\big(t,x(t)\big) + F\big(u(t),\lambda^*\big),\, v - u(t)\big\rangle
\end{aligned}$$

for all $v \in K(p^*)$ and a.e. $t \in [0,T]$. Combining this inequality with (4) implies $(x,u) \in \mathcal{S}(p^*,\lambda^*)$, i.e., $\mathcal{S}$ is closed. This completes the proof. □

Subsequently, we provide the following result on the upper semicontinuity of the set-valued operator $\mathcal{S}$ with respect to parameters.

Theorem 3. *Let the hypotheses of Theorem 2 hold. In addition, we suppose that* $G\colon (0,T)\times Y \to X^*$ *is Lipschitz continuous with respect to the second variable, and* $SD_u(p,\lambda)$ *is uniformly compact at* (p^*,λ^*)*. Then* $(p,\lambda)\mapsto \mathcal{S}(p,\lambda)$ *is upper semicontinuous at* $(p^*,\lambda^*)\in X_1\times X_2$.

Proof. Let $P\times V$ be a neighborhood of (p^*,λ^*) in $X_1\times X_2$. From the proof of Theorem 2, it follows that $\mathcal{S}(p,\lambda)\neq\emptyset$ for all $(p,\lambda)\in P\times V$. We proceed by contradiction and assume that the set-valued mapping $\mathcal{S}$ is not upper semicontinuous at $(p^*,\lambda^*)\in X_1\times X_2$. Hence, there exist two sequences $\{(p_n,\lambda_n)\}\subset X_1\times X_2$, $\{(x_n,u_n)\}\in C([0,T];Y)\times L^2(0,T;X)$ and an open set $\mathcal{O}$ in $C([0,T];Y)\times L^2(0,T;X)$ with

$$
\begin{aligned}
&(p_n,\lambda_n)\to(p^*,\lambda^*) \quad \text{as } n\to\infty,\\
&(x_n,u_n)\in\mathcal{S}(p_n,\lambda_n) \quad \text{for each } n\in\mathbb{N},\\
&\mathcal{S}(p^*,\lambda^*)\subset\mathcal{O}
\end{aligned}
$$

such that $(x_n,u_n)\notin\mathcal{O}$ for each $n\in\mathbb{N}$. Since $(x_n,u_n)\in\mathcal{S}(p_n,\lambda_n)$, we have

$$
\begin{aligned}
&u_n(t)\in K(p_n),\\
&\big\langle G\big(t,x_n(t)\big)+F\big(u_n(t),\lambda_n\big),\, v-u_n(t)\big\rangle \geqslant 0 \quad \text{for all } v\in K(p_n),\\
&\quad \text{a.e. } t\in[0,T]
\end{aligned}
\tag{5}
$$

and

$$
x_n(t)=T(t)x_0+\int_0^t T(t-s)Bu_n(s)\,\mathrm{d}s \quad \text{for all } t\in[0,T]. \tag{6}
$$

Invoking the assumption that $SD_u(p,\lambda)$ is uniformly compact at $(p^*,\lambda^*)\in X_1\times X_2$, we can see that there exists a subsequence of $\{u_n\}$, still denoted in the same way, such that $u_n\to u^*$ in $L^2(0,T;X)$ for some $u^*\in L^2(0,T;X)$. Passing to a further subsequence, if necessary, we may assume that $u_n(t)\to u^*(t)$ for a.e. $t\in[0,T]$ and there exists a function $h\in L^2_+(0,T)$ such that $\|u_n(t)\|_X\leqslant h(t)$ for a.e. $t\in[0,T]$. From the latter and the Lebesgue dominated convergence theorem, from (6) we infer that

$$
\begin{aligned}
x_n(t)&=T(t)x_0\\
&\quad+\int_0^t T(t-s)Bu_n(s)\,\mathrm{d}s\to T(t)x_0+\int_0^t T(t-s)Bu(s)\,\mathrm{d}s
\end{aligned}
\tag{7}
$$

in X for all $t\in[0,T]$. Furthermore, using the hypothesis that K is continuous with compact and convex values, by using the same argument as in proof of Theorem 2, we obtain $u^*(t)\in K(p^*)$ for a.e. $t\in[0,T]$. Since

$$
x^*(t)=T(t)x_0+\int_0^t T(t-s)Bu^*(s)\,\mathrm{d}s
$$

is continuous on $[0,T]$, $x_n \in C([0,T];Y)$, from (7), we deduce that $x_n \to x^*$ on $C([0,T];Y)$.

Next, we show that $u^*(t) \in SOL(K(p^*), G(t,x^*(t)) + F(\cdot,\lambda^*))$ for a.e. $t \in [0,T]$. Thanks to $u^*(t) \in K(p^*)$ for a.e. $t \in [0,T]$, we only to prove that

$$\langle G(t,x^*(t)) + F(u^*(t),\lambda^*),\ v - u^*(t)\rangle \geqslant 0 \tag{8}$$

for any $v \in K(p^*)$ and a.e. $t \in [0,T]$. By applying the continuity of K, we know that, for any $v \in K(p^*)$, there exists a sequence $\{v_n\} \subset X$ with $v_n \in K(p_n)$ such that $v_n \to v$ in X. Next, by (5) and the continuity of G and F, we get (8). Moreover, from Theorem 2 we know that the set $\mathcal{S}(p^*,\lambda^*)$ is closed, thus we obtain that $(x^*,u^*) \in \mathcal{S}(p^*,\lambda^*) \subset \mathcal{O}$. On the other hand, by assumptions, we know $(x_n,u_n) \notin \mathcal{O}$ for all $n \in \mathbb{N}$, which is a contradiction. This completes the proof of the theorem. □

Now, we provide the main result of the paper.

Theorem 4. *Let $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ be real, reflexive, and separable Banach spaces, (X_1,d_1) and (X_2,d_2) be metric spaces, and $(p^*,\lambda^*) \in X_1 \times X_2$. Assume that, for each $x \in X$, the function $t \mapsto G(t,x)$ is continuous, and for all $t \in [0,T]$, the function $x \mapsto G(t,x)$ is Lipschitz continuous, and the following conditions hold*:

(i) *there exists a neighborhood $P \times V \subset X_1 \times X_2$ of (p^*,λ^*) such that $SD_u(p,\lambda)$ is uniformly compact at (p^*,λ^*)*;

(ii) *$F\colon X \times X_2 \to X^*$ is continuous, and for each $\lambda \in V$, the function $u \mapsto F(u,\lambda)$ is uniformly strongly monotone*;

(iii) *$K\colon X_1 \to 2^X$ is a continuous set-valued mapping with nonempty, compact, and convex values.*

Then the mapping $(p,\lambda) \mapsto \mathcal{S}(p,\lambda)$ is continuous at (p^*,λ^*).

Proof. First, we shall prove that $\mathcal{S}(p,\lambda)$ is a singleton for every $(p,\lambda) \in P \times V$. To this end, let $(x_1,u_1), (x_2,u_2) \in \mathcal{S}(p,\lambda)$. Then, for $i = 1, 2$, we have

$$\begin{aligned} &u_i(t) \in K(p),\\ &\langle G(t,x_i(t)) + F(u_i(t),\lambda),\ v - u_i(t)\rangle \geqslant 0 \quad \text{for all } v \in K(p),\\ &\quad \text{a.e. } t \in [0,T] \end{aligned} \tag{9}$$

and

$$x_i(t) = T(t)x_0 + \int_0^t T(t-s)Bu_i(s)\,\mathrm{d}s \quad \text{for all } t \in [0,T]. \tag{10}$$

Taking $v = u_2(t)$ in (9) for $i = 1$ and $v = u_1(t)$ for $i = 2$, we obtain

$$\langle G(t,x_2(t)) - G(t,x_1(t)) + F(u_2(t),\lambda) - F(u_1(t),\lambda),\ u_1(t) - u_2(t)\rangle \geqslant 0,$$

and hence,

$$\begin{aligned}&\big\langle F\big(u_1(t),\lambda\big)-F\big(u_2(t),\lambda\big),\,u_1(t)-u_2(t)\big\rangle\\&\quad\leqslant\big\langle G\big(t,x_2(t)\big)-G\big(t,x_1(t)\big),\,u_1(t)-u_2(t)\big\rangle\end{aligned}$$

for a.e. $t\in[0,T]$. Since G is Lipschitz continuous with respect to the second argument and F is uniformly strongly monotone in the first variable, we get

$$m_F\big\|u_1(t)-u_2(t)\big\|_X^2\leqslant L_G\big\|u_1(t)-u_2(t)\big\|_X\big\|x_1(t)-x_2(t)\big\|_Y\quad\text{a.e. } t\in[0,T],$$

where $m_F>0$ and $L_G>0$. This implies that

$$\big\|u_1(t)-u_2(t)\big\|_X\leqslant\frac{L_G}{m_F}\big\|x_1(t)-x_2(t)\big\|_Y\quad\text{a.e. } t\in[0,T].\tag{11}$$

On the other hand, from (10) for $i=1,2$, we have

$$x_1(t)-x_2(t)=\int_0^t T(t-s)\big(Bu_1(s)-Bu_2(s)\big)\,\mathrm{d}s\quad\text{for all } t\in[0,T].$$

Hence

$$\big\|x_1(t)-x_2(t)\big\|_Y\leqslant\int_0^t\big\|T(t-s)\big\|\big\|Bu_1(s)-Bu_2(s)\big\|_Y\,\mathrm{d}s.\tag{12}$$

Since $T(t)$ be a C_0-semigroup, from [26, Thm. 2.2] we know that there exist constants $\omega>0$ and $M\geqslant1$ such that $\|T(t)\|\leqslant M\mathrm{e}^{\omega t}$ for any $0\leqslant t<\infty$. Then it follows from (11) and (12) that, for all $t\in[0,T]$,

$$\begin{aligned}\big\|x_1(t)-x_2(t)\big\|_Y&\leqslant M_A\int_0^t\big\|Bu_1(s)-Bu_2(s)\big\|_Y\,\mathrm{d}s\\&\leqslant\frac{M_AL_G\|B\|}{m_F}\int_0^t\big\|x_1(s)-x_2(s)\big\|_Y\,\mathrm{d}s,\end{aligned}$$

where $M_A=\sup_{t\in[0,T]}\|T(t)\|<\infty$. It is readily seen from the Gronwall inequality that

$$x_1(t)=x_2(t)\quad\text{for all } t\in[0,T].$$

So, we have $x_1=x_2$, and by (11), we infer that $u_1=u_2$. This completes the uniqueness part of the theorem.

Next, we verify that the set-valued mapping $(p,\lambda)\mapsto\mathcal{S}(p,\lambda)$ is continuous at $(p^*,\lambda^*)\in X_1\times X_2$. Let $\{(p_n,\lambda_n)\}\subset P\times V$ be such that $(p_n,\lambda_n)\to(p^*,\lambda^*)$ in $X_1\times X_2$. From hypothesis (iii) there exists a sequence $\{(x_n,u_n)\}$ in $C([0,T];Y)\times L^2(0,T;X)$ such that $(x_n, u_n) \in \mathcal{S}(p_n, \lambda_n)$. This means that

$$u_n(t) \in K(p_n),$$
$$\big\langle G\big(t, x_n(t)\big) + F\big(u_n(t), \lambda_n\big),\, v - u_n(t)\big\rangle \geqslant 0 \quad \text{for all } v \in K(p_n),$$
$$\text{a.e. } t \in [0,T],$$

and

$$x_n(t) = T(t)x_0 + \int_0^t T(t-s)Bu_n(s)\,\mathrm{d}s \quad \text{for all } t \in [0,T].$$

By assumption (i), $SD_u(p,\lambda)$ is uniformly compact at (p^*, λ^*), so we may assume that $\{u_n\}$ have a convergent subsequence, denoted in the same way, such that $u_n \to u^*$ in $L^2(0,T;X)$ for some $u^* \in L^2([0,T];X)$. Similarly as in the proof of Theorem 3, we also get $x_n \to x^* \in C([0,T];Y)$, where

$$x^*(t) = T(t)x_0 + \int_0^t T(t-s)Bu^*(s)\,\mathrm{d}s \quad \text{for all } t \in [0,T].$$

Furthermore, Theorem 2 ensures that $S(p,\lambda)$ is closed at (p^*,λ^*), so we conclude that $(x^*, u^*) \in \mathcal{S}(p^*, \lambda^*)$. Summing up, we have that $\mathcal{S}(p,\lambda)$ is a singleton for every $(p,\lambda) \in P \times V$, and

$$\mathcal{S}(p_n, \lambda_n) = (x_n, u_n) \to (x^*, u^*) = \mathcal{S}(p^*, \lambda^*).$$

This completes the proof of the theorem. □

3 An application

As mentioned in Section 1, differential variational inequalities and partial differential variational inequalities have many important applications in economical dynamics, dynamic traffic networks, and control systems. A typical application in economical dynamics is the following dynamic Nash equilibrium problem with shared constraints, which has been given by Chen and Wang in the [3]. Considering the completeness of our paper, we restate it here with a concise version and refer the readers to the reference [3] for details.

Let $e_l \in \mathbb{R}^{n_l}$, $u_l \in \mathbb{R}^{m_l}$ denote the l_{th} player's state and strategy variables, respectively, $e = (e_l)_{l=1}^N \in \mathbb{R}^n$, $u = (u_l)_{l=1}^N \in \mathbb{R}^m$ with $n = \sum_{l=1}^N n_l$, $m = \sum_{l=1}^N m_l$ denote the state and strategy variables of all players, and $[0,T]$ be the time interval considered.

For the l_{th} player, we consider the strategy set $K = \{u \in \mathbb{R}^m\colon f_l(u_l) \leqslant 0,\, g(u) \leqslant 0\}$ defined by functions $f_l(\cdot)\colon \mathbb{R}^{n_l} \to \mathbb{R}^{s_v}$, $g(u) = g(u_l, u_l) \in \mathbb{R}^\ell$, the cost functional $\theta_l\colon \mathbb{R}^{n+m} \to R$, the state dynamic $\varTheta_l\colon \mathbb{R}^{1+n_l+m_l} \to \mathbb{R}^{n_l}$, and the initial state by $e_{l,0} \in \mathbb{R}^{n_v}$. Then, the dynamic Nash equilibrium problem with shared constraints can be formulated as: find a state-control pair $(\tilde{e}, \tilde{u})$ such that, for each $l = 1, \dots, N$, $(\tilde{e}_l, \tilde{u}_l)$ is a solution to the following optimal control problem:

$$\begin{aligned}\min\big\{\theta_l(e_l, \tilde{e}_{-l}, u_l, \tilde{u}_{-l})\colon\ &\dot{e}_l(t) = \varTheta_l(t, e_l, u_l),\ e_l(0) = e_{l,0},\\ &f_l\big(u_l(t)\big) \leqslant 0,\ g\big(u_l(t), u_{-l}(t)\big) \leqslant 0\big\},\end{aligned} \tag{13}$$

where $e_{-l} = (e_{l'})_{l' \neq l} \in \mathbb{R}^{n-n_l}$ and $u_{-l} = (u_{l'})_{l' \neq l} \in \mathbb{R}^{m-m_l}$. Constantly, for simplicity of writing, we use notations $e = (e_l, e_{-l})$ and $u = (u_l, u_{-l})$.

With some assumptions on the cost functional θ_l, the state dynamic Θ_l and the functions f_l, g for the strategy set, the dynamic Nash equilibrium problem (13) can be transformed into the PDVI(1) with $x(t) = (e_l(t), \nu_l(t))_{l=1}^n$, $u(t) = (u_l(t), 0)_{l=1}^n$, A, B being two sparse matrices, and $G(t, x(t)) + F(u(t)) = -(\nabla_{u_l} H_l(t, \nu_l, e, u))_{l=1}^n$, where ν_l is the adjoint variable of the ODE constraint in player l's control problem, and $H_l(t, \nu_l, e, u)$ is the Hamiltonian of player l.

It is well known that the dynamic Nash equilibrium problem could be influenced by many kinds of factors coming from different practical problems. Among these, it is very important to study the behavior of solution set of optimal state-control pairs of DVI/PDVI when the admission constraint set and the functions involved are perturbed. Because it gives the information about the tolerances, which are permitted in the specification of the mathematical models, suggests ways to solve parametric problem, and also can be useful in the computational analysis of the problem. Therefore our results obtained in this paper provide a valid method to study the dynamic Nash equilibrium problem.

We end this section with a specific example in which all conditions in Theorem 4 are satisfied. And thus, Theorem 4 can be applied to obtain the continuity of the solution set with respect to the parameters.

Example 1. Let $X = Y = \mathbb{R}$, $X_1 = X_2 = \mathbb{R}$, $(t, x) \in [0, T] \times \mathbb{R}$, $(u, \lambda) \in \mathbb{R} \times \mathbb{R}$, $G(t, x) = t + x$, $F(u, \lambda) = \lambda u$, $K(p) = [1, p + 5]$ with $p \in \mathbb{R}$, $(p^*, \lambda^*) = (4, 4)$, and $A(x) = ax$, $B(u) = bu$ with $a, b \in \mathbb{R}$. The PPDVI problem (2) can be specified as follows:

$$\begin{aligned} &\dot{x}(t) = ax(t) + bu(t) \quad \text{for a.e. } t \in [0; T];\\ &u(t) \in \mathrm{SOL}\big(K(p),\, t + x(t) + \lambda(\cdot)\big) \quad \text{for a.e. } t \in [0; T];\\ &x(0) = x^*. \end{aligned} \tag{14}$$

Then the solution mapping of PPDVI (14): $(p, \lambda) \mapsto S(p, \lambda)$ is continuous at $(4, 4)$.

Proof. We use Theorem 4 to prove the conclusion of Example 1 by checking all conditions in Theorem 4.

First, let $P \times V = [1, 8] \times [2, 6]$ be the neighborhood of $(p^*, \lambda^*) = (4, 4)$. It is obvious that the function $G(t, x) = t + x$ is continuous with respect to t for any $x \in \mathbb{R}$ and Lipschitz continuous with respect to x for any $t \in [0, T]$, the function $F(u, \lambda) = \lambda u$ is continuous and uniformly strongly monotone with respect to u for any $\lambda \in [2, 6]$, and the set valued function $K(p) = [1, p + 5]$ is continuous with nonempty, compact and convex values on $[1, 8]$. Therefore, conditions (ii) and (iii) in Theorem 4 hold. We only need to check condition (i) to complete our proof.

Let $(p_n, \lambda_n) \in [1, 8] \times [2, 6]$ be a sequence, which converges to $(4, 4)$. By Theorem 1, it follows that there exists a mild solution $(x_n, u_n) \in S(p_n, \lambda_n)$ for the problem PPDVI(p_n, λ_n). This means that, for all $t \in [0, T] \setminus I$ with $m(I) = 0$, $u_n(t) \in K(p_n)$ and

$$\big(t + x_n(t) + \lambda_n u_n(t)\big)\big(y - u_n(t)\big) \geqslant 0 \quad \forall y \in K(p_n). \tag{15}$$

For $s,\ s+\delta \in [0,T]\setminus I$, letting $t=s$, $y=u_n(s+\delta)$ and $t=s+\delta$, $y=u_n(s)$ in (15), respectively, and adding the obtained inequalities yield

$$\bigl(u_n(s+\delta)-u_n(s),\ x_n(s)-x_n(s+\delta)-\delta\bigr) \geqslant \lambda_n\bigl(u_n(s+\delta)-u_n(s)\bigr)^2,$$

which implies that, for $s,\ s+\delta \in [0,T]\setminus I$,

$$\delta + \bigl|x_n(s+\delta)-x_n(s)\bigr| \geqslant \lambda_n\bigl|u_n(s+\delta)-u_n(s)\bigr|. \tag{16}$$

On the other hand, since $(x_n,u_n)\in S(p_n,\lambda_n)$, it follows that

$$x_n(t) = T(t)x^* + \int_0^t T(t-s)Bu_n(s)\mathrm{d}s, \quad t\in[0,T],$$

where $T(t)=\mathrm{e}^{tA}$ represents the C_0-semigroup. And thus,

$$\begin{aligned}
&x_n(t+\delta)-x_n(t)\\
&\quad = T(t+\delta)x^* - T(t)x^* + \int_0^{t+\delta} T(t+\delta-\tau)Bu_n(\tau)\,\mathrm{d}\tau\\
&\qquad - \int_0^t T(t-\tau)Bu_n(\tau)\,\mathrm{d}\tau\\
&\quad = T(t+\delta)x^* - T(t)x^* + \int_0^{\delta} T(t+\delta-\tau)Bu_n(\tau)\,\mathrm{d}\tau\\
&\qquad + \int_0^t T(t-\tau)Bu_n(\tau+\delta)\,\mathrm{d}\tau - \int_0^t T(t-\tau)Bu_n(\tau)\,\mathrm{d}\tau,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\bigl|x_n(t+\delta)-x_n(t)\bigr|\\
&\quad \leqslant \bigl|T(t+\delta)x^*-T(t)x^*\bigr| + \left|\int_0^{\delta} T(t+\delta-\tau)Bu_n(\tau)\,\mathrm{d}\tau\right|\\
&\qquad + \int_0^t \bigl|T(t-\tau)\bigl(Bu_n(\tau+\delta)-Bu_n(\tau)\bigr)\bigr|\,\mathrm{d}\tau.
\end{aligned} \tag{17}$$

It follows from (16) and (17) that there exist constants $C_1, C_2, C_3, C_4 > 0$ such that

$$\bigl|x_n(t+\delta)-x_n(t)\bigr| \leqslant C_1\delta + C_2\delta + C_3\delta + C_4\int_0^t \bigl|x_n(\tau+\delta)-x_n(\tau)\bigr|\,\mathrm{d}\tau.$$

Applying the Gronwall inequality, we get from the above inequality that there exists a constant $C_5 > 0$ such that

$$\left|x_n(t+\delta) - x_n(t)\right| \leqslant C_5\delta, \tag{18}$$

which indicates that $\{x_n\}$ is an equicontinuous family of functions and has a uniform bound. By Arzelà–Ascoli theorem, there exists a subsequence of $\{x_n\}$, denoted again by $\{x_n\}$, such that it converges to a function x_0 on $[0,T]$. For any $\varepsilon > 0$, taking $\delta = 4(\varepsilon)^{1/2}/(1+C_5)T^{1/2}$, it follows from (16) and (18) that

$$\begin{aligned}\int_0^t \left|u_n(s+\delta) - u_n(s)\right|^2 \mathrm{d}s &\leqslant \frac{1}{4}\int_0^t \left(\delta + \left|x_n(s+\delta) - x_n(s)\right|\right)^2 \mathrm{d}s \\ &\leqslant \frac{1}{4}\int_0^t (\delta + C_5\delta)^2 \mathrm{d}s = \varepsilon.\end{aligned}$$

Now, by Kolmogorov–Riesz–Fréchet theorem, there exists a convergent subsequence of $\{u_n\}$, denoted again by u_n, such that $u_n \to u_0 \in L^2([0,T];\mathbb{R})$, which implies by Theorem 2 that the solution mapping $S(p,\lambda)$ is closed at $(4,4)$, and thus condition (i) in Theorem 4 holds. □

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