The virtual cactus group and Littelmann paths

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Abstract

We define a *virtual cactus group* and show that the cactus group action on Littelmann paths is compatible with the virtualization map defined by Pan–Scrimshaw in [PS18]. Our *virtual cactus group* generalizes the group with the same name defined for the symplectic Lie algebra by Azenhas–Tarighat-Feller–Torres in [ATFT22].

Mathematics Subject Classifications: 05E10, 05E05, 17B37

1 Introduction

Let $\mathfrak{g}$ be a finite dimensional, complex, semisimple Lie algebra. Let D be the Dynkin diagram associated to the root system of $\mathfrak{g}$, R its root system, $\Delta = \{\alpha_i : i \in D\} \subset R$ the set of simple roots, $W = W(R)$ its Weyl group, generated by the simple reflections $\{r_i : i \in D\}$, and $w_0 \in W$ the longest element of the Weyl group. For a connected subdiagram $J \subseteq D$, of D, denote by $\theta_J : J \to J$ the unique Dynkin diagram automorphism that satisfies $\alpha_{\theta_J(j)} = -w_0^J \alpha_j$, for any node $j \in J$, where w_0^J is the longest element of the parabolic subgroup $W^J \subseteq W$ (the Weyl group for $\mathfrak{g}$ restricted to J) [BB05]. This leads to the following definition by Halacheva.

Definition 1. [Hal20] The *cactus group* J_D is the group with generators s_J, one for each connected subdiagram J of D, and relations given as follows:

1. $s_J^2 = 1$;
2. $s_I s_J = s_J s_I$ for $I, J \subseteq D$ connected subsets if the union $J \cup I$ is disconnected;
3. $s_I s_J = s_{\theta_I(J)} s_I$ if $J \subset I$.

Definition 1 is a generalization of the original definition of the cactus group defined by Henriques–Kamnitzer in [HJK04], which was denoted by J_n and which corresponds to the cactus group associated to the Dynkin diagram of type A_{n-1}.

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1.1 Main results and aim of the paper.

In this paper we will be concerned with pairs of Dynkin diagrams (X, Y) related by *folding*, that is, there is an injection of sets of nodes $X \hookrightarrow Y$ which induces an injection of the corresponding Lie algebras $\mathfrak{g}_X \hookrightarrow \mathfrak{g}_Y$ as described in [BS17]. The main result and aim of this paper is the "virtualization" of the cactus group J_X, as defined by Halacheva in [Hal20], and of its action on $\mathfrak{g}_X$-crystals, transferring certain results obtained for the case $C_n \hookrightarrow A_{2n-1}$ in [ATFT22] to the more general setup described above. This is carried out in Theorem 4 and Theorem 9. It consists in defining a group monomorphism $J_X \hookrightarrow J_Y$ compatible with the action of J_X and J_Y on $\mathfrak{g}_X$, respectively $\mathfrak{g}_Y$-crystals. Moreover, by using the virtualization map on Littelmann paths described by Pan–Scrimshaw in [PS18], instead of the Baker virtualization map used in [ATFT22] for Kashiwara–Nakashima tableaux, we obtain a simple rule to compute the partial Schützenberger–Lusztig involutions of Littelmann paths in $\mathfrak{g}_X$-crystals in terms of partial Schützenberger–Lusztig involutions of Littelmann paths in $\mathfrak{g}_Y$-crystals. This is carried out in Theorem 9.

2 The cactus group and crystals

Let Λ be the integral weight lattice and $\Lambda^+ \subset \Lambda$ be the *dominant weights*. Recall that irreducible finite-dimensional representations of $\mathfrak{g}$ are in one-to-one correspondence with the set of highest weights Λ^+. We now recall the definition of a semi-normal crystal as in [BS17].

Definition 2. A semi-normal $\mathfrak{g}$-crystal consists of a non-empty set B together with maps

$$\begin{aligned} \mathrm{wt} :& B \longrightarrow \Lambda \\ e_i, f_i :& B \longrightarrow B \sqcup \{0\}, i \in D \end{aligned}$$

such that for all $b, b' \in B$:

- $b' = e_i(b)$ if and only if $b = f_i(b')$,
- if $f_i(b) \neq 0$ then $\mathsf{wt}(f_i(b)) = \mathsf{wt}(b) - \alpha_i$;
  if $e_i(b) \neq 0$, then $\mathsf{wt}(e_i(b)) = \mathsf{wt}(b) + \alpha_i$, and
- $\varphi_i(b) - \varepsilon_i(b) = \langle \mathsf{wt}(b), \alpha_i^\vee \rangle$,

where

$$\begin{aligned} \varepsilon_i(b) &= \max\{a \in \mathbb{Z}_{\geqslant 0} : e_i^a(b) \neq 0\} \text{ and} \\ \varphi_i(b) &= \max\{a \in \mathbb{Z}_{\geqslant 0} : f_i^a(b) \neq 0\}. \end{aligned}$$

To each such crystal B is associated a *crystal graph*, a coloured directed graph with vertex set B and edges coloured by elements $i \in D$, where if $f_i(b) = b'$ there is an arrow $b \xrightarrow{i} b'$. We say that a crystal is irreducible if its corresponding crystal graph is connected and finite.

The finite irreducible semi-normal $\mathfrak{g}$-crystals are labeled by the dominant weights Λ^+. Given a highest weight $\lambda \in \Lambda^+$, the corresponding irreducible crystal is usually denoted by $B(\lambda)$. It encodes important information about the corresponding irreducible finite dimensional representation of $\mathfrak{g}$, $V(\lambda)$. For instance, $\dim(\mathrm{V}(\lambda))$ equals the cardinality of B, and, in the weight decomposition $V(\lambda) = \bigoplus_{\mu \leqslant \lambda} V(\lambda)_\mu$, $\dim(V(\lambda)_\mu)$ equals the cardinality of the set of $b \in B(\lambda)$ such that $\mathrm{wt}(b) = \mu$. Moreover, for a subinterval $J \subset D$, the crystal corresponding to the Levi restriction of $V(\lambda)$ corresponds to the $\mathfrak{g}_J$-crystal $B(\lambda)_J$ with crystal graph obtained from the graph for $B(\lambda)$ by deleting edges with labels $i \notin J$. In this paper, we will only deal with crystals whose crystal graphs decompose into connected components, each of which is isomorphic to crystals of the form $B(\lambda)$. These are also known in the literature as *normal* crystals.

Schützenberger–Lusztig involutions

There is an elegant internal action of the cactus group $J_\mathfrak{g}$ on crystals via partial Schützenberger–Lusztig involutions, which are generalizations of Schützenberger–Lusztig involutions originally studied by Berenstein–Kirillov [BK95] and generalized by Halacheva [Hal20]. For a subinterval $J \subset D$, the partial Schützenberger–Lusztig involution is defined as follows on $B(\lambda)$. Let $v \in B(\lambda)_J$ be a *highest weight* element, and let $v_{w_0^J} \in B(\lambda)_J$ be a *lowest weight* element. In particular $\mathrm{wt}(v_{w_0^J}) = w_0^J(\mathrm{wt}(v))$ Let $b = f_{i_r} \cdots f_{i_1}(v)$ for $i_j \in J, j \in [1, r]$. Then the partial Schützenberger–Lusztig involution is the unique involution $\xi_J : B(\lambda) \to B(\lambda)$ which satisfies for each $j \in J$:

$$\begin{aligned} \xi_J(e_j(b)) &= f_{\theta_J(j)}(\xi_J(b)) \\ \xi_J(f_j(b)) &= e_{\theta_J(j)}(\xi_J(b)) \text{ and} \\ \mathrm{wt}(\xi_J(b)) &= w_0^J(\mathrm{wt}(b)). \end{aligned}$$

In fact, $\xi_J(b) = e_{\theta_J(i_r)} \cdots e_{\theta_J(i_1)}(v)$. If $J = D$, ξ_J is known as the Schützenberger–Lusztig involution, and denoted simply by ξ. Each partial Schützenberger–Lusztig involution acts as the corresponding Schützenberger–Lusztig involution applied to each connected component of the Levi-branched crystal $B(\lambda)_J$. If our normal crystal B is not connected, partial Schützenberger–Lusztig involutions are defined in the same way as above, on each connected component.

Theorem 3 (Halacheva, [Hal20])**.** *Let B be a normal $\mathfrak{g}$-crystal. The cactus group $J_\mathfrak{g}$ acts on B via partial Schützenberger–Lusztig involutions, that is, for $J \subset D$ a subinterval, the assignment $s_J \mapsto \xi_J$ induces a group action.*

3 The virtual cactus group

Let $X \hookrightarrow Y$ be an embedding of a twisted Dynkin diagram X into a simply-laced Dynkin diagram Y given by folding. More precisely, there is a Dynkin diagram automorphism aut : $Y \to Y$ of Y such that there is an edge-preserving bijection $\sigma : X \to Y/\operatorname{aut}$. The injection of Dynkin diagrams is reflected on the Lie algebras as follows. Let $\mathfrak{g}_X$, respectively $\mathfrak{g}_Y$ be the complex simple Lie algebras with Dynkin diagram X, respectively Y. Then the Dynkin diagram automorphism aut induces a Lie algebra automorphism aut : $\mathfrak{g}_Y \to \mathfrak{g}_Y$. The set of fixed points under this automorphism has the structure of a Lie algebra isomorphic to $\mathfrak{g}_X$ [Kac90]. This induces an injection $\mathfrak{g}_X \hookrightarrow \mathfrak{g}_Y$. Below we list all such pairs, together with the values of θ_X and θ_Y. We use the numbering of the vertices given by [BS17].

X	**Y**	$\boldsymbol{\theta_X}$	$\boldsymbol{\theta_Y}$
$\boldsymbol{C_n}$	$\boldsymbol{A_{2n-1}}$	Id	$\boldsymbol{\theta_Y(i) = 2n - i}$
$\boldsymbol{B_{2n-1}}$	$\boldsymbol{D_{2n}}$	Id	Id
$\boldsymbol{B_{2n}}$	$\boldsymbol{D_{2n+1}}$	Id	$\theta_Y(i) = \begin{cases} i & \text{if } i < 2n \\ 2n, 2n+1 & \text{if } i = 2n+1, 2n \text{ resp.} \end{cases}$
$\boldsymbol{G_2}$	$\boldsymbol{D_4}$	Id	Id
$\boldsymbol{F_4}$	$\boldsymbol{E_6}$	Id	$\theta_Y(i) = \begin{cases} 6, 1 & \text{if } i = 1, 6 \text{ resp.} \\ 5, 3 & \text{if } i = 3, 5 \text{ resp.} \\ i & \text{otherwise} \end{cases}$

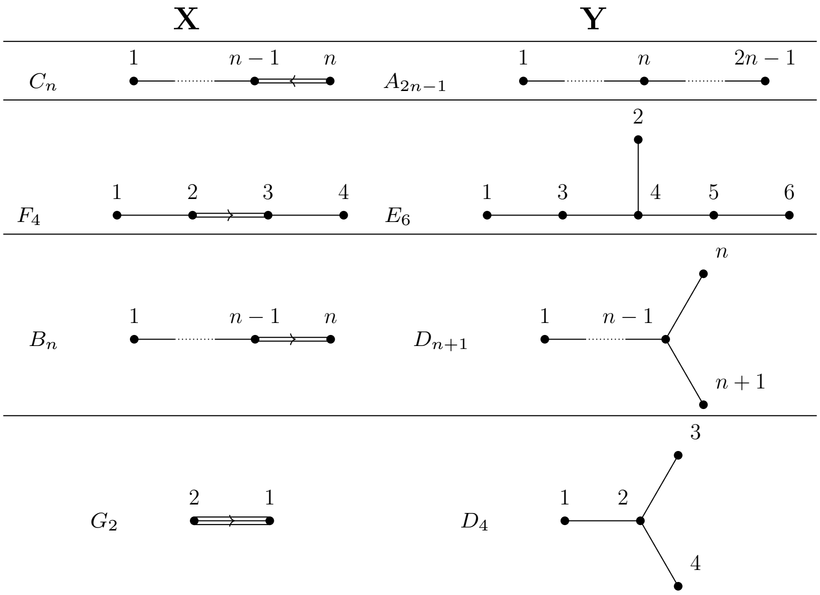
We have aut $= \theta_Y$, except for the cases where $Y = D_{2n}$, where

$$\operatorname{aut}(i) = \begin{cases} i & i < 2n - 1 \\ 2n & i = 2n - 1 \\ 2n - 1 & i = 2n. \end{cases}$$

We proceed to define a group monomorphism $J_X \hookrightarrow J_Y$. Its image will be isomorphic to what we call the virtual cactus group, generalizing the concept of the virtual symplectic cactus group defined in [ATFT22] for $X = C_n$ and $Y = A_{2n-1}$. We start by stating the following lemma, which immediately follows from the description in the previous section. We will abuse notation and consider the coset $\sigma(I) \in Y/\operatorname{aut}$, as a subset of Y, for $I \subset X$. Each non-simply laced Dynkin diagram we consider has what we will call in this note a *branching point* $x_0 \in X$, described in the table below.

X	x_0
$\boldsymbol{C_n}$	n
$\boldsymbol{F_4}$	2
$\boldsymbol{B_n}$	$n - 1$
$\boldsymbol{G_2}$	2

For the comfort of the reader we include the corresponding Dynkin diagrams as well below.



We now consider the following elements:

$$\tilde{s}_I = \prod s^Y_{\tilde{I}}$$

where $s^Y_{\tilde{I}}$ are the generators of the cactus group J_Y and the product is taken over the connected components $\tilde{I}$ of $\sigma(I)$. We will say that $\tilde{s}_I$ is the *virtual image* of s_I. Our aim for the rest of this section is to prove the following result.

Theorem 4. *The map defined by*

$$\begin{aligned} \Phi : J_X &\to J_Y \\ s_I &\mapsto \tilde{s}_I \end{aligned}$$

is a monomorphism of groups.

Lemma 5. *Let $I, J \subset X$ such that $J \subset I$. Then*

$$\tilde{s}_I \tilde{s}_J = \tilde{s}_{\theta_I(J)} \tilde{s}_I$$

Proof. First assume that $\theta_Y = \mathrm{Id}$. This means $Y = D_{2n}$ for some $n \geqslant 2$. If $I = X$ then $\sigma(I) = Y$, therefore the statement of Lemma 5 follows from $\theta_Y = \mathrm{Id}$ and the defining Relation 3 for the cactus group J_Y. If $I \subset X$ does not contain the branching point x_0 then $\sigma|_I : I \to \tilde{I} = \sigma(I)$ is an isomorphism, hence the statement follows trivially. If I is not X but contains the branching point, then either I is of type A, $\sigma(I) = \tilde{I}$ is of type A and $\sigma|_I : I \to \tilde{I}$ is an isomorphism, which implies the claim as in the previous case, or I is of type G_2, in which case the claim also follows easily since J is forced to consist of just one vertex.

Assume next that $\theta_Y = \mathrm{aut}$. If $I \subset X$ contains the branching point x_0, then $\theta_I = \mathrm{Id}_I$ and $\sigma(I) = \tilde{I}$ is connected. Let us then assume first that $x_0 \in I$. Now, if $x_0 \in J$ also, then $\sigma(J) = \tilde{J}$ is connected and $\theta_{\tilde{I}}(\tilde{J}) = \tilde{J}$. Now, if $J \subset I$ does not contain a branching point but I does, then either

- $\sigma(J) = \tilde{J}_1 \sqcup \tilde{J}_2$ has two isomorphic connected components, in which case $\theta_{\tilde{I}}(\tilde{J}_1) = \tilde{J}_2$ and $\theta_{\tilde{I}}(\tilde{J}_2)) = \tilde{J}_1$, or
- $\sigma(J) = \tilde{J}$ is connected and isomorphic to J, in which case $\theta_{\tilde{I}}(\tilde{J}) = \tilde{J}$.

We conclude then that if $x_0 \in I$ and $\sigma(J) = \tilde{J}$ is connected, then

$$\tilde{s}_I \tilde{s}_J = s^Y_{\tilde{I}} s^Y_{\tilde{J}} = s^Y_{\theta_{\tilde{I}}(\tilde{J})} s^Y_{\tilde{I}} = s^Y_{\tilde{J}} s^Y_{\tilde{I}} = \tilde{s}_J \tilde{s}_I = \tilde{s}_{\theta_I(J)} \tilde{s}_I,$$

as desired. Now, if $x_0 \in I$ and $\sigma_J = \tilde{J}_1 \sqcup \tilde{J}_2$, then we still have $\theta_I = \mathrm{Id}$, so $\theta_I(J) = J$. We have in this case

$$\tilde{s}_I \tilde{s}_J = s^Y_{\tilde{I}} s^Y_{\tilde{J}_1} s^Y_{\tilde{J}_2} = s^Y_{\theta_{\tilde{I}}(\tilde{J}_1)} s^Y_{\tilde{I}} s^Y_{\tilde{J}_2} = s^Y_{\theta_{\tilde{I}}(\tilde{J}_1)} s^Y_{\tilde{I}(\tilde{J}_2)} s^Y_{\tilde{I}} = \tilde{s}_J \tilde{s}_I = \tilde{s}_{\theta_I(J)} \tilde{s}_I.$$

This concludes the proof in the case $x_0 \in I$.

Now let us assume that $x_0 \notin I$. We have two cases: The case where $\sigma(I)$ is connected is trivial because since $\theta_Y = \mathrm{aut}$, we conclude that necessarily $\theta_{\sigma(I)} = \mathrm{aut}\,|_{\sigma(I)} = \mathrm{Id}_{\sigma(I)}$, also $\sigma(J) \subset \sigma(I)$ is connected for each $J \subset I$, and $\tilde{s}_J = s^Y_{\sigma(J)}$ for each $J \subset I$. It remains to consider the case where $\sigma(I)$ has two connected components $\sigma(I) = \tilde{I}_1 \sqcup \tilde{I}_2$. It follows that for each $J \subset I$ we have a decomposition into connected components $\sigma(J) = \tilde{J}_1 \sqcup \tilde{J}_2$, where $\tilde{J}_i \subset \tilde{I}_i, i = 1, 2$. The following identity holds by case-by-case analysis:

$$\sigma(\theta_I(J)) = \theta_{\tilde{I}_1}(\tilde{J}_1) \sqcup \theta_{\tilde{I}_2}(\tilde{J}_2). \tag{1}$$

Therefore we have in this case:

$$\begin{aligned}
\tilde{s}_I \tilde{s}_J &= s^Y_{\tilde{I}_1} s^Y_{\tilde{I}_2} s^Y_{\tilde{J}_1} s^Y_{\tilde{J}_2} \\
&= s^Y_{\tilde{I}_1} s^Y_{\tilde{J}_1} s^Y_{\tilde{I}_2} s^Y_{\tilde{J}_2} \\
&= s^Y_{\theta_{\tilde{I}_1}(\tilde{J}_1)} s^Y_{\tilde{I}_1} s^Y_{\theta_{\tilde{I}_2}(\tilde{J}_2)} s^Y_{\tilde{I}_2} \\
&= s^Y_{\theta_{\tilde{I}_1}(\tilde{J}_1)} s^Y_{\theta_{\tilde{I}_2}(\tilde{J}_2)} s^Y_{\tilde{I}_1} s^Y_{\tilde{I}_2} \\
&= \tilde{s}_{\theta_I(J)} \tilde{s}_I,
\end{aligned}$$

where the last equality follows from (1). This concludes the proof in the cases where $\theta_Y = \mathrm{aut}$ and therefore the whole proof.

$\square$

Definition 6. The virtual cactus group J_X^v is defined by generators $s_{\sigma(I)}$, for each $I \subset X$ connected subdiagram, and by the relations:

1. $s_{\sigma(I)}^2 = 1$;
2. $s_{\sigma(I)} s_{\sigma(J)} = s_{\sigma(J)} s_{\sigma(I)}$ if the union $J \cup I$ is disconnected;
3. $s_{\sigma(I)} s_{\sigma(J)} = s_{\sigma(\theta_I(J))} s_{\sigma(I)}$ if $J \subset I$.

It is clear from the definition that the virtual cactus group J_X^v is isomorphic to the cactus group J_X.

Proof of Theorem 4. To show that Φ is a group morphism, we need to show three relations:

1. $\tilde{s}_I^2 = Id$,
2. $\tilde{s}_I \tilde{s}_J = \tilde{s}_J \tilde{s}_I$,
3. $\tilde{s}_I \tilde{s}_J = \tilde{s}_{\theta_I(J)} \tilde{s}_I$.

Note that the third relation has already been established in Lemma 5. To prove (1), note that since the connected components of $\sigma(I)$ are disjoint, the commutation relation *2.* in Definition 1 implies

$$\tilde{s}_I^2 = \prod s_{\tilde{I}}^{Y\,2} = Id$$

To show the second relation, let $I, J \subset X$ be two disjoint, connected intervals. Then necessarily $\sigma(I)$ and $\sigma(J)$ are mutually disjoint. We have then

$$\tilde{s}_I \tilde{s}_J = \prod s_{\tilde{I}}^Y \prod s_{\tilde{J}}^Y = \prod s_{\tilde{J}}^Y \prod s_{\tilde{I}}^Y$$

where the third equality follows from relation *2.* for J_Y. Note that the image $\Phi(J_X)$ is a group isomorphic to the virtual cactus group $\tilde{J}_X$ via the isomorphism $\tilde{s}_I \mapsto s_{\sigma(I)}$, which is well defined because $\sigma(I) = \sigma(J) \iff I = J$. To see this assume that we have $r \in J_X$ such that $\Phi(r) = Id$ in J_Y. Now, r is a product of generators s_I of J_X and $\Phi(r)$ is a product of $\tilde{s}_I$ and therefore a product of $s_{\tilde{I}}^Y$, where for each $I \subset X$, one $s_{\tilde{I}}^Y$ appears for each connected component $\tilde{I}$ of $\sigma(I) \subset Y$. Now the relations satisfied by the $s_{\tilde{I}}^Y$'s are all relations in the cactus group J_Y. Moreover, from the previous parts of this proof, including the proof of Lemma 5, it follows by the case-by-case analysis carried out there that the relations satisfied by the $s_{\tilde{I}}^Y$ imply the same type of relations among the $\tilde{s}_I$ and therefore among the s_I as well. Therefore $r = Id$ in J_X. $\square$

4 Virtualization of the action of the cactus group on crystals of Littelmann paths

In this section we will borrow most of our notation from [PS18] for practical purposes as well as for the comfort of the reader. Let $\lambda \in \Lambda^+$. We consider $\mathcal{P}(\lambda)$ to be the Littelmann path model for λ with paths $\pi : [0,1] \to \Lambda_{\mathbb{R}}$ of the form

$$\pi(t) = \sum_{i \in D} H_{i,\pi}(t)\Lambda_i,$$

where $H_{i,\pi}(t) = \langle \pi(t), \alpha_i^\vee \rangle$ and where $\Lambda_i \in \Lambda^+$ are the fundamental weights for $i \in D$. The set $\mathcal{P}(\lambda)$ has the structure of a crystal isomorphic to $B(\lambda)$ with weight map $\mathrm{wt}(\pi) = \pi(1)$. We refer the reader to [PS18] for the definition of the crystal structure using the notation we use in this section. The original and standard reference of the topic is the paper [Lit95] by Littelmann.

Recall that in this paper we consider embeddings $X \hookrightarrow Y$ given by folding. Let Λ_X and Λ_Y be the corresponding integral weight lattices. The bijection $\sigma : X \to Y/\operatorname{aut}$ induces a map

$$\Psi : \Lambda_X \to \Lambda_Y$$

given by the assignment

$$\Lambda_i^X \mapsto \sum_{j \in \sigma(i)} \gamma_i (\Lambda^Y)_j,$$

where γ_i is given by Table 5.1 in [BS17] (included below) and where Λ_i^X and Λ_j^Y denote the fundamental weights in Λ_X, respectively Λ_Y.

X	γ_i
C_n	$\gamma_i = 1, 1 \leqslant i < r, \gamma_r = 2$
B_n	$\gamma_i = 2, 1 \leqslant i < r, \gamma_r = 1$
F_4	$\gamma_1 = \gamma_2 = 2, \gamma_3 = \gamma_4 = 1$
G_2	$\gamma_1 = 1, \gamma_2 = 3$

Definition 7. Let $\tilde{B}$ be a normal $\mathfrak{g}_Y$-crystal, and a subset $V \subset \tilde{B}$. The *virtual root operators* of type X are, for $i \in X$:

$$e_i^v = \prod_{j \in \sigma(i)} \tilde{e}_j^{\gamma_i} \tag{2}$$

$$f_i^v = \prod_{j \in \sigma(i)} \tilde{f}_j^{\gamma_i}, \tag{3}$$

where $\tilde{e}_i, \tilde{f}_i, i \in Y$ are the root operators for the $\mathfrak{g}_Y$-crystal $\tilde{B}$.
A *virtual crystal* is a pair $(V, \tilde{B})$ such that V has a $\mathfrak{g}_X$-crystal structure defined by

$$e_i := e_i^v f_i := f_i^v \tag{4}$$
$$\varepsilon_i := \gamma_i^{-1} \tilde{\varepsilon}_j \varphi_i := \gamma_i^{-1} \tilde{\varphi}_j, \tag{5}$$

where $\tilde{\varepsilon}_j, \tilde{\varphi}_j j \in Y$ denote the maps given by

$$\tilde{\varepsilon}_i(b) = \max\{a \in \mathbb{Z}_{\geqslant 0} : \tilde{e}_i^a(b) \neq 0\} \text{ and}$$
$$\tilde{\varphi}_i(b) = \max\{a \in \mathbb{Z}_{\geqslant 0} : \tilde{f}_i^a(b) \neq 0\}.$$

If $\mathfrak{g}_X$-crystal B is crystal isomorphic to a virtual crystal $V \subset \tilde{B}$ via an isomorphism $\phi : B \to V$, then the isomorphism ϕ is called a *virtualization* map.

For $\lambda \in \Lambda_X^+$, the weight $\psi(\lambda) \in \lambda_Y$, is dominant, that is, $\psi(\lambda) \in \Lambda_Y^+$. Given $\pi \in \mathcal{P}(\lambda)$, consider the path $\Psi(\pi) : [0, 1] \to \Lambda_Y$ defined by

$$\Psi(\pi)(t) = \sum_{i \in D} H_{i,\pi}(t) \psi(\Lambda_i) \tag{6}$$

One of the main results in [PS18] is the following theorem.

Theorem 8 (Pan–Scrimshaw, [PS18])**.** *The assignment $\pi \mapsto \Psi(\pi)$ induces a virtualization map*

$$\mathcal{P}(\lambda) \to \mathcal{P}(\psi(\lambda))$$
$$\pi \mapsto \Psi(\pi).$$

The principal aim of this section is to describe the action of the cactus group in terms of the virtualization map of Pan–Scrimshaw. For this, given a connected subdiagram $I \subset X$, let

$$\tilde{\xi}_{\sigma(I)} := \prod \xi_{\tilde{I}}^Y$$

where $\xi_{\tilde{I}}^Y$ are the partial Schützenberger–Lusztig involutions in $\mathcal{P}(\psi(\lambda))$ and the product is taken over the connected components $\tilde{I}$ of $\sigma(I)$. Our next aim is to prove the following result, which generalizes [ATFT22, Theorem 5, Theorem 6, Section 9.5].

Theorem 9. *Let $\lambda \in \Lambda_X^+$ and $\mathcal{P}(\lambda)$ the corresponding Littelmann path model. Then the following diagram commutes*

$$\begin{array}{ccc} \mathcal{P}(\lambda) & \xrightarrow{\quad\Psi\quad} & \mathcal{P}(\psi(\lambda)) \\ {\scriptstyle \xi_I^X}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{\xi}_{\sigma(I)}} \\ \mathcal{P}(\lambda) & \xrightarrow[\quad\Psi\quad]{} & \mathcal{P}(\psi(\lambda)) \end{array}.$$

Moreover, the left inverse Ψ^{-1} can be explicitly computed on $\tilde{\xi}_{\sigma(I)}^Y(\Psi(\mathcal{P}(\lambda)))$.

Proof. First note that since the Littelmann path model $\mathcal{P}(\psi(\lambda))$ is stable under the root operators $\tilde{e}_i, \tilde{f}_i$, it is also stable under the action of the operators $\tilde{\xi}^Y_{\sigma(I)}$ for $I \subset X$ connected. Therefore, all paths in $\tilde{\xi}^Y_{\sigma(I)}(\Psi(\mathcal{P}(\lambda)))$ must be of the form (6), so the left inverse Ψ^{-1} can be explicitly computed on $\tilde{\xi}^Y_{\sigma(I)}(\Psi(\mathcal{P}(\lambda)))$, simply by writing out the corresponding path in this form. We now proceed to show that the diagram commutes. Let $\pi_\nu \in \mathcal{P}(\lambda)_I$ be a highest weight path of weight $\mathrm{wt}(\pi_\nu) = \pi_\nu(1) = \nu$ and $\pi = f_{i_r} \cdots f_{i_1}\pi_\nu$ for $i_j \in I, j \in [1, r]$. Recall that

$$\xi^X_I(\pi) = e_{\theta_I(i_r)} \cdots e_{\theta_I(i_1)} \xi^X_I(\pi_\nu),$$

where $\xi^X_I(\pi_\nu)$ is the corresponding lowest weight path in the connected component of $\mathcal{P}(\lambda)_I$ with highest weight path π_ν. Therefore by Theorem 8 we have

$$\Psi(\xi^X_I(\pi)) = e^v_{\theta_I(i_r)} \cdots e^v_{\theta_I(i_1)} \Psi(\xi^X_I(\pi_\nu)).$$

Now, by Definition 7 and Theorem 8 we have

$$\begin{aligned}\tilde{\xi}_{\sigma(I)}(\Psi(\pi)) &= \prod \xi^Y_{\tilde{I}}(\Psi(\pi)) \\ &= \prod \xi^Y_{\tilde{I}}\Big(\prod_{j\in\sigma(i_r)} \tilde{f}_j^{\gamma_{i_r}} \cdots \prod_{j\in\sigma(i_1)} \tilde{f}_j^{\gamma_{i_1}}(\Psi(\pi_\nu))\Big)\end{aligned}$$

where the product is taken over the connected components $\tilde{I}$ of $\sigma(I)$. To continue our computations we consider two cases separately:

1. The subdiagram $\sigma(I) = \tilde{I} \subset Y$ is connected. Then $\theta_I = \mathrm{Id}$, we have $\gamma_{i_j} = 1$ if and only if $\sigma(i_j) = \{\tilde{i}^1_j, \tilde{i}^2_j\}$ or $\sigma(i_j) = \{\tilde{i}^1_j, \tilde{i}^2_j, \tilde{i}^3_j\}$ and $\gamma_{i_j} = 2, 3$ if and only if $\sigma(i_j) = \{\tilde{i}_j\}$. In case $\gamma_{i_j} = 1$ we have $\theta_{\tilde{I}}(\tilde{i}^1_j) = \tilde{i}^2_j$ and $\theta_{\tilde{I}}(\tilde{i}^2_j) = \tilde{i}^1_j$. Moreover, the root operators $\tilde{e}_{\tilde{i}^1_j}$ and $\tilde{e}_{\tilde{i}^2_j}$ commute. In case $\gamma_{i_j} = 2, 3$ we have $\theta_{\tilde{I}}(\tilde{i}_j) = \tilde{i}_j$. All together this implies:

$$\begin{aligned}\tilde{\xi}_{\sigma(I)}(\Psi(\pi)) &= \xi^Y_{\tilde{I}}(f^v_{i_r} \cdots f^v_{i_1}(\Psi(\pi_\nu))) \\ &= e^v_{\theta_I(i_r)} \cdots e^v_{\theta_I(i_1)} \xi^Y_{\tilde{I}}(\Psi(\pi_\nu)) \\ &= e^v_{\theta_I(i_r)} \cdots e^v_{\theta_I(i_1)}(\Psi(\xi^X_I(\pi_\nu))) \\ &= \Psi(\xi^X_I(\pi)).\end{aligned}$$

2. The subdiagram $\sigma(I) \subset Y$ is disconnected. Assume $\theta_Y = \mathrm{aut}$. In this case we must have $|\sigma(I)| = 2|I|$, that is, $\sigma(I) = \tilde{I}_1 \sqcup \tilde{I}_2$ is a disconnected union. In particular all root operators $\tilde{e}_s$, $\tilde{f}_t$ with $s, t \in \tilde{I}_1$ commute with the operators $\tilde{e}_u$, $\tilde{f}_v$, with $u, v \in \tilde{I}_2$. Moreover $\gamma_{i_j} = 1$ for all $j \in [1, r]$. Altogether, this implies:

$$
\begin{aligned}
\tilde{\xi}_{\sigma(I)}(\Psi(\pi)) &= \xi^Y_{\tilde{I}_1}\xi^Y_{\tilde{I}_2}(f^v_{i_r}\cdots f^v_{i_1}(\Psi(\pi_\nu))) \\
&= \xi^Y_{\tilde{I}_1}\xi^Y_{\tilde{I}_2}(\tilde{f}_{i^1_r}\tilde{f}_{i^2_r}\cdots\tilde{f}_{i^1_1}\tilde{f}_{i^2_1}(\Psi(\pi_\nu))) \\
&= \xi^Y_{\tilde{I}_1}\xi^Y_{\tilde{I}_2}(\tilde{f}_{i^2_r}\cdots\tilde{f}_{i^2_1}\tilde{f}_{i^1_r}\cdots\tilde{f}_{i^1_1}(\Psi(\pi_\nu))) \\
&= \xi^Y_{\tilde{I}_1}(\tilde{e}_{\theta_{\tilde{I}_2}(i^2_r)}\cdots\tilde{e}_{\theta_{\tilde{I}_2}(i^2_1)}\tilde{f}_{i^1_r}\cdots\tilde{f}_{i^1_1}(\xi^Y_{\tilde{I}_2}(\Psi(\pi_\nu)))) \\
&= \xi^Y_{\tilde{I}_1}(\tilde{f}_{i^1_r}\cdots\tilde{f}_{i^1_1}\tilde{e}_{\theta_{\tilde{I}_2}(i^2_r)}\cdots\tilde{e}_{\theta_{\tilde{I}_2}(i^2_1)}(\xi^Y_{\tilde{I}_2}(\Psi(\pi_\nu)))) \\
&= \tilde{e}_{\theta_{\tilde{I}_1}(i^1_r)}\cdots\tilde{e}_{\theta_{\tilde{I}}(i^1_1)}\tilde{e}_{\theta_{\tilde{I}_2}(i^2_r)}\cdots\tilde{e}_{\theta_{\tilde{I}_2}(i^2_1)}(\xi^Y_{\tilde{I}_1}\xi^Y_{\tilde{I}_2}(\Psi(\pi_\nu))) \\
&= \tilde{e}_{\theta_{\tilde{I}_1}(i^1_r)}\tilde{e}_{\theta_{\tilde{I}_2}(i^2_r)}\cdots\tilde{e}_{\theta_{\tilde{I}_1}(i^1_1)}\tilde{e}_{\theta_{\tilde{I}_2}(i^2_1)}(\xi^Y_{\tilde{I}_1}\xi^Y_{\tilde{I}_2}(\Psi(\pi_\nu))) \\
&= \Psi(\xi^X_I(\pi)).
\end{aligned}
$$

The case $\theta_Y = \mathrm{Id}$ occurs when $Y = D_{2n}$. In this case $\sigma(I)$ can only be disconnected in Y when I consists solely of the vertex in X corresponding to the small root. We have $\sigma(I) = \{2n-1, 2n\}$ for $n > 2$ (that is, $X = B_{2n-1}$ and $I = \{2n-1\}$) and $\sigma(I) = \{1, 3, 4\}$ for $n = 2$ (here $X = G_2$ and $I = \{1\}$). In the first case we have

$$
\begin{aligned}
\tilde{\xi}_{\sigma(I)}(\Psi(\pi)) &= \xi^Y_{\{2n\}}\xi^Y_{\{2n-1\}}(f^v_{2n-1})^d(\Psi(\pi_\nu)) \\
&= \xi^Y_{\{2n\}}\xi^Y_{\{2n-1\}}(\tilde{f}_{2n-1})^d(\tilde{f}_{2n})^d(\Psi(\pi_\nu)) \\
&= (\tilde{e}_{2n-1})^d(\tilde{e}_{2n})^d\xi^Y_{\{2n\}}\xi^Y_{\{2n-1\}}(\Psi(\pi_\nu)) \\
&= \Psi(\xi^X_I(\pi)).
\end{aligned}
$$

If $X = G_2$ then we have

$$
\begin{aligned}
\tilde{\xi}_{\sigma(I)}(\Psi(\pi)) &= \xi^Y_{\{1\}}\xi^Y_{\{3\}}\xi^Y_{\{4\}}(f^v_1)^d(\Psi(\pi_\nu)) \\
&= \xi^Y_{\{1\}}\xi^Y_{\{3\}}\xi^Y_{\{4\}}((\tilde{f}_1)^d(\tilde{f}_3)^d(\tilde{f}_4)^d(\Psi(\pi_\nu))) \\
&= (\tilde{e}_1)^d(\tilde{e}_3)^d(\tilde{e}_4)^d(\xi^Y_{\{1\}}\xi^Y_{\{3\}}\xi^Y_{\{4\}}(\Psi(\pi_\nu))) \\
&= \Psi(\xi^X_I(\pi)).
\end{aligned}
$$

□

Corollary 10. *The virtual cactus group J^v_X acts on $\mathcal{P}(\psi(\lambda))$ and preserves the image $\Psi(\mathcal{P}(\lambda))$ of Ψ.*

Example 11. Let $X = G_2$ and $Y = D_4$. The cactus group J_{G_2} has three generators: $s_{\{1\}}, s_{\{2\}}, s_{\{1,2\}}$ and relations: $s^2_{\{1\}} = 1, s^2_{\{2\}} = 1, s^2_{\{1,2\}} = 1$, $s_{\{2\}}s_{\{1,2\}} = s_{\{1,2\}}s_{\{2\}}$, $s_{\{1\}}s_{\{1,2\}} = s_{\{1,2\}}s_{\{1\}}$ and no relation between $s_{\{1\}}$ and $s_{\{2\}}$. Now, the virtual images of the generators of J_{G_2} in J_{D_4} are $\tilde{s}_{\{1\}} = s^{D_4}_{\{1\}}s^{D_4}_{\{3\}}s^{D_4}_{\{4\}}$, $\tilde{s}_{\{2\}} = s^{D_4}_{\{2\}}$ and $\tilde{s}_{\{1,2\}} = s^{D_4}_{\{1,2,3,4\}}$. It is clear that there is no relation between $\tilde{s}_{\{1\}}$ and $\tilde{s}_{\{2\}}$, and that the relations defining J_{G_2}

stated above are the only ones satisfied by the $\tilde{s}_I$. The second part of our example involves Littelmann paths. We calculate a Littelmann path model for the irreducible $\mathfrak{g}_{G_2}$-crystal of highest weight $\Lambda_1^{G_2}$ as well as its virtualization in the $\mathfrak{g}_{D_4}$-crystal of highest weight $\Lambda_1^{D_4} + \Lambda_3^{D_4} + \Lambda_4^{D_4}$. We use SageMath [The16] for this, following [PS18, Appendix A].

```
SageMath input:
G2 = RootSystem (['G',2]). weight_space()
LaG = G2.fundamental_weights()
A = crystals.LSPaths(LaG[1])
D4 = RootSystem (['D',4]).weight_space()
LaD = D4.fundamental_weights()
B = crystals.LSPaths( LaD[1]+ LaD[3] + LaD[4])
gens = B.module_generators
psi = A.crystal_morphism ( gens , codomain = B )
for x in A :
    print( " G2 : ", x)
    print(" D4 : ", psi(x))
SageMath output:
 G2 :  (Lambda[1],)
 D4 :  (Lambda[1] + Lambda[3] + Lambda[4],)
 G2 :  (-Lambda[1] + Lambda[2],)
 D4 :  (-Lambda[1] + 3*Lambda[2] - Lambda[3] - Lambda[4],)
 G2 :  (2*Lambda[1] - Lambda[2],)
 D4 :  (2*Lambda[1] - 3*Lambda[2] + 2*Lambda[3] + 2*Lambda[4],)
 G2 :  (-Lambda[1] + 1/2*Lambda[2], Lambda[1] - 1/2*Lambda[2])
 D4 :  (-Lambda[1] + 3/2*Lambda[2] - Lambda[3] - Lambda[4],
 Lambda[1] - 3/2*Lambda[2] + Lambda[3] + Lambda[4])
 G2 :  (-2*Lambda[1] + Lambda[2],)
 D4 :  (-2*Lambda[1] + 3*Lambda[2] - 2*Lambda[3] - 2*Lambda[4],)
 G2 :  (Lambda[1] - Lambda[2],)
 D4 :  (Lambda[1] - 3*Lambda[2] + Lambda[3] + Lambda[4],)
 G2 :  (-Lambda[1],)
 D4 :  (-Lambda[1] - Lambda[3] - Lambda[4],)
```

One can see the effect of the partial and virtual partial Schützenberger involutions by following the definitions in this case. The only i-string in the $\mathfrak{g}_{G_2}$-crystal of paths which has more than one arrow is the 1-string which consists of the three middle paths displayed above:

```
G2 :  (2*Lambda[1] - Lambda[2],)
G2 :  (-Lambda[1] + 1/2*Lambda[2], Lambda[1] - 1/2*Lambda[2])
G2 :  (-2*Lambda[1] + Lambda[2],)
```

Therefore $\xi^X_{\{1\}}$ sends the first element above to the last one. So in this case we see explicitly $\tilde{\xi}_{\sigma(I)}(\Psi(\pi)) = \Psi(\xi^X_I(\pi))$:

```
sage: psi(A[2]).f(1).f(1)
(-2*Lambda[1] - Lambda[2] + 2*Lambda[3] + 2*Lambda[4],)
sage: psi(A[2].f(1).f(1)) == psi(A[2]).f(1).f(3).f(4).f(1).f(3).f(4)
True
```

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