

# Boolean Algebra of One-Point Local Compactifications

*Artur Polański*

**ABSTRACT:** For a given locally compact Hausdorff space we introduce a Boolean algebra structure on the family of all its one-point local compactifications.

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## 1. Introduction

Every locally compact, noncompact Hausdorff space  $X$  has a well known one-point compactification (Alexandroff compactification, [1]). In this paper we consider the set  $\mathcal{B}(X)$  of all one-point local compactifications of  $X$  up to an equivalence. We prove that  $\mathcal{B}(X)$  is a partially ordered set such that the order  $\leq$  induces a Boolean algebra. Moreover, the elements 0 and 1 of  $\mathcal{B}(X)$  are respectively  $X$  and  $\omega X$ . Then we focus on describing the algebra we get using topological notions and convergence and we provide examples by computing the algebra in some special cases. We also note the connection with the topic of ends of manifolds (see [2, pages 110-118]), as for a noncompact, connected, second countable manifold  $L$  with  $n$  ends,  $n < \infty$ , we have  $|\mathcal{B}(L)| = 2^n$ .

## 2. Notation and terminology

- Throughout the paper, ZFC is assumed.
- Given a locally compact Hausdorff space  $X$  we denote by  $\omega X$  a one-point compactification of  $X$  if  $X$  is not compact and  $X$  otherwise,

- a clopen set is a set that is both closed and open,
- if  $Y$  is a one-point local compactification different from  $X$ , the unique point of  $Y \setminus X$  will be denoted by  $\infty_Y$ ,
- a filter  $\mathcal{F}$  of open sets in a topological space  $X$  is a non-empty family of sets open in  $X$  such that  $\emptyset \notin \mathcal{F}$  and, for all  $V_1, V_2 \in \mathcal{F}$  and an open  $V \subset X$  we have  $V_1 \cap V_2 \in \mathcal{F} \Rightarrow V \in \mathcal{F}$ .

### 3. Main results

**Definition 1.** If  $X$  is a locally compact Hausdorff space, we call  $(Y, f)$  an at most one-point local compactification of  $X$  iff  $Y$  is a locally compact Hausdorff and  $f : X \rightarrow Y$  is a homeomorphic embedding such that  $f(X)$  is dense in  $Y$  and  $|Y \setminus f(X)| \leq 1$ . If  $(Y, f)$  is an at most one point local compactification of  $X$  and  $|Y \setminus f(X)| = 1$ , we call  $(Y, f)$  a one-point local compactification of  $X$ .

For simplicity, we say that  $Y$  is a/an (at most) one-point local compactification of  $X$  iff  $(Y, \text{id}_X)$  is a/an (at most) one-point local compactification of  $X$ .

**Definition 2.** Let  $X$  be a locally compact Hausdorff space,  $(Y_1, f_1)$  and  $(Y_2, f_2)$  its at most one-point local compactifications. We will write  $(Y_1, f_1) \leq (Y_2, f_2)$  (or, for simplicity,  $Y_1 \leq Y_2$ ) iff one of the following conditions apply:

- $f_1(X) = Y_1$
- $Y_1 = f_1(X) \cup \{\infty_{Y_1}\}$ ,  $Y_2 = f_2(X) \cup \{\infty_{Y_2}\}$  and the function

$$Y_1 \ni x \mapsto \begin{cases} f_2(f_1^{-1}(x)), & x \in f_1(X) \\ \infty_{Y_2}, & x = \infty_{Y_1} \end{cases} \in Y_2$$

is continuous.

Note that  $\leq$  is reflexive and transitive, with  $0 = X$  and  $1 = \omega X$ . We can define an equivalence relation  $\equiv$  by

$$(Y_1, f_1) \equiv (Y_2, f_2) \text{ iff } (Y_1, f_1) \leq (Y_2, f_2) \text{ and } (Y_2, f_2) \leq (Y_1, f_1),$$

or, for simplicity,

$$Y_1 \equiv Y_2 \text{ iff } Y_1 \leq Y_2 \text{ and } Y_2 \leq Y_1.$$

We also define

$$\mathcal{B}(X) := \{Y\text{---one-point local compactification of } X\} / \equiv.$$

From now on instead of an equivalence class of  $Y$  in  $\mathcal{B}(X)$  we will just write  $Y$ .

We are now ready to state the first result where we will prove that  $\mathcal{B}(X)$  ordered by  $\leq$  is a Boolean algebra, by showing that it is in fact order isomorphic to a much simpler one.

**Theorem 1.** *Given a locally compact Hausdorff space  $X$ ,  $\mathcal{B}(X)$  is a partially ordered space with a lattice such that the order  $\leq$  induces a Boolean algebra, i.e., for  $Y_1, Y_2$  one-point local compactifications of  $X$ :*

- $Y_1 \vee Y_2 = \sup_{\leq} \{Y_1, Y_2\}$ ,
- $Y_1 \wedge Y_2 = \inf_{\leq} \{Y_1, Y_2\}$ ,
- $0 = X$ ,
- $1 = \omega X$ ,
- for any space  $Y \in \mathcal{B}(X)$  there exists a unique space  $\setminus Y \in \mathcal{B}(X) : Y \wedge \setminus Y = 0, Y \vee \setminus Y = 1$ .

*In particular,  $0 = 1$  iff  $X$  is compact.*

**Proof.** First consider  $\beta X$ , a Čech–Stone compactification of  $X$ . We define  $\mathcal{A}(X) := \{F \subset \beta X \setminus X : F \text{ clopen in } \beta X \setminus X\}$  (note that  $\beta X \setminus X$  is compact).  $\mathcal{A}(X)$  with standard set operations is a Boolean algebra. We will show an isomorphism between  $\mathcal{B}(X)$  and  $\mathcal{A}(X)$ , proving that  $\mathcal{B}(X)$  is also a Boolean algebra.

To this end, we will define  $f : \mathcal{B}(X) \rightarrow \mathcal{A}(X)$ . If  $X$  is compact, both  $\mathcal{B}(X)$  and  $\mathcal{A}(X)$  are trivial, therefore assume that  $X$  is not compact. Consider a clopen in  $\beta X \setminus X$  set  $F$  such that  $\emptyset \neq F \neq \beta X \setminus X$ . We can now identify  $F$  and  $(\beta X \setminus X) \setminus F$  with points, getting a compact space  $X \cup \{\{F\}\} \cup \{\{(\beta X \setminus X) \setminus F\}\}$ . Its subspace  $X \cup \{\{F\}\}$  is then a one-point local compactification of  $X$ . Conversely, for any one-point local compactification  $Y$  of  $X$  there exists a unique clopen in  $\beta X \setminus X$  set  $F_Y$  such that  $Y$  is equivalent with  $X \cup \{\{F_Y\}\}$  (from the universal property of  $\beta X$ ). We define  $f(X) = \emptyset$  and for every one-point local compactification  $Y$  of  $X$  we put  $f(Y) = F_Y$ , where  $Y$  is the unique clopen in  $\beta X \setminus X$  set such that  $Y$  is equivalent to  $X \cup \{\{F_Y\}\}$ . It can be easily seen that for one-point local compactifications  $Y_1, Y_2$  of  $X$  we have  $Y_1 \leq Y_2$  iff  $F_{Y_1} \subset F_{Y_2}$ , so  $f$  preserves the partial order and is indeed an isomorphism. Furthermore, for one-point local compactifications  $Y_1, Y_2$  of  $X$  we have:

1.  $Y_1 \vee Y_2 = X \cup \{\{F_{Y_1} \cup F_{Y_2}\}\}$ .
2.  $Y_1 \wedge Y_2 = X \cup \{\{F_{Y_1} \cap F_{Y_2}\}\}$  if  $F_{Y_1} \cap F_{Y_2} \neq \emptyset$  and  $Y_1 \wedge Y_2 = X$  otherwise.
3.  $\setminus Y = X \cup \{\{(\beta X \setminus X) \setminus F_Y\}\}$  for  $\emptyset \neq F_Y \neq \beta X \setminus X$ .

□

**Remark 1.** The proof of Theorem 1 shows that  $\mathcal{B}(X)$  is isomorphic (as a Boolean algebra) to the algebra of all clopen subsets of the remainder  $\beta X \setminus X$  of  $X$ . One easily concludes that the Stone space of  $\mathcal{B}(X)$  is homeomorphic to the space of all connected components of  $\beta X \setminus X$  (that is, the space obtained from  $\beta X \setminus X$  by identifying points that lie in a common connected component).

Now that we know that  $\mathcal{B}(X)$  is a Boolean algebra, we will focus on describing it without using  $\mathcal{A}(X)$ . If we add a point  $\{\infty_Y\}$  to a locally compact Hausdorff space  $X$  to get its one-point local compactification  $Y$ , we only need to know the neighborhood basis at  $\{\infty_Y\}$  to know its topology. To this end, let us introduce the following characterization. For simplicity, we will also use one more definition.

**Definition 3.** Let  $X$  be a locally compact Hausdorff space,  $Y$  its one-point local compactification. Then

$$\tau(Y) := \{U \setminus \{\infty_Y\} : U \text{ open neighborhood of } \infty_Y \text{ in } Y\}.$$

$\tau(Y)$  uniquely determines  $Y \neq X$ ,  $Y \in \mathcal{B}(X)$ .

**Proposition 1.** Let  $X$  be a locally compact Hausdorff space,  $Y_1, Y_2 \in \mathcal{B}(X)$ ,  $Y_1, Y_2 \neq 0$ ,  $Y_1, Y_2 \neq 1$ .

1.  $\tau(Y_1 \wedge Y_2) = \{U_1 \cap U_2 : U_1 \in \tau(Y_1), U_2 \in \tau(Y_2)\}$ , provided that the sets  $U_1 \cap U_2$  are nonempty for all  $U_1 \in \tau(Y_1), U_2 \in \tau(Y_2)$  and  $Y_1 \wedge Y_2 = 0$  otherwise.
2.  $\tau(Y_1 \vee Y_2) = \{U_1 \cup U_2 : U_1 \in \tau(Y_1), U_2 \in \tau(Y_2)\} = \tau(Y_1) \cap \tau(Y_2)$ .
3.  $\tau(\setminus Y_1) = \{X \setminus F : F \subset X, \text{ for any } U \in \tau(Y_1) \text{ } F \setminus U \text{ compact}\}$ .

Or, in terms of convergence:

- (a) A net  $(x_\gamma) \subset X$  in  $Y_1 \wedge Y_2$  is convergent to  $\infty_{Y_1 \wedge Y_2}$  iff  $(x_\gamma)$  is convergent to  $\infty_{Y_1}$  in  $Y_1$  and to  $\infty_{Y_2}$  in  $Y_2$ , and  $Y_1 \wedge Y_2 = 0$  if there is no such net.
- (b) A net  $(x_\gamma) \subset X$  in  $Y_1 \vee Y_2$  is convergent to  $\infty_{Y_1 \vee Y_2}$  iff every subnet of  $(x_\gamma)$  has a subnet convergent to  $\infty_{Y_1}$  in  $Y_1$  or to  $\infty_{Y_2}$  in  $Y_2$ .
- (c) A net  $(x_\gamma) \subset X$  in  $\setminus Y_1$  is convergent to  $\infty_{\setminus Y_1}$  iff  $(x_\gamma)$  has no convergent subnets in  $Y_1$ .

**Proof.** Again, let  $\beta X$  be a Čech–Stone compactification of  $X$ .

Note that if  $Y$  is a one-point local compactification of  $X$  and  $F_Y$  is a clopen set in  $\beta X \setminus X$  such that  $Y$  is equivalent with  $X \cup \{F_Y\}$ , then

$$\tau(Y) = \{X \cap U : U \supset F_Y \text{ and } U \text{ open in } \beta X\}. \quad (*)$$

Following this notation consider  $F_{Y_1}$  and  $F_{Y_2}$  such that  $Y_1$  and  $Y_2$  are equivalent to  $X \cup \{F_{Y_1}\}$  and  $X \cup \{F_{Y_2}\}$  respectively.

Property (2) follows easily from (\*).

To see that  $\{U_1 \cup U_2 : U_1 \in \tau(Y_1), U_2 \in \tau(Y_2)\} = \tau(Y_1) \cap \tau(Y_2)$ , take any  $U_1 \in \tau(Y_1), U_2 \in \tau(Y_2)$ .  $U_2 = (U_2 \cup \{\infty_{Y_2}\}) \cap X$  is open in  $X$ , and thus open in  $Y_1$ .  $U_1 \cup \{\infty_{Y_1}\}$  is also open in  $Y_1$  and thus so is  $U_1 \cup \{\infty_{Y_1}\} \cup U_2$ . Similarly,  $U_1 \cup \{\infty_{Y_2}\} \cup U_2$  is open in  $Y_2$ . The reverse inclusion is trivial.

We turn to (1). If  $F_{Y_1} \cap F_{Y_2} = \emptyset$  we have  $Y_1 \wedge Y_2 = 0$ , assume the contrary. Consider  $U$  open in  $\beta X$  such that  $F_{Y_1} \cap F_{Y_2} \subset U$  and take  $V_1, V_2$  open in  $\beta X$  such

that  $V_1 \cap V_2 = \emptyset$ , and we have  $F_{Y_1} \setminus U \subset V_1$  and  $F_{Y_2} \setminus U \subset V_2$ . Then  $U_1 := V_1 \cup U$  and  $U_2 := V_2 \cup U$  are open (in  $\beta X$ ) supersets of respectively  $F_{Y_1}$  and  $F_{Y_2}$  such that  $U_1 \cap U_2 = U$ , which gives us (1).

We are left with (3). To see that

$$\tau(\setminus Y_1) \subset \{X \setminus F : F \subset X, \text{ for any } U \in \tau(Y_1) \ F \setminus U \text{ compact}\},$$

consider  $V$  open in  $\beta X$  such that  $(\beta X \setminus X) \setminus F_{Y_1} \subset V$  and take any  $U$  open in  $\beta X$  such that  $F_{Y_1} \subset U$ . Then  $(X \setminus V) \setminus U = X \setminus (U \cup V) = \beta X \setminus (U \cup V)$  is a closed subset of  $\beta X$  contained in  $X$  and therefore compact.

For the reverse inclusion, let  $V_0$  and  $W_0$  be open sets with disjoint closures in  $\beta X$  such that  $(\beta X \setminus X) \setminus F_{Y_1} \subset V_0$  and  $F_{Y_1} \subset W_0$ . Consider  $F \subset X$  such that for any  $U \in \tau(Y_1)$  the set  $F \setminus U$  is compact. Take any  $x \in X$  and its closed (in  $X$ ) neighborhood  $G$  such that  $G$  is compact. Then  $X \setminus G \in \tau(Y_1)$ , so  $F \cap G$  is compact. Since  $x$  and its neighborhood  $G$  were arbitrary, this implies that  $F$  is closed in  $X$  (since if we take  $x$  from the boundary of  $F$ , we get that it must be in  $F$ ). Similarly, since  $F \cap \overline{V_0} \subset F \setminus W_0$  and  $W_0 \cap X \in \tau(Y_1)$ , we get that  $F \cap \overline{V_0}$  is compact which implies that  $F \cup F_{Y_1}$  is closed in  $\beta X$ . Therefore we have  $X \setminus F = X \cap (\beta X \setminus (F \cup F_0)) \in \tau(\setminus Y_1)$  which ends the proof of (3).

Properties (a) – (c) follow easily from (1) – (3). □

On the other hand, one can wonder when a family  $\mathcal{F}$  of sets open in a locally compact Hausdorff space  $X$  induces its one-point local compactification. The following proposition answers that question.

**Proposition 2.** *Let  $\mathcal{F}$  be a filter of open sets in a locally compact Hausdorff space  $X$ . Then  $\mathcal{F}$  induces a one-point local compactification  $Y$  of  $X$  such that  $\tau(Y) = \mathcal{F}$  iff:*

1.  $\bigcap \mathcal{F} = \emptyset$ ,
2. there exists  $U \in \mathcal{F}$  such that for every  $V \in \mathcal{F}$ ,  $\overline{U} \setminus V$  is compact,
3. for every  $U \in \mathcal{F}$  there exists  $V \in \mathcal{F}$  such that  $\overline{V} \subset U$ .

**Proof.** It follows from the definition of  $\tau(Y)$  and the definition of a locally compact Hausdorff space that those conditions are necessary. We will prove that they are also sufficient. We take  $Y := X \cup \{\infty_Y\}$ . A set is open in  $Y$  iff it is open in  $X$  or it is of the form  $U \cup \{\infty_Y\}$  for some  $U \in \mathcal{F}$ . It follows from (1) and (3) that the topology defined like that is Hausdorff. It remains to show that  $Y$  is locally compact. Take  $U \in \mathcal{F}$  such that for every  $V \in \mathcal{F}$   $\overline{U} \setminus V$  is compact and assume that  $\overline{U}$  (closure taken in  $Y$ ) is not compact. It follows that there exists a net  $(x_\gamma) \subset \overline{U}$  with no convergent subnets. In particular,  $(x_\gamma)$  is not convergent to  $\infty_Y$ , so there exists  $V_1$  a neighborhood of  $\infty_Y$  and  $(y_\gamma)$  a subnet of  $(x_\gamma)$  such that  $(y_\gamma) \subset \overline{U} \setminus V_1$  with no convergent subnets, a contradiction. □

We will now provide a characterization for  $\mathcal{B}(\mathbb{R}^n)$ . To this end, we will need facts about  $n$ -point Hausdorff compactifications (see [5] or [3, Theorem 6.8]).

**Theorem 2** (Theorem 2.1 in [5]). *The following statements concerning a space  $X$  are equivalent:*

1.  $X$  has a  $N$ -point compactification.
2.  $X$  is locally compact and contains a compact subset  $K$  whose complement is the union of  $N$  mutually disjoint, open subsets  $\{G_i\}_{i=1}^N$  such that  $K \cup G_i$  is not compact for each  $i$ .
3.  $X$  is locally compact and contains a compact subset  $K$  whose complement is the union of  $N$  mutually disjoint, open subsets  $\{G_i\}_{i=1}^N$  such that  $K \cup G_i$  is contained in no compact subset for each  $i$ .

Using this, we can prove the following facts.

**Lemma 1.** *Let  $X$  be a locally compact, noncompact Hausdorff space such that for any  $K \subset X$  compact there exists  $K_0$  compact such that  $K \subset K_0$  and  $X \setminus K_0$  has exactly  $n$  connected components (for some fixed  $n \in \mathbb{N}$  independent of the choice of  $K$ ), all of them are open and have noncompact (in  $X$ ) closures. Then  $X$  has an  $n$ -point Hausdorff compactification and does not have an  $(n + 1)$ -point Hausdorff compactification.*

**Lemma 2.** *Let  $n \in \mathbb{N}$  and  $X$  be a Hausdorff topological space that has an  $n$ -point Hausdorff compactification and does not have an  $(n + 1)$ -point Hausdorff compactification. Then  $X$  is locally compact and  $|\mathcal{B}(X)| = 2^n$ .*

We will start with Lemma 1.

**Proof.** Applying the assumption of the lemma to the empty set we get that there exists  $n \in \mathbb{N}$  and  $K_0$  compact such that  $X \setminus K_0$  has exactly  $n$  connected components, let us denote them by  $G_1, \dots, G_n$ . Therefore (by [5])  $X$  has an  $n$ -point Hausdorff compactification. Suppose that  $X$  has an  $(n + 1)$ -Hausdorff compactification. Again by [5], there exist  $H_1, \dots, H_{n+1}$  such that  $K_1 := X \setminus \bigcup_{i=1}^{n+1} H_i$  is compact, but for each  $i$  the set  $K_1 \cup H_i$  is not compact. Applying the assumption of the lemma again, this time to  $K_1$ , we get that there exists a compact set  $K_2$  such that  $K_1 \subset K_2$  and  $X \setminus K_2$  has  $n$  connected components, let us denote them by  $V_1, \dots, V_n$ . Then there exist  $i_0 \in \{1, \dots, n\}$  and  $j_1, j_2 \in \{1, \dots, n + 1\}$  such that  $j_1 \neq j_2$  and  $H_{i_0}$  has nonempty intersection with both  $V_{j_1}, V_{j_2}$ , so it cannot be connected, a contradiction.  $\square$

Now we turn to Lemma 2.

**Proof.** Since  $X$  has an  $n$ -point Hausdorff compactification, but does not have an  $n + 1$ -point Hausdorff compactification,  $\beta X \setminus X$  has exactly  $n$  connected components. From the proof of Theorem 1 we know that  $|\mathcal{B}(X)| = |\mathcal{A}(X)|$ . Each element of  $\mathcal{A}(X)$  is a union of some connected components of  $\beta X \setminus X$ , so  $|\mathcal{B}(X)| = |\mathcal{A}(X)| = 2^n$ .  $\square$

**Remark 2.** Note that if we assume that if  $X$  is a locally compact space such that  $|\mathcal{B}(X)| = 2^n$ , we also get that  $X$  has an  $n$ -point Hausdorff compactification and does not have an  $(n + 1)$ -point Hausdorff compactification (see also [3, Theorem 6.32]).

From the above lemmas we immediately get the following.

**Corollary 1.**

- $\mathcal{B}(\mathbb{R}) = \{\mathbb{R}, [-\infty, \infty), (-\infty, \infty], \mathbb{S}^1\}$ .
- $\mathcal{B}(\mathbb{R}^n) = \{\mathbb{R}^n, \mathbb{S}^n\}$  for  $n \geq 2$ .

We will now define the end of manifolds, as seen in [2].

**Definition 4.** Let  $L$  be a noncompact, connected manifold. Denote by  $\{K_\alpha\}_{\alpha \in \mathcal{K}}$  the family of all compact subsets of  $L$ . We consider descending chains

$$U_{\alpha_1} \supseteq U_{\alpha_2} \supseteq \cdots \supseteq U_{\alpha_n} \supseteq \cdots$$

where each  $U_{\alpha_k}$  is a connected component of  $L \setminus K_{\alpha_k}$ , has noncompact closure in  $L$ , satisfies  $U_{\alpha_k} \supseteq \overline{U_{\alpha_{k+1}}}$  and

$$\bigcap_{k=1}^{\infty} U_{\alpha_k} = \emptyset.$$

We say that two such chains  $\mathcal{U} = \{U_{\alpha_k}\}_{k=1}^{\infty}$  and  $\mathcal{V} = \{U_{\beta_k}\}_{k=1}^{\infty}$  are equivalent ( $\mathcal{U} \sim \mathcal{V}$ ) if for each  $k \geq 1$  there is  $n > k$  such that  $U_{\alpha_k} \supseteq V_{\beta_n}$  and  $V_{\beta_k} \supseteq U_{\alpha_n}$ . It is easy to check that  $\sim$  is an equivalence relation. If

$$\mathcal{U} = \{U_{\alpha_1} \supseteq U_{\alpha_2} \supseteq \cdots \supseteq U_{\alpha_n} \supseteq \cdots\}$$

is as above, we call its equivalence class under  $\sim$  an *end* of  $L$ .

**Corollary 2.**

If  $L$  is a noncompact, connected, second countable manifold with  $n$  ends,  $n < \infty$ , then  $|\mathcal{B}(L)| = 2^n$ .

**Proof.** Let

$$\begin{aligned} \mathcal{U}_1 &= \{U_{\alpha_1}^1 \supseteq U_{\alpha_2}^1 \supseteq \cdots\} \\ &\vdots \\ \mathcal{U}_n &= \{U_{\alpha_1}^n \supseteq U_{\alpha_2}^n \supseteq \cdots\} \end{aligned}$$

be representatives of the ends of  $L$ .

For every  $k \in \{1, 2, \dots\}, l \in \{1, 2, \dots, n\}$  let  $K_{\alpha_k}^l$  be a compact set such that  $U_{\alpha_k}^l$  is a connected component of  $L \setminus K_{\alpha_k}^l$ . We will show that by taking subsequences of  $\mathcal{U}_2, \dots, \mathcal{U}_n$  we can assume that  $U_{\alpha_k}^{l_2} \subset L \setminus K_{\alpha_k}^{l_1}$  for every  $l_2 > l_1$  (note that a subsequence of a representative of an end is a representative of the same end).

Consider  $K_{\alpha_1}^1$ . Then  $\{L \setminus \overline{U_{\alpha_1}^2}, L \setminus \overline{U_{\alpha_1}^3}, \dots\}$  is an open cover of  $K_{\alpha_1}^1$  so there exists  $N_1 > 0$  such that  $K_{\alpha_1}^1 \subset L \setminus \overline{U_{\alpha_{N_1}}^2} \subset L \setminus U_{\alpha_{N_1}}^2$ . Therefore  $U_{\alpha_{N_1}}^2 \subset L \setminus K_{\alpha_1}^1$ . Similarly, for each  $m > 1$ , we can define  $N_m > N_{m-1}$  such that  $U_{\alpha_{N_m}}^2 \subset L \setminus K_{\alpha_m}^1$ . Replacing  $U_{\alpha_m}^2$  by  $U_{\alpha_{N_m}}^2$  for each  $m > 0$  we get a subsequence we want for  $\mathcal{U}_2$ . Now we proceed similarly for  $\mathcal{U}_3, \dots, \mathcal{U}_n$ .

We will now show that by again taking subsequences we can assume that for every  $l_1 \neq l_2$  we have  $U_{\alpha_1}^{l_1} \cap U_{\alpha_1}^{l_2} = \emptyset$ . Assume the contrary. Then, without loss of generality, for each  $k > 0$  we have  $U_{\alpha_k}^1 \cap U_{\alpha_k}^2 \neq \emptyset$ . Since  $U_{\alpha_k}^2 \subset L \setminus K_{\alpha_k}^1$ , the set  $U_{\alpha_k}^2$  is connected,  $U_{\alpha_k}^1$  is a connected component of  $L \setminus K_{\alpha_k}^1$  and  $U_{\alpha_k}^1 \cap U_{\alpha_k}^2 \neq \emptyset$ , it follows that  $U_{\alpha_k}^2 \subset U_{\alpha_k}^1$  for each  $k > 0$ . Now consider  $K_{\alpha_k}^2$ . As before, there exists  $N_k > k$  such that  $K_{\alpha_k}^2 \subset L \setminus U_{\alpha_{N_k}}^1$ . It follows that  $U_{\alpha_{N_k}}^1 \subset L \setminus K_{\alpha_k}^2$ . If  $U_{\alpha_{N_k}}^1 \not\subset U_{\alpha_k}^2$  then  $U_{\alpha_{N_k}}^1 \cap U_{\alpha_k}^2 = \emptyset$ , so  $U_{\alpha_{N_k}}^1 \cap U_{\alpha_{N_k}}^2 = \emptyset$  (since  $U_{\alpha_{N_k}}^2 \subset U_{\alpha_k}^2$ ). Therefore  $U_{\alpha_{N_k}}^1 \subset U_{\alpha_k}^2$  and so  $\mathcal{U}_1$  and  $\mathcal{U}_2$  are representatives of the same end, a contradiction.

Now our aim is to use Lemmas 1 and 2, which will end the proof. To this end, we will construct a family of compact sets  $\{K_j\}_{j=1}^\infty$ . We will need some properties of manifolds, namely that a second countable manifold is metrizable and that the one-point compactification of a connected manifold is locally connected (see [4] or [6, page 104]). Let  $\omega L = L \cup \{\infty\}$  be the one-point compactification of  $L$ . Since  $L$  is second countable we can choose a countable basis of its topology  $\mathcal{B} = \{B_1, B_2, B_3, \dots\}$  consisting of open sets with compact closures. Take  $A_1 := K_{\alpha_1}^1 \cup \dots \cup K_{\alpha_1}^n \cup \overline{B_1}$ . Let  $K_1$  be a compact superset of  $A_1$  such that  $\omega L \setminus K_1$  is connected (it exists because  $\omega L$  is locally connected). Note that connected components of  $L \setminus K_1$  are all open and have noncompact (in  $L$ ) closures (because  $\infty$  is in the closure taken in  $\omega L$  of every one of them). Again, because  $L$  is locally compact we can take an open set  $A_2$  with compact closure such that  $K_1 \cup \overline{A_2} \subset A_2$ . Let  $K_2$  be a compact superset of  $A_2$  such that  $\omega L \setminus K_2$  is connected. As before, all connected components of  $L \setminus K_2$  are open and have noncompact (in  $L$ ) closures. Moreover, each of them is contained together with its closure in some connected component of  $L \setminus K_1$ . Note that since  $\omega L \setminus K_2$  has non-empty intersection with every connected component of  $L \setminus K_1$  (because  $\infty$  is in the closure taken in  $\omega L$  of every one of them), for every connected component of  $L \setminus K_1$  there is at least one connected component of  $L \setminus K_2$  contained in it. Continuing in this manner, we get  $\{K_j\}_{j=1}^\infty$ . Note that  $K_j$  is contained in the interior of  $K_{j+1}$  for each  $j \geq 1$  and  $\bigcup_{j=1}^\infty K_j = L$ . Moreover, when  $j$  increases the number of connected components of  $L \setminus K_j$  either increases or stays the same. Consider a connected component  $U_1$  of  $L \setminus K_1$ . We want to show that  $U_1 \cap U_{\alpha_1}^i \neq \emptyset$  for some  $i$ . Indeed, otherwise by choosing a connected component  $U_2$  of  $U_1 \setminus K_2$ , then a connected  $U_3$  of  $U_2 \setminus K_3$  etc. we would get a representative of an end that is not among  $\mathcal{U}_1, \dots, \mathcal{U}_n$ , a contradiction. Suppose that  $U_1 \cap U_{\alpha_1}^1 \neq \emptyset$ . Since  $K_{\alpha_1}^1 \subset K_1$  and  $U_{\alpha_1}^1, U_1$  are connected components of their complements we get  $U_1 \subset U_{\alpha_1}^1$ . The sets  $U_{\alpha_1}^i$  are pairwise disjoint, so  $L \setminus K_1$  has at least  $n$  connected components. Moreover, the number of connected components of  $L \setminus K_j$  cannot increase past  $n$  for any  $j$ . Indeed, if we had at least  $n + 1$  connected components of  $L \setminus K_j$  for some  $j$ , we could construct at least  $n + 1$  different ends (similarly as before) which again contradicts the fact that  $\mathcal{U}_1, \dots, \mathcal{U}_n$  are all of the ends in  $L$ . Lemma 1 ends the proof.  $\square$

From this and Remark 2 we also get the following.

**Corollary 3.** *If  $L$  is a noncompact, connected, second countable manifold with  $n$  ends,  $n < \infty$ , then  $L$  has an  $n$ -point Hausdorff compactification and does not have an  $(n + 1)$ -point Hausdorff compactification.*



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**Artur Polański**

email: [artur.polanski@uj.edu.pl](mailto:artur.polanski@uj.edu.pl)

ORCID: 0000-0002-3302-2551

Department of Functional Analysis

Institute of Mathematics of the Jagiellonian University

POLAND

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