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ONSAGERS FLUCTUATION THEORY AND NEW DEVELOPMENTS INCLUDING NON-EQUILIBRIUM LÉVY FLUCTUATIONS*

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The first part of the paper briefly reviews and explains basic ideas of the theory of Gaussian fluctuations and their relaxation developed in 1931 by Lars Onsager in the context of a general theory of irreversible processes. Motivated by Onsager's approach, we extend the theory to fluctuations including Lévy processes. We assume that deviations from Gaussian distributions, which are often observed in non-equilibrium systems, may be described by convoluted Gauss–Lévy distributions and their relation to stationary states by generalized Smoluchowski equations. The central part of the distributions we study here is determined by the Gaussian core with the wings decaying according to a power law characteristic for a Lévy-type contribution to statistics. Furthermore, we develop a generalization of Onsager's theory of linear relaxation processes to those which include statistically independent Gaussian fluctuations and (non-equilibrium) Lévy noises. We apply the generalized version of the fluctuation-dissipation theorem (FDT) to analyze regime of the linear response of the non-equilibrium system driven by Lévy (Cauchy) white noise and subject to thermal (Gaussian) fluctuations. In the last part, applications to non-Maxwellian velocity fluctuations and their relaxations are investigated.

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1. Introduction

On the basis of the early work by Boltzmann and Einstein, the modern theory of fluctuations was created in two fundamental works by Onsager [1, 2], which nearly 40 years later were pointed out when nominating the Author to the Nobel Prize for "the discovery of the reciprocal relations bearing his name, which are fundamental for the thermodynamics of irreversible processes".

The basic ideas of Onsager's fluctuation theory can be found in the second part of the eminent paper [2]. More than 20 years after its publication in 1931, Onsager's theory was extended and generalized in a fruitfull collaboration with Machlup [3].

In 2013 the 110th birthday anniversary of Lars Onsager takes place. On that occasion, let us briefly recollect several biographical facts from Onsager's works and life [4]. He was born in Oslo, November 27, 1903, to parents Erling Onsager, the barrister of the Supreme Court of Norway and Ingrid née Kirkeby. Young Lars Onsager received a quite liberal education: His main interests were focused on classical literature, philosophy and music. In 1920, Onsager became matriculated at the Norges Tekniske Hoiskole, Oslo in the field of chemical engineering. It is reported that beside the basic program of studies which he did not take too serious, his main interest was to study a textbook on mathematical functions written by Whittaker and Watson. In forthcoming years Onsager devoted himself to Debye's theory of electrolytes and in 1925 he visited Debye in Zürich. The biography sources say that he introduced himself to a prominent professor with the words : "Professor Debye, your theory is incorrect". Apparently, Debye indulged the impoliteness and offered him an assistenship. During the time spent in Zürich, he worked on irreversible processes in electrolytes which, at that time, were the best studied examples of linear irreversible processes. Onsager succeeded to develop a new, more symmetric and in this way, also more correct version of the Debye–Hückel theory, a work which found broad recognition worlwide [5].

In 1928, Onsager was appointed an associate in chemistry at the Hopkins University, but he eventually failed and was fired. Over the period 1928–33 he worked on the theory of kinetic rates of chemical reactions employed by C.A. Kraus at the Department of Chemistry, Brown University. In 1933 Onsager was appointed a postdoctoral Sterling and Gibbs Fellowship at Yale University. When the Chemistry Department found out that he had no Ph.D., Onsager had to do something. An outline of his results on the reciprocal relations submitted to the Trondheim University was rejected for a doctorate. His colleagues at Yale suggested that for the thesis any published work would do. However, Onsager felt that he should write something new and submitted a thesis on the solutions of the Mathieu equations which brought him the Ph.D. degree awarded in 1935. Already in 1934, he was appointed an Assistent Professor in the Chemistry Department at Yale, where he was to remain for the greater part of his life. Between 1936–1939, Onsager published several works on dielectric properties (claimed to be "unreadable", according to Debye) and theory of turbulence. Over next years, the subject of his scientific interest switched occasionally from theory of fluctuations to order–disorder transitions, theory of magnetization, quantized vorticity and lattice Ising problem. In 1968 Onsager was awarded the Nobel Prize in chemistry. A year later, he received the National Science Medal and became a honorary member of The Bunsen Society for Physical Chemistry. During spring 1970 he was Lorentz Professor in Leiden and in 1972–76, being already emeritus and living in Coral Gable/Miami he worked actively on various biophysical problems [4].

Apart from an elegant solution to the lattice Ising problem, the major Onsager's achievement is the fluctuation theory based on the assumption that the fluctuations follow linear laws and are described by Gaussian probability distribution functions. His theory predicted the symmetry relations for the coefficients in the linear laws of relaxation to equilibrium and provided tools to study general irreversible processes. A possible generalization of this point, specifically allowing for inclusion of non-Gaussian fluctuations is a main objective of our studies.

In the recent experimental work, an accumulating evidence demonstrates that in certain non-equilibrium systems, the distribution of fluctuating physical quantities possesses also, beside a Gaussian "core" part, a heavy-tailed wing typical for of Lévy-type distributions [6, 7]. Here, we will present an entirely phenomenological approach to this problem by including Lévy-type terms into the phenomenological Onsager theory of linear relaxation processes. This way, we aim to describe some realistic situations where Lévy flights are considered as external perturbations to weakly non-equilibrium (*i.e.* influenced by Gaussian thermal fluctuations) thermodynamic states. Our method is based on generalizations of the Smoluchowski–Fokker–Planck equation (SFPE), which describes a normal diffusion under the influence of an external force field, to situations modeling anomalous (super) diffusion [8–12]. The common SFPE can be then replaced by space-fractional equation which governs evolution of the probability density $p(x,t)$. This type of equation can be derived either from the generalized Master Equation with long-range jump length statistics [11], or from a suitable Langevin equation with additive, white Lévy noises [9].

There exist several theoretical approaches which connect Lévy-type distributions with Langevin– and Fokker–Planck equations [8, 9, 13–23]. We follow here a different way of reasoning. First, we start with fairly general Onsager's theory of linear relaxation and fluctuation processes [24, 25], and include two statistically independent white-noise sources. Apart from discussing relaxation properties of the system in terms of Gaussian thermal fluctuations [26], we analyze its response to additional external white Cauchy noise [24, 27].

Notably, the problem of various noise sources in a classical Langevin equation is well investigated, both for Gaussian, as well as for more general Lévy-type random forces [8, 24, 27]. Usually, non-Gaussian Lévy noise sources are characterized by a stability parameter $0 < \alpha < 2$ which determines their asymptotic properties, *i.e.* power law decay of the probability distribution function (PDF), $L_{\alpha,\beta}(y) \propto |y|^{-(\alpha+1)}$. Self-affine Lévy PDF are found ubiquitous in nature: Examples of super-diffusion include motion of fluid particles in fully developed turbulence, ion transport in tokamak plasmas [13], tracer particles in vortex arrays in a rotating flow [28], layered velocity fields [29], and Richardson diffusion [30]. Lévy superdiffusion and the so-called truncated Lévy flights, in which arbitrarily large steps are eliminated by a cutoff [31, 32], have been also used extensively to model stock markets. Accordingly, they have finite variance and are more suitable to address diffusive transport in physical systems, in which an unavoidable cutoff seems to be always present, due to *e.g.* finite size of the system. Moreover, in molecular spectroscopy and atmospheric radiative transfer, the combined effects of Doppler and pressure broadening lead to the so-called Voigt profile function which is the convolution of Gaussian (representing the Doppler broadening) and the Lorentzian (responsible for pressure broadening) distributions [33, 34].

Since the Gaussian and Maxwell distributions are the key distributions in equilibrium statistical mechanics, clearly use of more general, Léve-type PDFs requires extension to non-equilibrium situations [8, 9, 13, 16, 18–20]. In particular, the problem of velocity and energy distribution for exploding Coulomb clusters by using combination of Gaussian and Lévy-type stochastic forces has been addressed [17].

The goal of this work was stimulated by several observations that in many non-equilibrium fluids and plasmas, in particular in turbulent systems and in cell kinetics, non-Gaussian distributions and anomalous diffusion are observed [6, 7, 13, 15, 35–39]. One of possible causes of this could be, from our point of view, noise sources with a Gaussian and a non-Gaussian component. We will consider here systems with two noise sources with Gaussian and Lévy-type probability distribution functions. As a specific example, we will study Cauchy distributions, which are analytically most simple Lévy PDF. We consider the convolution of two distributions and the solution of Langevin and Smoluchowski equations with two noise sources.

2. Onsager's theory of fluctuations and linear relaxation processes in a nutshell

2.1. One relaxation variable

According to Einstein's postulate, any macroscopic quantity x may be considered as a fluctuating variable, which is determined by a certain probability distribution function $w(x)$. We consider a Gaussian distribution

$$p(x) = \sqrt{\frac{\Lambda}{2\pi}} \exp\left[-\frac{\Lambda(x - x_0)^2}{2}\right] . \tag{1}$$

The mean value is given by the first moment of the probability distribution $x_0 = \langle x \rangle = \int x \cdot w(x) dx$. In a stationary state, without loss of generality, one may shift the origin and assume $x_0 = 0$. The dispersion is then given by $\langle x^2 \rangle = \frac{1}{\Lambda}$. In view of the fact that at equilibrium state ($x = 0$) entropy assumes a maximum, the following relations hold

$$S(x = 0) \; = \; \max ; \tag{2}$$

$$\left(\frac{\partial S}{\partial x}\right)_{x=0} \; = \; 0 ; \qquad \left(\frac{\partial^2 S}{\partial x^2}\right)_{x=0} \leq 0 . \tag{3}$$

According to Onsagers's view, the relaxation dynamics of the variable x is determined by the first derivative of the entropy, which is different from zero outside equilibrium. Starting from a deviation from the equilibrium (with entropy value below its maximum), the spontaneous irreversible processes should drive the system towards increasing entropy, so that

$$\frac{d}{dt} S(x) = \frac{\partial S}{\partial x} \cdot \frac{dx}{dt} \geq 0 . \tag{4}$$

In this expression two factors appear which were interpreted by Onsager in a quite ingenious way. First, the derivative

$$X = -\frac{\partial S}{\partial x} \tag{5}$$

is considered — in the spirit of the Second Law — as the driving force of the relaxation to equilibrium. In irreversible thermodynamics this term is named in analogy to mechanics the *thermodynamic force* conjugated to a dynamic variable x. This analogy suggests that the (negative) entropy takes over the role of a potential. The second term

$$J = -\frac{dx}{dt} \tag{6}$$

is considered as the *thermodynamic flux* or *thermodynamic flow*. Onsager (1931) postulated a linear relation

$$J = LX = -\dot{x}\,. \tag{7}$$

The idea behind is that the thermodynamics force is the cause of the thermodynamic flow and both should disappear at the same time. The coefficient L is called *Onsager's phenomenological coefficient*, or *Onsager's kinetic coefficient*. From the Second Law it follows that the Onsager-coefficients are strictly positive

$$P \equiv \frac{d}{dt}S(x) = \dot{x}\frac{\partial S}{\partial x} = J \cdot X = L \cdot X^2 \geq 0\,. \tag{8}$$

Onsager's postulate about a linear connection between thermodynamic forces and fluxes has been the origin of the development of the thermodynamics of linear dissipative system, termed also *linear irreversible thermodynamics*. A remarkable property of the theory is the bilinearity of the entropy production $P = J \cdot X$. Moreover, since $p(x)dx$ is proportional to the number of accessible microscopic states of the system, by use of the Boltzmann identity $p(x)dx = \mathrm{const} \times e^{S(x)/k_{\mathrm{B}}}$, we find for the neighborhood of the equilibrium state the relation

$$X = -\frac{\partial S}{\partial x} = k_{\mathrm{B}} \Lambda x\,. \tag{9}$$

Using Eq. (7), we get finally the following linear relaxation dynamics

$$\dot{x} = -LX = -k_{\mathrm{B}} L \Lambda x = -\lambda x\,, \tag{10}$$

where $\lambda = L k_{\mathrm{B}} \Lambda$ stands for the so-called *relaxation coefficient* of the quantity x. Accordingly, the entropy function can be expressed as a Taylor series around the equilbrium value $x_0 = 0$

$$S(x) = S(x_0) - \tfrac{1}{2} k_{\mathrm{B}} \Lambda (x - x_0)^2 + \ldots \tag{11}$$

The linear kinetic equation (10) describes the relaxation of a thermodynamic system brought initially out of equilibrium. Starting with the initial state $x(0)$, dynamics of the variable x follows the trajectory

$$x(t) = x(0)\exp[-\lambda t]\,. \tag{12}$$

We see that $t_0 = \lambda^{-1}$ plays the role of the decay time of the linear deviations from the equilibrium. This way, we arrived for the first time at the so-called fluctuation-dissipation relation. *In his approach to relaxation dynamics, Onsager assumed that deviations of macroscopic observables from their equilibrium values and spontaneous fluctuations around the equilibrium state follow the same kinetics.*

The Onsager theory for one fluctuating variable may be formulated in a compact form by use of the Smoluchowski equation which we devise in two steps. First, we assume a continuity equation for the probability density

$$-\frac{\partial}{\partial t}p(x,t) = \nabla_x j(x,t)\,. \tag{13}$$

Here, $p(x,t)$ can be interpreted as a concentration of particles at a given position x and time t. In the next step, we assume

$$j = -\lambda x p(x,t) - D\nabla_x p(x,t)\,, \tag{14}$$

where D stands for the diffusion coefficient. The meaning of this relation is that there is at first a deterministic flow $\dot{x}p = -\lambda x p$ into the direction of the equilibrium state and an opposite compensating "diffusional flow" following the gradient of the concentration (alternatively, the gradient of the probability density). With these assumptions, we get the standard Smoluchowski equation which describes the relaxation of the probability density function to the stationary distribution

$$\frac{\partial}{\partial t}p(x,t) = \nabla_x\left(\lambda x p(x,t) + D\nabla_x p(x,t)\right) \tag{15}$$

with $\nabla_x = \partial/\partial x$. The stationary solution ($t \to \infty$) to the Smoluchowski equation reads

$$p_{\mathrm{ss}}(x,t) = \sqrt{\frac{\lambda}{2\pi D}}\exp\left[-\frac{\lambda x^2}{2D}\right]\,, \tag{16}$$

and the corresponding stationary dispersion (we have assumed $x_0 = 0$) is $\langle x^2\rangle = D/\lambda$. In order to be compatible with the Onsager approach, we identify

$$\Lambda = \frac{\lambda}{D}\,. \tag{17}$$

Formula (17) expresses a fluctuation-dissipation relation between the dispersion of fluctuations around equilibrium and the linear transport coefficient of the relaxation problem. The relaxation of the distribution to the stationary one is described by the Smoluchowski diffusion equation. At the same time, by use of Eq. (16), one can determine the time derivative of the correlation function $\langle x(t)x(0)\rangle = \int dx \int dx' x x' p(x,t|x',0)p_{\mathrm{ss}}(x')$

$$\frac{\partial}{\partial t}\langle x(t)x(0)\rangle = -\lambda\,\langle x(t)x(0)\rangle\,. \tag{18}$$

With the (equilibrium) initial condition $\langle x(0)^2\rangle = \int dx x^2 p_{\mathrm{ss}}(x)$, the solution to the above equation yields $\langle x(t)x(0)\rangle = \frac{D}{\lambda}\exp(-\lambda t)$.

2.2. Many coupled relaxation variables

The formalism of the preceding section can be easily extended to systems described by n forces X_i and conjugated displacements x_i which, in most general case, are cross-coupled and have to vanish at the state of equilibrium. Retaining terms up to the second order only, we get for the entropy

$$S(x_1,\dots,x_n) = S_{\max} - \tfrac{1}{2}k_{\mathrm{B}}\sum_{i,j}\Lambda_{ij}x_ix_j\,. \tag{19}$$

Following the Onsager approach, we get the forces and flows

$$X_i = k_{\mathrm{B}}\sum_j \Lambda_{ij}x_j\,, \qquad J_i = -\dot{x}_i\,. \tag{20}$$

The generalized linear Onsager-ansatz reads

$$J_i = \sum_j L_{ij}X_j\,, \qquad \sum L_{ij}X_iX_j \geq 0\,. \tag{21}$$

The positive definiteness of the matrix L_{ij} follows from the positivity of the entropy production which is now a bilinear expression in the fluxes and thermodynamic forces appearing in phenomenological equations for which the Onsager relations hold

$$P = \sum_i J_iX_i \geq 0\,. \tag{22}$$

By using Eqs. (20), (21), we get

$$\dot{x}_i = -\sum_j L_{ij}X_j\,, \tag{23}$$

and after introducing the matrix of relaxation coefficients, we end up with

$$\dot{x}_i = -\sum_j \lambda_{ij}x_j\,, \tag{24}$$

$$\lambda_{ij} = k_{\mathrm{B}}\sum_k L_{ik}\cdot\Lambda_{kj}\,. \tag{25}$$

Since the matrix Λ_{ij} determines the dispersion of the stationary fluctuations, we have found again a close relation between fluctuations and dissipation, *i.e.* we have got a fluctuation-dissipation relation for a set of fluctuating and relaxing variables.

The Smoluchowski equation assumes now the form

$$\frac{\partial}{\partial t} p(x_1, ..., x_n, t) = \sum_{ij} \nabla_i \left(\lambda_{ij} x_j p(x_1, ...x_n, t) + D_{ij} \nabla_j p(x_1, ...x_n, t) \right) \quad (26)$$

with the stationary solution

$$p_{\mathrm{s}}(x, t) = C \exp \left[-\tfrac{1}{2} \sum_{ij} \Lambda_{ij} x_i x_j \right] . \quad (27)$$

The fluctuation-dissipation relations between the matrices is given by Eq. (25). The linear kinetic equation (24) describes the relaxation processes which are characterized typically by an exponential decay in time. Further, the Smoluchowski equation (26) describes the relaxation of the probability distributions to the stationary solution. We underline that the information about the relaxation process contained in the Smoluchowski equation is more extended than the information contained in the relaxation dynamic (24), since the Smoluchowski equation describes the mean as well as the fluctuations. We will show that this description is the appropriate form to be generalized to fluctuations of a Lévy type.

3. Including Lévy flights into the relaxation-fluctuation theory

3.1. Linear response theorem

Let us first briefly review the linear relaxation theory in the context of fluctuation-dissipation theorem (FDT), as discussed for Markovian processes [24, 25, 40]. The theorem applies to any Markov process $x(t)$ whose dynamics depends on a set of parameters and for which a well-defined (non-equilibrium) stationary state exists.

We would like to study the linear response of the system to (weak) perturbations $f(t) = f_0 \Theta(-t)$ switched on at some distant past time and switched off at time t_0. Evolution of the dynamic variable $x(t)$ is governed by the propagator $p(x', t|x, 0)$

$$\begin{aligned} \langle x(t) \rangle &= \int dx' \int dx \;\; x' p(x', t|x, 0) \tilde{p}(x, 0) , \\ \tilde{p}(x, 0) &= \frac{e^{-\beta H(x)}}{\int dx' e^{-\beta H(x')}} = \frac{e^{-\beta [H_0(x) + x f_0]}}{\int dx' \; e^{-\beta H(x')}} \end{aligned} \quad (28)$$

with $\tilde{p}(x, 0)$ given by a canonical form, in which perturbation forces coupled to dynamic variable $x(t)$ contribute to the energy of the system and β stands for the reciprocal temperature $\beta = 1/k_{\rm B}T$. By performing approximation

$$\begin{aligned} e^{-\beta x f_0} &\approx 1 - \beta x f_0 \,, \\ \tilde{p}(x,0) &\approx p_0(x)(1 - \beta f_0(x - \langle x \rangle)) = p_0(x)(1 - \beta f_0 x) \,, \end{aligned} \tag{29}$$

the average value of $x(t)$ is given by

$$\begin{aligned} \langle x(t) \rangle &= \int dx' \int dx \;\; x' p(x',t|x,0) p_0(x)(1 - \beta f_0 x) \\ &= \langle x \rangle_0 - \beta f_0 \left\langle x(t) x(0) \right\rangle_0 \,. \end{aligned} \tag{30}$$

Here, subscript 0 denotes average taken with unperturbed form of the distribution $p(x) = \frac{e^{-\beta H_0(x)}}{\int dx' e^{-\beta H_0(x')}}$. We expect that response to (weak) perturbation can be expressed via linear relation

$$\langle x(t) \rangle = \langle x \rangle_0 + \int\limits_{-\infty}^{t} f(\tau) \chi(t-\tau) d\tau \,, \tag{31}$$

in which RHS integrated over time and compared with Eq. (30) leads to the identity

$$\begin{aligned} f_0 \int\limits_0^{\infty} d\tau \Theta(\tau - t) \chi(\tau) &= \beta f_0 \langle x(t) x(0) \rangle_0 \,, \\ -\chi(t) &= \beta \frac{d}{dt} \langle x(t) x(0) \rangle_0 \,. \end{aligned} \tag{32}$$

Hence the fluctuation-dissipation theorem (FDT) relates susceptibility $\chi(t)$ to correlations measured in the reference unperturbed state. In general, the variable *conjugate* to perturbations f_γ can be defined as

$$X_\gamma(x) = - \left. \frac{\partial \ln p_{\rm ss}\left(x; \vec{f}\right)}{\partial f_\gamma} \right|_{\vec{f} = \vec{f}_0} = \frac{\partial \phi}{\partial f_\gamma} \,. \tag{33}$$

Note that in the above definition $\phi \equiv - \ln p_{\rm ss}$ stands for a non-equilibrium potential [24, 25]. If the reference state is the Gibbs equilibrium state corresponding to a temperature $k_{\rm B}T = \beta^{-1}$ and a Hamiltonian $H(x; \vec{f})$, the stationary PDF $p_{\rm ss}(x; \vec{f})$ assumes the form $p_{\rm ss}(x; \vec{f}) = \exp[-\beta H(x; \vec{f})]/Z(\beta, \vec{f})$ and the conjugate variable reads

$$X_\gamma(x) = \frac{1}{k_{\rm B}T} \left. \frac{\partial \left[H\left(x; \vec{f}\right) - F\left(\beta, \vec{f}\right) \right]}{\partial f_\gamma} \right|_{\vec{f} = \vec{f}_0} \,, \tag{34}$$

where $F = -k_\mathrm{B}T\ln Z$ stands for the free energy. Accordingly, X_γ can be interpreted as the fluctuation of the quantity $\frac{\partial H(x;\vec{f}_0)}{\partial f_\gamma} \equiv \left.\frac{\partial H(x;\vec{f}\,)}{\partial f_\gamma}\right|_{\vec{f}=\vec{f}_0}$

$$X_\gamma(x) = \frac{1}{k_\mathrm{B}T}\left[\frac{\partial H\left(x;\vec{f}_0\right)}{\partial f_\gamma} - \left\langle \frac{\partial H\left(x;\vec{f}_0\right)}{\partial f_\gamma}\right\rangle_0\right]. \tag{35}$$

We see that if the control parameter responsible for deviations from stationary state appears in the Hamiltonian as $-f_\gamma x_\gamma$, then the conjugate variable $X_\gamma = -(x_\gamma - \langle x_\gamma\rangle)/(k_\mathrm{B}T)$ represents fluctuations of x_γ. Following the Onsager theory, the above relation holds true generally for any pair of conjugate thermodynamic variables (X_γ, f_γ) provided one can assume that the perturbation of the equilibrium state is linear.

In general, if the reference state $p_\mathrm{ss}(x;\vec{f}_0)$ is not an equilibrium state, the conjugate variables defined by Eq. (33) do not have any straightforward physical interpretation [24, 25, 41]. They do not have also any particular signature with respect to the time reversal [40]. In practice, it is therefore by no means a trivial problem to identify correct choice of generalized forces and "displacements" [25, 42].

We proceed to discuss response of the linear system (over-damped Lévy–Brownian particle) modeled by the corresponding Langevin equation

$$\dot{x}(t) = \mu_0 - \lambda x + f(t) + \xi_\mathrm{C}(t) + \xi_\mathrm{G}(t)\,, \tag{36}$$

where $\xi_\mathrm{C}(t)$ and $\xi_\mathrm{G}(t)$ stand for symmetric Lévy noises with stability indices $\alpha_\mathrm{C} = 1$ (for Cauchy) and $\alpha_\mathrm{G} = 2$ (for Gaussian case). Here, $f(t)$ represents additional deterministic, time-dependent force field. The noises are defined as the time derivatives of stationary Lévy processes, both described by means of characteristic functions

$$\varphi_\mathrm{G}(k,t) = e^{-\sigma_0^2|k|^2 t}\,, \tag{37}$$

$$\varphi_\mathrm{C}(k,t) = e^{-\gamma_0|k|t}\,. \tag{38}$$

Since the Langevin Eq. (36) is linear, its solution depends linearly on two independent stable processes. Accordingly, PDF of $x(t)$ attains the form of the convolution of two Lévy PDFs. The corresponding characteristic function is then a product of two characteristic functions

$$\hat{p}(k,t) = e^{ik\mu(t) - \sigma^2(t)|k|^2 - \gamma(t)|k|}\,. \tag{39}$$

In the above equation $\mu(t)$ stands for the time-dependent location parameter, whereas $\sigma^2(t)$, $\gamma(t)$ represent scale parameters (intensities) of the Gaussian and Lévy–Cauchy noises, accordingly.

Note, that Eq. (36) can be associated with fractional Fokker–Planck equation [14, 17–23] in which Gaussian and Cauchy terms appear as independent contributions

$$\begin{aligned}\frac{\partial p(x,t)}{\partial t} &= -\frac{\partial}{\partial x}\left[\mu_0 - \lambda x + f(t)\right] p(x,t) \\ &\quad + \sigma_0^2 \frac{\partial^2}{\partial x^2} p(x,t) + \gamma_0 \frac{\partial}{\partial |x|} p(x,t)\,. \end{aligned} \tag{40}$$

Here, the fractional (Riesz–Weyl) derivative is defined by its Fourier transform $\mathcal{F}\left[\frac{\partial^\alpha}{\partial |x|^\alpha} f(x)\right] = -|k|^\alpha \mathcal{F}\left[f(x)\right]$. Accordingly, Eq. (40) has the following Fourier representation

$$\begin{aligned}\frac{\partial \hat{p}(k,t)}{\partial t} &= -\lambda k \frac{\partial}{\partial k} \hat{p}(k,t) + ik\left[\mu_0 + f(t)\right] \hat{p}(k,t) \\ &\quad - \sigma_0^2 |k|^2 \hat{p}(k,t) - \gamma_0 |k| \hat{p}(k,t)\,, \end{aligned} \tag{41}$$

where $\hat{p}(k,t) = \mathcal{F}\left[p(x,t)\right]$. For $\mu_0 = \gamma_0 = f(t) = 0$ and $\sigma^2 \equiv D$, Eq. (40) attains a form of a standard Smoluchowski equation Eq. (15) discussed in Section 2.

We note that now the relation between the fluctuations and the transport coefficient is more complicated and cannot be expressed by a simple relation as Eq. (17). The Lévy jump length statistics leads to an inherently non-equilibrium situation: the stationary probability distribution of the process is of Lévy type with the inverse power-law character in the tails. Accordingly, the stationary PDF deviates in such cases from a common exponential Gibbs–Boltzmann form.

In particular, we have to take into account that the dispersion $\langle x^2 \rangle$ does not exist anymore. The distribution of fluctuations is characterized by a bulk Gaussian body and a heavy tail. Asuming an initial condition $f(k,0) = \delta(k-k_0)$, *i.e.* a wave-like perturbation, and $\mu_0 = 0$, we show in Fig. 1 how the Fourier modes of the distribution decay in time.

By using the ansatz Eq. (39) and FFPE (41), we can easily obtain evolution equations for parameters

$$\dot{\mu}(t) = -\lambda \mu(t) + f(t)\,, \tag{42}$$

$$\dot{\gamma}(t) = \gamma_0 - \lambda \gamma(t)\,, \tag{43}$$

$$\frac{d\sigma(t)^2}{dt} = \sigma_0^2 - 2\lambda \sigma(t)^2\,. \tag{44}$$

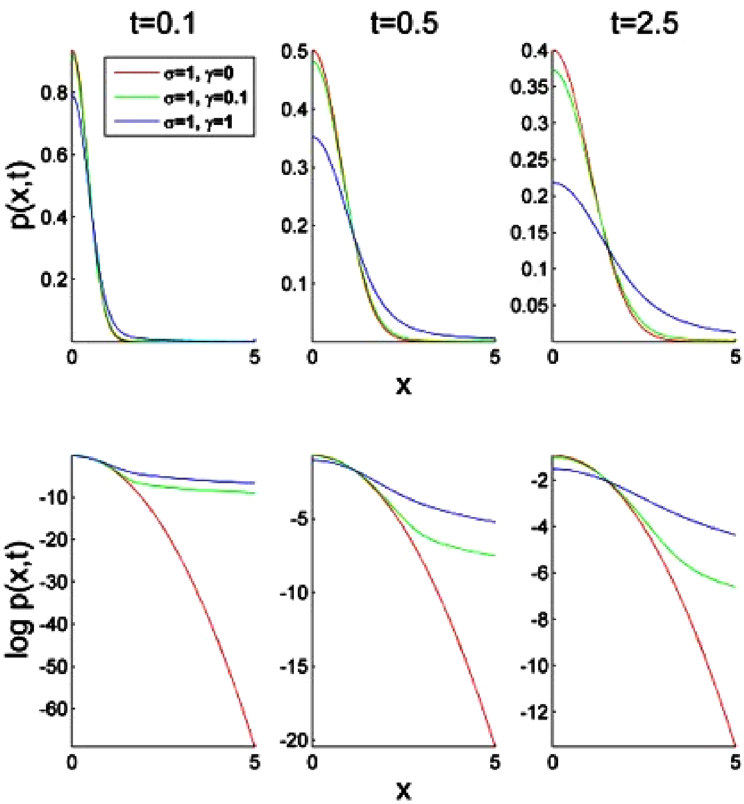

Fig. 1. Convoluted Cauchy–Gauss probability distribution functions compared at different times. Lower panel represents the logarithm of the non-equilibrium pseudo-potential $\phi(x,t)$ (*cf.* Eq. (52)).

In order to analyze the generalized susceptibility, it is sufficient to derive a stationary solution to FFPE for a constant force f

$$\mu_\infty := \lim_{t\to\infty} \mu(t) = \frac{f}{\lambda}\,, \tag{45}$$

$$\gamma_\infty := \lim_{t\to\infty} \gamma(t) = \frac{\gamma_0}{\lambda}\,, \tag{46}$$

$$\sigma^2_\infty := \lim_{t\to\infty} \sigma(t) = \frac{\sigma_0^2}{2\lambda}\,. \tag{47}$$

These results are the same as for the case with only one noise source (stable process) [24].

3.2. Stationary PDF

The characteristic function for a force-free case (hence, $\mu_\infty = 0$) reads

$$\hat{p}_{\rm ss}(k) = e^{-\sigma_\infty^2 |k|^2 - \gamma_\infty |k|} \,. \tag{48}$$

Although the corresponding PDF

$$p_{\rm ss}(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} dk \; \hat{p}_s(k) e^{-ikx} \tag{49}$$

cannot be expressed in terms of elementary functions, it can be nevertheless rewritten using the Faddeeva function (also known as the complex error function)

$$w(x) := e^{-x^2} \mathrm{erfc}(-ix) \,, \tag{50}$$

where $\mathrm{erfc}(x)$ is the complementary error function. Accordingly,

$$p_{\rm ss}(x) = \frac{1}{2\sqrt{\pi}\sigma_\infty} \Re \; w\left(\frac{-x + i\gamma_\infty}{2\sigma_\infty}\right) \,. \tag{51}$$

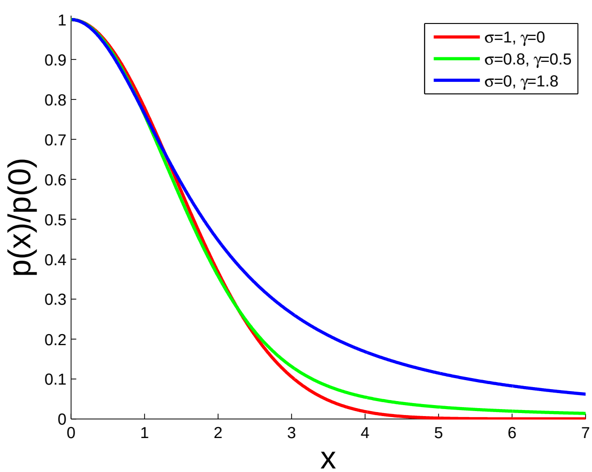

Fig. 2. Comparison of a Gauss distribution (fast decay of extreme cases) with a Cauchy distribution (slow decay) and a convoluted Gauss–Cauchy distribution. Graphs represent the ratio of $p(x)$ with respect to the maximal value at the origin. The case of the convoluted Gauss–Cauchy distribution describes a typical Voigt profile of a spectral line, see the main text [33, 34].

In Fig. 2, we show a comparison of (time-independent) Cauchy, Gauss and mixed Cauchy–Gauss distributions. We see, that in a convolution of Gauss and Lévy distributions the tail is determined by the Lévy part and the body by the Gauss part. Such distributions are widely used in spectroscopy for the description of line profiles [43]. If ions are at a non-zero temperature, their profiles are widened in accordance with the Doppler effect. The Doppler effect is associated with temperature; particles moving in random directions due to thermal motion away and towards the observing optics will produce a Gaussian profile. In addition to temperature broadening, the Stark effect also causes lines to broaden. The Stark effect results from the electric field imposed on the radiating particle by the charged particles surrounding it and causes spectral lines to have a Lorentzian (Cauchy) profile. If the line profile results from the convolution of two (statistically independent) broadening mechanisms, one observes typically Voigt profiles.

3.3. Conjugate variable

Non-equlibrium pseudo-potential of the system reads

$$\phi := -\ln p_{\mathrm{ss}}(x)\,, \tag{52}$$

where $p_{\mathrm{ss}}(x)$ is the stationary PDF (the stationary solution to the corresponding Langevin equation for a constant force f). Accordingly, the conjugate variable takes the form

$$X := \left.\frac{\partial \phi}{\partial f}\right|_{f=0} = X = \frac{\partial \ln p_{\mathrm{ss}}(x)}{\partial x}\,. \tag{53}$$

Using formula (51), it is easy to obtain the conjugate variable for the system subject to Cauchy and Gauss noises simultaneously

$$X_{\mathrm{GC}} = -\frac{x}{2\sigma_\infty^2\lambda} - \frac{\gamma_\infty}{2\sigma_\infty^2\lambda}\frac{\Im\, w\left(\frac{-x+i\gamma_\infty}{2\sigma_\infty}\right)}{\Re\, w\left(\frac{-x+i\gamma_\infty}{2\sigma_\infty}\right)}\,. \tag{54}$$

The limit of pure Gauss (Cauchy) driving can be obtained by taking the limit

$$\lim_{\gamma_0\to 0} X_{\mathrm{GC}} = -\frac{x}{2\lambda\sigma_\infty^2} = -\frac{x}{\sigma_0^2}\,, \tag{55}$$

$$\lim_{\sigma_0\to 0} X_{\mathrm{GC}} = -\frac{2\lambda x}{\gamma_0^2+\lambda^2x^2}\,. \tag{56}$$

In order to consider the effect of a weak Cauchy (Gauss) noise, one can expand (54) in series of $\frac{1}{\sigma_0}$ $(\frac{1}{\gamma_0})$

$$\begin{aligned} X_{\mathrm{GC}} &= -\frac{x}{\sigma_0{}^2}+\frac{\gamma_0\sqrt{\frac{2}{\pi}}x}{\sqrt{\lambda}\sigma_0{}^3}-\frac{\gamma_0{}^2(-2+\pi)x}{\lambda\pi\sigma_0{}^4} \\ &\quad +\frac{\gamma_0 x\left(-3\gamma_0{}^2(-4+\pi)+\lambda^2\pi x^2\right)}{3\sqrt{2}\lambda^{3/2}\pi^{3/2}\sigma_0{}^5}+O\left[\frac{1}{\sigma_0}\right]^6 . \end{aligned} \tag{57}$$

By expanding (54) in series of $\frac{1}{\gamma_0}$, one gets

$$X_{\mathrm{GC}}=-\frac{2(\lambda x)}{\gamma_0{}^2}+\frac{2\lambda^2 x\left(5\sigma_0{}^2+\lambda x^2\right)}{\gamma_0{}^4}+O\left[\frac{1}{\gamma_0}\right]^6 . \tag{58}$$

3.4. Susceptibility and response

We are now at position to compare the response of the system to external perturbation as calculated directly from the definition

$$\langle X(t)\rangle=\int\limits_{-\infty}^{\infty} X(x)p(x,t)dx\,, \tag{59}$$

or, otherwise determined by the generalized susceptibility

$$\chi(t)=\frac{d}{dt}\langle X(t)X(0)\rangle_0$$

within linear response theory

$$\langle X(t)\rangle_{\mathrm{LR}}=\int\limits_0^t \chi(t-s)f(s)ds\,. \tag{60}$$

The time-dependent average (59) can be calculated exactly with the probability density

$$p(x,t)=\int\limits_{-\infty}^{\infty} p(x,t|x_0,0)p(x_0)dx_0\,, \tag{61}$$

where $p(x_0)\equiv p_s(x=x_0)$ is an initial stationary distribution at zero force $(f(t)=0)$ and $p(x,t|x_0,0)$ is given by the inverse Fourier transform of the solution to the FFPE Eq. (41).

On the other hand, the FDT relates the susceptibility with the autocorrelation of the conjugate variables in the reference state, *i.e.*, for $f = 0$. The autocorrelation is defined as

$$\langle X(t)_{\mathrm{GC}} X(0)_{\mathrm{GC}} \rangle_0 \tag{62}$$

with $X(t)_{\mathrm{GC}}$ given by Eq. (54). From the above, the generalized susceptibility can be derived by differentiation with respect to time

$$\chi(t) = \frac{d}{dt} \langle X(t) X(0) \rangle_0 \,. \tag{63}$$

Note that, unlike Eq. (32) which describes relaxation of the system after a stepwise external field has been switched off, in the above formula, the response is related to the changes of quantity $X(t)$ when an external field is applied. The causality criterion for both cases is reflected in different sign of the derivative with respect to time, *cf.* Eqs. (32), (63).

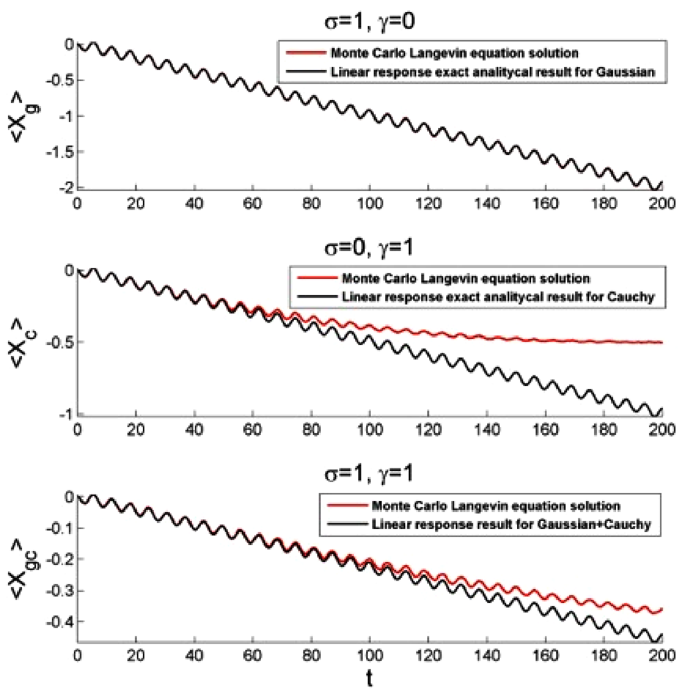

Fig. 3. Comparison of response to external driving $f(t) = \sin(t)/10 + t/100$ evaluated by Monte Carlo solution to Langevin equation and by use of the linear response theory. Deterministic relaxation time of the linear system has been set up to 1 ($\lambda = 1$).

For the mixed Gauss–Cauchy case, the integrals Eq. (60) have been evaluated numerically and the results are presented in Figs. 3, 4. Direct inspection of the above figures indicates that the range of the linear response is finite and "exact" (Monte Carlo) and LRT-approximated curves of response diverge for times $t > 50$, when the external perturbation $f(t) = \sin(t)/10 + t/100$ becomes stronger (all units of time in our model have been set up to 1). The results resemble closely the ones analyzed in [24], however, inclusion of convoluted noises results in "tuning" of the response caused by the action of relatively strong Cauchy noise (*cf.* upper and lower panels in Fig. 4).

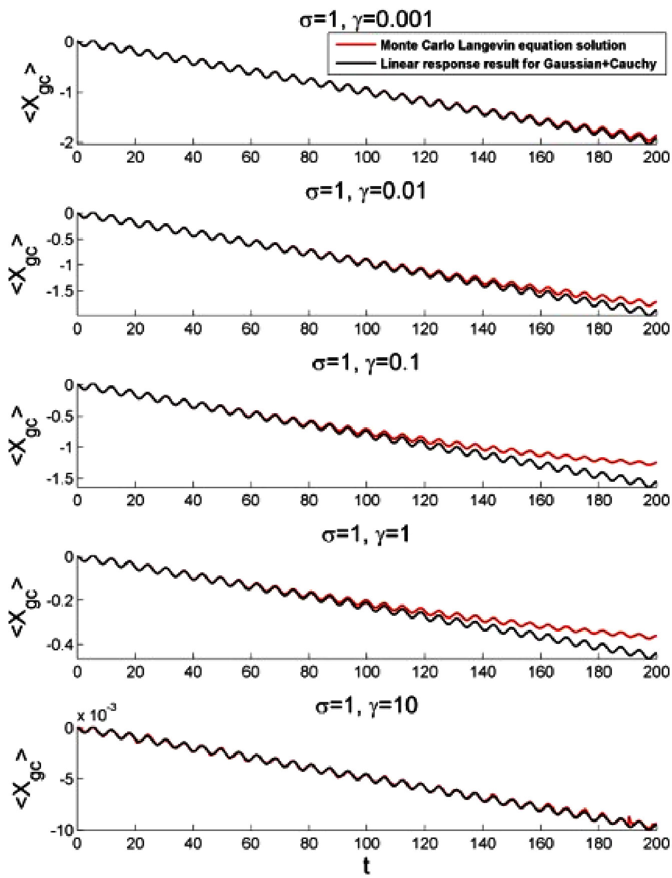

Fig. 4. Response of the system to external driving $f(t) = \sin(t)/10 + t/100$ evaluated by Monte Carlo solution to Langevin equation and by use of the linear response theory. Graphs represent results obtained for a constant scale parameter of Gaussian noise σ^2 and various intensities of the scale parameter γ characterizing Lévy–Cauchy noise. Deterministic relaxation time λ^{-1} has been set up to 1.

Closer examination of a relative error in approximated and exact response exhibits damping of those differences for increasing γ (results not shown). This observation can be traced back to a nonlinear character of the conjugate variable $X(t)_{\rm GC}$ coupled to the driving force $f(t)$ and will be a subject of a seperate analysis.

4. Applications to fluctuations of the velocity, its modulus and energy distributions of Lévy-type

4.1. Distributions of velocity and kinetic energy

One of the first distributions studied in physics was the Maxwell distribution of velocities of molecules of an ideal gas. In units in which the mass of a molecule is set $m = 1$, each component of the velocity vector $v_i, i = 1, 2, 3$ is distributed according to the PDF

$$p_{\rm M}(v) = \sqrt{\beta/2\pi}\exp\left(-\beta v_i^2/2\right) , \tag{64}$$

where $\beta = 1/k_{\rm B}T$. The corresponding distribution of the modulus of the velocity $v = |\boldsymbol{v}|$ reads

$$p_{\rm M}(|v|) = {\rm const}|v|^2\exp\left(-\beta|v|^2/2\right) , \tag{65}$$

and the kinetic energy $\epsilon = mv^2/2$ is, accordingly, distributed as

$$p_M(\epsilon) = {\rm const}\sqrt{\epsilon}\exp(-\beta\epsilon) . \tag{66}$$

However, deviations from Maxwell distributions are quite ubiquitously observed in nature: Inelastic collisions between particles in granular matter create corrrelations which are responsible for velocity distributions departing from the standard Maxwell PDF and, in consequence, overpopulation of high energy tails [44]. Temperature fluctuations of the cosmic microwave background radiation [45] have been reported to follow PDF with algebraic tails and experimental investigation of the edge turbulence in the fusion devices showed that plasma is characterized with non-Gaussian statistics and non-Maxwellian velocity distribution [13, 17, 39]. The Gaussian distribution, which we find most often under equilibrium conditions, is a special case of a more general class of Lévy-type distributions. Hereafter we assume that in appropriate units the components of the velocities may be distributed according to a Lévy alpha-stable, symmetric distribution in velocity space characterized by and index α and defined by the form

$$p_{\rm L}(v) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\exp\left[-itv - |t|^\alpha\right]dt = \frac{1}{\pi}\int\limits_{0}^{\infty}\cos(vt)\exp\left[-t^\alpha\right]dt . \tag{67}$$

The asymptotics of this distribution for large v and $\alpha < 2$ reads

$$p_{\mathrm{L}}(v) \sim \frac{\alpha \sin(\pi\alpha/2)\Gamma(\alpha)}{\pi|v|^{\alpha+1}} \,. \tag{68}$$

For $\alpha = 1$, the above formula represents the Cauchy–Lorentz distribution

$$p_{\mathrm{C}}(x) = \frac{1}{\pi}\frac{1}{v^2+1} \,, \tag{69}$$

whereas for $\alpha = 2$, it reduces to a Maxwellian

$$p_{\mathrm{M}}(v) = \frac{1}{\pi}\int\limits_0^\infty \cos(vt)\exp\left[-t^2\right] dt = \frac{1}{2\sqrt{\pi}}\exp\left(-v^2/4\right) \,. \tag{70}$$

We consider now a more complicated case. As discussed in preceding sections, the sum of two statistically independent random variables ξ and η is distributed according to a convolution. In other words, the sum $\zeta = \xi + \eta$ has the PDF

$$p_\zeta(y) = \int\limits_{-\infty}^{+\infty} p_\xi(y-x)p_\eta(x)d\eta \,. \tag{71}$$

Since convolution corresponds to a multiplication of the characteristic functions in Fourier domain, the convoluted Gauss–Lévy distribution of a velocity component attains the form [15, 16]

$$p_{\mathrm{GC}}(v) = \frac{\mathrm{const}}{\pi}\int\limits_{-\infty}^{+\infty} \cos(vt)\exp\left(-\gamma t^\alpha - \sigma^2 t^2\right) dt \,. \tag{72}$$

Here, the coefficients γ, σ^2 give the strength of each Lévy-type component. This example was analyzed in the context of a distribution in plasmas [15, 16]. Since the integrals are — in the general case — quite complicated, we consider in the following the simplest case of a Gauss–Cauchy distribution $\alpha = 1$. Then the integral may be expressed by error functions (*cf.* Eqs. (50), (51))

$$p_{\mathrm{GC}}(v) = \frac{1}{2\sqrt{\pi}\sigma}\Re\, w\left(\frac{-v+i\gamma}{2\sigma}\right) \,. \tag{73}$$

From the asymptotic representations of the error function, we get for $v \to \infty$ the asymptotic wing of the distribution in the form of a Cauchy distribution

$$p_{\mathrm{C}}(v) \simeq \frac{\gamma}{\pi\left(\gamma^2+v^2\right)} \,. \tag{74}$$

So far, we have considered only a one-component problem by looking just at one component of the fluctuating quantity, say *e.g.* a component of the velocity. However, in many applications we need to know besides the distribution of the components, also the distribution of the modulus and the distribution of the energy ($m = 1, n = 2, 3$)

$$|\boldsymbol{v}| = \sqrt{v_1^2 + \ldots + v_n^2}\,. \tag{75}$$

Finding an appropriate PDF is then not as trivial as for the Gaussian case. In calculating the distribution of the absolute value of the field, its components cannot be considered as independent [26]. We proceed here as follows: The distribution discussed above gives us a projection of the velocity field $\boldsymbol{v}$ onto an *arbitrary* direction defined by its unit vector $\boldsymbol{e}$, *i.e.* $p(y)$ with $y = v\cos\theta$ where $\theta = \arccos(\boldsymbol{v}\cdot\boldsymbol{e}/v)$. We are interested, however, in the distribution of the absolute value $x = v \geq 0$.

If the probability density $p(x)$ of x is known, it is not complicated to calculate the PDF of y assuming the angles between $\boldsymbol{v}$ and $\boldsymbol{e}$ are taken at random, *i.e.* $\boldsymbol{e}$ is homogeneously distributed on a unit sphere. In this case,

$$\begin{aligned} p_y(y) &= \frac{1}{4\pi}\int\limits_0^{2\pi} d\phi \int\limits_{-\pi/2}^{\pi/2} d\theta \sin\theta \int\limits_0^{\infty} dx\; p(x)\delta(y - x\cos\theta) \\ &= \frac{1}{2}\int\limits_{-1}^{1} d\cos\theta \int\limits_0^{\infty} dx p(x)\frac{1}{\cos\theta}\delta\left(\frac{y}{\cos\theta} - x\right) \\ &= \frac{1}{2}\int\limits_{-1}^{1} \frac{d\xi}{\xi} p\left(\frac{y}{\xi}\right)\,. \end{aligned}$$

Note that since $p(x) = 0$ for $x < 0$, the integrand vanishes on the left half-axis for $y > 0$ and on the right half-axis for $y < 0$. Since the distribution $p_y(y)$ is symmetric, it is enough to consider $y > 0$, *i.e.* to write

$$p_y(y) = \frac{1}{2}\int\limits_0^1 \frac{d\xi}{\xi} p\left(\frac{y}{\xi}\right)$$

for $y > 0$. Now we can make the change of variables in our integral, $z = y/\xi$, to get

$$p_y(y) = \frac{1}{2}\int\limits_y^{\infty} \frac{dz}{z} p\left(z\right)\,.$$

The rest is simple: Differentiating both parts of the equation in y, we get

$$\frac{d}{dy}p_y(y) = -\frac{1}{2y}p\left(y\right)$$

and, therefore,

$$p(x) = -2x\frac{d}{dx}p\left(x\right) = -2x\frac{d}{dx}p_{\mathrm{L}}\left(x,\alpha\right)\ , \tag{76}$$

where the corresponding Lévy distribution is considered only on the positive half-axis. At this stage, we can also check the normalization of the distribution

$$\begin{aligned}\int\limits_0^\infty p(x)dx &= -2\int\limits_0^\infty x\frac{d}{dx}p_{\mathrm{L}}\left(x,\alpha\right) \\ &= -2xp_{\mathrm{L}}\left(x,\alpha\right)\Big|_0^1 + 2\int\limits_0^\infty p_{\mathrm{L}}\left(x,\alpha\right)dx = 1\,,\end{aligned}$$

where we performed the integration by parts and used the fact of the symmetry of the Lévy distribution and its integrability.

Using Eq. (76) and the asymptotic form of the Lévy distribution Eq. (68), we get the following asymptotic decay form for the far tail of the distribution of the absolute value of the velocity

$$p(x) \simeq \frac{2\sin(\pi\alpha/2)\mathit{\Gamma}(\alpha+2)}{\pi|v|^{(\alpha+1)}}$$

with the same power-law asymptotics as the distribution of a single component.

Using these results, we get for the distribution of the modulus of the velocity

$$p(|v|) = \mathrm{const}\int\limits_0^\infty dk(k|v|)\sin(k|v|)\exp\left[-\gamma k^\alpha - \sigma^2k^2\right]. \tag{77}$$

The asymptotics is for large v and $\alpha < 2$ given by the Lévy part of the distribution

$$p(|v|;\mu) \sim \frac{\mathrm{const}}{\alpha|v|^{\alpha+1}}\,. \tag{78}$$

Figure 5 displays the stationary distributions of the modulus of the velocity for the case of pure Cauchy noise, pure Gaussian noise, and mixed, Cauchy–Gaussian convoluted noise. The corresponding distribution of kinetic energy $\epsilon = mv^2/2$ is given by

$$p(\epsilon;\alpha) = \text{const} \int_0^\infty dk \left(k\sqrt{2m\epsilon}\right) \sin\left(k\sqrt{2m\epsilon}\right) \exp\left[-\gamma k^\alpha - \sigma^2 k^2\right]. \quad (79)$$

The tail of this distribution of kinetic energy is given by

$$p(\epsilon;\alpha) \sim \frac{\text{const}}{\alpha\epsilon^{\alpha+1/2}}. \quad (80)$$

The corresponding cumulative distribution for finding energies higher than e_0 is

$$p_0(\epsilon > \epsilon_0;\mu) \sim \frac{\text{const}}{\alpha(\alpha+1)\epsilon^{\alpha-1/2}}. \quad (81)$$

For many physical processes, in particular for rate processes, the tail of the energy distribution and the cumulative distribution may be quite important [17].

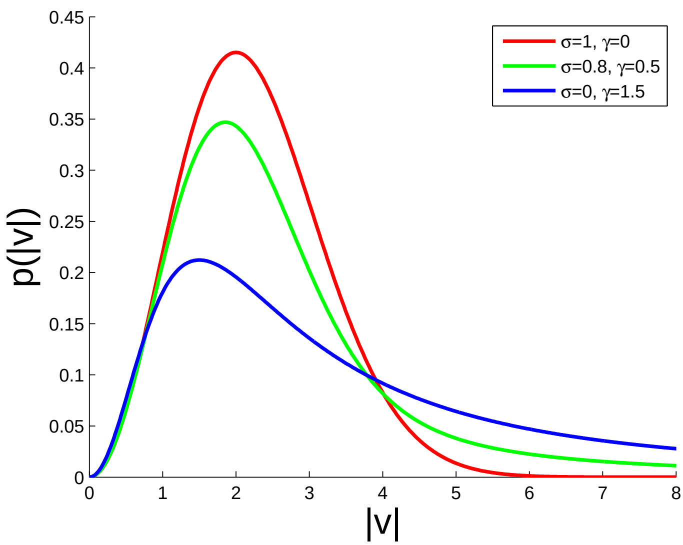

Fig. 5. Probability distribution of the modulus of the velocity $p_s(|v|)$ for the Cauchy, Gaussian and mixed Cauchy–Gaussian noises.

4.2. General relaxation kinetics

Lévy-type processes are most conveniently described in the Fourier-space. We introduce the 3d-Fourier transform by

$$p(\boldsymbol{v},t) = \frac{1}{(2\pi)^3}\int d\boldsymbol{k}\,\exp[i\boldsymbol{k}\cdot\boldsymbol{v}]\hat{p}(\boldsymbol{k},t)\,, \tag{82}$$

where $\boldsymbol{k}$ denotes the Fourier vector. Then, in the case with only one relaxation time ($\tau = \lambda^{-1}$) in conformity to the results of Section 3.1, we get the phenomenological Smoluchowski-type equation for the evolution of the probability density function in the k-space

$$\frac{\partial}{\partial t}\hat{p}(\boldsymbol{k},t) + \lambda\boldsymbol{k}\cdot\frac{\partial}{\partial\boldsymbol{k}}\hat{p}(\boldsymbol{k},t) = -D_2\boldsymbol{k}^2\hat{p}(\boldsymbol{k},t) - |\boldsymbol{k}|^\alpha D_\alpha\hat{p}(\boldsymbol{k},t)\,. \tag{83}$$

The solution is

$$\hat{p}(\boldsymbol{k},t) = \exp\left[-d_2(t)D_2\boldsymbol{k}^2 - d_\alpha(t)D_\alpha|\boldsymbol{k}|^\alpha\right]\,. \tag{84}$$

By generalizing the many-component relaxation equation Eq. (26) at this level of description, we get for the Fourier transform of the distribution function $p(v_1,\ldots,v_n,t)$

$$\frac{\partial}{\partial t}p(k_1,\ldots,k_n,t) + \sum_{i,j=1}^{n}k_i\lambda_{ij}\frac{\partial}{\partial k_j}p(k_1,\ldots,k_n,t) = R(k_1,\ldots,k_n,t)\,. \tag{85}$$

For the generalized r.h.s. of the stochastic equation including Levy terms, exist several possibilities, the easiest one is

$$R(k_1,\ldots,k_n,t) = -D_2k^2p(k_1,\ldots,k_n,t) - |k|^\alpha D_\alpha p(k_1,\ldots,k_n,t)\,. \tag{86}$$

A more complex “ansatz” with a tensorial diffusion reads

$$R(k_1,\ldots,k_n,t) = -\sum_{i,j=1}^{n}k_ik_jD_{ij}p(k_1,\ldots,k_n,t) - |k|^\alpha D_\alpha p(k_1,\ldots,k_n,t)\,. \tag{87}$$

4.3. Relaxation of 3-dimensional velocities including external forces

We study now the velocity relaxation of a particle in a fluid starting from a representation of the Langevin equation in velocity Fourier space [8, 9, 13, 16, 17, 22]. For the 3d-Fourier transform of a process including a constant external force $\boldsymbol{F}_0$ and a friction constant λ ($\boldsymbol{k}$ denotes the Fourier vector in the velocity space), we find [16, 17]

$$\frac{\partial}{\partial t}\hat{p}(\boldsymbol{k},t) = -i\frac{1}{m}\boldsymbol{k}\cdot\boldsymbol{F}_0\hat{p}(\boldsymbol{k},t) - \lambda\boldsymbol{k}\cdot\frac{\partial}{\partial\boldsymbol{k}}\hat{p}(\boldsymbol{k},t) - D_\alpha|\boldsymbol{k}|^\alpha\hat{p}(\boldsymbol{k},t) - D_2k^2\hat{p}(\boldsymbol{k},t)\,. \tag{88}$$

In the simplest case when $F_0 = 0$ and $D_2 = 0$ the explicit solution to the above equation reads

$$\hat{p}(\boldsymbol{k},t) = \exp\left[-d_\alpha(t)D_\alpha|\boldsymbol{k}|^\alpha\right], \tag{89}$$

$$d_\alpha(t) = \frac{1}{\alpha\lambda}\left(1-\exp(-\alpha\lambda t)\right)\,. \tag{90}$$

There are two easily obtained limits: the short time distribution for $t \ll 1/\lambda$, where the term proportional to λ may be neglected and the stationary solution for $t\to\infty$

$$\hat{p}(\boldsymbol{k},t) = \exp\left[-D_\alpha|\boldsymbol{k}|^\alpha t\right]; \qquad \hat{p}_{\mathrm{ss}}(\boldsymbol{k}) = \exp\left[-\frac{D_\alpha|\boldsymbol{k}|^\alpha}{\alpha\lambda}\right]. \tag{91}$$

In the general case, the time-dependent solution assumes the form

$$\hat{p}(\boldsymbol{k},t) = \exp\left[-i\frac{1}{m}\boldsymbol{k}\cdot\boldsymbol{F}_0d_1(t) - D_\alpha|\boldsymbol{k}|^\alpha d_\alpha(t) - D_2\boldsymbol{k}^2d_2(t)\right]. \tag{92}$$

For short times, this solution becomes

$$\hat{p}(\boldsymbol{k},t) = \exp\left[-D_\alpha|\mathbf{k}|^\alpha t - D_2\boldsymbol{k}^2t - i\frac{1}{m}\boldsymbol{k}\cdot\boldsymbol{F}_0t\right]. \tag{93}$$

We see that the tails have the longest relaxation times. The stationary distribution is the limit of long times and we find by back transformation again the stationary distribution in the velocity space

$$p_{\mathrm{ss}}(\boldsymbol{v}) = \frac{1}{(2\pi)^3}\int d\boldsymbol{k}\exp\left[i\boldsymbol{k}\cdot\left(\boldsymbol{v}-\frac{1}{m\gamma}\boldsymbol{K}_0\right)\right]\exp\left[-\frac{D_\alpha|\boldsymbol{k}|^\alpha}{\alpha\gamma}-\frac{D_2\boldsymbol{k}^2}{2\gamma}\right]. \tag{94}$$

This distribution conforms with the results of Section 3, where we have analyzed a propagator of the generalized Ornstein–Uhlenbeck process driven by independent Gauss and Cauchy white noises. As we discussed already above, in such case, the velocity distribution has a diverging mean square, and correspondingly is characterized by a long, slowly decaying tail. Figure 5 shows the stationary distributions of the modulus of the velocity for the case of pure Cauchy noise, pure Gaussian noise, and mixed noise.

At high velocities and correspondingly high energies which are large in comparison to the mean value, the slowly decaying tails determine the behavior. The long tails decay with the slowest relaxation time. This might be of interest, in particular, for rate processes. We must underline that, strictly speaking, all the distributions given above correspond to non-equilibrium. Since proper thermodynamic equilibrium is characterized by Gaussian Maxwell distributions, the PDFs obtained here may correspond only to stationary stationary stages of non-equilibrium processes. In some cases, the distribution may be considered as quasi-stationary, where the constants in the distributions are slowly changing time-dependent quantities. In particular, the friction λ may be a slowly changing phenomenological quantity. The most remarkable result are the long velocity tails for $\alpha < 2$. As demonstrated in [15, 16], long velocity tails may be responsible for a strong increase of reaction rates and in particular this refers to special fusion processes.

5. Discussion

After repeating briefly the main topics of Onsagers theory of Gaussian fluctuations and relaxation processes near to equilibrium including Smoluchowski equations, we have discussed extensions to Lévy-type distributions and the relaxation to stationary states. For clarity, let us summarize the main points of our strategy in extending the Onsager theory:

(i) The Gaussian equilibrium distributions in Onsagers theory are replaced by stationary distributions which are convolutions of Gauss and Lévy distributions. The Gaussian fluctuations determine the body and the Lévy contributions determine the tail of the distribution, *i.e.* the large fluctuations.

(ii) The role of entropy is taken over by a non-equilibrium potential, also called stochastic potential, which is (up to a constant) the log of the stationary non-equilibrium distribution.

(iii) The Onsager forces are replaced by derivatives of the stochastic potential.

(iv) Onsagers linear laws for the relaxation of fluctuations are replaced by Smoluchowski-type equations for the relaxation of the distribution function to the stationary distribution.

(v) The relaxation times are not fixed as in Onsagers theory but depend on the amplitude of the fluctuations. The Lévy fluctuations corresponding to the tail decay are much slower than the Gaussian fluctuation corresponding to the body of the distribution.

We have discussed several special cases and, in particular, mixed Gauss–Cauchy distributions. The body of the distribution is determined by the Gaussian component and the tails by Cauchy contributions to the fluctuations. The relaxation to stationary states is studied by solving generalized Smoluchowski equations in the Fourier space. We have shown that, in comparison to the Gaussian contributions around the center body of the PDF, the fluctuations in the tails have longer relaxation times. This is an essential difference to the Onsager theory, where all fluctuations have the same relaxation time (at least in the one-component case).

In the last part, we have studied distributions of the components of velocity vectors, the velocity modulus and the kinetic energy. We have investigated the relaxation of these fluctuations by solving Smoluchowski equations in the 3d-velocity Fourier space including in a systematic way Lévy-type fluctuations. We have shown that under certain conditions, in reality, long tails in the velocity distribution might exist. The appearance of high-energetic tails in the velocity and energy distribution, which we observe here as well may contribute to the understanding of several high rate processes which are determined by the tails in the velocity and the energy distribution. We underline again that strictly speaking, all the distributions given above are valid only for stationary non-equilibrium systems since fluctuations around the proper thermodynamic equilibrium are always characterized by Gaussian distributions of the fluctuating quantities. For that reason, the PDFs obtained here clearly correspond non-equilibrium processes. Our distributions and relaxation processes are related to fluctuations around certain stationary states. Looking at the time dependence of the PDFs, we see the relaxation of the non-equilibrium Lévy fluctuations characterizing the far tails of the fluctuations decay in a longer time $(\lambda\alpha)^{-1}$, in comparison to the decay time $(\lambda)^{-1}$ for Gaussian fluctuations.

Altogether, we conclude that Lévy statistics and, in particular, Lévy flights are suitable models of random phenomena where rare and large (extreme) events are important. Such statistics are *e.g.* relevant for description of turbulence and radio-wave scattering in the interstellar plasma. Admixture of Lévy component in white Gaussian noises driving dynamic systems at hand, results in scaling of temporal broadening of spectral profile in which large seldom fluctuations are observed to decay slower in time than their Gaussian analogues.

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