CHARACTERIZATION OF BOUNDARY PLURIPOLAR HULLS

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Contents

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Abstract

A pluripolar set in a domain $D \subset \mathbb{C}^n$ is a subset of D that lies in $-\infty$-locus of a plurisubharmonic function in D. Some properties and applications of such a set are known. In this thesis we will discuss boundary pluripolar set and hull.

A set in ∂D is called *boundary pluripolar* or $b-pluripolar$ for D if it is a subset of $-\infty$-locus of an upper semicontinuous function on $\overline{D}$ that is plurisubharmonic in D. We will discuss different possibilities to compute boundary pluripolar hull of a b-pluripolar set in the boundary of a domain.

We give some properties of boundary relative extremal function and use it to characterize boundary pluripolar hull in the domain.

We show the existence of a set that is b-pluripolar for the unit ball but not pluripolar in $\mathbb{C}^2$. We prove by two different methods that the hull is always trivial in the boundary of a B-regular domain.

After giving some approximation theorems of holomorphic maps we characterize boundary pluripolar sets in terms of analytic disc.

We review the definitions of the *boundary relative extremal*. For various domains we give an affirmative answer to the question of Sadullaev, [71], whether these extremal functions are equal.

We also show that certain versions of Edwards duality theorem do not hold in open sets.

By using Poletsky's theory we characterize the *thinness* of a subset in $\mathbb{C}^n$ by analytic discs.

Chapter 1

Introduction

1.1 Introduction

In mathematics, holomorphic functions are the central objects of study in complex analysis. A holomorphic function is a complex-valued function of one or more complex variables that is complex differentiable in a neighborhood of every point in its domain. The existence of a complex derivative in a neighborhood is a very strong condition. It implies that any holomorphic function is actually infinitely differentiable and equal to its own Taylor series.

By a classical result of Bremermann every plurisubharmonic function is given locally as the upper semicontinuous regularization $u^*(z)$ of

$$u(z) = \limsup_{j \to +\infty} \frac{1}{j} \log |f_j(z)|$$

for some sequence f_j of holomorphic functions. Despite of this strong relation there are lot of surprises surrounding the boundary behavior of plurisubharmonic functions.

In this thesis, we introduce the notion of *boundary pluripolar set* and compute *boundary pluripolar hull*. We also discuss the completeness of boundary pluripolar set. At the beginning we shall recall some known results about pluripolar hull of a pluripolar set in a bounded domain $D \subset \mathbb{C}^n$. We denote by $\mathrm{PSH}(D)$ the class of all plurisubharmonic functions in D and by $\mathrm{PSH}(D)^-$ the set of negative functions in $\mathrm{PSH}(D)$. A subset A of D is said to be locally pluripolar if for any $z \in A$ there is a neighborhood U of z in D and $u \in \mathrm{PSH}(U)$ such that $u \not\equiv -\infty$ and $A \cap U \subset \{y \in U; u(y) = -\infty\}$, and A is said to be globally pluripolar if there exists $u \in \mathrm{PSH}(D)$ such that $u \not\equiv -\infty$ and $A \subset \{y \in D; u(y) = -\infty\}$.
The question whether local pluripolarity is equivalent to global pluripolarity is called by Sadullaev [71] the Lelong first problem. A positive answer is given by Josefson in [43]. He proved that in $\mathbb{C}^n$ every pluripolar set is globally pluripolar see also Theorem 4.7.4 in [47]. Countable sets are pluripolar If f is a holomorphic function in a domain D, then the sets

$$A = \{z \in D; f(z) = 0\} \text{ and } \Gamma_f(D) = \{(z, f(z)); z \in D\}$$

are pluripolar respectively in D and in $D \times \mathbb{C}$. As we shall see pluripolar sets abound. The problem is not to produce them, but to try to characterize them. For instance if

$A \subset \mathbb{C}^n$ is pluripolar it can happen that any $u \in \mathrm{PSH}(D)$ that is $-\infty$ on $A \subset D$ is automatically $-\infty$ on a bigger set. To see it clearly, consider the set

$$A = \{z \in \mathbb{C}; |z| < 1\} \times \{0\}.$$

Remark that any $u \in \mathrm{PSH}(\mathbb{C}^2)$ that is $-\infty$ on A must be $-\infty$ on $\mathbb{C} \times \{0\}$. An interesting thing in this topic is to find the biggest pluripolar set A^* containing A such that any plurisubharmonic function in a neighborhood D of A that takes $-\infty$ on A assumes $-\infty$ on A^* also. A^* will be called a pluripolar hull of A and it is defined as follows

$$A^*_D = \{z \in D; u(z) = -\infty; u \in \mathrm{PSH}(D); u|A \equiv -\infty\}.$$

Zeriahi in his study of exceptional sets introduced the notion of pluripolar hull in [88]. If D is bounded one can define another hull $\hat{A}_D$ as follows

$$\hat{A}_D = \{z \in D; u(z) = -\infty; u \in \mathrm{PSH}(D); u < 0; u|A \equiv -\infty\}.$$

See [57] for relations between $\hat{A}_D$ and A^*_D it is proven that $\hat{A}_D = A^*_D$ if D is hyperconvex. In general it is hard to compute A^*_D. The simplest case is when $A^*_D = A$, in this situation we look for a $u \in \mathrm{PSH}(D)$ such that $A = \{z \in D; u(z) = -\infty\}$. If such a u exists then we say that A is *complete pluripolar* in D. Note that if A is complete pluripolar then A is a G_δ set and $A = A^*_D$. Conversely Zeriahi stated that for a pluripolar set A in a pseudoconvex domain D if $A = A^*$ and A is G_δ as well as F_σ then A is complete. Many authors worked on completeness of pluripolar sets, Armen Edigarian, Nguyen Quang Dieu, Phung Van Manh, Levenberg, Martin, Poletsky, Wiegerinck, Sadullaev .....

Recall that in [56] Levenberg-Martin-Poletsky proved that there is $f \in \mathscr{O}(\mathbb{D}, \mathbb{C})$ whose graph $\Gamma_f(\mathbb{D})$ is complete pluripolar ($\mathbb{D} \subset \mathbb{C}$ denotes the unit disk). This result was generalized by Dieu-Manh in [16] to polydisk $\mathbb{D}^n$ in $\mathbb{C}^n$. Levenberg, Martin and Poletsky conjectured that if f is a holomorphic function, which is defined on its maximal domain of definition $D \subset \mathbb{C}$ then its graph $\Gamma_f(D)$ is complete pluripolar in $\mathbb{C}^2$. The conjecture was disproved by Edigarian-Wiegerinck in [27]. They showed that there is a holomorphic function f defined on $\mathbb{D}$, which does not extend holomorphically across $\partial\mathbb{D}$, such that its $\Gamma_f(\mathbb{D})$ is not complete pluripolar in $\mathbb{C}^2$.

Though plurisubharmonic functions are quasicontinuous in its domain of definition $D \subset \mathbb{C}^n$, the problem that consists to approximate a plurisubharmonic function by functions that are continuous and plurisubharmonic on D is classical and it depends on the geometry of D. This problem is studied by F., Wikstrom in [85], Wikstrom and Dieu in [17], by Fornaess and Wiegerinck in [31]. In fact if u is plurisubharmonic in D and $U \subset\subset D$ by the main approximation theorem see Theorem 2.9.2 in [47] one can find a sequence $(u_j)_j \subset \mathrm{PSH}(U) \cap C(U)$ that decreases to u. Fornaess and Stensines proved that there is a case where u_j can not be defined in whole D. But Fornaess and Narasimhan in [29] showed that if D is pseudoconvex then u_j can be defined in the whole D. Frank Wikstrom via an example in [85] showed that u_j are not necessarily continuous on $\overline{D}$ and stated that if D is B-regular we can take u_j in $\mathrm{PSH}(D) \cap C(\overline{D})$. If D is a bounded domain with C^0-boundary then any $u \in \mathrm{PSH}(D) \cap C(\overline{D})$ can be approximated uniformly by functions that are continuous and plurisubharmonic in a neighborhood of $\overline{D}$. This property is known as Mergelyan property for plurisubharmonic functions.

Domains that have *PSH-Mergelyan property* were studied by Lisa Hed see [38].

1.2 Outline

This thesis is based on the following papers [19], [20] accepted respectively in :
Complex Variables and Elliptic Equations and **Mathematica Scandinavica**.

In Chapter 2 we give the necessary background in pluripotential theory. Section 2.1 includes harmonic functions, upper semicontinuity, subharmonic functions.

In Chapter 3 we give some properties of the boundary relative extremal function. We characterize the boundary pluripolar hull and its propagation inside the domain mainly we prove that the propagation of boundary pluripolar hull is stopped (in the boundary) by strong plurisubharmonic barriers. We infer that boundary pluripolar hull is always trivial in the boundary of a B-regular domain. We adapt Zeriahi's technique to boundary pluripolar set and prove the completeness of boundary pluripolar sets that are F_σ as well as G_δ.
We use a method of Bu and Schachermayer to give a disc formula for the boundary relative extremal function and infer a characterization of boundary pluripolar hull in terms of analytic discs.

In Chapter 4 we prove that the different definitions of the boundary relative extremal function given by Sadullaev in [71] are equivalent in ellipsoidal domains. This is done by exploiting Wikström works in [85]. By using Siciak's relative extremal function we show, in Section 4.4, that Edwards duality theorem does not hold in open sets.

In Chapter 5 we prove some properties of Siciak's relative extremal function and use it to characterize the *thinness* of a set in terms of analytic disc.

Chapter 2

Background

2.1 Basic definitions in pluripotential theory

In this section we state some definitions and well known results from pluripotential theory, these results are mainly from [47, 67]. We begin with the definition of *upper semicontinuous* functions. By a domain we mean an open and connected set.

2.1.1 Semicontinuity

Definition 2.1.1. Let X be a topological space. We say that a function $u : X \to \mathbb{R} \cup \{-\infty\}$ is upper semicontinuous if the set $\{x \in X; u(x) < \alpha\}$ is open in X for each $\alpha \in \mathbb{R}$. Also $v : X \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous if $-v$ is upper semicontinuous.

Proposition 2.1.2. *If φ is an upper semicontinuous function on a compact set K, then there exists $x \in K$ such that $\varphi(x) = \sup_K \varphi < +\infty$.*

Definition 2.1.3. Let $Y \subset X$ be a nonempty set and $u : Y \to \mathbb{R} \cup \{-\infty\}$ a function which is locally bounded around each point in the closure of Y. Then we define the upper semicontinuous regularization u^* of u by

$$u^*(x) = \limsup_{y \to x, y \in Y} u(y) = \lim_{r \to 0} \sup_{B(x,r) \cap Y} u.$$

The function u^* is upper semicontinuous on $\overline{Y}$, $u \leq u^*$ on Y, and it is the smallest upper semicontinuous function larger than u.

An important fact about upper semicontinuous functions is that they can be approximated by continuous functions from above.

Proposition 2.1.4. *Let u be an upper semicontinuous function on a metric space (X, d), and suppose that u is bounded from above on X. Then there exist continuous functions $\phi_n : X \to \mathbb{R}$ such that $\phi_1 \geq \phi_2 \geq \cdots \geq u$ on X and $\lim_{n\to\infty} \phi_n = u$*

The following is stated in Demailly [12].

Proposition 2.1.5 (Choquet's lemma)**.** *Let $D \subset \mathbb{C}^n$ be a bounded domain, $I \subset \mathbb{R}$ and $(u_\alpha)_{\alpha \in I}$ be a family of upper semicontinuous functions on D which is locally bounded from above. Then $(u_\alpha)_{\alpha \in I}$ has a countable subfamily $(u_{\alpha(j)})_{j \in \mathbb{N}}$ whose upper envelope v satisfies $v \leq u \leq u^* = v^*$, where u is the upper envelope of $(u_\alpha)_{\alpha \in I}$.*

Below we present Dini's theorem.

Theorem 2.1.6 ([70],Theorem 7.13)**.** *Suppose K is compact, and*

1. *$(f_m)_{m\in\mathbb{N}}$ is a sequence of continuous functions on K,*

2. *$(f_m)_{m\in\mathbb{N}}$ converges pointwise to a continuous function f on K,*

3. *$f_m(x) \geq f_{m+1}(x)$ for all $x \in K$, $m = 1, 2, 3, \cdots$*

Then $f_m \to f$ uniformly on K.

2.1.2 Subharmonic Functions

A reason to use subharmonic functions is that they are a lot more flexible than harmonic functions. For example, the maximum of two subharmonic functions is subharmonic. Via *Perron method* subharmonic functions can be used to construct harmonic ones.

Definition 2.1.7. Let U be an open subset of $\mathbb{C}$. Let $u : U \to \mathbb{R} \cup \{-\infty\}$ be a function that is not identically $-\infty$ on any connected component of U. We say that u is *subharmonic* if it is upper semicontinuous and satisfies the local submean inequality, i.e. given $w \in U$, there exists $\rho > 0$ such that

$$u(w) \leq \frac{1}{2\pi}\int_0^{2\pi} u(w + re^{it})dt \quad (0 < r < \rho).$$

Moreover $v : U \to \mathbb{R} \cup \{+\infty\}$ is superharmonic if $-v$ is subharmonic.

The integral above is to be interpreted as the difference of the corresponding integrals of u^+ and u^-. Note that u^+ is bounded on $\partial D(w, r)$ so its integral is finite. Thus the difference of the two integrals makes sense even though the integral of u^- may be infinite.

As the subharmonicity is defined via the submean inequality, it is a local property. This means that if (U_α) is an open cover of U then u is subharmonic in U if and only if it is subharmonic on each U_α.

Example 2.1.8. If f is holomorphic on an open set $U \subset \mathbb{C}$, then $\log|f|$, $|f|$, $|f|^m$ are subharmonic.

Further examples can be generated using the following elementary result, which is an immediate consequence of the above definition.

Theorem 2.1.9. *Let u and v be subharmonic functions on an open set $U \subset \mathbb{C}$. Then:*

- *$\max(u; v)$ is subharmonic on U,*

- *$\alpha u + \beta v$ is subharmonic on U for all α, $\beta \geq 0$.*

Corollary 2.1.10. *If $f : U_1 \to U_2$ is holomorphic between open sets U_1 and U_2 in $\mathbb{C}$, and if u is subharmonic on U_2 then $u \circ f$ is subharmonic on U_1.*

A number of fundamental theorems concerning holomorphic functions can be deduced from the corresponding properties of subharmonic functions. Such as The identity principle for holomorphic function, Liouville theorem for holomorphic function, maximum principle for holomorphic function.

2.1.3 Harmonic functions

Definition 2.1.11. Let $U \subset \mathbb{C}$ be a domain. We say that $u : U \to \mathbb{R}$ is *harmonic* if u and $-u$ are subharmonic in U.

Example 2.1.12. Let U be a domain in $\mathbb{C}$. If f is holomorphic on U, then $\operatorname{Re} f$ and $\operatorname{Im}(f)$ are harmonic on U.

Corollary 2.1.13. *If $f : U_1 \to U_2$ is a holomorphic map between open subsets U_1 and U_2 of $\mathbb{C}$, and if h is harmonic on U_2, then $h \circ f$ is harmonic on U_1.*

Theorem 2.1.14. *Let h be a function harmonic on an open neighborhood of the disc $\overline{D}(w, \rho)$. Then*

$$h(w) = \frac{1}{2\pi} \int_0^{2\pi} h(w + \rho e^{it}) dt.$$

Theorem 2.1.15 (Theo1.2.3, [67])**.** *Let $D \subset \mathbb{C}$ be a disc and $f \in C(\partial D)$. Then there is h harmonic in D and continuous on $\overline{D}$ so that $h = f$ on ∂D.*

Theorem 2.1.16. *Let U be an open subset of $\mathbb{C}$, and let $u : U \to \mathbb{R} \cup \{-\infty\}$ be an upper semicontinuous function. Then the following are equivalent.*

- *The function u is subharmonic on U.*
- *Whenever V is a relatively compact subdomain of U, and h a harmonic function on V satisfying*

  $$\limsup_{z \to \zeta} (u - h)(z) \leq 0 \quad (\zeta \in \partial V),$$

  then $u \leq h$ on V.

The following result is known as Harnack's theorem and it is demonstrated in [47].

Theorem 2.1.17. *Let Z be an open and connected subset of $\mathbb{R}^2$. If $(u_j)_j$ is an increasing sequence of harmonic function and $u = \lim_{j \to +\infty} u_j$, then either u is harmonic or $u \equiv \infty$.*

2.1.4 Plurisubharmonic functions

Let U be an open set in $\mathbb{C}^n$. A function $u : U \to \mathbb{R} \cup \{-\infty\}$ is called *plurisubharmonic* if it is upper semicontinuous, not identically $-\infty$ on any connected component of U and for every $a \in U$ and $b \in \mathbb{C}^n$ the function $z \mapsto u(a + zb)$ is subharmonic in a neighborhood of 0 in the complex plane. We let $\mathrm{PSH}(U)$ denote the family of plurisubharmonic functions on U which are not identically $-\infty$ on any connected component of U. A function u such that $-u$ is plurisubharmonic is called *plurisuperharmonic*.

Proposition 2.1.18. *The following are equivalent for an upper semicontinuous function u on a domain $D \subset \mathbb{C}^n$.*

- $u \in \mathrm{PSH}(D)$,
- *$u \circ f$ is subharmonic on $\mathbb{D}$ for every $f \in \mathscr{O}(\mathbb{D}, U)$.*

Plurisubharmonicity can also be defined using differential operators. We let d and d^c denote the real differential operators

$$d = \partial + \overline{\partial} \quad \text{and } d^c = i(\overline{\partial} - \partial),$$

where

$$\partial = \sum_{j=1}^{n} \frac{\partial}{\partial z_j} dz_j \quad \text{and } \overline{\partial} = \sum_{j=1}^{n} \frac{\partial}{\partial \overline{z}_j} d\overline{z}_j.$$

Proposition 2.1.19. *If $u \in \mathrm{PSH}(U)$ then $dd^c u \geq 0$ in a weak sense. Conversely, if u is a locally integrable function on U such that $dd^c u \geq 0$ in a weak sense, then there is a plurisubharmonic function $\tilde{u}$ on U which is equal to u almost everywhere.*

Most of the results concerning subharmonic functions carry over to the plurisubharmonic case.

Theorem 2.1.20. *Let D be a domain in $\mathbb{C}^n$.*

1. *The family $\mathrm{PSH}(D)$ is a convex cone, i.e. if α, β are non-negative numbers and $u, v \in \mathrm{PSH}(D)$, then $\alpha u + \beta v \in \mathrm{PSH}(D)$.*
2. *If D is connected and $\{u_j\}_{j \in \mathbb{N}} \subset \mathrm{PSH}(D)$ is a decreasing sequence, then $u = \lim_{j \to \infty} u_j \in \mathrm{PSH}(D)$ or $u \equiv -\infty$.*
3. *If $u : D \to \mathbb{R}$, and if $\{u_j\}_{j \in \mathbb{N}} \subset \mathrm{PSH}(D)$ converges to u uniformly on compact subsets of D, then $u \in \mathrm{PSH}(D)$.*
4. *Let $\{u_\alpha\}_{\alpha \in A} \subset \mathrm{PSH}(D)$ be such that its upper envelope $u = \sup_{\alpha \in A} u_\alpha$ is locally bounded above. Then the upper semicontinuous regularization u^* is plurisubharmonic in D.*

Definition 2.1.21 ([78]). A domain $D \subset \mathbb{C}^n$ is called a *domain of holomorphy* if there exists a function $f \in \mathcal{O}(D)$ which does not extend holomorphically to any point of D^c.

In [78] it is shown that if $D \subset \mathbb{C}^n$ is a domain of holomorphy then for any $a \in \partial D$ there is a holomorphic function $f_a \in \mathcal{O}(D)$ so that

$$\lim_{z \to a, z \in D} |f_a(z)| = +\infty.$$

Example 2.1.22. In $\mathbb{C}$ every domain is a domain of holomorphy.

A proof of the following theorem can be found in [Siciak [77]] (see also [Sibony [73]]).

Theorem 2.1.23 (Bremermann)**.** *Let $D \subset \mathbb{C}^n$ be a domain of holomorphy and $K \subset D$ be compact. Then for any $u \in \mathrm{PSH}(D) \cap C(D)$ there exist functions $f_1, \ldots, f_k$ holomorphic in D and positive integers $c_1, \ldots, c_k$ such that*

$$u(z) - \epsilon \leq \sup\{c_1 \log |f_1(z)|; \ldots ; c_k \log |f_k(z)|\} \leq u(z),\ z \in K.$$

The notion of maximality for plurisubharmonic functions was introduced by Sadullaev in [71].

Definition 2.1.24. A plurisubharmonic function $u : D \to \mathbb{R}$ is said to be maximal if for every relatively compact open subset G of D, and for each upper semicontinuous function v on $\overline{G}$ such that $v \in \mathrm{PSH}(G)$ and $v \leq u$ on ∂G, we have $v \leq u$ in G.

By $\mathrm{MPSH}(D)$ we denote the class of maximal plurisubharmonic functions on D. In the complex plane, as a direct consequence of the definition, we observe that, the class of maximal subharmonic functions is equal to the class of harmonic functions on D. Let $D \subset \mathbb{C}^n$ be a bounded domain and $f \in C(\partial D)$. To solve the *generalized Dirichlet problem* we need to find a continuous function $u : \overline{D} \to \mathbb{R}$ such that $(u|_D) \in \mathrm{MPSH}(D)$ and $u|_{\partial D} \equiv f$.

Definition 2.1.25. Let $f : \partial D \to \mathbb{R}$ be bounded and $z \in D$. Perron-Bremermann function for D and f is given by

$$u_f(z) = \sup\{v(z); v \in \mathrm{PSH}(D); v^*|_{\partial D} \leq f\}.$$

u_f was introduced in 1959 by Bremermann [7] in analogy to the classical Perron function used in (real) potential theory see [37].

A classical theorem of Walsh [81] states that if $f \in C(\partial D)$ and $\lim_{z \to z_0} u_f(z) = f(z_0)$ for all $z_0 \in \partial D$, then u_f is continuous on $\overline{D}$. This fact implies the following well known theorem see [75].

Theorem 2.1.26. *Let D be a bounded domain in $\mathbb{C}^n$. The following are equivalent*

1. *The generalized Dirichlet problem admits a solution for any $f \in C(\partial D)$.*
2. *For every $z \in \partial D$ there is $u \in \mathrm{PSH}(D) \cap C(\overline{D})$ such that $u(z) = 0$ and $u < 0$ on $\overline{D} \setminus \{z\}$.*
3. *If $f \in C(\partial D)$ then $u_f \in C(\overline{D})$.*

2.2 Definitions

In this section we collect some well known definitions that are used throughout the thesis.

Definition 2.2.1. We say that a domain $D \subset\subset \mathbb{C}^n$ has a *weak plurisubharmonic barrier* at a boundary point $z_0 \in \partial D$ if there exists a function $u \in \mathrm{PSH}(D)$ such that $u < 0$ on D and $\lim_{z \to z_0} u(z) = 0$.

Definition 2.2.2. We say that a domain $D \subset\subset \mathbb{C}^n$ has a *strong plurisubharmonic barrier* at a boundary point $z_0 \in \partial D$ if there exists a function $u \in \mathrm{PSH}(D)$ such that $u^* < 0$ on $\overline{D} \setminus \{z_0\}$ and $\lim_{z \to z_0} u(z) = 0$.

Definition 2.2.3. Let D be a bounded domain in $\mathbb{C}^n$, and let μ be a positive, regular Borel measure on $\overline{D}$. We say that μ is a *Jensen measure* with *barycentre* $z \in \overline{D}$ for continuous plurisubharmonic functions, if

$$u(z) \leq \int_{\overline{D}} u d\mu$$

for every function $u \in \mathrm{PSH}(D) \cap C(\overline{D})$.

$M(X)$ will be the set of Borel probability measures with compact support contained in X, where $X \subset \mathbb{C}^n$. For $D \subset \mathbb{C}^n$, $z \in D$ we set

$$J_z = J_z(\overline{D}) = \left\{\mu \in M(\overline{D}); u(z) \leq \int_{\overline{D}} u\, d\mu \text{ for all } u \in \mathrm{PSH}(D) \cap \mathrm{USC}(\overline{D})\right\}$$

$$J_z^c = J_z^c(\overline{D}) = \left\{\mu \in M(\overline{D}); u(z) \leq \int_{\overline{D}} u\, d\mu \text{ for all } u \in \mathrm{PSH}(D) \cap C(\overline{D})\right\}$$

Definition 2.2.4. Let $K \subset \mathbb{C}^n$ be a compact set. Its *polynomial hull* is defined as

$$\mathrm{Pol}\,\hat{K} = \{z \in \mathbb{C}^n; |p(z)| \leq \sup_K |p| \text{ for any polynomial } p\}.$$

Definition 2.2.5. A domain $D \subset \mathbb{C}^n$ is called a *Runge domain* if every function $f \in \mathscr{O}(D)$ can be approximated uniformly on compact subsets of D by polynomials in $\mathbb{C}^n$.

Definition 2.2.6. A compact set $K \subset \mathbb{C}^n$ is said to have a *Stein neighborhood basis* if for any domain V containing K there exists a pseudoconvex domain D_V such that $K \subset D_V \subset V$.

Definition 2.2.7. A domain $D \subset \mathbb{C}^n$ is called *strongly star shaped with respect to the origin* if $r\overline{D} \subset D$ for $r \in [0, 1[$.

Definition 2.2.8. $A \subset \partial D$ is *open*, it is *relatively* open in ∂D.

2.3 Domains

In this section we give definitions of some domains on which one can construct a plurisubharmonic function with prescribed boundary values. Observe that for $f : \partial D \to \overline{\mathbb{R}}$ it is in general not possible to find $u \in \mathrm{PSH}(D)$ such that $u^* = f$ on ∂D. This can be seen by considering the domain $D = \{z \in \mathbb{C}; 0 < |z| < 1\}$ and the function f that is zero on the unit circle $\mathbb{T}$ and 1 at the origin. For a given $D \subset \mathbb{C}^n$ and f defined on ∂D the problem whether there is $u \in \mathrm{PSH}(D)$ so that $u^* = f$ depends both on the continuity properties of f and on the geometry of ∂D.

2.3.1 Pseudoconvex domains

In [30] it is proven that there is a domain D and a function $u \in \mathrm{PSH}(D)$ such that there is no sequence of continuous plurisubharmonic functions u_j decreasing to u on D. But the situation is different if D is pseudoconvex.

Definition 2.3.1. A domain $D \subset \mathbb{C}^n$ is called *pseudoconvex* if there exists a continuous plurisubharmonic function φ on D such that for every $c \in \mathbb{R}$, we have

$$\{z \in D; \varphi(z) < c\} \subset\subset D.$$

Such a function φ is called an *exhaustion function* for D.

Example 2.3.2. Let $D = B(a, r)$ for some $a \in \mathbb{C}^n$ and $r > 0$. Set $\varphi(z) = -\ln d(z, \partial D)$. Observe that $\varphi(z) = -\ln \inf\{|z-x|; x \in \partial D\} = \sup\{-\ln|z-x|; x \in \partial D\}$. We observe that $\varphi \in \mathrm{PSH}(D)$, is an exhaustion function for D.

A proof of the following is given in [78].

Theorem 2.3.3. *A domain $D \subset \mathbb{C}^n$ (with $\partial D \neq \varnothing$) is pseudoconvex if and only if the function $-\ln d(z, \partial D)$ is plurisubharmonic in D.*

In $\mathbb{C}^n$ a domain is pseudoconvex if and only if it is a domain of holomorphy. This result is known as the *Levi problem*. It was solved by Norguet [61]. In a pseudoconvex domain D, elements of $\mathrm{PSH}(D) \cap C(D)$ and elements of $\mathscr{O}(D)$ are related (see for instance Theorem 2.1.23). Moreover, Gamelin and Sibony (see Theorem 1.2 in [32]) showed that if D is a bounded domain in $\mathbb{C}^n$ with smooth boundary, such that $\overline{D}$ has a Stein neighborhood basis, then any $u \in \mathrm{PSH}(D) \cap C(\overline{D})$ can be approximated uniformly on $\overline{D}$ by functions of the form $\max\{c_1 \log|f_1|; \ldots; c_m \log|f_m|\}$, where $c_1, \ldots, c_m > 0$ and $f_1, \ldots, f_m$ are analytic in a neighborhood of $\overline{D}$, (see also Proposition 1, [36], Corollary 10, [73]). The following theorem is due to Fornaess and Narasimhan (see [29] for a proof).

Theorem 2.3.4. *Let $D \subset \mathbb{C}^n$ be a pseudoconvex domain and $u \in \mathrm{PSH}(D)$. Then, there exists a sequence $(u_j)_{j=1}^{\infty} \subset \mathrm{PSH}(D) \cap C^{\infty}(D)$ that decreases to u.*

Reinhardt domains

Definition 2.3.5. Let $\pi : \mathbb{C}^n \to (\mathbb{R} \cup \{-\infty\})^n$ be the projection

$$\pi(z) = (\ln|z_1|, \ldots, \ln|z_n|).$$

For a set $S \subset \mathbb{C}^n$, we denote by S_R the corresponding set in $\overline{\mathbb{R}}^n$

$$S_R = \{x \in \mathbb{R}^n; x = \pi(z); z \in S\}.$$

An open set $D \subset \mathbb{C}^n$ is called a *Reinhardt domain* if

$$(z_1, \ldots, z_n) \in D \Longleftrightarrow (e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n) \in D,$$

for all real $\theta_1, \ldots, \theta_n$. A Reinhardt domain D is called *Logarithmically convex* if $(D^*)_R$ is a convex set in the real space $\mathbb{R}^n$ where

$$D^* = \{z = (z_1, \ldots, z_n) \in D;\ z_1 \ldots z_n \neq 0\}.$$

One reason for studying these kinds of domains is that logarithmically convex Reinhardt domains are the domains of convergence of power series in several complex variables see [59].

Theorem 2.3.6 ([9])**.** *Let D be a bounded pseudoconvex Reinhardt domain. Then the logarithmic image of D is convex.*

See [40, 89] for properties of Reinhardt domains.

Definition 2.3.7. A domain $D \subset \mathbb{C}^n$ is called *balanced* if and only if whenever $z \in G$ and $\lambda \in \mathbb{C}$, $|\lambda| \leq 1$, then $\lambda z \in D$.

Balanced domains are discussed in [24, 42].

2.3.2 Hyperconvex domains

In [13] Demailly proved that if D is a bounded pseudoconvex domain in $\mathbb{C}^n$ with Lipschitz boundary, then it admits a bounded exhaustion function.

Definition 2.3.8. A bounded domain $D \subset \mathbb{C}^n$ is called *hyperconvex* if there exists a non-positive plurisubharmonic function u such that $\{z \in D; u(z) < c\}$ is relatively compact in D for all $c < 0$.

One can use Jensen measures to characterize hyperconvex domains (see [85] for a proof).

Proposition 2.3.9. *For a bounded domain D in $\mathbb{C}^n$ the following are equivalent:*

- *D is hyperconvex.*
- *Every boundary point of D admits a weak plurisubharmonic barrier.*
- *For every $z \in \partial D$ and every Jensen measure $\mu \in J_z^c(\overline{D})$, μ is supported on ∂D.*

Hyperconvex domains are studied in [5] by Błocki. In [85] Wikström showed the following

Theorem 2.3.10. *Let $D \subset \mathbb{C}^n$ be a bounded hyperconvex domain. If $J_z = J_z^c$ for $z \in \overline{D}$, then for any $u \in \mathrm{PSH}(D)$ bounded from above, there exists a sequence $(u_j)_{j\in\mathbb{N}} \subset \mathrm{PSH}(D) \cap C(\overline{D})$ that decreases to u^* on $\overline{D}$.*

2.3.3 B-regular domains

Following Sibony we will say that a bounded domain $D \subset \mathbb{C}^n$ is $B - regular$ if every $f \in C(\partial D)$ can be extended to a plurisubharmonic function in D that is continuous on $\overline{D}$. Our next proposition was proven in [75] by Sibony.

Proposition 2.3.11. *The following are equivalent*

- *D is B-regular.*
- *For $z \in \partial D$ there is $u \in \mathrm{PSH}(D) \cap C(\overline{D})$ such that $u(z) = 0$ and $u < 0$ on $\overline{D} \setminus \{z\}$.*
- *There is $u \in \mathrm{PSH}(D) \cap C(\overline{D})$ such that $\lim_{z\to\partial D} u(z) = 0$ and $u(.) - |.|^2 \in \mathrm{PSH}(D)$.*
- *For every $z \in \partial D$ and every Jensen measure $\mu \in J_z^c(\overline{D})$, μ is supported on $\{z\}$.*

In [85] Wikström showed that the functions u_j in Fornaess and Narasimhan's theorem, Theorem 2.3.4, can not be extended continuously to $\overline{D}$ and proved the following.

Theorem 2.3.12. *Let $D \subset \mathbb{C}^n$ be a bounded B-regular domain and let u be a plurisubharmonic function on D that is bounded from above. Then there exists a sequence $(u_j)_{j\in\mathbb{N}} \subset \mathrm{PSH}(D) \cap C(\overline{D})$ that decreases to u^* on $\overline{D}$.*

In [15], Diederich and Fornaess proved the existence of a smooth bounded pseudoconvex domain, the so-called worm domain, whose closure does not have a Stein neighborhood basis. The question of when a Stein neighborhood basis exists has been studied by Sibony in [75]. Lisa Hed in [38] proved the following.

Theorem 2.3.13. *Let D be a bounded B-regular domain in $\mathbb{C}^n$ with C^1-boundary. Then $\overline{D}$ has a Stein neighborhood basis.*

2.4 Pluripolar sets

One of the most important class of sets which appear naturally in the theory of analytic functions are the *analytic varieties*, which locally are described as the common zeros of a finite family of analytic functions. More precisely, a subset A of a domain D is said to be an analytic variety (or an analytic set) in D if for each point $z_0 \in D$ there exists a connected neighborhood $U \subset D$ of z_0 and a finite family $\{f_1(z); \cdots ; f_k(z)\}$ of analytic functions in U such that

$$A \cap U = \{z \in U; f_1(z) = \cdots = f_k(z) = 0\}.$$

If $A \neq U$ then $\max(\log |f_1(z)|; \cdots ; \log |f_k(z)|)$ is a non-trivial plurisubharmonic function on U which equal $-\infty$ on $A \cap U$.
Recall that a subset A of D is said to be *locally pluripolar* if for any $z \in A$ there is a neighborhood U of z in D and $u \in \mathrm{PSH}(U)$ such that $u \not\equiv -\infty$ and

$$A \cap U \subset \{y \in U; u(y) = -\infty\}.$$

The following theorem is due to Josefson.

Theorem 2.4.1 ([43]). *A set $A \subset \mathbb{C}^n$ is locally pluripolar if and only if there is $u \in \mathrm{PSH}(\mathbb{C}^n)$, $u \not\equiv -\infty$ and $A \subset \{y \in \mathbb{C}^n; u(y) = -\infty\}$.*

Finite sets are pluripolar, a countable union of pluripolar sets is pluripolar. The zero set of a holomorphic function is pluripolar.

Proposition 2.4.2. *Pluripolar sets have Lebesgue measure equal to zero.*

Proposition 2.4.3. *Let D be an open subset of $\mathbb{C}^n$, and let F be a closed subset of D of the form $F = \{z \in D; v(z) = -\infty\}$, where $v \in \mathrm{PSH}(D)$. If $u \in \mathrm{PSH}(D \setminus F)$ is bounded from above, then u can be extended to a plurisubarmonic function in D.*

Let λ denotes the Lebesgue measure and $f_i : D \to \mathbb{R}$ be a $\lambda-$measurable function with $|f_i| < \infty$ $\lambda-$almost everywhere for every $i = 1, 2, \ldots$ We say that f_i converges to $f : D \to \mathbb{R}$ in measure in D, if

$$\lim_{i\to\infty} \lambda(\{x \in D; |f_i(x) - f(x)| \geq \epsilon\}) = 0$$

for every $\epsilon > 0$.

Theorem 2.4.4 ([45]). *Assume that $f_i \to f$ in measure. Then there exists a subsequence (f_{i_k}) such that $f_{i_k} \to f$ $\lambda-$almost everywhere.*

The following approximation theorem was studied by Sadullaev [72] and Conchar [34].

Theorem 2.4.5. *Let $D \subset \mathbb{C}^n$ be a pseudoconvex domain containing the origin. Then any $f \in \mathscr{O}(D)$ can be rapidly approximated with respect to the Lebesgue measure in D by rational functions if and only if $\mathbb{C}^n \setminus D$ is pluripolar. It means that there exists a sequence of rational functions r_k, deg $r_k \leq k$ such that*

$$|f - r_k|^{1/k} \to 0$$

in measure on D.

The pluripolar hull of a pluripolar set A in a domain D was defined by Zeriahi [88] as follows

$$A_D^* = \{z \in D; u(z) = -\infty \text{ for every } u \in \mathrm{PSH}(D) \text{ such that } u|_A \equiv -\infty\}.$$

In case where D is bounded Levenberg and Poletsky [57] introduced the set $\hat{A}_D$ as follows

$$\hat{A}_D = \{z \in D; u(z) = -\infty \text{ for every } u \in \mathrm{PSH}(D)^- \text{ such that } u|_A \equiv -\infty\}.$$

In [57] a relation between A_D^* and $\hat{A}_D$ is established.

Theorem 2.4.6 ([57])**.** *Let D be a pseudoconvex domain in $\mathbb{C}^n$. Let $\{D_j\}$ be an increasing sequence of relatively compact subdomains with $\cup_j D_j = D$. Let $A \subset D$ be pluripolar. Then*

$$A_D^* = \cup_j (\widehat{A \cap D_j})_{D_j}.$$

Theorem 2.4.7 ([57])**.** *Let D be a hyperconvex domain in $\mathbb{C}^n$. Let $\{D_j\}$ be an increasing sequence of relatively compact subdomains with $\cup_j D_j = D$. Let $A \subset D$ be pluripolar. Then*

$$\hat{A}_D = \cup_j (\widehat{A \cap D_j})_{D_j}.$$

Remark 2.4.8. Observe that if D is hyperconvex then $A_D^* = \hat{A}_D$.

Definition 2.4.9. We say that a subset A in a domain $D \subset \mathbb{C}^n$ is complete pluripolar if there exists $u \in \mathrm{PSH}(D)$ so that $A = \{z \in D; u(z) = -\infty\}$.

Notice that if $A \subset \mathbb{C}^n$ is complete pluripolar then it is a G_δ-set.

Example 2.4.10. Let $f \in \mathscr{O}(\mathbb{C}^n)$. Then its graph $\Gamma_f(\mathbb{C}^n)$ is complete pluripolar. This becomes clair by considering the plurisubharmonic function $\log|f(z) - w|$, $(z, w) \in \mathbb{C}^{n+1}$.

Example 2.4.11 ([28])**.** For any closed subset F of the complex plane there exists a continuous function on it whose graph is complete pluripolar in $\mathbb{C}^n$.

Theorem 2.4.12 ([88])**.** *Let $A \subset D$ be a closed pluripolar set. Then A is complete pluripolar if and only if $A = A_D^*$. In such a case one can find $u \in \mathrm{PSH}(D)$ continuous on $D \setminus A$ so that $A = \{z \in D; u(z) = -\infty\}$.*

2.4.1 Polar set in $\mathbb{C}$

In dimension one polar sets are well understood. The followings are proven in [50].

1. A is complete polar in D if and only if A is a G_δ polar set in D.
2. If A is complete polar in D, then A is complete polar in $\mathbb{C}$.

Theorem 2.4.13 ([10])**.** *Let $D \subset \mathbb{C}$ be a domain. The following are equivalent*

1. *For every $f \in \mathscr{O}(D)$, $\int_D |f|^2 dxdy < \infty$ implies $f \equiv 0$*
2. *$\mathbb{C} \setminus D$ is polar.*

2.5 Relative Extremal Functions

Siciak in his studies of Hartogs' theorem (on separate analyticity) introduced an extremal function called nowadays *relative extremal function*. His function is one of the main tools for studying pluripolar sets. Via Poletsky's theory it brings the geometry of analytic discs into pluripotential theory. For instance Levenberg and Poletsky used it to characterize in terms of analytic discs the pluripolar hull of a pluripolar set see [57] and in [63] Poletsky characterized the polynomial hull of a compact set in terms of analytic discs.

2.5.1 Definition and Properties

Let D be a bounded domain in $\mathbb{C}^n$ and E be a subset of D. The *relative extremal function* for E is defined as

$$u_{E,D}(z) = \sup\{u(z); u \in \mathrm{PSH}(D); u \leq -1 \text{ on } E \text{ and } u < 0 \text{ on } D\}.$$

Here are some basic properties.

Proposition 2.5.1. ,

- *If* $E_1 \subset D_1$ *and* $E_2 \subset D_2$ *are such that* $E_1 \subset E_2$ *and* $D_1 \subset D_2$ *then* $u_{E_1,D_1} \geq u_{E_2,D_2}$ *on* D_1.
- $u^*_{E,D}$ *is maximal in* $D \setminus \overline{E}$.
- $u^*_{P,D} \equiv 0$ *if and only if* P *is pluripolar.*
- $u^*_{E\cup P,D} = u^*_{E,D}$ *if* P *is pluripolar.*
- *If* K_j *is a sequence of compact subsets of* D *decreasing to* K, *then* $u_{K_j,D}$ *converges to* $u_{K,D}$.
- *If* $K \subset D$ *is compact and* D *is bounded and hyperconvex then the supremum in the definition of* $u_{K,D}$ *can be taken only over continuous functions. In such a case* $u_{K,D}$ *is continuous on* $\overline{D}$ *with* $u_{K,D} = 0$ *on* ∂D.

Theorem 2.5.2. *Let* $D \subset \mathbb{C}^n$ *be a domain and* $E \subset D$ *then*

$$u_{E,D} = \sup\{u_{V,D};\ E \subset V \subset D;\ V \text{ is open}\}.$$

Convexity of the sublevel sets of the *relative extremal function* is studied by Larusson-Lassere-Sigurdsson in [51].

Theorem 2.5.3. *Let* D *be a convex domain in* $\mathbb{C}^n$ *and let* E *be a convex subset of* D. *If* E *is either open or compact, then the sublevel sets*

$$\{z \in D; u_{E,D}(z) < \alpha\}$$

are convex for all $\alpha \in [-1, 0]$.

In [80] convex functions are used by Thorbiorson to construct the relative extremal function (see also [40]).

Theorem 2.5.4. *Suppose $D \subset \mathbb{C}^n$ is a bounded pseudoconvex Reinhardt domain and $E \subset\subset D$ is a Reinhardt domain. Then for $z \in D$ we have*

$$u_{E,D}(z) = \sup\{f \circ \pi(z); f \text{ convex on } D_R; f \leq -1 \text{ on } E_R; f < 0 \text{ in } D_R\}.$$

2.5.2 Applications

Here we mention a few results of Poletsky.

Theorem 2.5.5 ([57]). *Let $D \subset\subset \mathbb{C}^n$ be a domain and $E \subset D$ be pluripolar. Then the pluripolar hull $\hat{E}$ of E is given by*

$$\hat{E} = \{z \in D; \quad u_{E,D} < 0\}.$$

Theorem 2.5.6 ([63]). *Let $D \subset\subset \mathbb{C}^n$ be a Runge domain and $E \subset D$ be compact. Then the polynomial hull $\operatorname{Pol}\hat{E}$ of E is given by*

$$\operatorname{Pol}\hat{E} = \{z \in D; \quad u_{E,D} = -1\}.$$

In Chapter 5 we will see that a pluripolar set $E \subset D$ is complete if and only if there is a sequence $(E_j)_{j\in\mathbb{N}}$ of open sets so that $E = \bigcap_j E_j$ and the sequence $(u_{E_j,D})_{j\in\mathbb{N}}$ converges uniformly to zero on $D \setminus E_m$, $m > 0$.

2.6 Construction of plurisubharmonic functions

Here we recall some duality results.

2.6.1 Poletsky's theorem

Let $D \subset \mathbb{C}^n$ be a domain. Elements of $\mathscr{O}(\mathbb{D}, D)$ (resp. of $\mathscr{O}(\overline{\mathbb{D}}, D)$) are called analytic discs (reps. closed analytic discs) in D. A *disc functional* on D is a map $H : \mathscr{A} \to \mathbb{R} \cup \{-\infty\}$, where $\mathscr{A} \subset \mathscr{O}(\mathbb{D}, D)$.

Example 2.6.1. Let α be a non-negative function on $D \subset \mathbb{C}$. Define the functional H_1 by the formula

$$H_1(f) = \sum_{z\in\mathbb{D}} \alpha(f(z)) m_z(f) \log|z|, \quad f \in \mathscr{O}(\overline{\mathbb{D}}, D).$$

The sum is defined as the infimum of its finite partial sums. Here $m_z(f)$ denotes the multiplicity of f at z. H_1 is called the *Lelong disc functional.*

Example 2.6.2. Let v be a continuous plurisubharmonic function on $D \subset \mathbb{C}^n$. The functional H_2 defined by

$$H_2(f) = \frac{1}{2\pi} \int_{\mathbb{D}} \log|x| \Delta(v \circ f)(x) d\lambda(x).$$

is known as the *Riesz disc functional.*

If H is a disc functional defined on $\mathscr{A} \subset \mathscr{O}(\mathbb{D}, \mathbb{R})$ its *envelope* is the function $EH : D \to \mathbb{R} \cup \{-\infty\}$ defined by

$$EH(x) = \inf\{H(f); f \in \mathscr{A}; f(0) = x\}, \quad x \in D.$$

By works of Poletsky, we know that certain disc functionals have envelopes that are plurisubharmonic. This gave birth to a new theory called the *theory of disc functionals*. Its main goal is to provide *disc formulas* for important extremal plurisubharmonic functions in pluripotential theory, that is, to describe these functions as envelopes of disc functionals. This brings the geometry of analytic discs into pluripotential theory. Many authors worked on this theory: Poletsky, Edigarian, Sigurdsson, Magnusson, Larusson, Rosay and others.

Definition 2.6.3. Let $\mathscr{A}_1 \subset \mathrm{PSH}(D)$, $\mathscr{A}_2 \subset \mathscr{O}(\mathbb{D}, D)$, $H : \mathscr{A}_2 \to \mathbb{R} \cup \{-\infty\}$ and $x \in D$. The equality of the form

$$\sup\{u(x); u \in \mathscr{A}_1\} = \inf\{H(f); f(0) = x; f \in \mathscr{A}_2\}$$

is called a *disc formula*.

An example is given in Theorem 2.6.4.

Let $\varphi : D \to \mathbb{R} \cup \{-\infty\}$ be a Borel measurable function. The functional

$$\mathscr{O}(\overline{\mathbb{D}}, D) \ni f \mapsto \frac{1}{2\pi} \int_0^{2\pi} \varphi \circ f(e^{it}) dt$$

is called the *Poisson disc functional*.

We will refer to the theorem below as Poletsky's theorem. We should mention that it was proven also by Bu and Schahermayer in 1992 (see [8]).

Theorem 2.6.4 (Poletsky, [65])**.** *Let $D \subset \mathbb{C}^n$ be a domain, $\varphi \in \mathrm{USC}(D)$. For $x \in D$ we have*

$$\sup\{u(x); u \in \mathrm{PSH}(D); u \leq \varphi\} = \inf\left\{\frac{1}{2\pi} \int_0^{2\pi} \varphi \circ f(e^{it}) dt; f(0) = x; f \in \mathscr{O}(\overline{\mathbb{D}}, D)\right\}.$$

In 2003 Edigarian proved in [22] that Poletsky's theorem holds if φ is superharmonic on D. Magnusson 2011 in [59] proved that φ can be taken of the form $\varphi = \varphi_1 - \varphi_2$, where φ_1 is upper semicontinuous and φ_2 is plurisubharmonic on D. Disc formulas for *Lelong* functional, for *Riesz* functional and for *Poisson* functional are discussed by Larusson and Sigurdsson in [53]. Poletsky's theorem may fail if we assume that φ is merely lower semicontinuous (see [64]).

The theory of disc functional gives an alternative definition of certain notions of pluripotential theory. Poletsky in [63] proved the following

Theorem 2.6.5. *Let $D \subset \mathbb{C}^n$ be a Runge domain and let $K \subset D$ be a compact set. Then $z_0 \in \mathrm{Pol}\,\hat{K}$ if and only if for any $\epsilon > 0$ and any open neighborhood U of K, there exists an analytic disk $f : \overline{\mathbb{D}} \to D$ such that $f(0) = z_0$ and*

$$\sigma(f^{-1}(U) \cap \mathbb{T}) > 1 - \epsilon.$$

2.6.2 Edwards' theorem

Edwards' theorem is used to express upper envelopes of plurisubharmonic functions as lower envelopes of integrals with repsect to Jensen measures. The traditional method of constructing interesting plurisubharmonic functions is by taking upper envelopes over some class of plurisubharmonic functions and thus Edwards' theorem provides an alternative way of studying these constructions. By comparing different classes of Jensen measures, Edwards' theorem can be used to approximate a plurisubharmonic function by continuous plurisubharmonic functions (see [17]). It is also used to study plurisubharmonic functions on compact sets (see [38]).
Let $X \subset \mathbb{C}^n$ be a bounded set, and $F \subset \mathrm{USC}(X)$ be a convex cone of upper semicontinuous functions on X bounded from above. For $z \in X$ we set

$$J_z^F(X) = \left\{ \mu \in M(X);\ u(z) \leq \int_X u\, d\mu \text{ for all } u \in F \right\}.$$

For $g : X \to \mathbb{R}$ define

$$Sg(z) = \sup\{u(z); u \in F; u \leq g\}$$

and

$$Ig(z) = \inf \left\{ \int_X g\, d\mu;\ \mu \in J_z^F(X) \right\}.$$

The following theorem is due to Edwards (see [67] Theorem 2.1).

Theorem 2.6.6 (Edwards). *Assume that X is compact and g is a bounded Borel function on X. Then $Sg(z) \leq Ig(z)$. If g is lower semicontinuous, then $Sg = Ig$.*

One can find an example (see Wikstrom [85] and Ransford [67]) showing that if g is merely upper semi-continuous then $Sg \neq Ig$. One can also find an interesting example in Section 3 of [33].

Observe that $\mathrm{PSH}(D) \cap C(\overline{D})$ is a cone satisfying the conditions in Edwards' theorem. Hence we have the following:

Corollary 2.6.7 ([85]). *Let D be a bounded domain in $\mathbb{C}^n$, and let φ be a real-valued lower semicontinuous function on $\overline{D}$. Then, for every $z \in \overline{D}$,*

$$\inf \left\{ \int g\, d\mu; \mu \in J_z^c(\overline{D}) \right\} = \sup \left\{ v(z); v \in \mathrm{PSH}(D) \cap C(\overline{D}); v \leq g \right\}.$$

Edwards' theorem helps to understand relations between different classes of Jensen measures.

Theorem 2.6.8 ([85],Th 4.10). *Let D be a bounded strongly star-shaped domain in $\mathbb{C}^n$. Then, for every $z \in D$, we have $J_z^c = J_z$.*

Let $C_0(D)$ denotes the set of continuous real valued functions on D whose support is compact.

Definition 2.6.9. Let $(\mu_j)_j$ be a sequence of measures on D. We say that μ_j converges to a measure μ on D as $j \to \infty$, in the *weak*$^*-$ topology if

$$\int_D f d\mu_j \to \int_D f d\mu,$$

for every $f \in C_0(D)$.

Proposition 2.6.10 ([9]). *If $(\mu_j)_j$ is a sequence of positive measure such that μ_j converges weak*- to μ and if f is an upper semicontinuous function with compact support, then*

$$\limsup_{j\to\infty} \int f d\mu_j \leq \int f d\mu.$$

Proposition 2.6.11 ([9]). *Let D be a bounded domain in $\mathbb{C}^n$. Let $(z_m)_m \subset \overline{D}$ be a sequence of points converging to $z \in \overline{D}$. For each m, let $\mu_m \in J_{z_m}$. Then there is a subsequence μ_{m_j} and a measure $\mu \in J_z$ such that μ_{m_j} converges weak*- to μ.*

Set

$$A_z(D) = \{f_*\sigma; f \in \mathscr{O}(\overline{\mathbb{D}}, D); f(0) = z\}.$$

Wikström and Dieu in [17] proved that J_z is in the closure (in weak*-topology) of $A_z(D)$. Wikström [85] proved that if D is B-regular or is a polydisc then $J_z = J_z^c$ for $z \in D$.
A measure of the form $f_*\sigma$ with $f \in \mathscr{O}(\overline{\mathbb{D}}, D)$ and $f(0) = x$ is called a *disc measure* with center x. It is defined as follows for $u \in \mathrm{USC}(D)$,

$$\int_D u\, d(f_*\sigma) = \int_{\mathbb{T}} u \circ f d\sigma.$$

Definition 2.6.12. We say that D has the *weak approximation property* if for all $u \in \mathrm{PSH}(D)^-$ there is a sequence $(u_j)_j \subset \mathrm{PSH}(D) \cap C(\overline{D})$ that converges pointwise to u (no information about the monotony).

For $z \in D$ the problem whether $J_z^c = J_z$ is studied in [17] by Dieu and Wikström. They proved that if a domain D has the weak approximation property, then $J_z = J_z^c$ and the converse holds. And they gave examples of domains in which $J_z \neq J_z^c$.

Chapter 3

Characterizations of Boundary Pluripolar Hulls

We present some basic properties of the so called boundary relative extremal function. We introduce and discuss *boundary pluripolar* sets and *boundary pluripolar hulls*. For B-regular domains the boundary pluripolar hull is always trivial on the boundary of the domain. We present a "boundary version" of Zeriahi's theorem on the completeness of pluripolar sets.

3.1 Introduction

Boundary behavior of analytic functions in one or several complex variables is a classical subject, starting with the work of Fatou, and the literature on it is so vast that it seems justifiable to omit references. The boundary behavior of harmonic and subharmonic functions is also classical and well understood, e.g., [1]. The boundary behavior of plurisubharmonic functions, however, is less well understood. In this chapter we mainly study properties of the *boundary extremal function* $\omega(z, A, D)$, which is a generalization of the classical notion of harmonic measure.
Let D be a bounded domain in $\mathbb{C}^n$. For $z \in D$ and $A \subset \overline{D}$ one defines

$$\omega(z, A, D) = \sup\{u(z); u \in \mathrm{PSH}(D); u < 0; u^* \leq -1 \text{ on } A\}.$$

If $A \subset \partial D$ we call $\omega^*(., A, D)$ the *boundary relative extremal function* see [71, 63, 25] (note that [25] appeared as [26]). In that case ω^* is a special case of (the regularization of) the Perron-Bremermann function, hence is always maximal in D, cf. [71]. In Section 3.2, we will study $\omega(., A, D)$ and give some properties and applications.

For $A \subset \partial D$, it can happen that any $u \in \mathrm{PSH}(D)$ such that $u^*|_A = -\infty$ assumes the value $-\infty$ automatically on a bigger set in $\overline{D}$. For instance, set $\mathbb{B} = \{(z_1, z_2) \in \mathbb{C}^2; |z_1^2| + |z_2^2| < 1\}$. Let $A_1 \subset \mathbb{T}$ be the closure of a half-circle. Set $A = A_1 \times \{0\}$. Any $u \in \mathrm{PSH}(\mathbb{B})$ such that $u^* \equiv -\infty$ on A is identically $-\infty$ in $\{z \in \mathbb{C}; |z| < 1\} \times \{0\}$. The phenomenon is similar to the occurrence of pluripolar hull $\hat{E}_D$ of a pluripolar subset E of a domain D in $\mathbb{C}^n$ (see Section 2.4).

Definition 3.1.1. We will call a subset $A \in \partial D$ *b-pluripolar* (boundary pluripolar) if there exists a $u \in \mathrm{PSH}(D)$, $u < 0$, such that $A \subset \{u^* = -\infty\}$. Moreover, we will call a subset $A \subset \partial D$ *completely b-pluripolar* if there exists a $u \in \mathrm{PSH}(D)$, $u < 0$, $u \not\equiv -\infty$, such that

$$\{z \in \partial D; \quad u^*(z) = -\infty\} = A.$$

We will define the *boundary pluripolar hull* in Definition 3.3.3 and employ $\omega(., A, D)$ to describe this in Section 3.3 and 3.4. In Section 3.4 we will give a boundary version of Zeriahi's theorem. We will show that for B-regular domains the b-pluripolar hull $\hat{A} \subset \overline{D}$ of a b-pluripolar set A is contained in $A \cup D$. It is perhaps mildly surprizing that no hull is picked up at the boundary. In particular we will see in Corollary 3.4.6 that for B-regular domains every b-pluripolar set that is G_δ as well as F_σ, is completely b-pluripolar.

3.2 Properties of ω

The first proposition is a direct consequence of the definition of ω.

Proposition 3.2.1. *If $A_1 \subset A_2 \subset \partial D$, then*

$$\omega(., A_2, D) \leq \omega(., A_1, D).$$

If $D_1 \subset D_2$ and $A \subset \partial D_1 \cap \partial D_2$, then on D_1 we have

$$\omega(., A, D_2) \leq \omega(., A, D_1).$$

Proposition 3.2.2. *Let $D \subset \mathbb{C}^n$ be a bounded domain and $A \subset \partial D$. Then*

$$\omega(., A, D) = \sup\{\omega(., V, D); A \subset V; V \subset \partial D \text{ is open}\}$$

and there is a non-increasing sequence $(V_j)_j$ of open neighborhoods of A in ∂D such that

$$[\lim_{j\to\infty} \omega(z, V_j, D)]^* = [\omega(z, A, D)]^*. \tag{3.2.1}$$

Proof. Because of Proposition 3.2.1 it suffices to prove the existence of a sequence V_j such that (3.2.1) holds. By Choquet's lemma (see Lemma 2.1.5) there is an increasing sequence $(u_j)_j \subset \mathrm{PSH}(D)$ so that $u_j < 0$ on D, $u_j^* \leq -1$ on A and

$$[\lim_{j\to\infty} u_j]^* = [\omega(., A, D)]^*.$$

Set $V_1 = \{z \in \partial D; u_1^* - 1 < -1\}$ and $V_j = \{z \in \partial D; u_j^* - 1/j < -1\} \cap V_{j-1}$ for $j > 1$. Obviously $u_j - 1/j \leq \omega(., V_j, D)$ for all j. Hence we have (almost everywhere)

$$\omega(., A, D) = \lim_{j\to\infty} u_j \leq \lim_{j\to\infty} \omega(., V_j, D) \leq \omega(., A, D).$$

□

Proposition 3.2.3. *Let D be a domain in $\mathbb{C}^n$, let $A_1 \supset A_2 \supset A_3 \supset \ldots$ be a sequence of compact subsets of ∂D and $A = \bigcap_{j=1}^{\infty} A_j$. Then at each point in D*

$$\lim_{j\to\infty} \omega(., A_j, D) = \omega(., A, D).$$

Proof. Clearly, $\omega(., A_1, D) \leq \omega(., A_2, D) \leq \dots$, hence the limit exists. Take a negative function $v \in \mathrm{PSH}(D)$ such that $v^*|_A \leq -1$. As the set $V = \{z \in D; v(z) - \epsilon < -1\}$ is open and A is compact, we can find an open set U containing A such that $U \cap D \subset V$. There exists j_0 such that $A_j \subset U$ for $j \geq j_0$. Therefore $v - \epsilon \leq \omega(., A_j, D)$ for $j \geq j_0$. As a consequence, $v - \epsilon \leq \lim_{j\to\infty} \omega(., A_j, D)$, hence $\omega(., A, D) - \epsilon \leq \lim_{j\to\infty} \omega(., A_j, D)$, this for all $\epsilon > 0$. The opposite inequality is trivial. □

Following Edigarian and Sigurdsson [26] we define a *weakly regular* domain as follows

Definition 3.2.4. Let $D \subset \mathbb{C}^n$ be a bounded domain. We say that D is *weakly regular* if for every relatively open subset U of ∂D we have

$$\omega^*(., U, D) \leq -\chi_U \quad \text{on } \partial D.$$

In [26] it is shown that hyperconvex domains are weakly regular.

Proposition 3.2.5. *Let $D \subset \mathbb{C}^n$ be weakly regular and $A \subset \partial D$. Then for all x in the interior of A we have*

$$\omega^*(y, A, D) = -1.$$

Proof. Let $U = A^o$. Then for $y \in U$ we find

$$-1 \leq \omega^*(y, A, D) \leq \omega^*(y, U, D) \leq -\chi_U(y) = -1.$$

□

Corollary 3.2.6. *If $D \subset \mathbb{C}^n$ is a weakly regular domain and $A \subset \partial D$ is open, then*

$$\omega(., A, D) = \omega^*(., A, D).$$

In particular $\omega(.A, D) \in \mathrm{PSH}(D)$.

Proposition 3.2.7. *[25] Assume that $D \subset \mathbb{C}^n$ is weakly regular and $A_1 \subset A_2 \subset \dots \subset \partial D$ are open sets. Put $A = \bigcup_j A_j$. Then*

$$\lim_{j\to\infty} \omega(z, A_j, D) = \omega(z, A, D), \quad z \in D.$$

Proof. As A is open we have $\omega(., A, D) \in \mathrm{PSH}(D)$ see Corollary 3.2.6. For $z \in D$ set $u(z) = \lim_{j\to\infty} \omega(z, A_j, D)$. Note that the sequence is decreasing, so $u \in \mathrm{PSH}(D)$ and $u \geq \omega(., A, D)$. On the other hand, $u \leq \omega(., A_j, D)$ means that $u^* \leq -1$ on all A_j, hence $u^* \leq -1$ on A. That means u is in the family defining $\omega(., A, D)$. □

Proposition 3.2.8. *Let $D \subset \mathbb{C}^n$ be a weakly regular domain and $A_j \subset \partial D$ be an increasing sequence of sets. Then*

$$\lim_{j\to\infty} \omega(., A_j, D) = \omega(., A, D)$$

where $A = \bigcup_j A_j$.

Proof. It is clear that $\omega(.,A,D) \le \lim \omega(.,A_j,D)$. Let $\epsilon > 0$, $x \in D$. Then for all $j > 0$ there is u_j in the family defining $\omega(.,A_j,D)$ so that $\omega(x,A_j,D) \le u_j(x)+\epsilon$.
Set $V_j = \{u_j^* < -1+\epsilon\} \cap \partial D$. Recall that $u_j \le \omega(.,V_j,D)+\epsilon$. We get an open neighborhood V of A by setting $V = \bigcup_j V_j$. By Propositions 3.2.1 and 3.2.7 one gets

$$\omega(x,A,D) \le \lim_j \omega(x,A_j,D) \le \lim_{j\to\infty} \omega(x,V_j,D)+2\epsilon = \omega(x,V,D)+2\epsilon \le \omega(x,A,D)+2\epsilon.$$

This holds for all $x \in D$ and $\epsilon > 0$. □

Proposition 3.2.9. *Let D be a weakly regular domain in $\mathbb{C}^n$ and $A \subset \partial D$ be open. Suppose that $\{D_j\}$ is an increasing sequence of open subsets of D such that $D = \cup D_j$ and $A \subset \bigcap_j \partial D_j$. Then*

$$\lim_{j\to\infty} \omega(x,A,D_j) = \omega(x,A,D), \qquad \text{for } x \in D.$$

Proof. Set $v = \lim \omega(.,A,D_j)$. By Proposition 3.2.1, $\omega(.,A,D_{j+1}) \le \omega(.,A,D_j)$ and (because A is relatively open), $\omega(.,A,D_j) \in \mathrm{PSH}(D_j)$ (see Corollary 3.2.6). Hence $v \ge \omega(.,A,D)$. As v is the limit of a decreasing sequence of plurisubharmonic functions $v \in \mathrm{PSH}(D)$, and $v^* \le -1$ on A, therefore $v \le \omega(.,A,D)$. It follows that $v = \omega(.,A,D)$. □

Remark 3.2.10. We do not know if the condition that A be open can be dropped out.

Proposition 3.2.11. *Let $D \subset \mathbb{C}^n$ be B-regular and $A \subset \partial D$. Then $\omega^*(.,A,D) = 0$ on $\partial D \setminus (\overline{A})^o$.*

Proof. If $\partial D \setminus (\overline{A})^o$ is empty there is nothing to prove. If not let $x \in \partial D \setminus (\overline{A})^o$ and $r > 0$. Let $z \in B(x,r) \cap \partial D \setminus \overline{A}$ and let U be a neighborhood of $\overline{A}$ that does not contain z. Let $f \in C(\partial D, [-1,0])$ such that $f = -1$ on A and $f = 0$ on $\partial D \setminus U$. Then the Perron-Bremermann function u_f see definition 2.1.25 is less than $\omega(.,A,D)$. Thus

$$0 = u_f(z) \le \sup_{B(x,r)\cap D} u_f \le \sup_{B(x,r)\cap D} \omega(.,A,D) \le 0.$$

This holds for all $r > 0$. Hence $0 = \lim_{r\to 0} \sup_{B(x,r)\cap D} \omega(.,A,D) = \omega^*(x,A,D)$. □

3.3 Boundary pluripolar sets and boundary pluripolar hulls

As in the classical case the boundary relative extremal function can be used to describe boundary pluripolar sets. Levenberg-Poletsky's characterization of pluripolar hulls (in [57]) holds for b-pluripolar sets. As in the classical case a countable union of b-pluripolar set is b-pluripolar (Proposition 3.3.6). However, in contrast with the classical case where the relative extremal function $\omega^*(.,E,D)$ of a subset $E \subset D$ has the property that $\{z \in E; \omega^*(z,E,D) > -1\}$ is pluripolar, the set $\{z \in A; \omega^*(z,A,D) > -1\}$ is not in general b-pluripolar, see Example 3.3.4.

Definition 3.3.1. We say that a subset $A \subset \partial D$ is a *b-pluripolar set* if there exists a $u \in \mathrm{PSH}(D)$, $u < 0$, such that $u^* = -\infty$ on A.

It is well known that a compact set $K \subset \mathbb{T}$ in the boundary of the unit disc $\mathbb{D}$ is b-polar if and only if it has arc length 0 see [26], and that not all such sets are polar. Hence there exist b-polar sets that are not polar. This example can be modified to several variables case.

Example 3.3.2. Let K be a b-polar set in $\mathbb{T}$ that is not polar and let u be a subharmonic function on D such that $u \leq 0$ and $u^*|_K = -\infty$. Consider the function v on the unit ball $\mathbb{B} \subset \mathbb{C}^2$ defined by $v(z,w) = u(z^2 + w^2)$. Let

$$A = \{(z,w) \in \partial\mathbb{B}; z^2 + w^2 \in K\}.$$

Then $v^* = -\infty$ on A, hence A is b-pluripolar. Now if A were pluripolar we could find, by Josefson's theorem, (cf. [43]), $f \in \mathrm{PSH}(\mathbb{C}^2)$ so that $f|_A = -\infty$. For $\alpha \in [0, 2\pi)$ consider the function f_α on $\mathbb{C}$ defined by $f_\alpha(\zeta) = f(\zeta \cos\alpha, \zeta \sin\alpha)$. It is subharmonic or identically equal to $-\infty$. Take a branch $h(z)$ of $\sqrt{z}$ with branch cut not meeting K. Then $f_\alpha \circ h = -\infty$ on K. It follows that $f_\alpha \equiv -\infty$. In particular $f = -\infty$ on $\mathbb{R}^2 \subset \mathbb{C}^2$, which is not a pluripolar set. The conclusion is that A is not pluripolar.

Definition 3.3.3. Let $A \subset \partial D$ be b-pluripolar. The set

$$\{z \in \overline{D};\ u^*(z) = -\infty;\ \text{for all } u \in \mathrm{PSH}(D) \text{ with } \ u < 0;\ u^*|_A = -\infty\}$$

will be called the *b-pluripolar hull* of A and will be denoted by $\hat{A}$.

Example 3.3.4. Let $A_\alpha = \{(e^{i\phi}\cos\alpha, e^{i\psi}\sin\alpha); \phi, \psi \in [0, 2\pi)\} \subset \partial\mathbb{B}$ be the distinguished boundary of the polydisc Δ_α, where $\alpha \in \mathbb{R}$. We have $\omega^*(., A_\alpha, \mathbb{B}) \equiv 0$ on $\partial\mathbb{B}$, by Proposition 3.2.11. But every $u \in \mathrm{PSH}(\mathbb{B})$ such that $u^*_{|A_\alpha} \equiv -\infty$ is identically $-\infty$ on the polydisc. That means $u \equiv -\infty$ on $\mathbb{B}$ hence A_α is not b-pluripolar. Similarly, for $E_m = \bigcup_{j=1}^m A_{\alpha_j}$, we also find $\omega^*(., E_m, \mathbb{B}) \equiv 0$ on $\partial\mathbb{B}$. However, if $(\alpha_j)_j$ is a dense sequence in $(0, 2\pi)$ we get for $z \in \partial\mathbb{B}$

$$0 = \lim_{m\to\infty} \omega^*(z, E_m, \mathbb{B}) \neq \omega^*(z, \lim_m \cup_{i=1}^m E_i, \mathbb{B}) = -1.$$

Indeed, if $u \in \mathrm{PSH}(D)$ is negative and $u^* \leq -1$ on all E_m we have $u \leq -1$ on $\cup_j \Delta_{\alpha_j}$.

Proposition 3.3.5 (cf. [71, 57, 25])**.** *Let $D \subset \mathbb{C}^n$ be a bounded domain in $\mathbb{C}^n$ and $A \subset \partial D$. Then the following conditions are equivalent :*

1. $\omega^*(., A, D) \equiv 0$*;*
2. *A is b-pluripolar.*

In the case when A is b-pluripolar we have

$$\hat{A} \cap D = \{z \in D; \quad \omega(., A, D) < 0\}.$$

In particular, $\hat{A} \cap D$ is pluripolar.

Proof. 2. ⇒ 1. Assume that A is b-pluripolar. Take any $v \in \mathrm{PSH}(D)$, $v < 0$, $v \not\equiv -\infty$ such that $v^* = -\infty$ on A. Then $\epsilon v \leq \omega(., A, D)$ for all $\epsilon > 0$. Hence if for some z, $\omega(z, A, D) < 0$ then $v(z) = -\infty$, and it follows that $z \in \hat{A}$. Moreover, for all z such that $v(z) > -\infty$ we find $\omega(z, A, D) = 0$, hence $\omega^*(., A, D) \equiv 0$.
1. ⇒ 2. Assume now that $\omega^*(., A, D) \equiv 0$. Let $z \in D$ be such that $\omega(z, A, D) = 0$. For $j \in \mathbb{N}$ there is a negative $u_j \in \mathrm{PSH}(D)$ with $u_j^*|_A \leq -1$ and $u_j(z) > -2^{-j}$. Define

$$v(y) = \sum_{j=1}^{\infty} u_j(y), \; y \in D.$$

Observe that $v(z) > -1$, hence as a limit of a decreasing sequence of negative plurisubharmonic functions, v is a negative plurisubharmonic funtion. Moreover, $v^*|A \equiv -\infty$. We conclude that A is b-pluripolar and $z \notin \hat{A}$.
Finally, as $\hat{A} \cap D = \{z \in D; \omega(z, A, D) \neq \omega^*(z, A, D)\}$ is negligible, it is pluripolar. □

Proposition 3.3.6. *Let D be a bounded domain in $\mathbb{C}^n$. Suppose that $A = \cup_j A_j$, where $A_j \subset \partial D$ for $j = 1, 2, \ldots$. If $\omega^*(., A_j, D) \equiv 0$ for each j, then $\omega^*(., A, D) \equiv 0$. In particular, a countable union of b-pluripolar sets is b-pluripolar.*

Proof. By Proposition 3.3.5 we can choose $v_j \in \mathrm{PSH}(D)$ such that $v_j < 0$ on D and $v_j|A_j \equiv -\infty$. Take a point $a \in (D \setminus \bigcup_j v_j^{-1}(\{-\infty\}))$. Multiplying each of the functions v_j by a suitable positive constant, we may suppose that $v_j(a) > -2^{-j}$. As in the proof of the proposition above we check that $v = \sum_j v_j \in \mathrm{PSH}(D)$, $v < 0$ on D, $v \not\equiv -\infty$ on D and $v^* = -\infty$ on A. By Proposition 3.3.5, $\omega^*(., A, D) \equiv 0$ and A is b-pluripolar. □

Proposition 3.3.7. *Let $D \subset \mathbb{C}^n$ be a B-regular domain and $A \subset \partial D$ be b-pluripolar. Then*

$$\hat{A} \cap \partial D = A.$$

Proof. Obviously $A \subset \hat{A} \cap \partial D$. Let $z \in \partial D \setminus A$. As A is b-pluripolar there exists $u \in \mathrm{PSH}(D)$ such that $u < -1$, $u^* = -\infty$ on A. If $u^*(z)$ is finite, there is nothing to prove. Assume that $u^*(z) = -\infty$. We will construct a function $v \in \mathrm{PSH}(D) \cap C(\overline{D} \setminus \{z\})$ such that $(u + v)^*(z)$ is finite and $u + v$ is negative in D. This will show that $z \notin \hat{A}$.
Let

$$E_z(j) = \left\{ w \in \partial D; \; \frac{1}{4j+1} \leq |z - w| \leq \frac{1}{4j} \right\}.$$

Because u^* is upper semicontinuous on ∂D and A is b-pluripolar, while $E_z(j)$ is not b-pluripolar, u^* assumes at $w_j \in E_z(j)$ its maximum M_j with $-\infty < M_j \leq -1$. Let $f_j \leq 0$ be continuous on ∂D, $f_j > u^*$ and $f_j(w_j) < u^*(w_j) + 1$ and let $0 \leq \chi_j \leq 1$ be a smooth function on ∂D with $\chi_j(w_j) = 1$ and compactly supported in $E_z(j)$. Then

$$u^* \leq \sum_{j=1}^{\infty} f_j \chi_j \quad \text{on } \partial D \tag{3.3.1}$$

and

$$u^*(w_j) \geq \sum_{k=1}^{\infty} f_k \chi_k(w_j) - 1 \quad \text{for every } j. \tag{3.3.2}$$

Let F_j be a harmonic function on D, continuous on ∂D with boundary values $-f_j\chi_j$. The series $\sum_{j=1}^{\infty} F_j$ represents a monotonically increasing sequence of harmonic functions that are continuous up to ∂D. By choosing the support of χ_j sufficiently small, we can achieve, in view of Harnack's theorem, that the series converges uniformly on compact sets in $\overline{D} \setminus \{z\}$ and represents a harmonic function on D that is continuous on $\overline{D} \setminus \{z\}$ and has boundary values $\sum_{j=1}^{\infty} -f_j\chi_j$.

Let $v_j = u_{F_j}$ be the Perron-Bremermann function of $-f_j\chi_j$. Then $0 \leq v_j = v_j^* \leq F_j$ on $\overline{D}$ with equality on ∂D because D is B-regular, and v_j is a continuous plurisubharmonic function. It follows that the series $v = \sum_{j=1}^{\infty} v_j$ is also uniformly convergent on compact sets in $\overline{D} \setminus \{z\}$, hence it represents a plurisubharmonic function that is continuous up to $\partial D \setminus \{z\}$ with boundary values $\sum_{j=1}^{\infty} F_j$ on $\partial D \setminus \{z\}$. Then by (3.3.1) and (3.3.2) we have

$$u^* + v = \lim_{k\to\infty} \left(u^* + \sum_{j=1}^{k} v_k \right) \leq 0 \quad \text{ and } \quad (u^* + v)(w_j) \geq -1 \text{ for all } j.$$

Because $u^* + v^*$ is upper semicontinuous, we have that $(u^* + v^*)(z) \geq -1$. □

Theorem 3.3.8. *Let $D \subset \mathbb{C}^n$ be B-regular and $A \subset \partial D$ be a b-pluripolar set. Then*

$$\hat{A} = A \cup \{z \in D; \quad \omega(z, A, D) < 0\}.$$

Proof. Combine Proposition 3.3.5 and Proposition 3.3.7. □

Remark 3.3.9. Of course, if the domain is not B-regular, Proposition 3.3.7 is no longer valid. Take for D the standard open bidisc and let $A = \{(z, 1); |z| = 1\}$. Then A is b-pluripolar.

3.4 Completeness of b-pluripolar sets

Definition 3.4.1. We say that a subset $A \subset \partial D$ is *completely b-pluripolar* if there exists a $u \in \mathrm{PSH}(D)$, $u < 0$ such that $\{z \in \partial D; u^*(z) = -\infty\} = A$.

Zeriahi in [88] gave conditions under which a pluripolar set is completely pluripolar. Here we adapt Zeriahi's result to boundary pluripolar sets. Our approach requires minor adaptations.

Proposition 3.4.2. *Let $D \subset \mathbb{C}^n$ be a B-regular domain and $A \subset \partial D$ be a b-pluripolar set. Suppose that F and K are compact subsets of $\overline{D}$ with $F \subset \hat{A}$ and $K \subset \overline{D} \setminus \hat{A}$. Then for all $C > 1$ there exists $\psi_K \in \mathrm{PSH}(D) \cap C(\overline{D})$ so that $\psi_K < 0$, $\psi_K < -C$ on F, and $\psi_K > -1$ on K.*

Proof. Let $a \in K \subset \overline{D} \setminus \hat{A}$. Then there exists a negative $u \in \mathrm{PSH}(D)$ so that $u^* = -\infty$ on $\hat{A}$ and $u^*(a) > -\infty$. Set $M = \sup\{u^*(z) - u^*(a); z \in \overline{D}\}$. Then

$$w(z) = \frac{u(z) - u^*(a)}{2(|M| + 1)} - 1/2, \quad z \in D,$$

is plurisubharmonic and $w < 0$ on D, $w^*|_{\hat{A}} = -\infty$, $w^*(a) = -1/2$. By Theorem 2.3.12, we can find a sequence in $\mathrm{PSH}(D) \cap C(\overline{D})$ that decreases to $\max\{w^*; -2C\}$ on $\overline{D}$. In particular, in view of Dini's theorem (see Theorem 2.1.6) the convergence is uniform on F. Hence there is a negative $f_a \in \mathrm{PSH}(D) \cap C(\overline{D})$ such that $f_a < -C$ on F and $f_a(a) \geq w^*(a) = -1/2 > -1$. Then there exists a neighborhood V_a of a so that $f_a(z) > -1$ for all $z \in V_a$. By compactness we can find a finite subset of $I \subset K$ such that

$$K \subset \bigcup_{a \in I} V_a.$$

Set $\psi_K = \max\{f_a; a \in I\}$. Then

$$\psi_K < 0, \quad \psi_K \in \mathrm{PSH}(D) \cap C(\overline{D}), \quad \psi_K < -C \text{ on } F, \quad \text{and } \psi_K > -1 \text{ on } K.$$

□

Lemma 3.4.3. *Let D be a domain in $\mathbb{C}^n$. Let $A \subset \partial D$ be b-pluripolar and let $K \subset \partial D \setminus A$ be compact. Then there exists an $L \subset D \setminus \hat{A}$ such that $K \subset \overline{L}$ and $L \cup K$ is compact.*

Proof. If K is empty there is nothing to prove. As K is compact then for every $j \in \mathbb{N}$ there exist N_j points $z_{jl} \in K$, $1 \leq l \leq N_j$ such that

$$K \subset \bigcup_{l=1}^{N_j} B(z_{jl}, 1/j).$$

Because of Proposition 3.3.5 $\hat{A}$ has empty interior in D and we can find a point $w_{jl} \in (B(z_{jl}, 1/j) \cap D) \setminus \hat{A}$. Now let $L = \{w_{jl}; 1 \leq l \leq N_j; j \in \mathbb{N}\}$. Then the limit points of L belong to K; hence $K \cup L$ is compact. If $z \in K \cap B(z_{lj}, 1/j)$, then $|z - w_{lj}| < 2/j$, therefore z is a limit of a subsequence of L. □

Theorem 3.4.4. *Let D be a B-regular domain in $\mathbb{C}^n$. Let $A \subset \partial D$ be b-pluripolar, F an F_σ set, G a G_δ set in $\overline{D}$ such that $F \subset \hat{A} \subset G$. Then there exists a set $E \subset \overline{D}$ and a negative function $\psi \in \mathrm{PSH}(D)$ such that $F \subset E \subset G$, where*

$$E = \{z \in \overline{D};\ \psi^*(z) = -\infty\}.$$

Proof. Let $f \in \mathrm{PSH}(D)$ be negative, $f^*|_A = -\infty$ and $f \not\equiv -\infty$. Then $G_0 = \{f^* = -\infty\} \subset \overline{D}$ is a G_δ set containing $\hat{A}$ with the property that $\partial D \setminus G_0$ and $D \setminus G_0$ are non-empty. Replacing G by $G \cap G_0$, we have $\partial D \setminus G$ and $D \setminus G$ are non-empty. Set $F = \cup_j F_j$ where $(F_j)_{j\geq 1}$ is an increasing sequence of compact sets in $\hat{A}$, and $\overline{D} \setminus G = \cup_j \tilde{K}_j$ where $(\tilde{K}_j)_j$ is an increasing sequence of compact sets in $\overline{D} \setminus G$. Applying Lemma 3.4.3 to $\tilde{K}_j \cap \partial D$, each $\tilde{K}_j$ can be enlarged to a compact set $K_j \subset \overline{D} \setminus \hat{A}$ with the property that $K_j \cap \partial D \subset \overline{K_j \cap D}$. Replacing K_{j+1} by $K_{j+1} \cup K_j$, if necessary, we can assume $K_j \subset K_{j+1}$. By Proposition 3.4.2 for each $j > 0$ there exists $\psi_j \in \mathrm{PSH}(D) \cap C(\overline{D})$ with

$$\psi_j \leq -2^j \text{ on } F_j, \quad \text{and} \quad \psi_j \geq -1 \text{ on } K_j. \tag{3.4.1}$$

The function $\psi = \sum_{j=1}^{\infty} 2^{-j}\psi_j$ is negative. For $z \in D \setminus G$ there is $j_0 > 0$ so that $z \in K_{j_0}$ and we find that

$$\psi(z) = \sum_{j=1}^{j_0} 2^{-j}\psi_j(z) + \sum_{j=1+j_0}^{\infty} 2^{-j}\psi_j(z) \geq \inf_{K_{j_0}} \sum_{j=1}^{j_0} 2^{-j}\psi_j - 1 > -C_{j_0} > -\infty, \tag{3.4.2}$$

where C_{j_0} depends only on K_{j_0}, in view of the continuity of the ψ_j. It follows that ψ is plurisubharmonic on D as the limit of a decreasing sequence of plurisubharmonic functions. It satisfies $\psi^* \equiv -\infty$ on F because of (3.4.1). Finally if $z \in \partial D \setminus G$, then $z \in \overline{K_j \cap D}$ for some j. By (3.4.2) $\psi^*(z) > C_j$, hence $\psi^* > -\infty$ on $\overline{D} \setminus G$. Set $E = \{z \in \overline{D}; \psi^*(z) = -\infty\}$. Then $F \subset E \subset G$. □

Remark 3.4.5. In Proposition 3.4.2 and Theorem 3.4.4 we used B-regularity because B-regular domains have the *approximation property.* These results hold in any domain that has the approximation property.

Corollary 3.4.6. *Let D be a B-regular domain. Every b-pluripolar set $A \subset \partial D$ that is a G_δ as well as an F_σ is completely b-pluripolar. If, moreover, $\hat{A}$ is a G_δ, then $\hat{A} = \{z \in \overline{D}; \psi^*(z) = -\infty\}$. In particular, $\hat{A} \cap D$ is completely pluripolar.*

Proof. By Proposition 3.3.7 $\hat{A} \cap \partial D = A$. We apply Theorem 3.4.4 with $F = A$ and $G = A \cup D$. The theorem gives us a negative $\psi \in \mathrm{PSH}(D)$ with $A = \{z \in \partial D$ with $\psi^*(z) = -\infty\}$. In particular, $\psi \not\equiv -\infty$ on D, because it has finite boundary values on $\partial D \setminus A$, and hence A is completely b-pluripolar.

If moreover $\hat{A}$ is G_δ, we apply Theorem 3.4.4 with $F = A$, $G = \hat{A}$ and obtain a function ψ such that

$$A \subset \{\psi^* = -\infty\} \subset \hat{A}. \tag{3.4.3}$$

Now $\psi^*|_A = -\infty$ implies $\hat{A} \subset \{\psi^* = -\infty\}$, hence the last inclusion in (3.4.3) is an equality. □

3.5 Further observations

Looking at Remark 3.3.9 one can suspect that there is no propagation of the hull on the distinguished boundary. This fact suggests the following.

Theorem 3.5.1. *Let $D \subset \mathbb{C}^n$ be a bounded domain, $A \subset \partial D$ and $z \in \partial D \setminus A$. If there is a strong plurisubharmonic barrier at z then $z \notin \hat{A}$.*

Proof. Let $m > 0$ and v_m be a strong plurisubharmonic barrier at z such that $v_m \leq -1$ on $\overline{D} \setminus B(z, 1/m)$. Fix a sequence $(z_j)_j \subset D \setminus \hat{A}$ converging to z. By continuity of v_m^* at z there is j_0 so that $v_m^*(z_j) > -2^{-m}$ for $j \geq j_0$. Let $u \in \mathrm{PSH}(D)$ be negative so that $u^* \equiv -\infty$ on A and $u(z_j) > -2^{-m}$ for $j = 1, 2, \ldots, j_0$. Set

$$u_m = \max\{v_m; u\}.$$

Note that $u_m > -2^{-m}$ on $(z_j)_j$ and $u_m^* \leq -1$ on $A \setminus B(z, 1/m)$. Set

$$w = \sum u_m.$$

We have $-1 \leq w(z_j)$ for all j that means $w^*(z) \geq -1$ and $w^* \equiv -\infty$ on A. We conclude that $z \notin \hat{A}$. □

3.6 Boundary Relative Extremal Function and Analytic discs

For any subset $A \subset \overline{D}$, $z \in D$ we put

$$\Omega(z, A, D) = \inf\{-\sigma(\mathbb{T} \cap f^{-1}(A)); f(0) = z; f \in \mathscr{O}(\mathbb{D}, D) \cap C(\overline{\mathbb{D}}, \overline{D})\}.$$

Our main goal is to compute boundary pluripolar hulls by using Ω (see Theorem 3.6.8). In Theorem 3.6.3 we present an approximation theorem that will be used to prove that $\Omega(., A, D) = \omega(., A, D)$ for A relatively open in ∂D.

3.6.1 Approximation of analytic discs

We start with a pointwise approximation.

Theorem 3.6.1. *Let $D \subset \mathbb{C}^n$ be a domain, $f \in \mathscr{O}(\overline{\mathbb{D}}, D)$ and $V \subset D$ an open neighborhood of $f(0)$, $\delta_0 > 0$. Assume that V is path connected and the set $f^{-1}(V) \cap \mathbb{T}$ is nonempty. Then for all $t \in f^{-1}(V) \cap \mathbb{T}$ there is $g \in \mathscr{O}(\overline{\mathbb{D}}, V)$ such that $f(0) = g(0)$ and $||g(t) - f(t)|| < \delta_0$.*

Proof. Let $t_0 \in f^{-1}(V) \cap \mathbb{T}$, $x = f(0)$ and define $f_x(z) = x$ for $z \in \overline{\mathbb{D}}$. Extend f_x to a continuous map $f_x : \overline{\mathbb{D}} \cup [1, 2] \to V$ such that $f_x|_{[1,2]}$ is a path in V with $f_x(2) = f(t_0)$. Take $0 < \delta < \delta_0$. By Mergelyan's theorem there is a polynomial $P : \mathbb{C} \to \mathbb{C}^n$ so that

$$||P(z) - f_x(z)|| < \delta/2 \text{ for } z \in \overline{\mathbb{D}} \cup [1, 2].$$

We may assume that $P(0) = x$. Choose δ small so that $P \in \mathscr{O}(\overline{\mathbb{D}} \cup [1, 2], V)$. Let $W \subset\subset \mathbb{C}$ be a domain containing $\overline{\mathbb{D}} \cup [1, 2]$. By uniform continuity of P on $\overline{W}$ there is $\delta_P > 0$ such that

$$||P(z_1) - P(z_2)|| < \delta/2 \text{ for } z_1, z_2 \in W \text{ with } ||z_1 - z_2|| < \delta_P. \tag{3.6.1}$$

Let $\Omega \subset W$ be a Jordan domain of points within distance less than $\delta_P/2$ to $\overline{\mathbb{D}} \cup [1, 2]$. Choose δ_P small so that $P(\overline{\Omega}) \subset V$. By Caratheodory's theorem (conformal mapping) there is $q : \overline{\mathbb{D}} \longrightarrow \overline{\Omega}$ with $q(0) = 0$ and $q(\overline{\mathbb{D}}) = \overline{\Omega}$ hence there is $z_0 \in \mathbb{T}$ such that $|2 - q(z_0)| \le \delta_P/2$. Take $0 < r < 1$ such that

$$|2 - q(rz_0)| < \delta_P. \tag{3.6.2}$$

Choose $\gamma \in \mathbb{T}$ such that $\gamma t_0 = z_0$ and set $g(t) = P \circ q(r\gamma t)$ for $t \in \overline{\mathbb{D}}$. It follows from the construction that $g \in \mathscr{O}(\overline{\mathbb{D}}, V)$, $g(0) = f(0)$ and

$$||f(t_0) - g(t_0)|| = ||f_x(2) - P \circ q(r\gamma t_0)||$$

$$\le ||f_x(2) - P(2)|| + ||P(2) - P \circ q(rz_0)|| \le \delta < \delta_0, \text{ by (3.6.1) and (3.6.2).}$$

□

The following result due to Bu-Schachermayer is the core of the proof of Theorem 3.6.3. For a proof see [58] and [8].

Lemma 3.6.2. *Let A be a compact subset of $\mathbb{T}$ and $\psi \in C(\overline{\mathbb{D}})$. Then there exists a sequence (p_k) of polynomials $p_k : \mathbb{C} \to \mathbb{C}$ satisfying*

(i) $p_k(\mathbb{D}) \subset \mathbb{D}$ *and* $p_k(0) = 0$,

(ii) $p_k \to 0$ *uniformly on every compact subset of* $\overline{\mathbb{D}} \setminus A$,

(iii) $\int_A \psi \circ p_k(t) d\sigma(t) \longrightarrow \sigma(A) \int_{\mathbb{T}} \psi(t) d\sigma(t)$.

In Theorem 3.6.3 we give a uniform approximation of analytic discs on the unit circle. Recall that Theorem 3.6.3 is stated by Edigarian-Sigurdsson in [25] and a similar result is given by Poletsky in [63]. Here we present an alternative proof that is based on Bu-Schachermayer's lemma.

Theorem 3.6.3. *Let $D \subset \mathbb{C}^n$ be a domain, $\delta > 0$, and $\{B_j\}$ be a countable family of open subsets in D of diameter less than $\delta/2$. Assume that U and V are open subsets of D and*

$$V \subset \bigcup_j \{x \in B_j; \omega(x, U \cap B_j, B_j) < -a\},$$

where $a \in]0,1[$. Let $h \in \mathscr{O}(\overline{\mathbb{D}}, D)$, and assume that $\Delta \subset h^{-1}(V) \cap \mathbb{T}$ is a nonempty open set. Then for every $\epsilon \in]0,1[$ there exist $g \in \mathscr{O}(\overline{\mathbb{D}}, D)$ and an open set $\tilde{\Delta} \subset \Delta$ such that

1) $g(0) = h(0)$,

2) $||g - h|| = \sup_{z \in \overline{\mathbb{D}}} ||g(z) - h(z)|| \leq \delta + \epsilon$,

3) $\sigma(\tilde{\Delta}) \geq (1 - \epsilon) a \sigma(\Delta)$, *and*

4) $g(\tilde{\Delta}) \subset U$.

Proof. Fix $0 < \epsilon' < a\sigma(\Delta)\epsilon/8$ and $\Delta' \subset \Delta$ compact such that $\sigma(\Delta') > (1 - \epsilon/2)\sigma(\Delta)$. Take $t_0 \in \Delta'$. Then $x_0 = h(t_0) \in V$, so there is j_0 such that $h(t_0) \in B_{j_0}$. By the assumption we have

$$\omega(x_0, U \cap B_{j_0}, B_{j_0}) < -a.$$

Poletsky's theorem (Theorem 2.6.4) gives

$$\inf\{-\sigma(\mathbb{T} \cap f^{-1}(U \cap B_{j_0})); f(0) = x_0; f \in \mathscr{O}(\overline{\mathbb{D}}, B_{j_0})\} < -a.$$

Hence there is $f_0 \in \mathscr{O}(\overline{\mathbb{D}}, B_{j_0})$ such that $f_0(0) = x_0$ and

$$-\sigma(\mathbb{T} \cap f^{-1}(U \cap B_{j_0})) = \int_{\mathbb{T}} -\chi_{U \cap B_{j_0}} \circ f_0(z) d\sigma(z) < -a.$$

Set $f_0 = h(t_0) + g_0$, where $g_0 \in \mathscr{O}(\overline{\mathbb{D}}, \mathbb{C}^n)$ and $g_0(0) = 0$. Observe that $h(t_0) + g_0 \in \mathscr{O}(\overline{\mathbb{D}}, B_{j_0})$ and

$$\int_{\mathbb{T}} -\chi_{U \cap B_{j_0}}(h(t_0) + g_0(z)) d\sigma(z) < -a.$$

By Proposition 2.1.4 there is a continuous function ψ_0 on D bigger than $-\chi_{U \cap B_{j_0}}$ such that

$$\int_{\mathbb{T}} \psi_0(h(t_0) + g_0(z)) d\sigma(z) < -a.$$

We may assume that ψ_0 is non-positive. As B_{j_0} is open and ψ_0 is continuous, there is an open arc I_0 containing t_0 such that

$$\{h(t)+g_0(z); t\in I_0; z\in\overline{\mathbb{D}}\}\subset\subset B_{j_0},$$
$$|\psi_0(h(t)+g_0(z))-\psi_0(h(t_0)+g_0(z))|<\epsilon' \text{ for } t\in I_0, z\in\mathbb{T},$$
$$|\psi_0\circ h(t)-\psi_0\circ h(t_0)|<\epsilon' \text{ for } t\in I_0.$$

By a compactness argument there are N points $t_1,\dots,t_N\in\Delta'$, open arcs $I_1,\dots,I_N$ in $\mathbb{T}$, holomorphic maps $g_1,\dots,g_N\in\mathscr{O}(\overline{\mathbb{D}},\mathbb{C}^n)$ and $\psi_1,\dots,\psi_N$ continuous functions on D bigger respectively than $-\chi_{U\cap B_{j_1}},\dots,-\chi_{U\cap B_{j_N}}$ such that

$$t_i\in I_i,\quad g_i(0)=0,\quad h(t_i)+g_i\in\mathscr{O}(\overline{\mathbb{D}},B_{j_i})\quad \text{ and } \Delta'\subset\cup I_i,$$
$$\{h(t)+g_i(z);\quad t\in I_i;\quad z\in\overline{\mathbb{D}}\}\subset\subset B_{j_i}$$

$$\int_{\mathbb{T}}\psi_i(h(t_i)+g_i(z))d\sigma(z)<-a, \tag{3.6.3}$$

$$|\psi_i(h(t)+g_i(z))-\psi_i(h(t_i)+g_i(z))|<\epsilon'. \tag{3.6.4}$$

Choose $\beta_0>0$ small enough such that for all i

$$\left\{h(t)+g_i(z)+x;\quad ||x||<\beta_0;\quad t\in I_i;\quad z\in\overline{\mathbb{D}}\right\}\subset\subset B_{j_i}.$$

Take an open set $K\subset\subset D$ containing

$$\bigcup_{i=1}^{N}\left\{h(t)+g_i(z)+x;\quad ||x||<\beta_0;\quad t\in I_i;\quad z\in\overline{\mathbb{D}}\right\}\cup h(\overline{\mathbb{D}}).$$

Choose disjoint closed arcs $I_i'\subset I_i$ such that

$$\sigma(\Delta')<\sigma(\cup I_i')+\epsilon'. \tag{3.6.5}$$

By uniform continuity of ψ_i on $\overline{K}$ there is $0<\beta<\min\{\epsilon,\beta_0\}$ such that

$$|\psi_i(x_1)-\psi_i(x_2)|<\epsilon',\text{ for all } x_1,x_2\in K \text{ with } ||x_1-x_2||<\beta \text{ for all } i. \tag{3.6.6}$$

Take a small open neighborhood V_i of I_i' such that

$$\left(\bigcup_{m=1,i\neq m}^{N} I_m'\right)\cup\{0\}\subset\mathbb{C}\setminus V_i.$$

Set $K_i=\overline{\mathbb{D}}\setminus V_i$. By Lemma 3.6.2 for each $i=1,\dots,N$ there is a polynomial p_i such that:

- $p_i(0)=0,$

- $p_i(\mathbb{D}) \subset \mathbb{D}$,
- $||g_i \circ p_i(z)|| < \beta/N$ for all $z \in K_i$, and

$$\int_{I'_i} \psi_i(h(t_i) + g_i \circ p_i(t)) d\sigma(t) < \sigma(I'_i) \int_{\mathbb{T}} \psi_i(h(t_i) + g_i(t)) d\sigma(t) + \epsilon'/N. \qquad (3.6.7)$$

Set

$$g(z) = h(z) + \sum_{i=1}^{N} g_i \circ p_i(z) \text{ for all } z \in \overline{\mathbb{D}}.$$

Then g is holomorphic in a neighborhood of $\overline{\mathbb{D}}$, $g(\overline{\mathbb{D}}) \subset K$ and $g(0) = h(0)$. We have

$$\begin{aligned}
\int_{\mathbb{T}} -\chi_U \circ g(t) d\sigma(t) &\leq \sum_{i=1}^{N} \int_{I'_i} -\chi_{U \cap B_{j_i}} \circ g(t) d\sigma(t) \\
&= \sum_{i=1}^{N} \int_{I'_i} -\chi_{U \cap B_{j_i}} \left(h(t) + g_i \circ p_i(t) + \sum_{m=1, m \neq i}^{N} g_m \circ p_m(t) \right) d\sigma(t) \\
&\leq \sum_{i=1}^{N} \int_{I'_i} \psi_i \left(h(t) + g_i \circ p_i(t) + \sum_{m=1, m \neq i}^{N} g_m \circ p_m(t) \right) d\sigma(t) \\
&\leq \sum_{i=1}^{N} \int_{I'_i} \psi_i \left(h(t) + g_i \circ p_i(t) \right) d\sigma(t) + \epsilon' && \text{(because of (3.6.6))} \\
&\leq \sum_{i=1}^{N} \int_{I'_i} \psi_i \left(h(t_i) + g_i \circ p_i(t) \right) d\sigma(t) + 2\epsilon' && \text{(because of (3.6.4))} \\
&\leq \sum_{i=1}^{N} \sigma(I'_i) \int_{\mathbb{T}} \psi_i \left(h(t_i) + g_i(t) \right) d\sigma(t) + 3\epsilon' && \text{(by (3.6.7))} \\
&\leq -\sum_{i=1}^{N} \sigma(I'_i) a + 3\epsilon' && \text{(because of (3.6.3))} \\
&\leq -a\sigma(\Delta') + 4\epsilon' && \text{(because of (3.6.5))} \\
&\leq -a(1 - \epsilon/2)\sigma(\Delta) + 4\epsilon' && \text{(by the choice of } \Delta') \\
&\leq -a(1 - \epsilon/2)\sigma(\Delta) + a\sigma(\Delta)\epsilon/2 && \text{(by the choice of } \epsilon') \\
&= -a(1 - \epsilon)\sigma(\Delta).
\end{aligned}$$

Set $\tilde{\Delta} = \Delta \cap g^{-1}(U)$. Then

$$\sigma(\tilde{\Delta}) \geq (1 - \epsilon) a \sigma(\Delta).$$

It is clear that $g(\tilde{\Delta}) \subset U$, $g(0) = h(0)$ and $||g(z) - h(z)|| \leq \delta + \beta \leq \delta + \epsilon$ for all $z \in \overline{\mathbb{D}}$. □

3.6.2 Characterization of b-pluripolar hulls by analytic discs

Recall that our main goal is Theorem 3.6.8. To prove it we need some results demonstrated by Edigarian-Sigurdsson in [25]. For the reader's convenience we include their results. By weakly regular domain we refer to Definition 3.2.4.

Lemma 3.6.4 ([25]). *Let $D \subset \mathbb{C}^n$ be a weakly regular domain and $A \subset \partial D$ be open. For every $x_0 \in A$ and $\epsilon > 0$ there exists $r > 0$ such that*

$$B(x_0, r) \cap D \subset \{x \in D; \omega(x, A, D) < -1 + \epsilon\}.$$

Proof. Assume that for any $n \in \mathbb{N}$ there exists $x_n \in B(x_0, 1/n) \cap D$ such that $\omega(x_n, A, D) \geq -1+\epsilon$. Then $x_n \to x_0$ and $\omega^*(x_0, A, D) \geq -1+\epsilon$. But $\omega^*(., A, D) \leq -\chi_A$ on ∂D, a contradiction. □

Lemma 3.6.5 ([25]). *Let $D \subset \mathbb{C}^n$ be a weakly regular domain, $A \subset \partial D$ be open. Assume that $V \subset D$ is an open set such that for any $x_0 \in A$ there is $r > 0$ with $B(x_0, r) \cap D \subset V$. Then*

$$\omega(., V, D) \leq \omega(., A, D) + \epsilon.$$

Proof. We have $\omega^*(., V, D) \leq -\chi_A$ on ∂D. Hence $\omega(., V, D) \leq \omega(., A, D)$ on D.

□

Proposition 3.6.6 ([25]). *Let $D \subset \mathbb{C}^n$ be a weakly regular domain and $A \subset \partial D$ be open. Then $\Omega(., \overline{A}, D) \leq \omega(., A, D)$.*

Proof. Let $x \in D$ and $a \in]0, 1[$ so that $\omega(x, A, D) < -a$. We will show that $\Omega(x, \overline{A}, D) \leq -a$. This inequality will follow if we prove that for every $\epsilon \in]0, 1[$ there is $h \in \mathscr{O}(\mathbb{D}, D) \cap C(\overline{\mathbb{D}}, \overline{D})$ such that $h(0) = x$ and $\sigma(\mathbb{T} \cap h^{-1}(\overline{A})) > (1 - \epsilon)a$.
Take $(\epsilon_m)_m$ decreasing to zero such that $\prod_m (1 - \epsilon_m)^2 \geq 1 - \epsilon$. For every m we find a covering $\{B_j^m\}$ of A by countable many balls of diameter less than ϵ_m. Set

$$A_m = \bigcup_j \{z \in D \cap B_j^m; \omega(z, A \cap B_j^m, D \cap B_j^m) < -1 + \epsilon_m\}.$$

Since A is open A_m is open as well and we have

$$\omega(., A_{m+1} \cap B_j^m, D \cap B_j^m) \leq \omega(., A \cap B_j^m, D \cap B_j^m) \text{ on } D \cap B_j^m. \tag{3.6.8}$$

Indeed this is obvious by using Lemma 3.6.4 and Lemma 3.6.5. Inequality (3.6.8) implies that

$$A_m \subset \bigcup_j \{z \in D \cap B_j^m; \omega(z, A_{m+1} \cap B_j^m, D \cap B_j^m) < -1 + \epsilon_m\}.$$

We have

$$-a > \omega(x, A, D) \geq \omega(x, A_1, D) = \Omega(x, A_1, D).$$

So there is $h_1 \in \mathscr{O}(\overline{\mathbb{D}}, D)$ and $\Delta_1 \subset \mathbb{T}$ such that $h_1(0) = x$ and $-\sigma(\Delta_1) < -a$ with $\Delta_1 = \mathbb{T} \cap h_1^{-1}(A_1)$. We apply the Theorem 3.6.3 to construct inductively a sequence $h_m \in \mathscr{O}(\overline{\mathbb{D}}, D)$ and a decreasing sequence Δ_m of subsets of $\mathbb{T}$ such that

$$h_m(0) = x, h_m(\Delta_m) \subset A_m,\ \sigma(\Delta_{m+1}) > (1 - \epsilon_m)^2 \sigma(\Delta_m), ||h_{m+1} - h_m|| < 2\epsilon_m.$$

The last condition implies that h_m converges uniformly on $\overline{\mathbb{D}}$ to some $h \in \mathscr{O}(\mathbb{D}, D) \cap C(\overline{\mathbb{D}}, \overline{D})$. We set $\Delta = \bigcap_m \Delta_m$. Since $h_m(\Delta_m) \subset A_m$ and the points of A_m are at a distance less than or equal to ϵ_m from A, we have $h(\Delta) \subset \overline{A}$ and since $\sigma(\Delta_{m+1}) > (1-\epsilon_m)^2\sigma(\Delta_m)$ we get

$$\sigma(\mathbb{T} \cap h^{-1}(\overline{A})) \geq \sigma(\Delta) \geq \prod_m (1-\epsilon_m)^2 \sigma(\Delta_1) \geq (1-\epsilon)a.$$

□

One can find a proof of the result below in [63].

Theorem 3.6.7 ([25])**.** *Let $D \subset \mathbb{C}^n$ be a weakly regular domain and $A \subset \partial D$ be an open set. Then $\omega(., A, D) = \Omega(., A, D)$.*

Proof. Let $z \in D$. As $\omega(., A, D) \in \mathrm{PSH}(D)$, for all $h \in \mathscr{O}(\mathbb{D}, D) \cap C(\overline{\mathbb{D}}, \overline{D})$ with $h(0) = z$ we have

$$\omega(z, A, D) \leq \int_{\mathbb{T}} \omega(., A, D) \circ h d\sigma \leq \int_{\mathbb{T} \cap h^{-1}(A)} \omega(., A, D) \circ h d\sigma \leq -\sigma(\mathbb{T} \cap h^{-1}(A)).$$

Hence $\omega(z, A, D) \leq \Omega(z, A, D)$ for all $z \in D$. Take open sets $A_1 \subset\subset A_2 \subset\subset \cdots \subset\subset A$ such that $A = \cup_m^\infty A_m$. By Proposition 3.6.6 we have $\Omega(., A, D) \leq \Omega(., \overline{A}_m, D) \leq \omega(., A_m, D)$ on D for any $m \geq 1$. By tending m to ∞ we get by Proposition 3.2.8 that $\Omega(., A, D) \leq \omega(., A, D)$ on D. □

The following result is inspired by Levenberg-Poletsky in [57].

Theorem 3.6.8. *Let $D \subset \mathbb{C}^n$ be a bounded B-regular domain and $A \subset \partial D$ be b-pluripolar. Fix $z \in D$. Then $z \in \hat{A} \cap D$ if and only if there exists $M > 0$ such that, for any open set $V \subset \partial D$ containing A there is $f \in \mathscr{O}(\mathbb{D}, D) \cap C(\overline{\mathbb{D}}, \overline{D})$ with $f(0) = z$ and*

$$\sigma(\mathbb{T} \cap f^{-1}(V)) > M.$$

Proof. Let $z \in D$ assume that such M does exist. Then

$$\omega(z, V, D) \leq -M < 0$$

for all open neighborhood V of A. Hence by Proposition 3.2.2 we have

$$\omega(z, A, D) \leq -M < 0.$$

Then $z \in \hat{A} \cap D$ by Proposition 3.3.5. Conversely, assume $z \in \hat{A} \cap D$ but for all $M > 0$ there exists V an open neighborhood of A so that for any $f \in \mathscr{O}(\mathbb{D}, D) \cap C(\overline{\mathbb{D}}, \overline{D})$ with $f(0) = z$ we have $-\sigma(\mathbb{T} \cap f^{-1}(V)) > -M$. That means $\Omega(z, V, D) > -M$. By Theorem 3.6.7 we have

$$-M \leq \omega(z, V, D) \leq \omega(z, A, D) \leq 0.$$

Hence when M tends to zero we get $\omega(z, A, D) = 0$, which contradicts Proposition 3.3.5. □

Chapter 4

On a question of Sadullaev concerning boundary relative extremal functions

We study the relation between certain alternative definitions of the boundary relative extremal function. For various domains we give an affirmative answer to the question of Sadullaev posed in [71], whether these extremal functions are equal.

4.1 Introduction

Let $D \subset \mathbb{C}^n$ be a bounded C^1 smooth domain and $A \subset \partial D$. For $z \in D$ we define

1. $\omega^c(z, A, D) = \sup\{u(z); u \in \mathrm{PSH}(D)^- \cap C(\overline{D}); u|_A \leq -1\}$.
2. $\omega^n(z, A, D) = \sup\{u(z); u \in \mathrm{PSH}(D)^-; \limsup_{y \to \zeta, y \in n_\zeta} u(y) \leq -1 \text{ for } \zeta \in A\}$ where n_ζ is the inward normal to ∂D at ζ.
3. $\omega^R(z, A, D) = \sup\{u(z); u \in \mathrm{PSH}(D)^-; \limsup_{r \to 1^-} u(rx) \leq -1; x \in A\}$, if D is strongly star shaped with respect to the origin.

Sadullaev in [71] studied ω, ω^c and ω^n. Note that, smoothness is needed only to define ω^n. It is clear that

$$\omega^c(., A, D) \leq \omega(., A, D) \leq \omega^n(., A, D).$$

This chapter is motivated by the following question (Problem 27.4 in [71]): Suppose $A \subset \partial D$ is closed, in what situation the upper semicontinuous regularizations of $\omega^c(., A, D)$ and $\omega^n(., A, D)$ coincide? The same question is asked for $[\omega(., A, D)$ and $\omega^c(., A, D)]$ and for $[\omega(., A, D)$ and $\omega^n(., A, D)]$. The answer apparently depends on the geometry and convexity properties of D and the choice of the compact set $A \subset \partial D$. For instance we showed in [19] that Sadullaev's question has a positive answer when D is a smooth pseudoconvex Reinhardt domain and A is multi-circular. The result in [19] exploits the relation between relative extremal functions and convex functions in a Reinhardt domain. We answer in Section 4.3 the question affirmatively for ellipsoidal domains D_H, which are biholomorphic to the unit ball via a linear transformation. Here we exploit an idea of Wikström [85] and use Edwards' duality theorem. We attempted to use the version of Edwards' theorem in [33] and found that their result is not correct. In Section 4.4 we give two pertaining counterexamples.

Some basic properties of the boundary relative extremal function are given in [19, 26, 63, 71]. Depending on the way the boundary is approached, plurisubharmonic function may have different boundary values. Wikström in [85] considered the compact set $A = \mathbb{T}\times\{0\}$ and the function $u \in \mathrm{PSH}(\mathbb{B})$:

$$u(z) = \log \frac{|z_2|^2}{1 - |z_1|^2}.$$

He showed that $u^* = 0$ on A. The radial limit of u, $u^R = -\infty$ on A and the non-tangential limit of u, $u^\alpha = \log(1 - 1/2\alpha)$ on A [85, Example 5.5]. Let's recall the definition of u^α. If $\alpha > 1$ and $z_0 \in \partial\mathbb{B}$ we put

$$D_\alpha(z_0) = \{z \in \mathbb{B}; |1 - \langle z, z_0\rangle| < \alpha(1 - |z|^2)\},$$

$$u^\alpha(z_0) = \limsup_{z\to z_0, z\in D_\alpha(z_0)} u(z).$$

4.2 Notations and definitions

Let $D \subset \mathbb{C}^n$ be a bounded domain.

If ∂D is C^1-smooth and D has a defining function ρ, we define for $z \in \overline{D}$ and $t \in \mathbb{R}$

$$n(z, t) = z - t\left(\frac{\partial\rho}{\partial\overline{z}_1}(z), \ldots, \frac{\partial\rho}{\partial\overline{z}_n}(z)\right).$$

For $z \in \partial D$, the normal line n_z passing through z is parametrized by $\{n(z, t); t \in \mathbb{R}\}$. Let $u : D \to \mathbb{R} \cup \{-\infty\}$ be bounded from above and $z \in \partial D$. We define u^n at z as

$$u^n(z) = \limsup_{t\downarrow 0} u \circ n(z, t).$$

Extend u^n to $\overline{D}$ by setting $u^n(z) = u(z)$ if $z \in D$.

If D is strongly star shaped with respect to the origin, then for $z \in \partial D$ set

$$u^R(z) = \limsup_{r\uparrow 1} u(zr).$$

Extend u^R to $\overline{D}$ by setting $u^R(z) = u(z)$ if $z \in D$. By strongly star shaped we mean strongly star shaped with respect to the origin. For $z \in \overline{D}$ we consider four classes of positive measures J_z, J_z^c, J_z^n and J_z^R where

1. $J_z^n = J_z^n(\overline{D}) = \{\mu \in M(\overline{D}); u^n(z) \le \int_{\overline{D}} u^n\, d\mu$ for all $u \in \mathrm{PSH}(D); \sup_{\overline{D}} u^n < \infty\}$,
2. $J_z^R = \{\mu \in M(\overline{D}); u^R(z) \le \int_{\overline{D}} u^R\, d\mu$ for all $u \in \mathrm{PSH}(D); \sup_{\overline{D}} u^R < \infty\}$, in case D is strongly star shaped with respect to the origin.

Clearly for $z \in D$, $J_z^n, J_z^R \subset J_z \subset J_z^c$. Wikström studied these classes and proved that $J = J^c = J^R$ on D if D is strongly star shaped, see [85, Proposition 5.4].

4.3 Applications of Wikström's results

We use equalities between different classes of Jensen measures to prove the equivalence of different definitions. This is done by applying Edwards' theorem, Theorem 2.6.6, to the convex cone $\mathrm{PSH}(D) \cap C(\overline{D})$ and the associated Jensen measures J^c.

Proposition 4.3.1. *Let $D \subset \mathbb{C}^n$ be a bounded domain with C^1-boundary, $A \subset \partial D$ be compact. If $J_z^c = J_z^n$ for all $z \in D$, then*

$$\omega^c(z, A, D) = \omega^n(z, A, D).$$

Proof. We know that $\omega^c(., A, D) \leq \omega^n(., A, D)$. Let us prove that $\omega^n(., A, D) \leq \omega^c(., A, D)$. Let u be in the family defining ω^n. Set $g = -\chi_A$. Note that $u^n \leq g$ on $\overline{D}$. For $z \in D$ one has

$$u^n(z) \leq \inf \left\{ \int g \, d\mu; \mu \in J_z^n \right\} = \inf \left\{ \int g \, d\mu; \mu \in J_z^c \right\}, \text{ because } J_z^c = J_z^n.$$

Because g is lower semicontinuous on $\overline{D}$, by Corollary 2.6.7 we have

$$u^n(z) \leq \inf \left\{ \int g \, d\mu; \mu \in J_z^c \right\} = \sup \left\{ v(z); v \in \mathrm{PSH}(D) \cap C(\overline{D}); v \leq g \right\} \leq \omega^c(z, A, D).$$

As u was taken arbitrarily in the family defining ω^n we infer that $\omega^n(z, A, D) \leq \omega^c(z, A, D)$ for all $z \in D$. Thus $\omega^c(., A, D) = \omega^n(., A, D)$. □

The proof above applies to the next two propositions.

Proposition 4.3.2. *Let $D \subset \mathbb{C}^n$ be a bounded strongly star shaped domain and $A \subset \partial D$ compact. If $J_z^c = J_z^R$ for all $z \in D$, then*

$$\omega^c(z, A, D) = \omega^R(z, A, D).$$

Proposition 4.3.3. *Let $D \subset \mathbb{C}^n$ be a bounded domain and $A \subset \partial D$ compact. If $J_z = J_z^c$ for all $z \in D$, then $\omega(z, A, D) = \omega^c(z, A, D)$ for $z \in D$.*

Corollary 4.3.4. *If D is strongly star shaped, then $\omega(z, A, D) = \omega^c(z, A, D)$ for $z \in D$.*

Proof. In this domain $J_z = J_z^c$ for $z \in D$, see Theorem 2.6.8. Then Proposition 4.3.3 gives the result. □

For H a positive definite hermitian $n \times n$-matrix, let $\rho_H(z) = \overline{z}^T H z$ on $\mathbb{C}^n$ and set $D_H = \{z \in \mathbb{C}^n; \rho_H(z) < 1\}$.

Proposition 4.3.5. *On D_H we have $J_z^n = J_z^c = J_z$ for all $z \in D_H$.*

Proof. Set $D = D_H$. Let $z \in D$. Because for $u \in \mathrm{PSH}(D) \cap C(\overline{D})$, $u = u^n$ on $\overline{D}$, we have $J_z^n \subset J_z^c$. Let $\mu \in J_z^c$ and $u \in \mathrm{PSH}(D) \cap \mathrm{USC}(\overline{D})$. Let $0 < r < 1$. Observe that in case of D the map $n(., r)$ is holomorphic and maps $\overline{D}$ into D. For $y \in \overline{D}$ set

$$u_r(y) = u \circ n(y, r).$$

Then u_r is plurisubharmonic in a neighborhood of $\overline{D}$, hence u_r can be approximated monotonically from above by functions in $\mathrm{PSH}(D) \cap C(\overline{D})$. By the monotone convergence theorem $u_r(z) \leq \int u_r \, d\mu$ for all $r \in]0,1[$. By Fatou's lemma

$$\limsup_{r \to 0} u_r(z) \leq \limsup_{r \to 0} \int_{\overline{D}} u_r(y) \, d\mu.$$

For $y \in D$ one has $\limsup_{r \to 0} u_r(y) = u^n(y)$. Because the interval $[0,1]$ is not thin at 0, see Theorem 2.7.2 in [47], we have

$$u^n(z) = u(z) = \limsup_{r \to 0} u_r(z) \leq \int \limsup_{r \to 0} u_r(y) \, d\mu \leq \int_{\overline{D}} u^n(y) \, d\mu.$$

Thus $\mu \in J_z^n$ it follows that $J_z^c \subset J_z^n$. Hence $J_z^c = J_z^n \subset J_z \subset J_z^c$. □

The unit ball, i.e. the case where $H = Id$, was done in [85]. Our proof is a slight modification of Wikström's.

Theorem 4.3.6. *For all $z \in D_H$ one has $\omega(z, A, D_H) = \omega^n(z, A, D_H) = \omega^R(z, A, D_H) = \omega^c(z, A, D_H)$ for all $A \subset \partial D_H$ compact.*

Proof. By Proposition 4.3.5 $J^c = J^n = J$ and by Proposition 4.3.1 and Proposition 4.3.3 $\omega^c = \omega^n = \omega$. As D_H is strongly star shaped with respect to the origin, $J^c = J = J^R$ see the proof of Proposition 5.4 in [85]. By Proposition 4.3.2 the equality $\omega^c = \omega^R$ follows. □

4.4 Non-compact version of Edwards' theorem

We attempted to apply the non-compact version of Edwards' duality theorem stated in [33] to prove equalities between boundary extremal functions. However, we noticed that this version of Edwards' theorem as stated, does not hold. This section contains some counterexamples.

Let $D' \subset \mathbb{C}^n$ be a bounded set and $F \subset C(D')$ be a convex cone containing constants. $M(D')$ denotes the set of Borel probability measures with compact support in D'. For $z \in D'$ set

$$J_z^F(D') = \left\{ \mu \in M(D'); \ u(z) \leq \int_{D'} u \, d\mu \text{ for all } u \in F \right\}.$$

Let $g : D' \to \mathbb{R}$ define

$$Sg(z) = \sup\{u(z); u \in F; u \leq g\}$$

and

$$Ig(z) = \inf \left\{ \int_{D'} g \, d\mu; \ \mu \in J_z^F(D') \right\}.$$

The following theorem was claimed to be proven by Gogus, Perkins and Poletsky.

Theorem 4.4.1 ([33])**.** *Let D' be a locally compact Hausdorff space countable at infinity. If $g \in C(D')$ then either*

$$Sg(z) = \inf \left\{ \int_{D'} g\, d\mu; \mu \in J_z^F(D') \right\}$$

or $Sg \equiv -\infty$.

We will show that the result does not hold if D' is open.

Counterexample 4.4.2. Seeking for a contradiction, assume that
Theorem 4.4.1 holds for all open sets D' i.e

$$\sup\{u(z); u \in F; u \leq g\} = \inf \left\{ \int_{D'} g\, d\mu; \mu \in J_z^F(D') \right\}, \tag{4.4.1}$$

where $z \in D'$, $g \in C(D')$, $F \subset C(D')$ is a convex cone containing constants.
Let $D \subset\subset \mathbb{C}^n$ be a B-regular domain and $V \subset\subset D$ be an open ball. We recall the definition of Siciak's function

$$u_{D,V}(z) = \sup\{u(z); u \in \mathrm{PSH}(D); u \leq -\chi_V\}.$$

Let $u \in \mathrm{PSH}(D)^-$ be so that the set $\{u = -\infty\}$ is dense in V. For $m > 0$ set

$$U_m = \left\{\frac{u}{m} < -1\right\} \cap V \text{ and } F = \mathrm{PSH}(D) \cap C(\overline{D}).$$

Observe that the function $g_m = -\chi_{U_m}$ is continuous in the open set $D \setminus \partial U_m$ and that F is a convex cone in $C(D \setminus \partial U_m)$ containing the constants. By (4.4.1) we obtain for $z \in D \setminus \partial U_m$ the following equality (we take for D' the set $D \setminus \partial U_m$)

$$\inf \left\{ \int_{D\setminus\partial U_m} g_m\, d\mu; \mu \in J_z^F(D \setminus \partial U_m) \right\} = \sup\{v; v \in F; v \leq g_m\} \text{ on } D \setminus \partial U_m.$$

If $v \in F$ and $v \leq g_m$, then $v \leq -\chi_V$ because $\overline{U}_m = \overline{V}$ hence $v \leq u_{D,V}$, thus

$$\inf \left\{ \int_{D\setminus\partial U_m} g_m\, d\mu; \mu \in J_z^F(D \setminus \partial U_m) \right\} = \sup\{v; v \in F; v \leq g_m \text{ on } D \setminus \partial U_m\} \leq u_{D,V}.$$

As $J_z^F(D \setminus \partial U_m) \subset J_z^c$ we have on $D \setminus \partial U_m$

$$\inf \left\{ \int_{\overline{D}} g_m\, d\mu; \mu \in J_z^c \right\} \leq \inf \left\{ \int_{D\setminus\partial U_m} g_m\, d\mu; \mu \in J_z^F(D \setminus \partial U_m) \right\} \leq u_{D,V}.$$

Note that $J_z = J_z^c$ because D is B-regular, see Corollary 4.3 in [85]. It follows that

$$\inf \left\{ \int_{\overline{D}} g_m\, d\mu; \mu \in J_z \right\} = \inf \left\{ \int_{D} g_m\, d\mu; \mu \in J_z^c \right\} \leq u_{D,V} \text{ on } D \setminus \partial U_m.$$

Now $\frac{u}{m}$ is plurisubharmonic and $\frac{u}{m} \leq g_m$, hence for all $m > 0$ one has

$$\frac{u}{m}(z) \leq \inf \left\{ \int_{\overline{D}} g_m\, d\mu; \mu \in J_z \right\} \leq u_{D,V}(z) \text{ for } z \in D \setminus \partial U_m.$$

As $D \setminus \overline{V} \subset D \setminus \partial U_m$ we have for all $m > 0$ that

$$\frac{u}{m} \leq u_{D,V} \text{ on } D \setminus \overline{V}.$$

This is impossible since

$$0 \equiv \left(\sup_m \frac{u}{m}\right)^* \leq u_{D,V} < 0 \text{ on } D \setminus \overline{V}.$$

The conclusion is that equality (4.4.1) is false in open sets D'.

Next we prove that the version of Edwards' theorem stated in Theorem 2.6.6 does not hold for (open) B-regular domains.

Counterexample 4.4.3. Let D be a bounded B-regular domain and $V \subset \partial D$ be relatively open. Then $\overline{V}$ is not b-pluripolar, see Propositions 3.3.5 and 3.2.6. There exists a countable $L \subset D$ so that $L \cup \overline{V}$ is compact in $\overline{D}$ by Lemma 3.4.3. Set $g = -\chi_L$ and $F = \text{PSH}(D) \cap C(\overline{D})$. As L is non empty and does not have any accumulation point in D, g is lower semicontinuous in D . If Theorem 2.6.6 held in D we would get for $z \in D$

$$\inf\left\{\int_D g\,d\mu; \mu \in J_z^F(D)\right\} = \sup\{u(z); u \in F; u \leq g\} \leq \omega(z, V, D),$$
$$\inf\left\{\int_{\overline{D}} g\,d\mu; \mu \in J_z^c\right\} \leq \inf\left\{\int_D g\,d\mu; \mu \in J_z^F(D)\right\} \leq \omega(z, V, D),$$

because $J_z^F(D) \subset J_z^c$,

$$\inf\left\{\int_{\overline{D}} g\,d\mu; \mu \in J_z\right\} = \inf\left\{\int_{\overline{D}} g\,d\mu; \mu \in J_z^c\right\} \leq \omega(z, V, D),$$

because $J_z = J_z^c$. Therefore

$$\sup\{u(z); u \in \text{PSH}(D); u \leq g\} \leq \inf\left\{\int_{\overline{D}} g\,d\mu; \mu \in J_z\right\} \leq \omega(z, V, D).$$

Finally, because L is countable and therefore pluripolar, we would get

$$0 = (\sup\{u(z); u \in \text{PSH}(D); u \leq g\})^* \leq \omega(z, V, D).$$

This is impossible since V is not b-pluripolar. The conclusion is that Edwards' theorem does not hold in D.

Approximating g by continuous functions one can show that Theorem 4.4.1 does not hold in B-regular domains.

These counterexamples make it unlikely that a useful non-compact version of Edwards' theorem can be found. We have not been able to pinpoint the problematic points in ([33, Thm.1.3]).

Chapter 5

A characterization of thinness of a set

We use Siciak's relative extremal function to characterize complete pluripolar sets and to characterize the thinness of a set in terms of analytic discs.

5.1 Introduction

Let $D \subset \mathbb{C}^n$ be a bounded domain and $E \subset D$ be a subset. The relative extremal plurisubharmonic function of E in D will be denoted by $u_{E,D}$ (see Section 2.5 and [47, 77, 87]) for definition and properties. Thinness of a set is studied in [3] by Bedford and Taylor. In Section 5.3.2 we propose another definition of thinness of a set (see [18]). Complete pluripolar sets are discussed in Section 5.3.1 for definition and examples of such a set (see Section 2.4 and [28]). Roughly speaking we give a condition under which a set is complete pluripolar. Section 5.2 contains some properties of the relative extremal function.

5.2 Preliminaries

The relative extremal plurisubharmonic function is a precise tool to study pluripolar sets. It is well known that $u^*_{E,D}$ is plurisubharmonic and if E is open then $u_{E,D} = u^*_{E,D}$. For any set $E \subset D$ the set $\{u_{E,D} < u^*_{E,D}\}$ is pluripolar see [3]. Recall that one can use dualities to define u_E.

Proposition 5.2.1 (Poletsky, [63]). *If $E \subset D \subset\subset \mathbb{C}^n$ are open, then*

$$u_{E,D}(z) = \inf\left\{\int_{\mathbb{T}} -\chi_E \circ f d\sigma;\ f \in \mathscr{O}(\mathbb{D}, D) \cap C(\overline{\mathbb{D}}, D); f(0) = z\right\}.$$

Proposition 5.2.2 (Cegrell, [11]). *Let $D \subset \mathbb{C}^n$ be a bounded domain, $E \subset D$ be compact. Then*

$$u_{E,D}(z) = \inf\left\{\int_{\overline{D}} -\chi_E d\mu;\ \mu \in J_z\right\}.$$

Observe that for an arbitrary set $E \subset D$ we have

$$u_{E,D} \le \inf\left\{\int_{\overline{D}} -\chi_E d\mu;\ \mu \in J_z\right\} \le \inf\left\{\int_{\mathbb{T}} -\chi_E \circ f d\sigma;\ f \in \mathscr{O}(\mathbb{D}, D) \cap C(\overline{\mathbb{D}}, D); f(0) = z\right\} \tag{5.2.1}$$

For $E \subset D$ we say that E is *pluriregular* if $\{u_{E,D} = -1\} = E$. Edigarian and Sigurdsson proved in [26] that the inequalities in (5.2.1) are equalities in the case where E is pluriregular.
For $E \subset \mathbb{C}^n$ and $z \in \mathbb{C}^n$ we define

$$V(z, E) = \sup\{u(y); u \in \mathrm{PSH}(\mathbb{C}^n); u|_E \leq 0; u(.) \leq C_u + \log(1 + |.|); C_u = const \in \mathbb{R}\}.$$

If $E_1 \subset E_2$ then $V(z, E_2) \leq V(z, E_1)$.

Proposition 5.2.3. *If $E \subset \mathbb{C}^n$ is open then $V(., E) = V^*(., E) \in \mathrm{PSH}(\mathbb{C}^n)$.*

Proof. Note that $V^*(x, E) = \limsup_{y \to x} V(y, E) \leq 0$ for $x \in E$. Let $a \in E$ and $r > 0$ so that $F = \overline{B}(a, r) \subset E$. Then by properties **2.4, 2.6** in [76] we have for all $x \in \mathbb{C}^n$

$$V(x, E) \leq V(x, F) = \log^+ |x - a|/r.$$

Hence there is a constant $M \in \mathbb{R}$ such that $V(., E) \leq M + \log(1 + |.|)$ in $\mathbb{C}^n$. It follows that $V^*(., E) \leq M + \log(1 + |.|)$ in $\mathbb{C}^n$, that means $V^*(., E)$ is in the family defining $V(., E)$. Hence $V^*(., E) \leq V(., E) \leq V^*(., E)$.

□

Proposition 5.2.4. *[71] For any open set $E \subset \mathbb{C}^n$ and any $\alpha > 0$ the equality*

$$\frac{V(z, E) - \alpha}{\alpha} = u_{E \cap D_\alpha, D_\alpha}(z)$$

holds in $D_\alpha = \{z \in \mathbb{C}^n; V^*(z, E) < \alpha\}$.

Proof. Let $\epsilon > 0, z \in D_\alpha$ and u be in the family defining $V(., E)$. Observe that $(u - \alpha)/\alpha \leq u_{E \cap D_\alpha, D_\alpha}$ for any u in the family defining $V(., E)$ hence

$$\frac{V(z, E) - \alpha}{\alpha} \leq u_{E \cap D_\alpha, D_\alpha}(z).$$

Note that $V(., E) = V^*(., E)$. The function

$$w(z) = \begin{cases} \max\{\alpha(1 + u_{E \cap D_\alpha, D_\alpha}(z)); V(z, E)\} & \text{if} \quad z \in D_\alpha, \\ V(z, E) & \text{if} \quad z \notin D_\alpha \end{cases}$$

is in the family defining $V(., E)$, hence $\alpha(1 + u_{E \cap D_\alpha, D_\alpha}(z)) \leq V(., E)$. It follows that

$$u_{E \cap D_\alpha, D_\alpha} \leq \frac{V(., E) - \alpha}{\alpha}.$$

□

Proposition 5.2.4 is connected to Problem 12.2 in [71].

Proposition 5.2.5. *Assume that $D \subset \mathbb{C}^n$ is a bounded domain and $E_1 \subset E_2 \subset \cdots \subset D$ are open sets. Put $E = \bigcup_j E_j$. Then*

$$\lim_{j \to \infty} u_{E_j, D}(z) = u_{E,D}, \quad z \in D.$$

Proof. As E is open then $u_{E,D} \in \mathrm{PSH}(D)$. Set $u(z) = \lim_{j\to\infty} u_{E_j,D}(z)$ for $z \in D$. Note that the sequence is decreasing, so $u \in \mathrm{PSH}(D)$ and $u \geq u_{E,D}$. On the other hand, $u \leq u_{E_j,D}$ means that $u \leq -1$ on all E_j, hence $u \leq -1$ on E. That means u is in the family defining $u_{E,D}$. Hence $u = u_{E,D}$. □

Proposition 5.2.6. *[Problem 11.3 in [71]]. Let $D \subset \mathbb{C}^n$ be a bounded domain and $E_j \subset D$ be an increasing sequence of sets. Then*

$$\lim_{j\to\infty} u_{E_j,D} = u_{E,D}$$

where $E = \bigcup E_j$.

Proof. It is clear that $u_{E,D} \leq \lim u_{E_j,D}$. Let $\epsilon > 0$, $x \in D$. Then for all $j > 0$ there is $u_j \in \mathrm{PSH}(D)$ in the family defining $u_{E_j,D}$ such that $u_{E_j,D}(x) \leq u_j(x) + \epsilon$.
Set $V_j = \{u_j < -1 + \epsilon\}$. Observe that $u_j \leq u_{V_j,D} + \epsilon$. We get an open neighborhood V of E by setting $V = \bigcup_j V_j$. By Propositions 5.2.5 one gets

$$u_{E,D}(x) \leq \lim_j u_{E_j,D}(x) \leq \lim_{j\to\infty} u_{V_j,D}(x) + 2\epsilon = u_{V,D}(x) + 2\epsilon \leq u_{E,D}(x) + 2\epsilon.$$

This holds for all $x \in D$ and $\epsilon > 0$.

□

5.3 Main results

We state our main theorems that are to characterize the thinness of a set and the completeness of a pluripolar set in $\mathbb{C}^n$.

5.3.1 Characterization of complete pluripolar sets

Here we will characterize complete pluripolar set and infer that a countable intersection of complete pluripolar sets is complete pluripolar.

Theorem 5.3.1. *Let $D \subset \mathbb{C}^n$ be a bounded domain and $E \subset D$ be a pluripolar set. Then the following are equivalent*

1) *E is complete,*
2) *there is a decreasing sequence $(E_j)_j$ of open sets so that $E = \bigcap_j E_j$ and the sequence $(u_{E_j,D})_j$ converges uniformly to zero on $D \setminus E_m$, $m > 0$.*

Proof. 1) ⇒ 2) Let $v \in \mathrm{PSH}(D)^-$ so that $E = \{v = -\infty\}$. For $j > 0$ set $E_j = \{v < -j\}$. It is clear that the sequence E_j decreases to E. Fix $\epsilon > 0$ and $m > 0$. Note that $-m \leq v$ on $D \setminus E_m$. Take j_0 positive so that $m/j < \epsilon$. For all $j > j_0$, we get $-\epsilon < -m/j \leq v/j$ on $D \setminus E_m$. As $v/j \leq u_{E_j,D}$ on D then $-\epsilon < u_{E_j,D} < 0$ on $D \setminus E_m$ for all $j > j_0$. Hence $(u_{E_j,D})_j$ converges uniformly to zero on $D \setminus E_m$ for all $m > 0$.
1) ⇐ 2) For all $m > 0$ the assumption 2) ensures the existence of a j_m so that $u_{E_{j_m},D} > -2^{-m}$ on $D \setminus E_m$. Set $u = \sum_m u_{E_{j_m},D}$. Then $u = -\infty$ on E and $u > -\infty$ on $D \setminus E$. □

Proposition 5.3.2. *Let $(E_j)_j$ be a sequence of complete pluripolar sets in a bounded domain $D \subset \mathbb{C}^n$. Then $E = \cap_j E_j$ is complete pluripolar.*

Proof. As E_j is complete pluripolar, there is $u_j \in \mathrm{PSH}(D)^-$ so that $\{u_j = -\infty\} = E_j$. For $m > 0$ set $E_{jm} = \{u_j < -2^m\}$. Then E_{jm} is open and $E_j = \bigcap_m E_{jm}$. Set $A_m = \bigcap_{j=1}^m E_{jm}$ and $A = \bigcap_m A_m$. Observe that $E = A$. Indeed $E \subset A_m$ for all $m > 0$ that means $E \subset A$. Conversely $A \subset E_{jm}$ for all $m > 0$ that implies $A \subset E_j$ for all $j > 0$ hence $A \subset E$. It remains to prove that $(u_{A_m,D})_m$ converges uniformly on $D \setminus A_i$ where $i > 0$ is fixed.
Let $\epsilon > 0$ take $m_0 > i$ so that $2^{-m+i} < \epsilon$ for all $m > m_0$. By definition of E_{jm} we have $u_j < -2^m$ on E_{jm}. This means that $2^{-m}u_j$ is in the family defining $u_{A_m,D}$ for $j \leq m$ (because $A_m \subset E_{jm}$ for $j < m$), hence

$$2^{-m}u_j \leq u_{A_m,D}. \tag{5.3.1}$$

By definition of E_{ji} we have $-2^i \leq u_j$ on $D \setminus E_{ji}$ for all $j > 0$. In particular, for $j < m$ we obtain by the choice of m and (5.3.1)

$$-\epsilon \leq u_{A_m,D} \text{ on } D \setminus E_{ji}.$$

Hence

$$-\epsilon \leq u_{A_m,D} \text{ on } \bigcup_{j=1}^{i} \{D \setminus E_{ji}\} = D \setminus A_i.$$

This means that $(u_{A_m,D})_m$ converges uniformly to zero on $D \setminus A_i$ for all $i > 0$. Then Theorem 5.3.1 ensures that A is complete pluripolar. □

5.3.2 Thinness

Let u be a function plurisubharmonic on a neighborhood of $z_0 \in \mathbb{C}^n$. Even though u may be discontinuous at z_0, it is still always true that

$$\limsup_{z \to z_0} u(z) = u(z_0).$$

By upper semicontinuity, we have $\limsup_{z \to z_0} u(z) \leq u(z_0)$, and if the inequality were strict, then u would violate the submean inequality on a neighborhood of z_0. The situation may change if we take the limit along a set U whose closure contains z_0. For instance let $(z_n)_{n>1}$ be a sequence converging to z_0 and $U = \{z_1; z_2; \ldots\}$. It is easy to find $u \in \mathrm{PSH}(\mathbb{C}^n)$ such that

$$\limsup_{z \to z_0, z \in U} u(z) < u(z_0).$$

We say that U is *thin* at z_0.

Definition 5.3.3. Let Y be a subset of $\mathbb{C}^n$ and $x \in \mathbb{C}^n$. Then Y is *non-thin* at x if $x \in \overline{Y \setminus \{x\}}$ and if, for every plurisubharmonic function u defined on a neighborhood of x one has

$$\limsup_{z \to x,\ z \in Y \setminus \{x\}} u(z) = u(x).$$

If $h \in \mathscr{O}(\overline{\mathbb{D}}, \mathbb{C}^n)$ then the set $h([0,1])$ is not *thin* at any of its points see Corollary 4.8.5 in [47]. In this section we give a characterization of the thinness of a set at a given point in $\mathbb{C}^n$ in terms of analytic discs.

Theorem 5.3.4. *Let $U \subset \mathbb{C}^n$ be open and $x \in \mathbb{C}^n$. Then the following are equivalent:*

i) U is non-thin at x;

ii) For all $\epsilon > 0$, V neighborhood of x there is $f \in \mathscr{O}(\overline{\mathbb{D}}, V)$ such that $f(0) = x$ and

$$\sigma(\mathbb{T} \cap f^{-1}(V \cap U \setminus \{x\})) > 1 - \epsilon.$$

Proof. Assume *i*). Let $\epsilon > 0$ and V a neighborhood of x. By Proposition 5.2.1, the function $u_{(U\setminus\{x\})\cap V,V}$ is plurisubharmonic in V, where

$$u_{(U\setminus\{x\})\cap V,V}(x) = \inf\{-\sigma(\mathbb{T} \cap f^{-1}((U \setminus \{x\}) \cap V)); f \in \mathscr{O}(\overline{\mathbb{D}}, V); f(0) = x\}.$$

Since U is *non-thin* at x we have

$$u_{(U\setminus\{x\})\cap V,V}(x) = \limsup_{z\to x,\ z\in U\setminus\{x\}} u_{(U\setminus\{x\})\cap V,V}(z) = -1.$$

Thus there is $f \in \mathscr{O}(\overline{\mathbb{D}}, V)$ such that $f(0) = x$ and

$$-\sigma(\mathbb{T} \cap f^{-1}(V \cap U \setminus \{x\})) < -1 + \epsilon.$$

Suppose *ii*). Let $r_0 > 0$ and $u \in \operatorname{PSH}(B(x, r_0))$. For any $0 < r < r_0$ we set

$$c_r = \sup\{u(z); z \in \overline{B}(x, r) \cap (U \setminus \{x\})\}.$$

Take $M > |\sup_{\overline{B}(x,r)} u|$. For any $\epsilon > 0$ there is $f_\epsilon \in \mathscr{O}(\overline{\mathbb{D}}, B(x, r))$ with $f_\epsilon(0) = x$ such that

$$\sigma(\mathbb{T} \cap f_\epsilon^{-1}(B(x, r) \cap U \setminus \{x\})) > 1 - \epsilon.$$

Set $A = \mathbb{T} \setminus (\mathbb{T} \cap f_\epsilon^{-1}((U \setminus \{x\}) \cap B(x, r)))$ thus we have

$$\begin{aligned} u(x) \le \int_{\mathbb{T}} u \circ f_\epsilon(t)d\sigma(t) &\le \int_{\mathbb{T}\setminus A} u \circ f_\epsilon(t)d\sigma(t) + \int_A u \circ f_\epsilon(t)d\sigma(t) \\ &\le c_r(1 - \sigma(A)) + M\sigma(A) \le c_r + (|c_r| + M)\epsilon. \end{aligned}$$

This holds for all $\epsilon > 0$, hence when $\epsilon \to 0$ we get $u(x) \le c_r$. As r was taken arbitrarily then

$$u(x) \le \inf_{r>0} c_r = \inf_{r>0} \sup\{u(z); z \in \overline{B}(x, r) \cap U \setminus \{x\}\} = \limsup_{z\to x, z\in U\setminus\{x\}} u(z) \le \limsup_{z\to x} u(z) = u(x).$$

This holds for all u plurisubharmonic in a neighborhood of x. Hence U is *non-thin* at x. □

In the light of Corollary 4.8.3 in [47] we have the following.

Corollary 5.3.5. *Let $Y \subset \mathbb{C}^n$ and $x \in \mathbb{C}^n$. Then the following conditions are equivalent*

i) Y is non-thin at x;

ii) For every $\epsilon > 0$, neighborhood V of x and every open set U containing $Y \setminus \{x\}$ there exists $f \in \mathscr{O}(\overline{\mathbb{D}}, V)$ such that $f(0) = x$ and

$$\sigma(\mathbb{T} \cap f^{-1}(V \cap U \setminus \{x\})) > 1 - \epsilon.$$

Notations

$\mathbb{N}$: the set of natural numbers $\{0, 1, 2, \ldots\}$.

$\mathbb{R}$: the set of real numbers $]-\infty, \infty[$.

$\mathbb{C}$: the set of complex numbers $\{z = x + iy : x, y \in \mathbb{R}\}$.

$\mathbb{D}$: the unit disk in $\mathbb{C}$.

$\mathbb{T}$: the unit circle.

σ : arc length measure on $\mathbb{T}$.

$D(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$: disk of center $a \in \mathbb{C}$ and radius $r > 0$.

$B(b, r) = \{z \in \mathbb{C} : |z - b| < r\}$: ball of center $b \in \mathbb{C}^n$ and radius $r > 0$.

$\mathbb{B}$: the unit ball in $\mathbb{C}^n$, $n > 1$.

$M(Y)$: the set of Borel probability measures with compact support contained in Y, where $Y \subset \mathbb{C}^n$ is a subset.

χ_U : the characteristic function of U, where $U \subset \mathbb{C}^n$.

$\mathrm{PSH}(Y)$: the set of functions that are plurisubharmonic in a neighborhood of $Y \subset \mathbb{C}^n$.

$\mathrm{PSH}(Y)^-$: a subset of $\mathrm{PSH}(Y)$ formed by non-positive functions.

$\mathscr{O}(Y)$: the set of $\mathbb{C}$-valued functions that are holomorphic in a neighborhood of $Y \subset \mathbb{C}^n$.

$\Gamma_f(Y) = \{(z, f(z)) : z \in Y\}$, where $f \in \mathscr{O}(Y)$.

$C(Y)$: the set of $\mathbb{R}$-valued continuous functions defined on Y.

$C_0(Y)$: the set of $\mathbb{R}$-valued continuous functions with compact support contained in $Y \subset \mathbb{C}^n$, $n > 0$.

$\mathrm{USC}(Y)$: the set of $\overline{\mathbb{R}}$-valued upper semicontinuous functions defined on $Y \subset \mathbb{C}^n$.

$\mathscr{O}(\overline{\mathbb{D}}, Y)$: the family of maps $f : \overline{\mathbb{D}} \to Y$ which are holomorphic in a neighborhood of the closure $\overline{\mathbb{D}}$ of the unit disk $\mathbb{D}$.

$C(\overline{\mathbb{D}}, Y)$: the family of maps $f : \overline{\mathbb{D}} \to Y$ which are continuous on the closure $\overline{\mathbb{D}}$ of the unit disk $\mathbb{D}$.

$d(x, y) = |x - y|$: distance between two points $x, y \in \mathbb{C}^n$, $n > 0$.

$d(C, B) = \inf\{|x - y| : x \in C, y \in B\}$: distance between two non-empty sets $C, B \subset \mathbb{C}^n$, $n > 0$.

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