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MINIMAL PROJECTIONS ONTO SPACES OF SYMMETRIC MATRICES

by Dominik Mielczarek

Abstract. Let X_n denote the space of all $n \times n$ matrices and $Y_n \subset X_n$ its subspace consisting of all $n \times n$ symmetric matrices. In this paper we will prove that a projection $P_a\colon X_n \to Y_n$ given by the formula $P_a(A) = \frac{A+A^T}{2}$ is a minimal projection, if the norm of matrix A is an operator norm generated by symmetric norm in the space $\mathbb{R}^n$. We will show that the assumption about the symmetry of the norm is essential. We will also prove that this projection is the only minimal projection if our operator norm is determinated by the l_2 norm in the space $\mathbb{R}^n$.

1. Introduction. Let X be normed space over the field of real numbers and let Y be a linear subspace of X. A bounded linear operator $P\colon X \to Y$ is called a projection if $P\big|_Y = Id$. The set of all projections from X onto Y will be denoted by $P(X,\ Y)$. A projection P_0 is called minimal if

$$\|P_0\| = \inf\Big\{\|P\| \ \ : \ \ P \in P(X,\ Y)\Big\}.$$

Analogously, P_0 is said to be co-minimal if

$$\|Id - P_0\| = \inf\Big\{\|Id - P\| \ \ : \ \ P \in P(X,\ Y)\Big\}.$$

The constant

$$\Lambda(X,\ Y) = \inf\Big\{\|P\| \ : \ P \in P(X,\ Y)\Big\}$$

is called the relative projection constant.
The problem of finding formulas for minimal projections is related to the Hahn–Banach Theorem, as well as to the problem of producing a "good" linear replacement of an $x \in X$ by a certain element from Y, because of the inequality

$$\|x - P(x)\| \le (1 + \|P\|)\mathrm{dist}(x, Y),$$

where $P \in P(X,\ Y)$.

For more information concerning minimal projections, the reader is referred to [**1234512345123451234512345**], [**89101112138910111213891011121389101112138910111213**

In this paper we are interested in finding a minimal projection from $X_n = L(\mathbb{R}^n)$ equipped with operator norms onto its subspace Y_n consisting of all symmetric $n \times n$ matrices. The main result here is Theorem 2.1, which shows that if the norm in X_n is generated by a symmetric norm in $\mathbb{R}^n$, then the averaging operator $P_a(A) = \frac{A+A^T}{2}$ is a minimal projection. However, this is not true in general (see Theorem 2.3). We also show that if the norm in X_n is generated by the Euclidean norm in $\mathbb{R}^n$, then P_a is the only minimal projection (Theorem 3.1), some other results concerning P_a also will be presented. In the sequel we need

Theorem 1.1. (see, e.g., [**1**]). *Let* $Y = \bigcap_{i=1}^{k} ker(f_i)$, *where* $\{f_1, \ldots, f_k\}$ *is linearly independent subset of* X^*. *If* P *is a projection of* X *onto* Y, *then there exist* $y_1, \ldots, y_k \in X$ *such that* $f_i(y_j) = \delta_{ij}$ *and*

$$P(x) = x - \sum_{i=1}^{k} f_i(x) y_i,$$

for $x \in X$.

Now assume that G is a compact topological group such that each element $g \in G$ induces isometry A_g in the space X. Let $Y \subset X$ be subspace of X invariant with respect to A_g, $g \in G$, which means that $A_g(Y) \subset Y$ for each $g \in G$. We shall assume that the map $(g, x) \to A_g x$ from $G \times X$ into X is continous. We will write g instead of A_g and u$\|g\|$ instead of $\|A_g\|$.

Theorem 1.2. (see, e.g., [**14**]). *Let group a* G *fulfil the above conditions. If* $P(X, Y) \neq \emptyset$, *then there exists a projection* P *which commutes with group* G, *i.e. for each* $g \in G$, $A_g P = P A_g$.

Moreover, if $Q \in P(X, Y)$ then P can be defined as

$$P = \int_G g^{-1} Q g \, dg,$$

where dg is the probabilistic Haar measure on G.
If there exists exactly one projection witch commutes with G, we can state

Lemma 1.1. (see, e.g., [**6**]). *If a projection* $P \colon X \to Y$ *is the only projection which commutes with* G, *then* P *is minimal and cominimal.*

Proof. Let $Q \in P(X, Y)$. We show that

$$\|P\| \leq \|Q\|.$$

Since P is the only projection which commutes with G, by Theorem 1.2

$$P = \int_G g^{-1} Q g dg,$$

where dg is the probabilistic Haar measure on G. Making use of the properties of Bochner's integral, we obtain the following estimate

$$\|P\| = \left\| \int_G g^{-1} Q g dg \right\| \leq \int_G \|g^{-1}\| \|Q\| \|g\| dg \tag{1}$$

$$= \int_G dg \|Q\| = \|Q\|. \tag{2}$$

Therefore, P is minimal. Now we shall show that P is also co-minimal. Let $Q \in P(X,\ Y)$. Then ther occur inequalities hold:

$$\|Id - P\| = \left\| Id - \int_G g^{-1} Q g dg \right\|$$

$$= \left\| \int_G g^{-1}(Id - Q) g dg \right\| \leq \|Id - Q\|,$$

and thus the projection P is also co-minimal. □

2. Minimality of averaging projection. Let us denote

$$X_n = L(\mathbb{R}^n).$$

Then the space X_n, after fixing a base in $\mathbb{R}^n$, can be treated as the set of all real square matrices of dimension n. Set

$$Y_n = \left\{ A \in X_n \ : \ A = A^T \right\}.$$

DEFINITION 2.1. A projection $P_a \colon X_n \to Y_n$ given by

$$P_a(A) = \frac{A + A^T}{2},$$

for $A \in X_n$ is called the averaging projection.

DEFINITION 2.2. We will state that the norm $\| \cdot \|$ in the space $\mathbb{R}^n$ is symmetric if there exists a base $\{v_1,\ \ldots,\ v_n\}$ of $\mathbb{R}^n$ such that

$$\left\| \sum_{i=1}^n a_i v_i \right\| = \left\| \sum_{i=1}^n \varepsilon_i a_{\sigma(i)} v_i \right\|,$$

for any $a_i \in \mathbb{R}$, any permutation σ of the set $\{1,\ \ldots,\ n\}$ and $\varepsilon_i \in \{-1,\ 1\}$.

Now we show that if the operator norm of $A \in X_n$ is generated by symmetric norm in $\mathbb{R}^n$, that is

$$\|A\| = \sup_{\|x\|_0=1} \|Ax\|_0,$$

where $\|\cdot\|_0$ is a symmetric norm in the space $\mathbb{R}^n$, then projection P_a is minimal. Let us define

$$N = \{1, \ldots, n\} \times \{1, \ldots, n\},$$
$$L = \Big\{(i, i) \in N \ : \ i \in \{1, \ldots, n\}\Big\},$$
$$S = N \setminus L \quad \text{ and}$$
$$M = \{(i, j) \in N \ : \ i < j\}.$$

For $l, p, k \in N$

$$\delta_l(p) := \begin{cases} 1 & \text{if } \ p = l, \\ 0 & \text{if } \ p \neq l, \end{cases}$$
$$\delta_{lk}(p) := \delta_l(p) + \delta_k(p).$$

Then $\operatorname{codim} Y = \# M$ and

$$Y_n = \bigcap_{z \in M} \ker(f_z),$$

where for $z = (i, j) \in M$, $A = (a_{ij})_{(i, j) \in N} \in X_n$, $f_{ij}(A) = a_{ij} - a_{ji}$. By Theorem 1.1, there exists a sequence of matrices $\{B_z\}_{z \in M} \subset X_n$, such that

$$f_w(B_z) = \delta_{wz}, \quad \text{ and}$$
$$P_a(\cdot) = Id(\cdot) - \sum_{z \in M} f_z(\cdot) B_z.$$

Lemma 2.1. *Let*

$$P_a(\cdot) = Id(\cdot) - \sum_{z \in M} f_z(\cdot) B_z.$$

Then, for each $z = (i, j) \in M$ *the matrix* $B_z = (b^z_{lk})_{(l, k) \in N}$ *has the form*

$$b^z_{lk} = \begin{cases} \frac{1}{2} & \textit{if} \ \ (l, k) = (i, j), \\ -\frac{1}{2} & \textit{if} \ \ (l, k) = (j, i), \\ 0 & \textit{if} \ \ (l, k) \neq (i, j). \end{cases}$$

Proof. Let $z = (i, j) \in M$. Since for any $A \in X_n$ $P_a(A) = P_a(A^T)$,

$$0 = P_a(B_z) = P_a(B_z^T) = B_z^T + B_z.$$

Therefore, $B_z = -B_z^T$. Hence

$$b_{lk}^z + b_{kl}^z = 0,$$

for each $(l,\ k) \in N$.
Since $f_w(B_z) = \delta_{wz}$, for $w,\ z \in M$,

$$b_{lk}^z = \begin{cases} \frac{1}{2} & \text{if} \quad (l,\ k) = (i,\ j), \\ -\frac{1}{2} & \text{if} \quad (l,\ k) = (j,\ i), \\ 0 & \text{if} \quad (l,\ k) \neq (i,\ j), \end{cases}$$

which completes the proof. □

Now, for any $z = (i,\ j) \in S$, define

$$I_{ij}(\varepsilon_1,\ \varepsilon_2) = \sum_{l \in \{1,\ \ldots,\ n\} \setminus \{i,\ j\}} \delta_{ll} + \varepsilon_1 \delta_{(i,\ j)} + \varepsilon_2 \delta_{(j,\ i)},$$

$\varepsilon_1,\ \varepsilon_2 \in \{-1,\ 1\}$.
For any $(i,\ j),\ \in S$, set

$$I_{ij} = I_{ij}(1,\ 1),$$
$$I_{ij}^- = I_{ij}(-1,\ 1).$$

It is easy to see that for any $(i,\ j) \in S$, $\varepsilon_1,\ \varepsilon_2 \in \{-1,\ 1\}$

(3) $$I_{ij}(\varepsilon_1,\ \varepsilon_2) = I_{ji}(\varepsilon_2,\ \varepsilon_1),$$

(4) $$I_{ij}(1,\ 1) = I_{ij}(1,\ 1)^T,$$

(5) $$I_{ij}(-1,\ -1) = I_{ij}(-1,\ -1)^T,$$

(6) $$I_{ji}(-1,\ 1) = I_{ij}(-1,\ 1)^T \quad \text{and}$$

(7) $$I_{ij}(\varepsilon_1,\ \varepsilon_2) I_{ij}(\varepsilon_1,\ \varepsilon_2)^T = I_{ij}(\varepsilon_1,\ \varepsilon_2)^T I_{ij}(\varepsilon_1,\ \varepsilon_2) = Id.$$

It is also easy to verify that if a norm in the space $\mathbb{R}^n$ is symmetric, then every map $I_{ij}(\varepsilon_1,\ \varepsilon_2)$ is an isometry in $\mathbb{R}^n$.
Let G be the group generated by the matrices of this form. Each element I of G induces a linear isomorphism Ψ_I in the space X_n, given by the formula

$$\Psi_I(A) = I^T A I,$$

for $A \in X_n$.
If the norm in X_n is generated by a symmetric norm in $\mathbb{R}^n$, then it is easy to verify that for any $I \in G$ Ψ_I is an isometry in X_n. It is obvious that Y_n is invariant under Ψ_I, $I \in G$. By Theorem 1.2, there exists a projection Q which commutes with our group G. In order to show that the projection P_a is minimal it is enough to show that P_a is the only projection commuting with the group G. For the purpose of simplification, let $\Phi_{ij} = \Psi_{I_{ij}}$ i $\Phi_{ij}^- = \Psi_{I_{ij}^-}$. We are ready to state the main result of this paper

THEOREM 2.1. *If an operator norm in X_n is generated by a symmetric norm in $\mathbb{R}^n$, then the averaging projection P_a is minimal.*

PROOF. We will show that P_a is the only projection witch commutes with G. By Theorem 1.2, it follows that there exists a projection $Q\colon X_n \to Y_n$ commuting with G. It is sufficient to show that $Q = P_a$. By Theorem 1.1,

$$Q(\cdot) = Id(\cdot) - \sum_{z \in M} f_z(\cdot) B_z,$$

where $f_w(B_z) = \delta_{wz}$, for $w,\ z \in M$.
In order to show $Q = P_a$ it is sufficient to prove that, for every $z = (i,\ j) \in M$, $B_z = (b^z_{lk})_{(l,\ k) \in N}$ has the form

$$b^z_{lk} = \begin{cases} \frac{1}{2} & \text{if} \quad (l,\ k) = (i,\ j), \\ -\frac{1}{2} & \text{if} \quad (l,\ k) = (j,\ i), \\ 0 & \text{if} \quad (l,\ k) \neq (i,\ j). \end{cases}$$

Since the projection Q commutes with G, for any $(i,\ j) \in M$ there is:

$$Q\Phi_{ij} = \Phi_{ij} Q, \tag{8}$$
$$Q\Phi^-_{ij} = \Phi^-_{ij} Q. \tag{9}$$

By definition of Φ_{ij}, it follows that

$$\Phi_{ij}\delta_{(i,\ j)} = \delta_{(j,\ i)}, \tag{10}$$
$$\Phi^-_{ij}\delta_{(i,\ j)} = -\delta_{(j,\ i)}, \tag{11}$$

for each $(i,\ j) \in M$. Hence we obtain

$$Q\Phi_{ij}\delta_{(i,\ j)} = \Phi_{ij} Q \delta_{(i,\ j)}, \quad \text{and} \tag{12}$$
$$Q\delta_{(j,\ i)} = \Phi_{ij}\Big(Id(\delta_{(i,\ j)}) - \sum_{z \in M} f_z(\delta_{(i,\ j)}) B_z\Big). \tag{13}$$

Hence after simple re-formations

$$\delta_{(j,\ i)} + B_{(i,\ j)} = \delta_{(j,\ i)} - \Phi_{ij}(B_{(i,\ j)}), \tag{14}$$

and therefore,

$$B_{(i,\ j)} = -\Phi_{ij}(B_{(i,\ j)}). \tag{15}$$

Since each Φ_{ij} exchanges the i-th row with j-th row, and the i-th column with j-th column for any $A \in X_n$, by (15) we obtain:

$$b_{lk} = 0, \tag{16}$$

$$b_{ik} + b_{jk} = 0, \tag{17}$$

$$b_{ki} + b_{kj} = 0, \tag{18}$$

$$b_{ii} + b_{jj} = 0, \tag{19}$$

$$b_{ij} + b_{ji} = 0, \tag{20}$$

for $k,\ l \in \{1,\ \ldots,\ n\} \setminus \{i,\ j\}$.
Since $f_{(i,\ j)}(B_{(i,\ j)}) = 1$, by (20)

$$\begin{aligned} b_{ij} - b_{ji} &= 1, \\ b_{ij} + b_{ji} &= 0. \end{aligned}$$

Therefore, $b_{ij} = \frac{1}{2},\ b_{ji} = -\frac{1}{2}$.
Making use of equation (9), we get

$$Q\Phi_{ij}^{-}\delta_{(i,\ j)} = \Phi_{ij}^{-}Q\delta_{(i,\ j)}. \tag{21}$$

By (21), there is

$$-\,Q\delta_{(j,\ i)} = \Phi_{ij}^{-}\Big(Id(\delta_{(i,\ j)}) - \sum_{z\in M} f_z(\delta_{(i,\ j)})B_z\Big). \tag{22}$$

Hence

$$B_{(i,\ j)} = \Phi_{ij}^{-}(B_{(i,\ j)}). \tag{23}$$

Because the map Φ_{ij}^{-} exchanges, in any matrix from X_n, i-th row with j-th row, and the i-th column with the j-th column, as well as multiplies i-th row and the i-th column by -1, then

$$\begin{aligned} b_{ik} - b_{jk} &= 0, \\ b_{ki} - b_{kj} &= 0, \\ b_{ii} - b_{jj} &= 0, \end{aligned}$$

for $k \in \{1,\ \ldots,\ n\} \setminus \{i,\ j\}$.
From equations (16), (17), (18), (19), we obtain that $B_{(i,j)}$ has the form

$$b_{lk} = \begin{cases} \frac{1}{2} & \text{if} \quad (l,\ k) = (i,\ j), \\ -\frac{1}{2} & \text{if} \quad (l,\ k) = (j,\ i), \\ 0 & \text{if} \quad (l,\ k) \neq (i,\ j). \end{cases}$$

Hence $Q = P_a$, as required. $\square$

Now we show that the assumption about the symmetry of the norm in Theorem 2.1 is essential. For $x = (x_1,\ \dots,\ x_n) \in \mathbb{R}^n$ define

$$\|x\|_{bv} = |x_1| + \sum_{i=2}^{n} |x_i - x_{i-1}|,$$

and for $A \in X_n$ set

$$\|A\|_{bv} = \sup_{\|x\|_{bv}=1} \|Ax\|_{bv}.$$

In the sequel we need

Theorem 2.2. (see, e.g., [**14**]). *Let X be the normed space, Y be the closed subspace of finite codimension n. Then*

$$\Lambda(X,\ Y) \le \sqrt{n} + 1.$$

Theorem 2.3. *In the normed space $(X,\ \|\cdot\|_{bv})$, the averaging projection is not a minimal projection.*

Proof.
Since codim $Y_n = \frac{n(n-1)}{2}$, by Theorem 2.2,

$$\Lambda(X_n,\ Y_n) \le \sqrt{\frac{n(n-1)}{2}} + 1.$$

Hence we need to show that

$$\sqrt{\frac{n(n-1)}{2}} + 1 < \|P_a\|_{bv}.$$

Let

$$A = \sum_{i=1}^{n} \delta_{(i,\ 1)}.$$

A simple calculation shows that $\|A\|_{bv} = 1$ and $\|P_a(A)\|_{bv} \ge \frac{2n+1}{2}$. Hence

$$\|P_a\|_{bv} \ge \frac{2n+1}{2}.$$

Obviously, for any natural number n,

$$\sqrt{\frac{n(n-1)}{2}} + 1 < \frac{2n+1}{2}.$$

Consequently P_a is not minimal.
Let

$$\|A\|_1 = \sup_{\|x\|_1=1} \|Ax\|_1,$$

where $\|x\|_1 = \sum_{i=1}^{n} |x_i|\ x = (x_1,\ \dots,\ x_n) \in \mathbb{R}^n$.

THEOREM 2.4. *In the space* $(X_n,\ \|\cdot\|_1)$ *the relative projection constant is equal to* $\frac{n+1}{2}$.

PROOF. By Theorem 2.1,

$$\Lambda(X_n,\ Y_n) = \|P_a\|_1,$$

where P_a is the averaging projection.
To conclude this proof, it is enough to show that

$$\|P_a\|_1 = \frac{n+1}{2}.$$

Note that for $A = (a_{ij})_{(i,\ j)\in N} \in X_n$,

$$\|A\|_1 = \max_{j=1,\ \ldots,\ n} \Big\{ \sum_{i=1}^{n} |a_{ij}| \Big\},$$

and $\left\| \frac{A+A^T}{2} \right\| \le \frac{n+1}{2}$. But for any $i \in \{1,\ \ldots,\ n\}$ and

$$A = \sum_{j=1}^{n} \delta_{(i,\ j)},$$

$\left\| \frac{A+A^T}{2} \right\| = \frac{n+1}{2}$. This concludes the proof of Theorem 2.2. $\square$

Let for any $A \in X_n$

$$\|A\|_\infty = \sup_{\|x\|_\infty=1} \|Ax\|_\infty,$$

where $\|x\|_\infty = \max_{i\in\{1,\ \ldots,\ n\}} |x_i|$, for $x = (x_1,\ \ldots,\ x_n) \in \mathbb{R}^n$.

THEOREM 2.5. *In the space* $(X_n,\ \|\cdot\|_\infty)$ *the relative projection constant is equal to* $\frac{n+1}{2}$.

PROOF. By Theorem 2.1,

$$\Lambda(X_n,\ Y_n) = \|P_a\|_\infty,$$

Note that for any $A = (a_{ij})_{(i,\ j)\in N} \in X_n$,

$$\|A\|_\infty = \max_{i=1,\ \ldots,\ n} \Big\{ \sum_{j=1}^{n} |a_{ij}| \Big\},$$

and $\left\| \frac{A+A^T}{2} \right\| \le \frac{n+1}{2}$. Let for $j \in \{1,\ \ldots,\ n\}$

$$A = \sum_{i=1}^{n} \delta_{(i,\ j)},$$

then $\|A\|_\infty = 1$ and $\left\| \frac{A+A^T}{2} \right\| = \frac{n+1}{2}$. This concludes the proof of Theorem 2.3. $\square$

3. The unique minimality of averaging projection in the space $(X_n,\ \|\cdot\|_2)$. For $A \in X_n$ let

$$\|A\|_2 = \sup_{\|x\|_2=1} \|Ax\|_2,$$

where $\|x\|_2 = \Big(\sum_{i=1}^{n} |x_i|^2\Big)^{\frac{1}{2}}$, for $x = (x_1,\ \ldots,\ x_n) \in \mathbb{R}^n$. It is well-known that for $A \in X_n$

$$\|A\|_2 = \sqrt{r(A^T A)},$$

where $r(A)$ is the spectral radius of A. In particular

$$\left\| A^T \right\|_2 = \|A\|_2.$$

Hence $\|P_a\|_2 = 1$.
In this section we will show that P_a is the unique norm-one projection.
Define

$$A_{ij}(\theta) := \sum_{z\in L\setminus\{(i,\,i),\,(j,\,j)\}} \delta_z + \sin(\theta)(\delta_{(i,\,i)} + \delta_{(j,\,j)}) + \cos(\theta)(\delta_{(i,\,j)} - \delta_{(j,\,i)}),$$

for fixed $(i,\ j) \in M,\ \theta \in \mathbb{R}$.
It is easy to show that $A_{ij}(\theta)$ is orthogonal, that is

$$A_{ij}(\theta)^T A_{ij}(\theta) = A_{ij}(\theta) A_{ij}(\theta)^T = Id.$$

Hence $\|A_{ij}(\theta)\|_2 = 1$, for each $(i,\ j) \in M,\ \theta \in \mathbb{R}$.

Theorem 3.1. *In the normed space $(X_n,\ \|\cdot\|_2)$, P_a is the unique norm-one projection.*

Proof. We will show that $Q \in P(X_n,\ Y_n)$ is a norm-one projection, then $Q = P_a$. Applying Theorem 1.1, we obtain:

$$Q(\cdot) = Id(\cdot) - \sum_{z\in M} f_z(\cdot) B_z,$$

where $f_w(B_z) = \delta_{wz}$, for $w,\ z \in M$.
In order to complete the proof of Theorem 3.1, by Lemma 2.1, it is sufficient to show that any matrix $B_z = (b^z_{lk})_{(l,\,k)\in N}$ is given by

$$b^z_{lk} = \begin{cases} \ \ \frac{1}{2} & \text{if} \quad (l,\ k) = (i,\ j), \\ -\frac{1}{2} & \text{if} \quad (l,\ k) = (j,\ i), \\ \ \ 0 & \text{if} \quad (l,\ k) \neq (i,\ j). \end{cases}$$

Let $(i,\ j) \in M$. For any $\theta \in \mathbb{R}$

$$Q(A_{ij}(\theta)) = Id(A_{ij}(\theta)) - \sum_{z\in M} f_z(A_{ij}(\theta))B_z = \tag{24}$$

$$= A_{ij}(\theta) - 2\cos(\theta)B_{(i,\ j)}. \tag{25}$$

Since $\|Q\|_2 = 1$,

$$\|A_{ij}(\theta) - 2\cos(\theta)B_{(i,\ j)}\|_2 \leq 1, \tag{26}$$

for every $\theta \in \mathbb{R}$.
Fix $z = (l,\ l) \in L \setminus \{(i,\ i),\ (j,\ j)\}$. We will show that $b^z_{ll} = 0$.
By (26), we obtain

$$\|(A_{ij}(\theta) - 2\cos(\theta)B_{(i,\ j)})e_l\|_2 \leq 1, \tag{27}$$

for any $\theta \in \mathbb{R}$. It implies that

$$|1 - 2\cos(\theta)b^z_{ll}| \leq 1, \tag{28}$$

for any $\theta \in \mathbb{R}$. Consequently, $b^z_{ll} = 0$ and

$$b^z_{lk} = 0, \tag{29}$$

$$b^z_{kl} = 0, \tag{30}$$

for any $k \in \{1,\ \ldots, n\}$.
Now we will show that $b^z_{ii} = b^z_{jj} = 0$.
By (26),

$$\|(A_{ij}(\theta) - 2\cos(\theta)B_{(i,\ j)})\sin(\theta)e_i\|_2 \leq 1,$$

and consequently,

$$-1 \leq \sin^2\theta - 2\sin\theta\cos\theta b^z_{ii} \leq 1, \tag{31}$$

for any $\theta \in \mathbb{R}$.
From (31) one can easily get

$$\frac{\sin^2\theta - 1}{\sin 2\theta} \leq b^z_{ii} \leq \frac{\sin^2\theta + 1}{\sin 2\theta} \text{ dla } \theta \in \Big(0,\ \frac{\pi}{2}\Big) \quad \text{and} \tag{32}$$

$$\frac{\sin^2\theta + 1}{\sin 2\theta} \leq b^z_{ii} \leq \frac{\sin^2\theta - 1}{\sin 2\theta} \text{ dla } \theta \in \Big(-\frac{\pi}{2},\ 0\Big). \tag{33}$$

Hence

$$\lim_{\theta\to\frac{\pi}{2}^-} \frac{\sin^2\theta - 1}{\sin 2\theta} \leq b^z_{ii} \quad \text{and} \tag{34}$$

$$b^z_{ii} \leq \lim_{\theta\to-\frac{\pi}{2}^+} \frac{\sin^2\theta - 1}{\sin 2\theta}, \tag{35}$$

therefore, $b^z_{ii} = 0$. Analogously we obtain $b^z_{jj} = 0$.
To end the proof, it is necessary to show that $b^z_{ij} = \frac{1}{2}$, $b^z_{ji} = -\frac{1}{2}$.
Set $a := 1 - 2b^z_{ij}$. It is easy to check that the characteristic polynomial φ of the matrix $A_{ij}(\theta) - 2\cos(\theta)B_{(i,\, j)}$ is

$$\varphi(\lambda) = (\lambda - 1)^{n-2}((\lambda - \sin\theta)^2 - \cos^2\theta a^2).$$

Since $A_{ij}(\theta) - 2\cos(\theta)B_{(i,\, j)}$ is symmetric,

$$\|A_{ij}(\theta) - 2\cos(\theta)B_{(i,\, j)}\|_2 = r(A_{ij}(\theta) - 2\cos(\theta)B_{(i,\, j)}).$$

Straightforward calculations show that the zeros of polynomial φ are 1, $\sin\theta - |\cos\theta||a|$ and $\sin\theta + |\cos\theta||a|$.
Since $\|A_{ij}(\theta)\| \leq 1$, for each $\theta \in \mathbb{R}$ there is

$$\sin\theta + |\cos\theta||a| \leq 1. \tag{36}$$

Hence

$$|a| \leq \frac{1 - \sin\theta}{\cos\theta},$$

for $\theta \in \Big[0,\ \frac{\pi}{2}\Big)$, which gives

$$0 \leq |a| \leq \lim_{\theta \to \frac{\pi}{2}-} \frac{1 - \sin\theta}{\cos\theta} = 0.$$

Consequently, $b^z_{ij} = \frac{1}{2}$. The proof is complete. □

4. The unique minimality of averaging projection in l_p-norm. For any $1 \leq p < \infty$, $A = (a_{ij})_{(i,\, j) \in N} \in X_n$ define

$$\|A\|_p = \Big(\sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|^p\Big)^{\frac{1}{p}},$$

and

$$\|A\|_\infty = \max_{(i,\, j) \in N} |a_{ij}|.$$

It is easy to see that

$$\|P_a\|_p = \sup\Big\{\|P_a(A)\|_p\ :\ \|A\|_p = 1,\Big\} = 1$$

for any $1 \leq p \leq \infty$.
Hence P_a is a minimal projection. We will show that P_a is the only minimal projection if and only if $1 \leq p < \infty$. To do this, recall that a normed space $(X,\ \|\cdot\|)$ is called smooth if for any $x \in X$, $\|x\| = 1$ there exists exactly one functional f_x, $\|f_x\| = 1$ such that $f_x(x) = 1$.

THEOREM 4.1. (see, e.g., [**7**]). *Let X be a smooth Banach space and Y be linear subspace of X. Then if there exists a norm-one projection from X onto Y, then this projection is the unique minimal projection.*

Since for $1 < p < \infty$, space $(X_n, \ \|\cdot\|_p)$ is a smooth Banach space and $\|P_a\| = 1$, then P_a is the only norm-one projection. Now we consider the two remaining cases, $p = 1$ and $p = \infty$.

THEOREM 4.2. *In the space $(X_n, \ \|\cdot\|_1)$, P_a is the unique norm-one projection.*

PROOF. Let $Q\colon X_n \to Y_n$ be a projection and $\|Q\|_1 = 1$. We will show that $Q = P_a$. By Theorem 1.1,

$$Q(\cdot) = Id(\cdot) - \sum_{z\in M} f_z(\cdot)B_z,$$

where $f_w(B_z) = \delta_{wz}$, for $w, \ z \in M$.
Since $\|Q\|_1 = 1$, then

$$\|Q(\delta_{(i,\,j)})\|_1 \le 1, \tag{37}$$

$$\|Q(\delta_{(j,\,i)})\|_1 \le 1, \tag{38}$$

for fixed $(i, \ j) \in M$. This leads to

$$\sum_{(l,\,k)\in N\setminus\{(i,\,j),\,(j,\,i)\}} |b^z_{lk}| + |1 - b^z_{ij}| + |b^z_{ji}| \le 1, \quad \text{and} \tag{39}$$

$$\sum_{(l,\,k)\in N\setminus\{(i,\,j),\,(j,\,i)\}} |b^z_{lk}| + |b^z_{ij}| + |1 + b^z_{ji}| \le 1. \tag{40}$$

Since $b^z_{ij} - b^z_{ji} = 1$,

$$2|b^z_{ji}| = |1 - b^z_{ij}| + |b^z_{ji}| \le 1, \tag{41}$$

$$2|b^z_{ij}| = |1 + b^z_{ji}| + |b^z_{ij}| \le 1. \tag{42}$$

Hence

$$|b^z_{ji}| \le \frac{1}{2}, \tag{43}$$

$$|b^z_{ij}| \le \frac{1}{2}. \tag{44}$$

Therefore, there must be $b^z_{ij} = \frac{1}{2}, \ b^z_{ji} = -\frac{1}{2}$. $\square$

Theorem 4.3. *In the space* $(X_n, \ \|\cdot\|_\infty)$ P_a, *is not the only norm-one projection.*

Proof. For $A = (a_{ij})_{(i,\,j)\in N} \in X_n$, define a projection $Q\colon X_n \to Y_n$, by $Q(A) = (\overline{a}_{ij})_{(i,\,j)\in N}$, where

$$\overline{a}_{ij} = \begin{cases} a_{ij} & \text{if } (i,\ j) \in M, \\ a_{ji} & \text{if } (i,\ j) \in N \setminus M. \end{cases}$$

It is easy to show that the operator Q is a projection of X_n onto Y_n, $\|Q\|_\infty$=1, and $Q \neq P_a$. The proof is complete. □

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AGH University of Science and Technology
Faculty of Applied Mathematics
al. Mickiewicza 30
30-059 Kraków, Poland
e-mail: dmielcza@wms.mat.agh.edu.pl