Jagiellonian University
Faculty of Mathematics and Computer Science
Institute of Mathematics

Anna Tatarczak

Properties of orthogonal polynomials and typically real functions related to generalized Koebe function

PhD dissertation

Supervisor
dr hab. Stanisława Kanas, prof. UR

Krakow 2014

Abstract

As the title suggests, this thesis is concerned with the connection between orthogonal polynomials and typically real function, both strictly related to the Koebe function. Orthogonal polynomials appear in many areas of mathematics and have been the subject of interest of many mathematicians. In recent years this interest has often arisen from outside the subject of orthogonal polynomials, after their connection with the class of analytic functions. Our purpose is to investigate mathematical properties of some generalization of Koebe function, which is associated to the generalized Chebyshev polynomials of the second kind, as well as to the class of typically real functions that is defined by extended Koebe function. We approach the problem from both perspectives. By looking for a generating function of orthogonal polynomials and by using orthogonal polynomials as a tool to describe the properties of a class of functions.

First, the aim of thesis is a comprehensive study of generalized Koebe function. We present the motivation of research undertaken with a background leading to the proposed generalization. Next, we find the geometric description of the image of the unit disc under that function, which in conclusion directs to the solution of extremal problems in the related class.

Next, our consideration is focused on determining properties of generalized Chebyshev polynomials of the first and second kind, sparking interest in constructing a theory similar to the classical one. This studies highlight some important results and connections between this two types.

Combining generalized Koebe function and generalized Chebyshev polynomials we define and characterize the class of generalized typically real functions. Specifically, we present interesting geometric interpretation of the class, and solve the generalized Zalcman conjecture. Additionally, we solve several extremal problems; problem of modulus of function and its derivative, the coefficients problems, etc. We underline the fact that obtained results are sharp.

Finally, we present Generalized Meixner - Pollaczek polynomials with its special cases leading to a symmetric and quasi-symmetric case. In this context we find the Fisher information which was first introduced in the framework of statistical estimation theory, where it plays a key role.

Keywords Geometric function theory, Univalent functions, Typically real functions, Koebe function, Zalcman conjecture, Orthogonal polynomials, Meixner - Pollaczek polynomials, Chebyshev polynomials.

Mathematics Subject Classification 30C10, 30C45, 33C47, 33C45, 33D45.

Contents

Introduction

Complex analysis is one of the classical branches in mathematics with a history dating back to 19th century which interacts with many other part of mathematics, having many physical applications and used throughout analytic number theory. In recent years, it has become very popular through a new results from complex dynamics and fractals formed by iterating holomorphic functions. It is an essential tool in applications including algebra (theory of fields and equations), algebraic geometry and complex manifolds, Lie gropus, dynamics (iterated rational maps), number theory and automorphic forms, theory of Riemann surface, several complex variable, harmonic functions, elliptic equations and distributions.

Leading mathematicians associated with complex analysis include Euler, Gauss, Riemann, Cauchy, Weierstrass, Koebe, Bieberbach and many more in the 20th century. The latter two would have a significant impact on the development of complex analysis of the twentieth century; Koebe initiated in 1907 the univalent functions study, while Bieberbach presented in 1916 what would soon become a famous conjecture, described by Browder as "one of the most celebrated conjecture in classical analysis, one that has stood as a challenge to mathematicians for a very long time". The Bieberbach conjecture was finally proved in 1984 by L. de Branges; who in fact proved the Milin conjecture, which established the Robertson conjecture, which in turn established the Bieberbach conjecture. The conjecture is at heart an assertion about extremality of the Koebe function; it states that the Taylor coefficients of any univalent functions do not exceed the coefficients of the Koebe function. Although, bounds of coefficients were obtained in the meantime much more easily for some subclasses of univalent functions than for the full class.

One of the classes for which the Bieberbach conjecture holds true is the class of starlike or typically real functions. In both classes the Koebe function appears as the extremal. It resulted a significant increase of the interest in this function itself and its possible generalizations. As one of the earliest extension of Koebe function we mention Eguchi and Owa [44], and Okuyama [110], described in designation of a radius of starlikeness and convexity of order α, and next involved in the problem of α-spirallikeness of some order. Another form can be found in paper by Noor [108]; in the same form it appeared in Goodman work [62]. The complex parameter was introduced in the generalized Koebe function by Campbell and Pfaltzgraff in [27]. For other generalizations the reader is referred to Siregar [134], Yamashita [157] (nonunivalent case; in considering conformality and semiconformality at a boundary point), Starkov [138] (as the maximizing a problem of coefficients in the universal linearly invariant family) and Pommerenke [114]. Generalization in the other direction proposed Gasper in [58] in his study of some extension of the Löwner theory and de Brange's inequalities. In the dissertation we propose the generalization of Gasper type, but extended to the two parametric case. Due to this context a class of generalized typically real functions is defined.

The origins of the studies of typically real function date back to the years thirties of the twentieth century and are associated with the name Rogosinski [126] and Robertson [121]. The idea of the definition of such functions is connected with the name of the class, so that $\mathcal{T}_{\mathbb{R}}$ is a class of functions f, normalized by $f(0) = f'(0) - 1 = 0$, and analytic in the unit disk $\mathbb{D}$ that map the unit disk into a complex plane and take the real values for real arguments and nonreal values elsewhere. This beautiful interpretation allows to formulate analytic condition $\mathrm{Im}\,\{f(z)\}\,\mathrm{Im}\,\{z\} \geq 0$ in $\mathbb{D}$. The first results for $\mathcal{T}_{\mathbb{R}}$ are due to Rogosinski and Robertson, likewise. They provide, among other, integral Herglotz representation that gave rise for obtaining further results.

On the other hand the generalized Koebe function is associated with the generating function of Meixner - Pollaczek (MP), and Chebyshev polynomials. The Meixner - Pollaczek polynomials were discovered by Meixner [99] and later studied by Pollaczek. The major properties were discussed by Chihara [34], Koekoek and Swarttouw [83]. Chen and Ismail in 1997 [29] investigated the asymptotic behavior of the extreme zeros of the MP polynomials, as well as the asymptotic distribution of zeros in a symmetric case. Li and Wong in 2001 [92] obtained an asymptotic expansion of the MP polynomials in terms of the parabolic cylinder functions. They also obtained the improved asymptotic behaviors of the zeros. Krasovsky in 2003 [86] investigated the asymptotic distribution of zeros of MP polynomials on the approach of difference equations. Dominici [41] and Friden [53], described huge impact of Fisher information of polynomials to the science. The problem of generalization of the Meixner - Pollaczek polynomials for the complex case arose as a result of study Araaya work [10] and a knowledge about the generalized Koebe function.

The Chebyshev polynomials (of the first and second kind), named after Chebyshev [31], form a sequence of orthogonal polynomials which are defined recursively and are related to de Moivre's formula. Those are polynomials with the largest possible leading coefficient, and with absolute value bounded by 1. Chebyshev polynomials are variant of polynomials especially suited for approximating other functions. They are widely used in many areas of numerical analysis; uniform approximation, least-squares approximation, numerical solution of ordinary and partial differential equations (the so-called spectral or pseudospectral methods), and so on. Therefore several its extensions occur. Akhiezer [3, 4], and Akhiezer and Tomčuk [5] introduced orthogonal polynomials of two intervals which generalize the Chebyshev polynomials, and require the use of elliptic functions. In the case of more than two intervals, Tomčuk [152], investigated their Bernstein-Szegö asymptotics, with the theory of hyperelliptic integrals, and found expressions in terms of a certain Abelian integral of the third kind. In 1984 Al-Salam, Allaway, and Askey [8] introduced sieved ultraspherical polynomials which are orthogonal with respect to an absolutely continuous measure, but the weight function vanishes at k + 1 points. Ismail [70] observed that the vanishing of the weight function means that the polynomials are orthogonal on several adjacent intervals. In particular his polynomials include analogues of the Chebyshev polynomials of the first and second kind. Peherstorfer in [111] was carried out Chebyshev type polynomials as extremal polynomials that are orthogonal on several intervals. The Chebyshev type polynomials satisfy similar extremal properties to the classical Chebyshev polynomials on $[-1, 1]$. The extremal polynomials also have the property that they are orthogonal with respect to some weight function. For other generalization the reader is referred to [30]. Some of generalized Chebyshev polynomials are associated with generalized Koebe function, as was observed in [148]. Due to its properties can be easily demonstrate the specific properties of the class $\mathcal{T}_{\mathbb{R}}$. These polynomials are interesting themselves and are the object of interest from the possibilities of their use. A similar relationship with generalized Koebe function combines Meixner - Pollaczek polynomials.

The purpose of this thesis is to combine all of these relations in the one coherent whole. We present the fundamental interrelationships, dependencies, similarities and differences between Meixner - Pollaczek and Chebyshev polynomials and generalized typically real functions, indicating new methods of investigation. A significant topic in the theory of univalent functions is to find the geometric interpretation of several analytic problems. The importance of geometry is that it establishes a correspondence between geometric and analytic properties that makes it possible to reformulate problems in geometry as equivalent problems in analysis, and vice versa; the methods of either subject can then be used to solve problems in the other. In the dissertation a geometric interpretation of the class of typically real functions is provided.

The thesis consists of five chapters. Chapter 1 is structured in two sections in which are presented some fundamental definitions and results that constitutes the backgrounds

for the remaining chapters. The part of complex analysis (Section 1.1) includes with the presentation of some notions and fundamental results from the geometric theory of functions of one complex variable. First, we present the basic notions and elementary results about the class of normalized and univalent functions on the unit disc, then we consider various subclasses of univalent functions, including not only the starlike and convex functions, but also, among others, the class of positive real part and, especially the class of typically real functions. In Section 1.2 general problems concerning orthogonal polynomials are presented with their special cases like Meixner - Pollaczek and Chebyshev polynomials. We present the fundamental knowledge of that theory and a brief account of the results.
The Chapter 2 focuses on some original results on generalization of Koebe function. First in Section 2.1 we present a motivation of the research in this direction. Next, we find the radius of starlikeness of order α and radius of convexity. In Sections 2.2 we discuss in detail the generalized Koebe function. In particular geometric image of the unit disk is described, with the analogue of 'Koebe 1/4 theorem'. We describe the extremal values of the modulus, real and imaginary part of the image of unit disk. Helpfully, we present the graphs of the image of the unit disk under considered map.
Chapter 3 deals with the polynomials connected with extension of the class of typically real functions. Extending the analytic tool in these directions is not only beneficial to the topics considered in the thesis but can also contribute to other problems mentioned above, where the class of typically real functions and orthogonal polynomials may be of an interest.
Our consideration in Chapter 3 focuses on determining properties of special sequences of generalized Chebyshev type polynomials of the first and second kind, sparking interest in constructing a theory similar to the classical one. The generalization of Chebyshev polynomials of first kind occurs first in [79], where it was proposed to study the polynomials with one parameter. Another example is work of Freund [69]. In the thesis author is concerned with a classical inequality due to Bernstein which estimates the norm of polynomials on any given ellipse in terms of their norm on any smaller ellipse with the same foci. These Bernstein type inequalities are closely connected with certain constrained Chebyshev approximation problems on ellipses. The authors introduce an analogy to the Chebyshev polynomials. This chapter highlights some important results and connections between this two types.
Combining generalized Koebe function and generalized Chebyshev polynomials we define and characterize in Chapter 4 the class of generalized typically real functions. Specifically, we present interesting geometric interpretation of the class, and solve the generalized Zalcman conjecture. Additionally, we solve several extremal problems;

problem of modulus of function and its derivative, the coefficients problems, etc. We underline the fact that obtained results are sharp.
Finally, the last chapter is dedicated to the study of generalized Meixner - Pollaczek (GMP) polynomials; the motivation to the introduction and research of this new classes of orthogonal polynomials was presented in Section 2.1. We prove the orthogonality of the GMP in Section 5.1.2, we find this property very useful in our investigation.
In Section 5.2 we investigate the Symmetric Generalized Meixner - Pollaczek polynomials (SGMP); specially its extension on the complex parameters. We show that SGMP polynomials have many important and useful properties: closed form expression, explicit formula for the exponential generating function, recurrence relation, integral representation and asymptotic expansion, and satisfy differential equation. We summarize the asymptotic expansions and the limit relations as well as orthogonality of the polynomials in some strip.
Finally, Section 5.3 is devoted to quasi-symmetric case; we find the Fisher information which is interesting itself but also provides the interesting applications in information-theoretic measures of the probability distributions associated with them.
The bibliography contains 116 titles, 7 signed by the author.

Acknowledgments. I would like to express my warm and sincere thanks to my thesis supervisor Professor S. Kanas for her scientific and personal support, her open mind and patience, that she always believes in my abilities and take care off all aspect of my scientific development.

Chapter 1

Preliminary results

1.1 Complex analysis

The theory of univalent function was initiated by Koebe in 1907. Beginning with the classical Riemann Mapping Theorem, there is a lot of existence theorems for canonical conformal mappings. On the other hand there is an extensive theory of qualitative properties of conformal mappings, concerning mainly prior estimates, including the Bieberbach conjecture with the proof of the de Branges. Here a starting point was the classical Schwarz Lemma, and the Koebe's distortion theorem.

In this subsection, we begin with the presentation of some notions and fundamental results from geometric function theory of one complex variable. We present first the basic properties of the class $\mathcal{S}$ of normalized univalent functions on the unit disc, then we consider various subclasses of univalent functions, such as the well-known starlike and convex functions, but also typically real functions.

1.1.1 Notions and elementary results from the theory of univalent functions

Let $\mathbb{C}$ be the open complex plane, $\mathbb{R}$ the set of real numbers, and $\mathbb{N}$ the set of all integers. Consider the following notations which will be used further

- $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$,
- $\mathbb{D}(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$,
- $\partial\mathbb{D}(z_0, r) = \{z \in \mathbb{C} : |z| = r\}$.

Definition 1.1 [43, Duren, p. 2] *A function f of a complex variable z is analytic (holomorphic, regular) in a domain Ω if it has a derivative at each point in the domain*

Ω, and f is analytic at a point z_0 if there exists a neighborhood $\mathbb{D}(z_0, r)$, with $r > 0$, such that f is analytic in $\mathbb{D}(z_0, r)$.

Let $\mathcal{H}(\mathbb{D})$ be the set of holomorphic functions in $\mathbb{D}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$ denote

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(\mathbb{D}) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...\},$$
$$\mathcal{A}_n = \{f \in \mathcal{H}(\mathbb{D}) : f(z) = z + a_{n+1} z^{n+1} + ...\},\ (\mathcal{A}_1 = \mathcal{A}).$$

Thus each $f \in \mathcal{A}$ has a Taylor series representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!}.$$

Definition 1.2 [43, Duren, p. 5] *A single valued function f is said to be univalent (schlicht) in a domain $D \subset \mathbb{C}$ if it never takes the same value twice.*

If f is univalent and has the form (1.1) then it is called normalized univalent function. The class of all normalized univalent functions that are analytic in the disc $\mathbb{D}$ is denoted by $\mathcal{S}$, that is

$$\mathcal{S} = \{f \in \mathcal{A} : \ f \text{ univalent in } \mathbb{D}\}.$$

For analytic functions f, the condition $f'(z) \neq 0$ is equivalent to a local univalence at z_0. It is obvious that any univalent function f is locally univalent, so the condition $f'(z) \neq 0$ is necessary for univalence of f on $\mathbb{D}$, but not sufficient. For example, the function $f(z) = \exp z$ is not univalent in $\mathbb{C}$, being a periodic function, although $f'(z) = \exp z \neq 0$, for any $z \in \mathbb{C}$.
Since a locally univalence function preserves angles and orientation, an univalent function is also referred to as *a conformal mapping* or *a conformal equivalence.*

Lemma 1.1 [43, Schwarz Lemma, Duren, p. 197] *Let f be analytic in $\mathbb{D}$, with $f(0) = 0$ and $|f(z)| < 1$ in $\mathbb{D}$. Then $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ in $\mathbb{D}$. Strict inequality for some non-zero z in both estimates occurs iff f is a rotation of the disk $f(z) = e^{i\theta} z$ ($\theta \in \mathbb{R}$).*

Definition 1.3 [114, Pommerenke, p. 33] *If f and g are two functions analytic in $\mathbb{D}$, we say that f is subordinate to g, written as*

$$f \prec g \quad \text{or} \quad f(z) \prec g(z),$$

if there exists a Schwarz function ω (i.e. analytic in $\mathbb{D}$, with $\omega(0) = 0$ and $|\omega(z)| < 1$, for all $z \in \mathbb{D}$) such that

$$f(z) = g(\omega(z)) \quad (z \in \mathbb{D}).$$

Furthermore, if g is univalent in $\mathbb{D}$, then we have the following equivalence:

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{D}) \subset g(\mathbb{D}).$$

As early as 1851, Riemann enunciated the basic theorem that every simply connected domain can be mapped conformally onto the unit disk.

Theorem 1.2 [43, Riemann mapping theorem, Duren, p. 11] *Every simple connected domain E, which is not equal to $\mathbb{C}$, can be mapped conformally onto the unit disc $\mathbb{D}$.*

The leading example of a function of class $\mathcal{S}$ is the Koebe function

$$k(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + ... = \sum_{n=1}^{\infty} nz^n.$$

The Koebe function maps the disk $\mathbb{D}$, one to one and conformally onto the entire plane minus the slit $(-\infty, -\frac{1}{4}]$ on the real axis. This is best seen by writing

$$k(z) = \frac{1}{4}\left(\frac{1+z}{1-z}\right)^2 - \frac{1}{4},$$

and observing that the function

$$L(z) = w = \frac{1+z}{1-z}$$

maps $\mathbb{D}$ conformally onto the right half plane $\mathrm{Re}(w) > 0$.

Other example of functions in class $\mathcal{S}$ are:

$$f(z) = z, \quad f(z) = \frac{z}{1-z}, \quad f(z) = \frac{1}{2}\log\frac{1+z}{1-z}, \quad f(z) = z - \frac{1}{2}z^2.$$

The sum of two functions in $\mathcal{S}$ need to be univalent. However, the class $\mathcal{S}$ is preserved under some elementary transformations such as

Conjugation: If $f \in \mathcal{S}$ and $g(z) = \overline{f(\overline{z})} = z + \overline{a_2}z^2 + ...$, then $g \in \mathcal{S}$.

Rotation: If $f \in \mathcal{S}$ and $g(z) = e^{-i\theta}f(e^{i\theta}z)$ then $g \in \mathcal{S}$.

Dilation: If $f \in \mathcal{S}$ and $g(z) = r^{-1}f(rz)$, where $0 < r < 1$, then $g \in \mathcal{S}$.

Disk automorphism: If $f \in \mathcal{S}$, α is such that $|\alpha| < 1$, and

$$g(z) = \frac{f\left(\frac{z+\alpha}{1+\overline{\alpha}z}\right) - f(\alpha)}{(1-|\alpha|^2)f'(\alpha)}, \quad \text{then } g \in \mathcal{S}.$$

Each function $f \in \mathcal{S}$ is an open mapping with $f(0) = 0$, so its range contains a disk centered at the origin. In 1907, Koebe [82] discovered that the ranges of all functions in $\mathcal{S}$ contain a common disk $|w| < \rho$, where ρ is an absolute constant. The Koebe function shows that $\rho \leq \frac{1}{4}$, and Bieberbach later [15] established Koebe's conjecture that ρ may be taken to be $\frac{1}{4}$.

Theorem 1.3 [43, Koebe One-Quarter Theorem, Duren, p. 31] *The range of every function of class $\mathcal{S}$ contains the disk $\{w : |w| < \frac{1}{4}\}$.*

The idea of growth of analytic function f refers to the size of the image domain, that is $|f(z)|$. The term, distortion, arises from the geometric interpretation of $|f'(z)|$ as the infinitesimal magnification factor of the arclength under the mapping f. These concepts tell much about the boundedness of these functions and their derivatives. For the class $\mathcal{S}$ we have the following.

Theorem 1.4 [114, Koebe distortion theorem, Pommerenke, p. 21] *If $f \in \mathcal{S}$ then for $z \in \mathbb{D}$*

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2},$$

and

$$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3}.$$

In each case equality holds if and only if f is a suitable rotation of the Koebe function.

It is unknown how to completely characterize the coefficients of functions in the class $\mathcal{S}$. On the other, there are sharp inequalities which such coefficients satisfy. Historically, there interrelated problems (Bieberbach conjecture, Robertson conjecture and Milin conjecture) have played a central role in this subject. In 1916 Bieberbach [15] formulated the following conjecture:

Theorem 1.5 [23, Bieberbach Conjecture - de Branges Theorem] *The coefficients of each functions $f \in \mathcal{S}$ satisfy $|a_n| \leq n$, for $n = 2, 3, \ldots$. The inequality is sharp, the equality holds for Koebe function or one of its rotations.*

The above conjecture remained unsolved until 1985, when it was proved by de Branges [23], by means of the method of modified Löwner chains.

1.1.2 Convex and starlike functions

If we know that a univalent function maps $\mathbb{D}$ onto a domain with some nice property, then we have the means for a more penetrating study of the function. A convex domain is an outstanding example of a domain with nice properties. Another example is a domain that is starlike with respect to some point.

Definition 1.4 [65, Goodman, p. 107] *A set $D \subset \mathbb{C}$ is called convex if for every pair of points w_1 and w_2 in the interior of D, the line segment joining w_1 and w_2 is also in the interior of D. If a function f maps D onto a convex domain, then f is called a convex functions. The class of all convex, normalized and univalent functions we denote by $\mathcal{S}^{\mathcal{C}}$.*

We give next the well known analytical characterization for convex functions.

Theorem 1.6 [114, Pommerenke, p. 47] *Let the function f be analytic in the unit disk, $f(0) = 0, f'(0) \neq 0$. Then function $f \in \mathcal{S}^{\mathcal{C}}$ in $\mathbb{D}$ if and only if*

$$\mathrm{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > 0 \quad (z \in \mathbb{D}).$$

A generalization of a notion of the convexity is a convexity of order α, below.

Definition 1.5 [65, Goodman, p. 133] *Let $0 \leq \alpha \leq 1$. We denote*

$$\mathcal{S}^{\mathcal{C}}(\alpha) = \{f \in \mathcal{S} :\ \mathrm{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > \alpha \quad (z \in \mathbb{D})\},$$

the class convex functions of order α.

The next theorem gives the bounds for the coefficients for functions in $\mathcal{S}^{\mathcal{C}}$.

Theorem 1.7 [65, Goodman, p. 117] *If $f \in \mathcal{S}^{\mathcal{C}}$, where f is of the form* (1.1), *then $|a_n| \leq 1$ for any $n \in \mathbb{N},\ n \geq 2$. Equality holds if and only if $f(z) = \dfrac{z}{1 + e^{i\theta}z}$ $(z \in \mathbb{D},\ \theta \in \mathbb{R})$.*

For convex functions from the class $\mathcal{S}^{\mathcal{C}}$ we have the following growth and distortion theorem.

Theorem 1.8 [65, Goodman, p. 118] *If $z \in \mathbb{D}$ with $|z| = r$, and f belongs to $\mathcal{S}^{\mathcal{C}}$, then the following sharp estimates hold*

$$\frac{r}{1+r} \leq |f(z)| \leq \frac{r}{1-r},$$
$$\frac{1}{(1+r)^2} \leq |f'(z)| \leq \frac{1}{(1-r)^2}.$$

Equality holds at any given point other than 0 for the functions $f(z) = \dfrac{z}{1 - \lambda z}, z \in \mathbb{D}$ for a suitable choice of $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.

Definition 1.6 [43, Duren, p. 40] *A set $D \subset \mathbb{C}$ is said to be starlike with respect to w_0, which is an interior point of D, if each ray with initial point w_0 intersects the interior of D in a set that is either a line segment or a ray. If a function f maps D onto a domain that is starlike with respect to w_0, then we say that f is starlike with respect to w_0. In the special case $w_0 = 0$, we say that f is a starlike function. The class of all starlike functions we denote by $\mathcal{S}^*$.*

The class of starlike function with respect to origin was first studied by Alexander [7] and Nevanlinna [107]. The following theorem gives an analytic characterization of starlike functions.

Theorem 1.9 [65, Goodman, p. 111] *Let the function $f \in \mathcal{S}$. The function f is starlike in $\mathbb{D}$ if and only if*

$$Re\frac{zf'(z)}{f(z)} > 0 \quad (z \in \mathbb{D}).$$

From Theorem 1.9 we have $\mathcal{S}^* \subset \mathcal{S}$. However, the inequality $Re\dfrac{zf'(z)}{f(z)} > 0 \ \ (z \in \mathbb{D})$, is not sufficient for univalence of the function f, as it can easily be seen from the example $f(z) = z^2 \ (z \in \mathbb{D})$.
Because $\mathcal{S}^* \subset \mathcal{S}$ and the Koebe function and its rotations belongs to the class $\mathcal{S}^*$; it follows that the distortion and growth results for the full class $\mathcal{S}$ also hold and are sharp for functions in $\mathcal{S}^*$.

Theorem 1.10 [65, Goodman, p. 117] *If $z \in \mathbb{D}$ with $|z| = r$, and f belongs to $\mathcal{S}^*$, then the following sharp estimates hold*

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2},$$
$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}.$$

Equality occurs in each of these estimates if and only if f is the Koebe function or one of its rotation.

The next theorem, which gives the coefficient bounds for functions in $\mathcal{S}^*$ was proved, independently, by Löwner [94] in 1917 and Nevanlinna [107] in 1920.

Theorem 1.11 [114, Pommerenke, p. 46] *If $f \in \mathcal{S}^*$, where f is of the form* (1.1), *then $|a_n| \leq n$ for any $n = 2, 3, \ldots$. Equality holds if and only if $f(z) = \dfrac{z}{(1+e^{i\theta}z)^2}$* $(z \in \mathbb{D},\ \theta \in \mathbb{R})$.

Remark 1.1 *We have $\mathcal{S}^{\mathcal{C}} \subset \mathcal{S}^* \subset \mathcal{S}$ and*

$$f \in \mathcal{S}^{\mathcal{C}} \Leftrightarrow zf'(z) \in \mathcal{S}^*.$$

Similarly to the convexity of order α a notion of starlikeness of order α was introduced by Robertson [123].

Definition 1.7 [65, Goodman, p. 133] *Let $0 \leq \alpha \leq 1$. We denote*

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{S} :\ \mathrm{Re}\frac{zf'(z)}{f(z)} > \alpha \quad (z \in \mathbb{D}), \right\}$$

the class starlike functions of order α.

1.1.3 Class of functions with positive real part

The class of functions with positive real part plays a crucial rule in the Geometric Function Theory. Its significance can be seen from the fact that any subclasses of the class of univalent functions have been defined by using the concept of the class of functions with positive real part. The functions with positive real part constitute an important part in problem from signal theory, in moment problems and in constructing quadrature formulas and the references cited therein for some recent applications. In this section, we define the class of functions with positive real part and we presents here some of its interesting properties.

Definition 1.8 [65, Goodman, p. 78] *The class $\mathcal{P}$ is the set of all functions of the form*

$$\varphi(z) = 1 + c_1 z + c_2 z^2 + ... = 1 + \sum_{n=1}^{\infty} c_n z^n, \tag{1.2}$$

that are analytic in $\mathbb{D}$, and such that for $z \in \mathbb{D}$, $\operatorname{Re}\varphi(z) > 0$. Any function in $\mathcal{P}$ is called a function with positive real part in $\mathbb{D}$.

We give next the distortion and growth results for functions with positive real part.

Theorem 1.12 [65, Goodman, p. 81] *If $\varphi \in \mathcal{P}$ and $|z| = r < 1$, then*

$$\frac{1-r}{1+r} \leq |\varphi(z)| \leq \frac{1+r}{1-r},$$
$$\frac{1-r}{1+r} \leq \operatorname{Re}\varphi(z) \leq \frac{1+r}{1-r},$$
$$|\varphi'(z)| \leq \frac{2\operatorname{Re}\varphi(z)}{1-r^2} \leq \frac{2}{(1-r)^2}.$$

In each of the above inequality, equality holds for $p(z) = (1 + \lambda z)/(1 - \lambda z)$ $(z \in \mathbb{D})$, where $\lambda \in \mathbb{C}$ is such that $|\lambda| = 1$.

Theorem 1.13 [114, Pommerenke, p. 35] *Let $\varphi \in \mathcal{P}$ be of the form* (1.2)*. Then*

$$|c_n| \leq 2 \quad (n \geq 1),$$
$$\left| c_2 - \frac{c_1^2}{2} \right| \leq 2 - \frac{|c_1|^2}{2}.$$

The estimates are sharp. Equality holds for $\varphi(z) = (1 + \lambda z)/(1 - \lambda z)$ $\quad (z \in \mathbb{D})$, where $\lambda \in \mathbb{C}$ and $|\lambda| = 1$.

It should be noted that φ is not necessarily required to be univalent. For this purpose consider the function $\varphi(z) = 1 + z^n$ which is in $\mathcal{P}$ for any integer $n \geq 0$, but if $n \geq 2$, the function is not univalent.
Just as the Koebe function plays a central role in the class $\mathcal{S}$, the Möbius function

$$L(z) = \frac{1+z}{1-z} = 1 + 2z + 2z^2 + ... = 1 + 2\sum_{n=1}^{\infty} z^n, \tag{1.3}$$

plays a central role in the class $\mathcal{P}$. The function defined by (1.3) is analytic and univalent in $\mathbb{D}$, and it maps $\mathbb{D}$ onto the right half plane. There is one notable difference in the character of L and k. In many extremal problems for the class $\mathcal{S}$, the Koebe function is the unique solution (apart from a rotation). In contrast, the function L does maximize c_n in the class P, but if $n \geq 2$, there are infinitely many other function in P for which $c_n = 2$, and no one of these is obtained from any other by a rotation. According to the famous Herglotz representation theorem, proved in 1911 by Herglotz [67], every positive harmonic function is the Poisson integral of a positive measure.

Theorem 1.14 [65, Herglotz representation formula for analytic functions with positive real part, Goodman, p. 96] *Let φ be an analytic function with positive real part in $\mathbb{D}$. Then φ has the form*

$$\varphi(z) = \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} d\mu(t) \quad (|z| < 1),$$

where μ is a positive measure on $[0, 2\pi]$.

1.1.4 Class of typically real functions

Definition 1.9 [65, Goodman, p. 184] *If the function $f \in \mathcal{A}$, and satisfies for every $z \in \mathbb{D}$ the condition*

$$(\mathrm{Im} f(z))(\mathrm{Im} z) \geq 0, \tag{1.4}$$

then f is said to be typically real in $\mathbb{D}$. The class of all such functions is denoted by $\mathcal{T}_{\mathbb{R}}$.

This definition can be modified to include other normalizations, domains different from $\mathbb{D}$, or using the meromorphic functions.
From (1.4) it follows that all coefficients are real. In the opposite direction, every univalent function with real coefficients is typically real because $\mathrm{Im} f(z) = 0$ implies $f(z) = \overline{f(z)} = f(\overline{z})$ and therefore $z = \overline{z}$, that is $\mathrm{Im} z = 0$.

Theorem 1.15 [114, Pommerenke, p. 54] *Let $f \in \mathcal{A}$ have real coefficients. Then the following three conditions are equivalent*

1. *f is typically real,*

2. *the function*

$$(1 - z^2)\frac{f(z)}{z} = 1 + a_2 z + \sum_{n=2}^{\infty} (a_{n+1} - a_{n-1}) z^n$$

   has positive real part in $\mathbb{D}$,

3. *there exists an increasing function $\gamma(t) \quad (-1 \leq t \leq 1)$ such that*

$$f(z) = \int_{-1}^{1} \frac{z}{1 - 2z\cos t + z^2} d\gamma(t).$$

Example 1.1 *The function*

$$f(z)=\frac{(1+z^2)z}{(1-z^2)^2}=z+3z^3+5z^5+...$$

is typically real by Theorem 1.15, but is not univalent in $\mathbb{D}$.

Theorem 1.16 [114, Pommerenke, p. 54] *If* $f\in\mathcal{A}$ *be typically real, then*

$$|a_{n+1}-a_{n-1}|\leq 2,$$
$$|a_n|\leq n \quad (n=2,3,...)\ .$$

Ma in [95] proved the generalized Zalcman conjecture for typically real functions, as follows.

Theorem 1.17 [95, Ma] *Let* $f\in\mathcal{A}$ *be typically real. Then for* $n,m=2,3,...,$ *the sharp estimates hold:*

$$|a_na_m-a_{n+m-1}|\leq\begin{cases}n+1, & \textit{if } m=2, n=2,4,6,...,\\ m+1, & \textit{if } n=2, m=2,4,6,...,\\ (n-1)(m-1), & \textit{otherwise.}\end{cases}$$

Goluzin [61] first proved that inside of the curve

$$C=\left\{z:\Big((|z+i|=\sqrt{2})\cap(\mathrm{Im}z\geq 0)\Big)\cup\Big((|z-i|=\sqrt{2})\cap(\mathrm{Im}z\leq 0)\Big)\right\}$$

all function $f\in\mathcal{T}_{\mathbb{R}}$ are univalent, see Fig. 1.1.

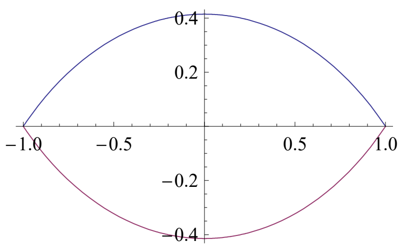

Fig. 1.1. The domain of univalence of the class $\mathcal{T}_{\mathbb{R}}$.

With the help of the other methods Brannan and Kirwan [24] and Goodman [63] proved that the Goluzin domain of univalence is maximal for the class $\mathcal{T}_{\mathbb{R}}$. Brannan and Kirwan [24] also proved that the image of the disk $\mathbb{D}$ under every function $f\in\mathcal{T}_{\mathbb{R}}$ in the w-plane covers the disk $|w|<1/4$. Goodman [64] found the Koebe domain for the class $\mathcal{T}_{\mathbb{R}}$, i.e. the largest possible domain that is covered under the image of the disk $\mathbb{D}$ be every function $f\in\mathcal{T}_{\mathbb{R}}$ in the w-plane, where the Brannan, Kirwan disk $|w|<1/4$ is maximal with a center $w=0$ in the Goodman Koebe domain.

Theorem 1.18 [65, Goodman, p. 191] *For $f \in \mathcal{T}_{\mathbb{R}}$ and $z \in \mathbb{D} \setminus \{0\}$ we have*

$$|f(z)| \leq \begin{cases} \left|\dfrac{z}{(1-z)^2}\right| & \text{if} \quad \mathrm{Re}\frac{1+z^2}{z} \geq 2, \\ \dfrac{1}{\left|\mathrm{Im}\frac{1+z^2}{z}\right|} & \text{if} \quad \left|\mathrm{Re}\frac{1+z^2}{z}\right| < 2, \\ \left|\dfrac{z}{(1+z)^2}\right| & \text{if} \quad \mathrm{Re}\frac{1+z^2}{z} \leq -2. \end{cases}$$

$$\arg\frac{z}{(1+z)^2} \leq \arg f(z) \leq \arg\frac{z}{(1-z)^2} \quad (z \in \mathbb{D}_+),$$

$$\arg\frac{z}{(1-z)^2} \leq \arg f(z) \leq \arg\frac{z}{(1+z)^2} \quad (z \in \mathbb{D}_-),$$

where $\mathbb{D}_+$ and $\mathbb{D}_-$ denotes the upper and lower part of the unit disk $\mathbb{D}$, respectively.

Todorov [151] gave an estimates for the functional $\mathrm{Re}(zf'(z)/f(z))$ in $\mathcal{T}_{\mathbb{R}}$.

Theorem 1.19 [151, Todorov] *For each typically real function we have*

$$\mathrm{Re}\frac{zf'(z)}{f(z)} \geq \frac{1-6r^2+r^4}{1-r^4} \quad (2-\sqrt{3} \leq r = |z| < 1)$$

with equality for function $f(z) = z(1+z^2)/(1-z^2)^2$ at the points $z = \pm ir$,

$$\mathrm{Re}\frac{zf'(z)}{f(z)} \geq \frac{1-r}{1+r} \quad (0 \leq r = |z| \leq 2-\sqrt{3})$$

with equality for the functions $f(z) = z/(z \pm 1)^2$ at the points $\pm r$.

1.1.5 Radius problems

If we suppose that some transformations or geometric conditions fail to preserve univalence (for instance) in the unit disk, then it is natural to ask if such transformations (or conditions) could preserve it in any subdisk of $\mathbb{D}$. Problems of this sort became known as radius problems. In Section 2.2.1 we determine the radii of starlikness of order α and radii of convexity for Koebe function. Next, in Section 4.6 we compute the radii of local univalence and of univalence. Before we state this results, we must introduce additional terminology and notation.

Definition 1.10 [65, Goodman, p.119] *The number $R_{\mathfrak{P}}$ is called the radius for the property $\mathfrak{P}$ in the class $\mathcal{M}$ if $R_{\mathfrak{P}}$ is the least upper bound of all numbers r such that every $f(z)$ in $\mathcal{M}$ has the property $\mathfrak{P}$ in the disk $\mathbb{D}$.*

Theorem 1.20 [65, Goodman, p. 119] *Let $f \in \mathcal{S}$. Then for each $r \leq 2-\sqrt{3}$, the image of $|z| = r$ is a simple closed convex curve. The number $R_{\mathcal{S}\mathcal{C}} = 2-\sqrt{3}$ is sharp.*

Specially, a radii of convexity and starlikeness were studied.

Definition 1.11 [65, Goodman, p. 119] *The number* $R_{\mathcal{S}c} = 2-\sqrt{3}$ *is called the radius of convexity for the set* $\mathcal{S}$.

Definition 1.12 [43, Duren, p. 95] *We say that* $r_{\mathcal{S}^*}(\mathcal{S})$ *is the radius of starlikeness in the class* $\mathcal{S}$, *if it is the maximum of numbers* r *such that the inequality* $\mathrm{Re}(zf'(z)/f(z)) \geq 0$ *holds in* $\mathbb{D}$ *for each function* $f \in \mathcal{S}$.

The radius of starlikeness for $\mathcal{S}$ is

$$r_{\mathcal{S}^*}(\mathcal{S}) = \frac{1-\exp\pi/2}{1+\exp\pi/2} = \tan\frac{\pi}{4},$$

and this fact was first discovered by Grunsky [66].

Definition 1.13 [135, Sobczak - Kneć] *The set* $G \subset \mathbb{D}$ *is called the set of local univalence in the class* $\mathcal{S}$, *if*

$$\forall_{f\in\mathcal{S}}\ \ \forall_{z\in G}:\ f'(z)\neq 0 \quad \textit{and} \quad \forall_{z\in\mathbb{D}\setminus G}\ \ \exists_{f\in\mathcal{S}}:\ f'(z)=0.$$

Definition 1.14 [135, Sobczak - Kneć] *We say that* $r_{\mathcal{LS}}(\mathcal{S})$ *is the radius of local univalence in the class* $\mathcal{S}$, *if it is the maximum of numbers* r *such that every function* $f \in \mathcal{S}$ *is locally univalence in* $\mathbb{D}$.

Definition 1.15 [135, Sobczak - Kneć] *We say that* $r_{\mathcal{S}}(\mathcal{S})$ *is the radius of univalence in the class* $\mathcal{S}$, *if it is the maximum of the numbers* r *such that every function* $f \in \mathcal{S}$ *is univalence in* $\mathbb{D}$.

In the class S the following inequality is satisfied

$$r_{\mathcal{S}^*}(\mathcal{S}) \leq r_{\mathcal{S}}(\mathcal{S}) \leq r_{\mathcal{LS}}(\mathcal{S})$$

1.1.6 Linearly invariant families of holomorphic functions

Definition 1.16 [113, Pommerenke] *A family* $\mathfrak{M}$ *of functions* f *of the form* (1.1), *holomorphic in the unit disc* $\mathbb{D}$, *is called a linearly invariant family if each function* $f \in \mathfrak{M}$ *satisfies the following conditions*

1. $f'(z) \neq 0$ *in* $\mathbb{D}$ *(local univalence),*
2. *for all* $f \in \mathfrak{M}$ *and* $\theta \in \mathbb{R}$, $e^{-i\theta}f(ze^{i\theta}) \in \mathfrak{M}$
3. *for all* $f \in \mathfrak{M}$ *and* $a \in \mathbb{D}$ $f_a(z) := \dfrac{f(\frac{z+a}{1+\bar{a}z}) - f(a)}{f'(a)(1-|a|^2)} = z + \ldots \in \mathfrak{M}$.

Definition 1.17 [113, Pommerenke] *The number*

$$ord\, f = \sup_{a\in\mathbb{D}} \frac{|f_a''(0)|}{2}$$

is called the order of a locally univalent function f, and the number

$$ord\,\mathfrak{M} = \sup_{f\in\mathfrak{M}} ord\, f$$

the order of the family $\mathfrak{M}$. Moreover,

$$\bigcup\{\mathfrak{M} : ord\ \mathfrak{M} \le \alpha\} := \mathcal{U}_\alpha$$

is called the universal linearly invariant family.

Linearly invariant families play an important role in the theory of conformal mappings. The examples of the well known linearly invariant families are the class $\mathcal{S}$, with ord $\mathcal{S} = 2$ [14, 15], $\mathcal{S}^{\mathcal{C}}$,with ord $\mathcal{S}^{\mathcal{C}} = 2$ [116], and $\mathcal{U}_\alpha$, ord $\mathcal{U}_\alpha = \alpha$.

Starkov in [137] (see also [139]) introduced a linear-invariant family $\mathcal{U}'_\alpha \subset \mathcal{U}_\alpha$ $(\alpha \ge 1)$ with ord $\mathcal{U}'_\alpha = \alpha$, defined by

$$f \in \mathcal{U}'_\alpha \Longleftrightarrow f'(z) = \exp\left(-2\int_0^{2\pi} \log\left(1 - ze^{-it}\right) d\mu(t)\right),$$

where $\mu(t)$ is an arbitrary complex-valued function of bounded variation on $[0, 2\pi]$ such that

$$\left|\int_0^{2\pi} d\mu(t) - 1\right| + \int_0^{2\pi} |d\mu(t)| \le \alpha.$$

1.2 Orthogonal polynomials

This introductory section gives a brief account on the standard theory of orthogonal polynomials. We give definitions, notation and results from orthogonal polynomials and hypergeometric series that will be using later in dissertation.

The subject of orthogonal polynomials finds its origins in the 18th century, thanks to the works of Legendre, Laplace and Lagrange. Although these three brilliant mathematicians are best remembered for their work in elliptic functions, the theory of differential equations and mathematical astronomy, they also developed the first examples of orthogonal polynomials, before any general theory existed. The development of the general theory began in the 19th century after investigations into Stieltjes continued fractions [143, 144] by Chebyshev [31].

Other important results, independent of the general theory were given by Gauss, Abel, Jacobi, Hermite and Laguerre, of whom the latter three gave their name (Jacobi, Hermite, Laguerre) to what became the classical orthogonal polynomials. Each of these sets of polynomials is an example of family of polynomials that are orthogonal with respect to an inner product that is included by a positive weight function on an interval of the real line. The classical orthogonal polynomials were the first families of orthogonal polynomials to be established and are important because they were discovered to posses many more properties than other orthogonal polynomials systems of the time. Looking more closely at the properties that these families have, it can be shown that they all have a generating function, a recursion and differential relation and a Rodrigues' formula.

Orthogonal polynomials have been found to have connections with trigonometric, hypergeometric, Bessel and elliptic functions. They have significance in helping to solve certain problems in quantum mechanics and mathematical statistics.

Up until the late 20th century, there were only a few authoritative texts on the subject of orthogonal polynomials. These included the book by Szegö [147], that covered most of the general theory along with all standard formulae for the three classical orthogonal polynomials. The monograph by Freud [51] also gave a detailed view on the classical orthogonal polynomials in the context of asymptotics. The text by Chihara [34] was focused on the elementary theory. Recently though, there has been a renewed interest in orthogonal polynomials, especially since the connection with integrable systems has been found. Among these we mention the books by Simon [132, 133], which has developed the general theory, [147] as an authoritative texts on orthogonal polynomials on the unit circle, and the monograph [70], which approaches orthogonal polynomials from the viewpoint of special functions.

The connection of orthogonal polynomials with other branches of mathematics is truly impressive. We only mention here continued fractions, operator theory (Jacobi operators), analytic functions (Bieberbachs conjecture), interpolation, approximation theory, numerical analysis, statical quantum mechanics, special functions, number theory (irrationality and transcendence), graph theory, combinatorics, random matrix, stochastic processes, data sorting and compression and computer tomography.

1.2.1 Hypergeometric series

The hypergeometric series arises in the theory of differential equations, and can also be used for the representation of several important sets of orthogonal polynomials.

Definition 1.18 [117, Rainville, p. 45] *For complex numbers a, b, and c ($c \neq 0, -1, -2, \cdots$), the Gaussian hypergeometric function ${}_2F_1(z)$ is defined by*

$$ {}_2F_1(a, b, c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!} \quad (z \in \mathbb{D}), \tag{1.5} $$

where $(a)_n$ *is the Pochhammer symbol or shifted factorial defined by*

$$\begin{aligned}(a)_0 &= 1,\\ (a)_n &= a(a+1)(a+2)...(a+n-1) \quad (n \in \mathbb{N}).\end{aligned}$$

Notice that ${}_2F_1(z)$ is symmetric with respect to a and b and the series terminates if either a or b is zero or a negative integer. In general, the series ${}_2F_1(z)$ is absolutely convergent in $\mathbb{D}$. If $\mathrm{Re}(c-a-b) > 0$, it is also convergent on $\partial\mathbb{D}$, and it is known that

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad (c \neq 0, -1, -2, ...).$$

Hypergeometric series are built from Pochammer symbols so it is natural that in extending the hypergeometric series an extension of the Pochammer symbol is obtained. This is given by the q-shifted factorial.

Definition 1.19 [83, Koekoek, p. 11] *The q-shifted factorials are defined as*

$$(a;q)_0 := 1, \quad (a;q)_n := \prod_{k=1}^{n}(1-aq^{k-1}) \quad (n = 1, 2, ...)$$

and the multiple q-shifted factorials are defined by

$$(a_1, a_2, ..., a_k; q)_n := \prod_{j=1}^{k}(a_j; q)_n.$$

The basic hypergeometric series is defined as

$${}_r\phi_s(a_1, a_2, ..., a_r; b_1, b_2, ..., b_s; q, z) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, ..., a_r; q)_n}{(q, b_1, b_2, ..., b_s; q)_n} z^n(-q^{\frac{n-1}{2}})^{n(s+1-r)}. \quad (1.6)$$

In formula (2.7) *we use a particulary case of* (1.6) *which is*

$${}_1\phi_0(a; -; q, z) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q,q)_n} z^n \quad (z \in \mathbb{D}).$$

1.2.2 General properties of orthogonal polynomials

In this section we mention some basic result for polynomials $P_n(x)$. Throughout the thesis when referring to a continuous Riemann integrable function w satisfying $w(x) > 0$ for $x \in (a, b)$, it will be assumed that

$$\int_a^b x^n w(x)dx < \infty \quad (n = 0, 1, 2, ...).$$

Lemma 1.21 [34, Chihara, p. 2] *Let* $P_n(x)$ *be an arbitrary real polynomial of degree* n *and* $w(x)$ *be continuous and positive function on* (a, b)*. The functional defined by*

$$\mu(P_n(x)) = \int_a^b P_n(x)w(x)dx$$

on the space of real polynomials (i.e. polynomials with real coefficients) is linear.

Definition 1.20 *An inner product $\langle\cdot,\cdot\rangle$ is a bilinear function of elements of a vector space, if for every $f, g \in V$ that satisfies the axioms:*

1. $\langle f, f\rangle \geq 0$ *with equality if and only if* $f \equiv 0$;
2. $\langle f, g\rangle = \langle g, f\rangle$;
3. $\langle f + g, h\rangle = \langle f, h\rangle + \langle g, h\rangle$;
4. $\langle \alpha f, g\rangle = \alpha\langle f, g\rangle$ *for any scalar* $\alpha \in \mathbb{R}$.

Definition 1.21 [117, Rainville, p. 150] *In the space of real polynomials define $\langle\cdot,\cdot\rangle$ for the functional μ, by*

$$\langle P_n(x), P_m(x)\rangle = \mu(P_n(x)P_m(x)),$$

where $P_n(x)$ and $P_m(x)$ are arbitrary real polynomials of degree m and n, respectively and $x \in \mathbb{R}$.

Lemma 1.22 [2, Akhiezer, p. 2] *The product $\langle\cdot,\cdot\rangle$ is an inner product on the space of real polynomials of real variable.*

With the inner product established it follows by application of the Grama - Schmidt process that there is a set of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ where $P_n(x)$ has degree n ($n = 0, 1, 2, ...$), such that

$$\langle P_n(x), P_m(x)\rangle = \delta_{nm}.$$

Definition 1.22 [117, Rainville, p. 148] *Let $P_n(x)$ denote the real polynomial of degree n. A set of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ satisfying*

$$\langle P_n(x), P_m(x)\rangle = h_n\delta_{nm}$$

where $h_n > 0$ and $\langle\cdot,\cdot\rangle$ is generated by $w(x) > 0$, is called a set of orthogonal polynomials with respect to the weight function $w(x)$.

In the general case of an orthogonal polynomials set, it is not necessary to assume $h_n > 0$. However, the most interesting work has been done under that assumption, this assumption is entailed by the choice $w(x) > 0$.

Definition 1.23 [34, Chihara, p. 7] *In the case where $h_n = 1$ the polynomials $P_n(x)$ are said to be orthonormal.*

Definition 1.24 [34, Chihara, p. 10] *In the case where the leading coefficient of each polynomials in a set of orthogonal polynomials is 1, the polynomials are referred to as monic orthogonal polynomials.*

Definition 1.25 [117, Rainville, p. 147] *If $\{P_n(x)\}_{n=0}^{\infty}$ is a set of polynomials such that $P_n(x)$ has degree n, for each $n = 0, 1, 2, ...,$ then $\{P_n(x)\}_{n=0}^{\infty}$ is called a simple set.*

An equivalent and useful form of the orthogonality relation can be introduced as follows.

Theorem 1.23 [117, Rainville, p. 148] *Let $\{P_n(x)\}_{n=0}^{\infty}$ be a simple set of polynomials. This set is an orthogonal set with respect to the weight function $w(x)$, continuous and positive on (a, b), if and only if it satisfies*

$$\int_a^b P_n(x)x^k w(x)dx = 0 \quad (k = 0, 1, 2, ..., n-1),$$

or, using the inner product

$$\langle P_n(x), x^k \rangle = 0 \quad (k = 0, 1, 2, ..., n-1).$$

Orthogonal polynomials satisfy several useful identities, one of which is the three-term recurrence relation.

Theorem 1.24 [9, Andrews, p. 244] *Let $\{P_n(x)\}_{n=0}^{\infty}$ be a set of orthogonal polynomials corresponding to the functional μ (or the weight function $w(x)$ which generates μ), and let k_n be the leading coefficient of $P_n(x)$. Then, there exist real sequences $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ and $\{c_n\}_{n=0}^{\infty}$, such that for $n \geq 1$ the three-term recurrence relation*

$$\begin{aligned} P_{-1}(x) &= 0, \\ P_0(x) &= k_0, \\ P_{n+1}(x) &= (a_n x + b_n)P_n(x) - c_n P_{n-1}(x), \end{aligned}$$

holds. Here $a_n a_{n-1} c_n > 0$ for $n = 0, 1, 2, ...,$ and if h_n is as in Definition 1.22 then

$$a_n = \frac{k_{n+1}}{k_n}, \quad c_{n+1} = \frac{a_{n+1}}{a_n}\frac{h_{n+1}}{h_n}.$$

Definition 1.26 [34, Chihara, p. 21] *The sequences of orthogonal polynomials are symmetric if $P_n(x) = (-1)^n P_n(-x)$ for all n or that b_n in Theorem 1.24 are all zero.*

In the modern theory the following are referred to as the classical orthogonal polynomials: Hermite polynomials, Laguerre polynomials, Jacobi or hypergeometric polynomials. The classical orthogonal polynomials can be defined [34] as those orthogonal polynomials satisfying the properties

1. $\{P_n'(x)\}$ is a system of orthogonal polynomials,

2. $y(x) = P_n(x)$ satisfies differential equation of the form

$$A(x)y'' + B(x)y' + \lambda_n y = 0,$$

   where $A(x)$ and $B(x)$ are independent of n, and λ_n is independent of x,

3. $P_n(x)$ satisfy a generalized Rodrigues' Formula
$$P_n(x) = \frac{1}{K_n w(x)} \frac{d^n}{dx^n}(w(x)X^n),$$
where K_n is a constant and X is a polynomials in x, whose coefficients are independent of n.

However, in recent times there have been a number of families that satisfy these conditions, but are not called classical. It is an interesting problem to compute of new orthogonal polynomials out of old ones. If the measure of the new orthogonal polynomials is the measure of the old ones multiplied by a rational function, one talks about modified orthogonal polynomials.

1.2.3 Meixner - Pollaczek polynomials

The Chapter 5. of this thesis is mainly concerned about the Meixner - Pollaczek polynomials. These are the polynomials first discovered by Meixner [99] and are known in the literature as the Meixner polynomials of the second kind (see Chihara [34]). These polynomials were later studied by Pollaczek [112]. The Meixner - Pollaczek polynomials are denoted by $p_n^{(\lambda)}(x,\phi)$, and have a hypergeometric representation
$$p_n^{(\lambda)}(x,\phi) = \frac{(2\lambda)_n}{n!} e^{in\theta}\ {}_2F_1(-n, \lambda + ix, 2\lambda; 1 - e^{-2i\phi}) \quad (\lambda > 0, 0 < \phi < \pi).$$
The polynomials $p_n^{(\lambda)}(x,\phi)$ are completely described by the recurrence formula
$$\begin{aligned} & p_{-1}^{(\lambda)}(x,\phi) = 0, \quad p_0^{(\lambda)}(x,\phi) = 1, \\ & (n+1)p_{n+1}^{(\lambda)}(x,\phi) - 2[x\sin\phi + (n+\lambda)\cos\phi]p_n^{(\lambda)}(x,\phi) \\ + \quad & (n+2\lambda-1)p_{n-1}^{(\lambda)}(x,\phi) = 0 \quad (n \geq 1), \end{aligned}$$
and are represented by generating function
$$(1 - e^{i\phi}t)^{-\lambda+ix}(1 - e^{-i\phi}t)^{-\lambda-ix} = \sum_{n=0}^{\infty} p_n^{(\lambda)}(x,\phi)t^n.$$
Some of the main properties of these polynomials are presented by Erdélyi et al. [45], Chihara [34], Askey and Wilson [11], and in the report by Koekoek and Swarttouw [83]. Detailed analysis with applications of these polynomials are also made by several others. Asymptotic properties of these polynomials and their zeros are studied by Li and Wong [92]. The connection between the Heisenberg algebra and Meixner-Polaczek polynomials were studied by Bender, Mead and Pinsky [13] and Koornwinder [85]. The combinatorial interpretation of the linearization coefficients was discussed by Zeng [159]. The interpretation of the Meixner - Pollaczek polynomials as overlap coefficients in the positive discrete series representation of the Lie algebra were discussed by Koelink and Van der Jeught [84].

1.2.4 Chebyshev polynomials of the first and second kind

The impact of the work russian mathematician P. L. Chebyshev (1821-1894) and his studen Markov has already been describe by Krein [87]. A few particular orthogonal polynomials were known before Chebyschev. It was Chebyshev who saw the possiblity of a general theory and its applications. His work arose out of the theory of least squares approximation and probability. He discovered the discrete analogue of the Jacobi polynomials but their importance was not recognized until 20th century. They were rediscovered by Hahn and named after him upon their rediscovered. Nowadays the notion of Chebyshev polynomials is well known. The sequence of polynomials $\{T_n(x)\}_{n=0}^{\infty}$ appearing in approximation theory [118], geometry [57], combinatorics [130], number theory, statistics, numerical integration ([49], [102]), and differential equations (Rivlin [120] gives numerous examples). Several generalizations have been found and investigated, see e.g. [136, 155].

For easy reference, let us first state the definitions and basic properties of Chebyshev polynomials.

Definition 1.27 [98, Mason, p. 2] *The Chebyshev polynomials of the first kind are defined on the interval* $[-1, 1]$ *by*

$$\begin{aligned} T_0(x) &= 1, \\ T_1(x) &= x, \\ T_n(x) &= 2xT_{n-1}(x) - T_{n-2}(x) \quad (n \geq 2). \end{aligned}$$

The first few $T_n(x)$ polynomials are

$$\begin{aligned} T_0(x) &= 1, \\ T_1(x) &= x, \\ T_2(x) &= 2x^2 - 1, \\ T_3(x) &= 4x^3 - 3x, \\ T_4(x) &= 8x^4 - 8x^2 + 1, \\ T_5(x) &= 16x^5 - 20x^3 + 5x. \end{aligned}$$

The most widely used definition of Chebyshev polynomials can be given by the following formula:

$$T_n(\cos(x)) = \cos(nx) \quad (n = 0, 1, ...).$$

This definition is useful for calculations with Chebyshev polynomials.

If we define the inner product using the interval $[-1, 1]$ and weight function

$$w(x) = (1 - x^2)^{-\frac{1}{2}},$$

then we find that the first kind Chebyshev polynomials satisfy

$$\langle T_i, T_j \rangle = \int_{-1}^{1} \frac{T_i(x)T_j(x)}{\sqrt{1 - x^2}} dx = \int_{0}^{\pi} \cos(i\theta)\cos(j\theta) d\theta.$$

Hence

$$\langle T_i, T_j \rangle = 0 \ (i \neq j),$$

and $\{T_i(x), i = 0, 1, ...\}$ forms an orthogonal polynomials systems on $[-1, 1]$ with respect to the given weight $(1 - x^2)^{-\frac{1}{2}}$.
The norm of T_i is given by

$$\begin{aligned} ||T_i||^2 &= \langle T_i, T_i \rangle = \int_0^\pi (\cos i\theta)^2 d\theta = \frac{1}{2} \int_0^\pi (1 + \cos 2i\theta) d\theta \\ &= \frac{1}{2} \Big[\theta + \frac{\sin 2i\theta}{2i} \Big]_0^\pi = \frac{1}{2}\pi \quad (i \neq 0), \end{aligned}$$

while

$$||T_0||^2 = \langle T_0, T_0 \rangle = \langle 1, 1 \rangle = \pi.$$

Definition 1.28 [98, Mason, p. 4] *The Chebyshev polynomials of the second kind are defined by*

$$\begin{aligned} U_0(x) &= 1, \\ U_1(x) &= 2x, \\ U_n(x) &= 2xU_{n-1}(x) - U_{n-2}(x) \quad (n \geq 2). \end{aligned}$$

The first few $U_n(x)$ polynomials are

$$\begin{aligned} U_0(x) &= 1, \\ U_1(x) &= 2x, \\ U_2(x) &= 4x^2 - 1, \\ U_3(x) &= 8x^3 - 4x, \\ U_4(x) &= 16x^4 - 12x^2 + 1, \\ U_5(x) &= 32x^5 - 32x^3 + 6x. \end{aligned}$$

Another definition of Chebyshev polynomials of the second kind is given by the following formula

$$U_n(x) = \sin{(n+1)\theta} / \sin\theta \quad \text{when} \quad x = \cos\theta \quad (n = 0, 1, 2, ...).$$

The second kind Chebyshev polynomials form also orthogonal systems on $[-1, 1]$, with respect to the weight functions $w(x) = (1 - x^2)^{\frac{1}{2}}$.
The normalization of U_i is given by

$$\langle U_i, U_i \rangle = ||U_i||^2 = \int_0^\pi \sin^2{(i+1)\theta} d\theta = \frac{1}{2}\pi.$$

Now we recall here the differential equations, differentiation formulae and generating functions for Chebyshev polynomials of the first and second kind.

$$\begin{aligned} (1 - x^2)T_n''(x) - xT_n'(x) + n^2 T_n(x) &= 0, \\ (1 - x^2)U_n''(x) - 3xU_n'(x) + n(n+2)U_n(x) &= 0, \end{aligned}$$

$$\begin{aligned}
\left((1-x^2)\frac{d}{dx}+nx\right)T_n(x) &= nT_{n-1}(x),\\
\left((1-x^2)\frac{d}{dx}+nx\right)U_n(x) &= (n+1)U_{n-1}(x),
\end{aligned}$$

$$\begin{aligned}
\frac{1-xt}{1-2xt+t^2} &= \sum_{n=0}^{\infty} T_n(x)t^n,\\
\frac{1}{1-2xt+t^2} &= \sum_{n=0}^{\infty} U_n(x)t^n.
\end{aligned}$$

The Chebyshev polynomials generate many fundamental sequences, including the constant sequences, the sequence of integers, and the Fibonacci numbers.

Chapter 2

Generalized Koebe function

The Koebe function is the heart of the thesis because it appears so often, that a theorem in this subject becomes very interesting if it does not use the Koebe function or some modification of it. We will mention some of these theorems.

As early as 1916, Bieberbach discovered that the second Taylor coefficient a_2 of any function in $\mathcal{S}$ satisfies the inequality $|a_2| \leq 2$, and that equality occurs only for the Koebe function

$$k(z) = \frac{z}{(1-z)^2} = z + \sum_{n=2}^{\infty} nz^n \quad (z \in \mathbb{D}),$$

or its rotations $e^{-i\theta}k(e^{i\theta}z)$.

Bieberbach's results can be used to derive the *growth theorem*

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2} \quad (0 \neq |z| = r < 1) \tag{2.1}$$

and the *distortion theorem*

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3} \quad (0 \neq |z| = r < 1) \tag{2.2}$$

for $f \in \mathcal{S}$. Again, equality in either (2.1) or (2.2) hold, if f is the Koebe function or one of its rotations. Similary, the argument of the derivative of a function $f \in \mathcal{S}$ admits the estimate

$$|\arg f'(z)| \leq 2 \log \frac{1+|z|}{1-|z|} \quad (z \in \mathbb{D}).$$

This result, however, is not sharp (later on a sharp estimate was obtained).

In the light of these results, Bieberbach conjectured in 1916 that the coefficients of each functions $f \in \mathcal{S}$ satisfy $|a_n| \leq n \quad (n = 2, 3, ...)$, and that strict inequality holds unless f is the Koebe function or a rotation of the Koebe function.

With the above motivation concerning the importance of Koebe function our attention in the present chapter is particulary directed to present several generalizations of Koebe function, which can be found in literature. We begin our exposition in Section

2.1 with a brief summary of some known generalization of Koebe function so that we draw motivation for the future research. Next, in Section 2.2 we discuss in detail one of generalizations of Koebe function, namely $k_{p,q}$. We describe a geometric properties of that function and find some its specific bound. The motivation for this research is that $k_{p,q}$ function have been found to have many connections with class of typically real functions. By considering $k_{p,q}$ function, we gain further insight the connection between class of univalent functions and coefficients problems as well as with orthogonal polynomials.

2.1 Backround and motivation

Let us now turn to several generalization of Koebe functions, which can be found in literature, for instance:

1. One of the earliest generalizations was achieved in [114, Theorem 1.5], where the extremal functions for the elementary estimates $|a_2^2 - a_3| \leq 1$ if $f \in \mathcal{S}$ are
$$f(z) = \frac{z}{(1-\lambda z)(1-\mu z)} \quad (|\mu| = |\lambda| = 1, z \in \mathbb{D}).$$
This estimate follows from the area theorem.

2. By virtue of the extremal function f for $\mathcal{S}^*(\alpha)$, a new function of Koebe type is considered by Eguchi and Owa in [44], namely
$$f(z) = \frac{z}{(1-z)^k} \quad (k \in \mathbb{R}, z \in \mathbb{D}).$$
The object of this paper was to derive radii for starlikeness of order α, and for convexity of order α for the function of Koebe type. Using the extremal functions for the classes of α-spiral like of order β and of α-convex like of order β, Eguchi and Owa also consider the following function of the generalized Koebe type
$$f(z) = \frac{z}{(1-z)^{ke^{i\alpha}}} \quad (k \in \mathbb{R},\ 0 \leq \alpha \leq 2\pi, z \in \mathbb{D}).$$

3. Another generalization of Koebe function was introduced by Kamali and Srivastava [73], which is
$$k(z) = \frac{z}{(1-z^n)^{b \exp i\alpha}} \quad (n \in \mathbb{N}, b \in \mathbb{R}, -\pi \leq \alpha \leq \pi, z \in \mathbb{D}).$$

4. Noor [108] in his work obtained the extremal function F_k defined by
$$F_k(z) = \frac{1}{k+2}\left(\left(\frac{1+k}{1-z}\right)^{\frac{1}{2}k+1} - 1\right) \quad (z \in \mathbb{D}, k \geq 2).$$
We also note that the function F_k is the same function as F_β defined by equation (2.6) in [62]. As Goodman has pointed out that this function is sometime referred to as the generalized Koebe function.

5. Okuyama [110] called
$$f_\beta(z) = z(1-z)^{-2e^{i\beta}\cos\beta} \quad (\beta \in (-\pi/2, \pi/2),\ z \in \mathbb{D})$$
the β-spiral Koebe function. Note that $f_0(z)$ is the Koebe function. The β-spiral Koebe function conformally maps the unit disk onto the complement of the β-logarithmic spiral $\{f_\beta(-e^{-2i\beta})exp(-e^{i\beta}) \quad (t \leq 0)\}$ in $\mathbb{C}$.

6. In paper [134] it was considered the class $M(\alpha, \lambda)_b$ for $\alpha \geq 0, \lambda > 0$, defined as the class of function $f \in \mathcal{A}$ such that
$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}(1-\alpha+\alpha(1-\lambda)\frac{zf'(z)}{f(z)}) + \alpha\lambda(1+\frac{zf''(z)}{f'(z)})\right) > 0,$$
with the extremal function f_b, that is Koebe type function
$$f_b(z) = \frac{z}{(1-z^n)^b} \quad (b \geq 0, n = 1, 2, ...).$$
The authors derived sufficient conditions for starlikness of the class $M(\alpha, \lambda)_b$ of $n-$fold symmetric analytic functions of Koebe type, which obviously corresponds to the familiar Koebe function when $n = 1$ and $b = 2$.

7. The family of analytic functions of the form
$$F_t(z) = \frac{(1-t)z}{(1-z)^2} + \frac{tz}{1-z} = \frac{z-tz^2}{(1-z)^2} \quad (0 \leq t \leq 1, z \in \mathbb{D}),$$
was considered in [42], which for $t = 0$ becomes the Koebe function.

8. The definition of k_α was extended for a nonzero complex number α by Yamashita [157]
$$k_\alpha(z) = \frac{1}{2\alpha}\left(\left(\frac{1+z}{1-z}\right)^\alpha - 1\right) \quad (z \in \mathbb{D}).$$
The classical result of Hille [68] ascertains that k_α is univalent in $\mathbb{D}$ if and only if $\alpha \neq 0$ is in the union A of the closed disks $\{|z+1| \leq 1\}$ and $\{|z-1| \leq 1\}$. Making use of a geometric properties Yamashita [157] described how k_α tends to be univalent in a whole $\mathbb{D}$ as α tends to each boundary point of A from outside.

9. The properties of generalized Koebe function $f_c(z) = \frac{1}{2c}\log k_c'(z)$, where
$$\begin{aligned} k_c(z) &= \left(\frac{1+z}{1-z}\right)^c - 1, \quad (c \in \mathbb{C} \setminus \{0\}), \\ k_0(z) &= \frac{1}{2}\log\frac{1+z}{1-z}, \end{aligned}$$
were studied in [27] by Campbell and Pfaltzgraff. In this case for $c = 1$ and $c = 2$ we have the familiar $k_1(z) = z/(1-z)$ and $k_2(z) = z/(1-z)^2$.

10. It is well known that for every $\alpha > 1$, the class U_α contains functions which are infinitely valent in $\mathbb{D}$ [114], for example
$$\widetilde{f}(z) = \frac{1}{2i\gamma}\left[\left(\frac{1+z}{1-z}\right)^{i\gamma} - 1)\right],$$

for which
$$\widetilde{f}'(z)=\frac{1}{(1+z)^{1-i\gamma}(1-z)^{1+i\gamma}}\quad(\gamma=\sqrt{\alpha^2-1}).$$
Another example of such a function was presented in [138]
$$f_0(z)=\frac{1}{(e^{it_2}-e^{it_1})i\sqrt{\alpha^2-1}}\left(\left(\frac{1-ze^{it_1}}{1-ze^{it_2}}\right)^{i\sqrt{\alpha^2-1}}-1\right)\quad(t_1\neq t_2+2k\pi),\tag{2.3}$$
for which
$$f_0'(z)=\frac{1}{(1-ze^{it_1})^{1-i\sqrt{\alpha^2-1}}(1-ze^{it_2})^{1+i\sqrt{\alpha^2-1}}}.\tag{2.4}$$
Function of the form
$$f(z)=z+a_2z^2+\ldots\quad(z\in\mathbb{D})\tag{2.5}$$
appears to be extremal for long lasting problems
$$\max_{f\in U'_\alpha}|a_3|\ \text{ and }\ \max_{f\in U'_\alpha}|\arg f'(z)|,$$
recently solved by Starkov [138, 140], who proved that the extremal function for $\max|a_3|$ is of the form (2.4) with $t_1=\theta, t_2=-\theta$, where
$$e^{i\theta}=\sqrt{\frac{(3-\alpha^2)+3i\sqrt{\alpha^2-1}}{\alpha\sqrt{\alpha^2+3}}}.$$
Moreover, the extremal function (see [138]) f_0 for $\max\limits_{f\in U'_\alpha}|\arg f'(z)|$ is of the form (2.3) with
$$\begin{aligned}t_1&=\pi-\arctan\frac{1}{\alpha}-\arctan\frac{r}{\alpha},\\ t_2&=-\pi+\arcsin\frac{1}{\alpha}-\arcsin\frac{r}{\alpha}\quad(r=|z|<1,\ t_1\neq -t_2).\end{aligned}$$
We see that the extremal function for $\max\limits_{f\in U'_\alpha}|a_3|$ has a special form leading to Meixner - Pollaczek polynomials, but the extremal function for $\max\limits_{f\in U'_\alpha}|\arg f'(z)|$ leads to Generalized Meixner - Pollaczek polynomials, defined below. This fact gives us motivation for the definition and study of the generalized Meixner - Pollaczek (GMP) polynomials.

Definition 2.1 *Let the Generalized Meixner - Pollaczek polynomials (GMP) $P_n^\lambda(x;\theta,\psi)$ of a variable $x\in\mathbb{R}$, and parameters $\lambda>0$, $\theta\in(0,\pi)$, $\psi\in\mathbb{R}$ be defined by a generating function*
$$G^\lambda(x;\theta,\psi;z)=\frac{1}{(1-ze^{i\theta})^{\lambda-ix}(1-ze^{i\psi})^{\lambda+ix}}=\sum_{n=0}^{\infty}P_n^\lambda(x;\theta,\psi)z^n\quad(z\in\mathbb{D}).\tag{2.6}$$

Obviously, we have $P_n^\lambda(x;\theta,-\theta) = p_n^\lambda(x;\theta)$.

By the above we see that an investigation of the properties of Koebe function is a natural problem associated to the general study of orthogonal polynomials and we will consider this problem in Chapter 5.

11. Another generalization of the Koebe function was proposed by Gasper in [58]. Namely, observing that
$$k_2(z) = \frac{z}{(1-z)^2} = z\ {}_1F_0(2,-;z),$$
he proposed some extension of the Löwner theory, and de Brange's inequalities, in which the natural extension of Koebe function is
$$k_q(z) = \frac{z}{(1-z)(1-qz)} = z\ {}_1\phi_0\left[q^2;q,z\right] \quad (z \in \mathbb{D}),$$
where $-1 \le q \le 1$ and ${}_1\phi_0$ denote basic hypergeometric series.

We propose an (p,q)-extension which is more symmetric, namely
$$k_{p,q}(z) = \frac{z}{(1-pz)(1-qz)} = z\ {}_1\phi_0\left[\frac{q^2}{p^2};\frac{q}{p},pz\right], \tag{2.7}$$
where $z \in \mathbb{D}$, $-1 \le p,q \le 1$. The case $p = q = 1$ leads to the famous standard Koebe function therefore, we will understand the function $k_{p,q}(z)$ as its generalization. Motivated essentially by the aforementioned earlier works, we aim here at study the properties of Koebe function (2.7).

Remark 2.1 *The functions*
$$z, \quad \frac{z}{1\pm z}, \quad \frac{z}{1\pm z^2}, \quad \frac{z}{(1\pm z)^2}, \quad \frac{z}{1\pm z+z^2}$$
are the only nine functions which are starlike, univalent and have integer coefficients in $\mathbb{D}$, (see [55] *for details). For apriopriate choice of p and q in $k_{p,q}$ we can get seven of these functions.*

This provides us with another motivation to study $k_{p,q}$.

2.2 Properties of generalized Koebe function $k_{p,q}$

In this section aside from examining the radius problem for $k_{p,q}$, we consider a related problem of examining the image of unit disc by $k_{p,q}$. The generalized Koebe function also naturally leads us to define object such as generalized class of typically real functions, which will important in our later analysis.

2.2.1 Radius of starlikness and convexity of order α

We begin with a statement of the result concerning the radii of starlikeness of order α and radii of convexity for the function $k_{p,q}$.

Theorem 2.1 *The function $k_{p,q}$ is α-starlike in $\mathbb{D}$ with*

$$\alpha = \alpha(p,q) = \frac{1}{2}\left(\frac{1-|p|}{1+|p|} + \frac{1-|q|}{1+|q|}\right).$$

Proof. By a simple calculation, we have

$$\frac{zk'_{p,q}(z)}{k_{p,q}(z)} = 1 + \frac{pz}{1-pz} + \frac{qz}{1-qz},$$

Using the obvious inequalities

$$\frac{1-r}{1+r} \le \mathrm{Re}\frac{1+z}{1-z} \le \frac{1+r}{1-r} \quad (|z| = r < 1),$$

we find that

$$Re\frac{zk'_{p,q}(z)}{k_{p,q}(z)} = \frac{1}{2}\mathrm{Re}\frac{1+pz}{1-pz} + \frac{1}{2}\mathrm{Re}\frac{1+qz}{1-qz} \ge \frac{1}{2}\frac{1-|p|}{1+|p|} + \frac{1}{2}\frac{1-|q|}{1+|q|}.$$

□

The class $\mathcal{S}^*(\frac{1}{2})$ is particulary important. Marx [97] and Strohäcker [145] proved that if $f(z)$ maps $\mathbb{D}$ onto a convex domain, then $f(z) \in \mathcal{S}^*(\frac{1}{2})$. Later, Gabriel [56] showed that the functions of the class $\mathcal{S}^*(\frac{1}{2})$ played an important role in the solution of certain differential equations. For $|pq| = 1$ the function $k_{p,q}$ is 1/2-starlike in $\mathbb{D}$, and according to result of Silverman [131] we have the following.

Remark 2.2 *For $|pq| = 1$ we have*

$$\left|\left(\frac{1 + zk''_{p,q}(z)/k'_{p,q}(z)}{zk'_{p,q}/k'_{p,q}(z)}\right) - 1\right| \le 1 \quad (|z| < (2\sqrt{3} - 3)^{1/2}).$$

The result is sharp.

Lemma 2.2 *Let $-1 \le p, q \le 1$, $|pq| \ne 1$. The function $k_{p,q}$ is convex of order α in $\mathbb{D}$ for $\alpha < \alpha(p,q)$, where*

$$\alpha(p,q) = \frac{2(1-|pq|)}{(1+|p|)(1+|q|)} - \frac{1+|pq|}{1-|pq|}.$$

Proof. We note that

$$1 + \frac{zk''_{p,q}(z)}{k'_{p,q}(z)} = \frac{1+pz}{1-pz} + \frac{1+qz}{1-qz} - \frac{1+pqz^2}{1-pqz^2},$$

hence, applying

$$\frac{1-r}{1+r} \le \mathrm{Re}\frac{1+z}{1-z} \le \frac{1+r}{1-r} \quad (|z| = r < 1),$$

we find that

$$\mathrm{Re}\left(1 + \frac{zk''_{p,q}(z)}{k'_{p,q}(z)}\right) \ge \frac{1-|p|}{1+|p|} + \frac{1-|q|}{1+|q|} - \frac{1+|p||q|}{1-|p||q|}.$$

The function is convex of order α, if the inequality

$$\frac{1-|p|}{1+|p|}+\frac{1-|q|}{1+|q|}-\frac{1+|p||q|}{1-|p||q|}>\alpha$$

holds, it means if $\alpha<\alpha(p,q)$, where

$$\alpha(p,q)=\frac{2(1-|pq|)}{(1+|p|)(1+|q|)}-\frac{1+|pq|}{1-|pq|}.$$

Since the right hand side is less than 1, the assertion follows. □

Corollary 2.3 *The function $k_{p,q}(z)$ is convex of order $\alpha=-\frac{1}{2}$ in $\mathbb{D}$ for $(p,q)\in\Omega_1\cup\Omega_2$, where $\Omega_1=\{|p|<\sqrt{17}-4,\ |q|<1\}$, and*

$$\Omega_2=\left\{\sqrt{17}-4<|p|<1,\ |q|<\frac{(1+|p|)\sqrt{9|p|^2+58|p|+1}-3|p|^2-12|p|-1}{2|p|(3-|p|)}\right\}.$$

Theorem 2.4 [100, Miller] *Assume that $f\in\mathcal{A}$, $f'(0)\neq 0$, and that it satisfies*

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>-\frac{1}{2}\quad(z\in\mathbb{D}).$$

If $g\prec f$, then

$$L[g]\prec L[f],$$

where

$$L[f](z)=\frac{2}{z}\int_0^z f(t)dt$$

is the Libera operator [90].
If $f(z)=a_0+a_1z+a_2z^2+\ldots$, then we have

$$\begin{aligned}L[f](z) &= \frac{2}{z}\int_0^z f(t)dt=\sum_{n=0}^{\infty}\frac{2a_n}{n+1}z^n\\ &= f(z)*\left(\frac{2}{z}\log\frac{1}{1-z}\right).\end{aligned}$$

Theorem 2.5 *Let $(p,q)\in\Omega_1\cup\Omega_2$, where Ω_1, Ω_2 are given in Corrolary 2.3, and $f\in\mathcal{A}$, then*

$$f\prec k_{p,q}(z)\Rightarrow L[f]\prec L[k_{p,q}(z)]. \tag{2.8}$$

Proof. Applying Corollary 2.3 and Theorem 2.4 we get (2.8). □

2.2.2 Image of unit disc by $k_{p,q}$

Now, we give a geometric characterization for $k_{p,q}$ function.

The function $k_{p,q}$

$$k_{p,q}(z)=\frac{z}{(1-pz)(1-qz)}=\frac{1}{2(p+q)}\left[\frac{(1+pz)(1+qz)}{(1-pz)(1-qz)}-1\right]\ (z\in\mathbb{D})$$

maps the unit disk onto a domain symmetric with respect to real axis. Indeed, the functions for the cases $p = q = -1$, as well as $p = q = 1$ map the unit disk onto the complex plane without the slit $(-\infty, -1/4]$ along the real axis, and slit $[1/4, \infty)$, respectively. When p or q are zero, we get the function $k_{p,0}(z) = z/(1-pz)$ that maps the unit disk onto a disk, which becomes the halfplane in the limiting cases $p = \pm 1$. The case $q = p$ leads to the function $k_{p,p}(z) = z/(1-pz)^2$ that maps the unit disk onto the ordinary Pascal snail (scaling in the direction of the imaginary axis by some factor) which starts from the disk (small p), through dimpled cardioid that becomes a plane with the single slit in the limiting cases $p \to \pm 1$. The other special case $k_{p,-p}(z) = z/(1-p^2z^2)$ maps the unit disk onto the Cassini oval which in the limiting cases $p \to \pm 1$ reduces to the plane with two disjunctive slits from $i/2$, and $-i/2$ to infinity.

Consider now the cases $p, q \neq \pm 1$. Let

$$h_{p,q}(z) = \frac{(1-pz)(1-qz)}{z}.$$

It is easy to find that $h_{p,q}$ maps the unit disk onto the interior of the ellipse

$$\mathcal{E} = \left\{ w = u + iv : \frac{(u+(p+q))^2}{(1+pq)^2} + \frac{v^2}{(1-pq)^2} = 1 \right\},$$

that has a center at $(-(p+q), 0)$, eccentricity $\varepsilon = 2\sqrt{|pq|}/(1+pq)$ $(pq \neq -1)$, and intersects the real axis at the points $((1-p)(1-q), 0), (-(1+p)(1+q), 0)$. The inverse transformation $T(w) = 1/w$ maps the ellipse onto the curve

$$\mathcal{E}' = \left\{ u + iv : \frac{(u+(p+q)(u^2+v^2))^2}{(1+pq)^2} + \frac{v^2}{(1-pq)^2} = (u^2+v^2)^2 \right\},$$

known as *Pascal snail (limaçon of Pascal).*

In the case when one of the parameter p or q is zero, say $q = 0$, the curve $\mathcal{E}'$ is the circle with the center at $S = (p/(1-p^2), 0)$ and the radius $R = 1/(1-p^2)$. In the other special case, when $p + q = 0$, we obtain the hippopede $(u^2+v^2)^2 = cu^2 + dv^2$, with $c = 1/(1+pq)^2, d = 1/(1-pq)^2$, that is the bicircular rational algebraic curve of degree 4, symmetric with respect to both axes. When $d > 0$ (that is our case) such a curve is known as an elliptic leminiscate of Booth. For the case $|p+q| = 1$ the Pascal snail is the conchoid of the circle, known also as a cardioid. For no case the hippopede is the Bernoulli leminiscate because it corresponds the case $d = -c$.

The equation of Pascal snail can be also transformed onto a form

$$\left[u^2 + v^2 - \frac{p+q}{(1-p^2)(1-q^2)} u \right]^2 = \frac{1}{(1-p^2)(1-q^2)} \left(u^2 + \frac{(1+pq)^2}{(1-pq)^2} v^2 \right). \tag{2.9}$$

Below, we present some images of the unit disk by $k_{p,q}$, for some special choice of the parameters.

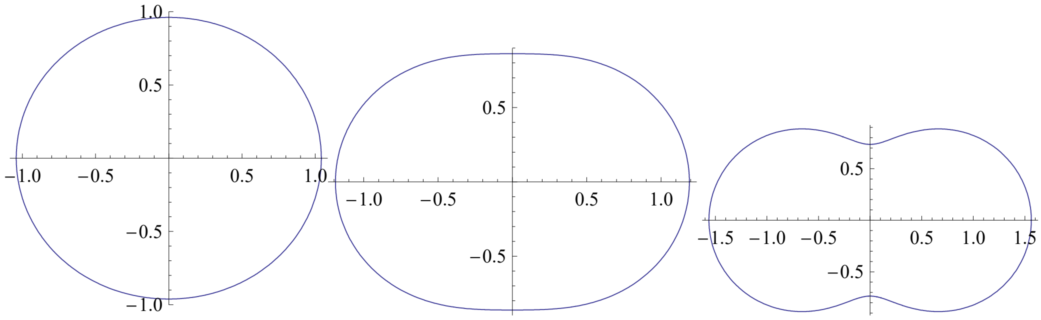

Fig. 2.1. The images of the unit disk by $k_{p,q}$ for $p + q = 0$.

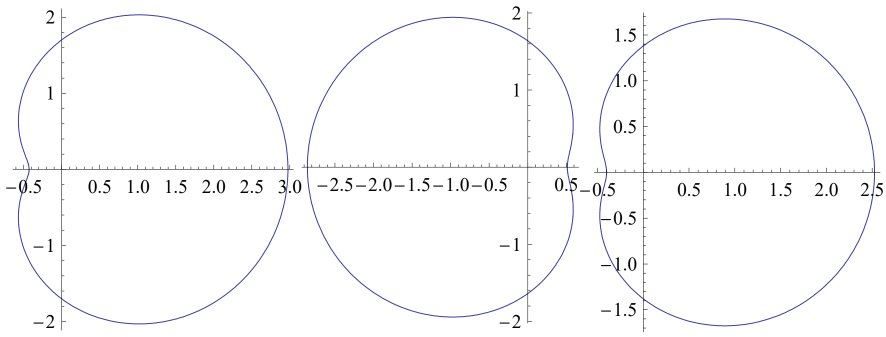

Fig. 2.2. The images of the unit disk by $k_{p,q}$ for $p + q = 1; p + q < 1; p + q > 1$.

The special case of the Pascal snail, it means an elliptic lemniscate (named later Booth lemniscate), was a topic of investigation by Booth [17–20], apparently given this name by Loria [93]. It has many interesting applications, for example, in mechanical linkages [161] and fluid physics [119]. It also appears in a solid geometry as an intersection of a plane with a spindle torus, or with Fresnel's elasticity surface [146].

We remain also, that in the case $q = 1/p$ and $p \in (0, 1)$, the function $k_{p,q}$ maps the unit disk onto a complement of the interval $[-p/(1-p)^2, -p/(1+p)^2]$ on the negative axis. In the family $S(p)$ of meromorphic univalent functions with the normalization $f(0) = 0, f'(0) = 1$, and $f(p) = \infty$ the function

$$k_{p,\frac{1}{p}} = \frac{z}{\left(1 - \frac{1}{p}z\right)(1 - pz)}$$

plays role of the Koebe functions. For example, Jenkins [72] showed that, if $f \in S(p)$ is of the form (1.1), and

$$k_{p,1/p}(z) = z + \sum_{n=2}^{\infty} A_n z^n \quad (|z| < p),$$

then $|a_n| \le A_n$ for any $n \ge 2$. Fenchel [46] solved the minimum modulus problem and showed that

$$\min_{|z| \to r} |f(z)| \ge |k_{p,1/p}(-r)| \quad \text{for all } f \in S(p).$$

However, when $q = 1/p$ and $p \in (0, 1)$, we get $q > 1$ so that, this case is outside the considered interval.

2.2.3 Some extremal problems

Now, we discuss some particular properties of $k_{p,q}$. By the properties of the ellipse $\mathcal{E}$ it is easy to see that the points of the intersection with the real axis are $\left(-\frac{1}{(1+p)(1+q)},0\right)$ and $\left(\frac{1}{(1-p)(1-q)},0\right)$. Therefore we may expect that they are realize min and max of the real part. Although, as we can see on Fig. 2.1. and Fig. 2.2. this is not the case.

Proposition 2.6 *Let $-1<p,q<1$. The values of $\max\limits_{0\le t\le 2\pi} \operatorname{Re} k_{p,q}(e^{it})$, $\min\limits_{0\le t\le 2\pi} \operatorname{Re} k_{p,q}(e^{it})$, are the following*

$$\max_{0\le t\le 2\pi} \operatorname{Re} k_{p,q}(e^{it}) = \begin{cases} \dfrac{1}{1-p} & for \quad q=0,\\ \dfrac{1}{1-q} & for \quad p=0,\\ \dfrac{(1+pq)^2}{2(1-pq)[2\sqrt{pq(1-p^2)(1-q^2)}-(p+q)(1-pq)]} & for \quad (p,q)\in B_2,\\ \frac{1}{(1-p)(1-q)} & otherwise, \end{cases}$$

$$\min_{0\le t\le 2\pi} \operatorname{Re} k_{p,q}(e^{it}) = \begin{cases} \dfrac{-1}{1+p} & for \quad q=0,\\ \dfrac{-1}{1+q} & for \quad p=0,\\ \dfrac{-(1+pq)^2}{2(1-pq)[2\sqrt{pq(1-p^2)(1-q^2)}+(p+q)(1-pq)]} & for \quad (p,q)\in B_1,\\ \dfrac{-1}{(1+p)(1+q)} & otherwise, \end{cases}$$

where

$$B_1=\{0<p<1,\ q_1(p)<q<1\}\,,\quad B_2=\{-1<p<0,\ -1<q<q_2(p)\}\,,$$

with

$$\begin{aligned} q_1(p) &= \frac{(1+p)\sqrt{p^2+14p+1}-(p^2+6p+1)}{2p(1-p)},\\ q_2(p) &= \frac{(1-p)\sqrt{p^2-14p+1}-p^2+6p-1}{2p(1+p)}. \end{aligned}$$

Proof. Observe that

$$\operatorname{Re} k_{p,q}(e^{it}) = \frac{-(p+q)+(1+pq)\cos t}{(1-2p\cos t+p^2)(1-2q\cos t+q^2)} =: R(t),$$

so that the extremes of $\operatorname{Re} k_{p,q}(e^{it})$ are located at the zeros of the derivative of R with respect to t. By the symmetry, we restrict our consideration to the interval $t\in[0,\pi]$. The equation $R'(t)=0$ is equivalent to the equation

$$\sin t\left\{4pq(1+pq)\cos^2 t-8pq(p+q)\cos t+(1+pq)\left[(p+q)^2-(1-pq)^2\right]\right\}=0,$$

therefore the obvious critical points are $t=0, t=\pi$ and the solutions of the equation

$$4pq(1+pq)\cos^2 t-8pq(p+q)\cos t+(1+pq)\left[(p+q)^2-(1-pq)^2\right]=0. \tag{2.10}$$

If p or q are zero, say $q = 0$, then the only critical points remain $t = 0$ or $t = \pi$. For such t we have $\operatorname{Re} k_{p,q}(e^{it}) = 1/(1-p)$ and $-1/(1+p)$, respectively, and these are the extremes.

Set

$$R_\pi = R(\pi) = \frac{-1}{(1+p)(1+q)}, \quad R_0 = R(0) = \frac{1}{(1-p)(1-q)}.$$

Denote the polynomial (2.10) by $V(t)$, and assume now, $pq \neq 0$. Then V is a nondegenerate quadratic polynomial with the discriminant $\Delta = 16pq(1-p^2)(1-q^2)(1-pq)^2$ which is negative for $pq < 0$, and positive when $pq > 0$. In the first case ($pq < 0$) we are going back to the critical points $t = 0$ or $t = \pi$, and for such t we obtain $\max \operatorname{Re} k_{p,q}(e^{it}) = R_0$, and $\min \operatorname{Re} k_{p,q}(e^{it}) = R_\pi$, respectively. In the second case (when $pq > 0$), there are two roots of V. We have to check that they satisfy the condition $|\cos t| \leq 1$. The roots of V have the form

$$\cos t_{1,2} = \frac{p+q}{1+pq} \pm \frac{1-pq}{2(1+pq)} \sqrt{\frac{(1-p^2)(1-q^2)}{pq}}, \tag{2.11}$$

and let t_1 corresponds to the expression with the minus sign, while t_2 with the plus sign inside. Denote

$$\begin{aligned} q_1(p) &= \frac{(1+p)\sqrt{p^2+14p+1} - (p^2+6p+1)}{2p(1-p)}, \\ q_2(p) &= \frac{(1-p)\sqrt{p^2-14p+1} - p^2+6p-1}{2p(1+p)}, \end{aligned}$$

and

$$B_1 = \{0 < p < 1,\ q_1(p) < q < 1\}, \quad B_2 = \{-1 < p < 0,\ -1 < q < q_2(p)\}.$$

Then a condition $|\cos t_{1,2}| < 1$ holds, if $(p,q) \in B_1 \cup B_2$, $\cos t_1 = -1$ for $q = q_1(p)$, $p \in (0,1)$, and $\cos t_2 = 1$ for $q = q_2(p)$, $p \in (-1,0)$, see Fig 2.3.

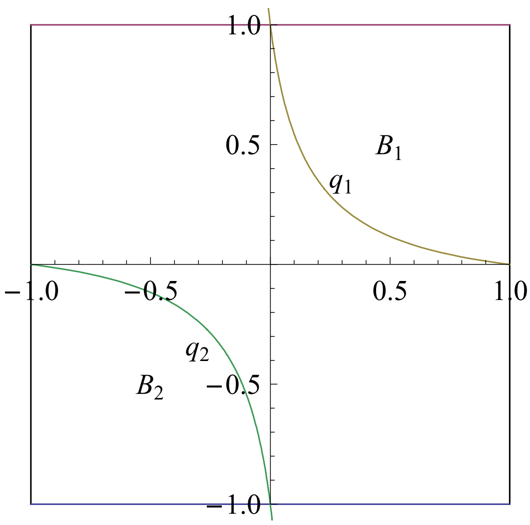

Fig. 2.3. The range of the parameters p, q for $\operatorname{Re} k_{p,q}$.

Assume now $(p,q) \in B_1$. Then there exists a single $t_1 \in (0,\pi)$, given by (2.11), such that $V(t_1)=0$. The value of $\operatorname{Re} k_{p,q}(e^{it})$ at t_1 is

$$R_1 = R(t_1) = \frac{-(1+pq)^2}{2(1-pq)[2\sqrt{pq(1-p^2)(1-q^2)}+(p+q)(1-pq)]},$$

and $R_1 < R_\pi < 0,\ R_0 > 0$. Therefore, $\max \operatorname{Re} k_{p,q}(e^{it})$ is R_0, and $\min \operatorname{Re} k_{p,q}(e^{it}) = R_1$.

If $(p,q) \in B_2$, then $V(t_2)=0$, and

$$R_2 = R(t_2) = \frac{(1+pq)^2}{2(1-pq)[2\sqrt{pq(1-p^2)(1-q^2)}-(p+q)(1-pq)]}.$$

In that case we have $R_2 > R_0 > 0,\ R_\pi < 0$, so that $\max \operatorname{Re} k_{p,q}(e^{it}) = R_2$, and $\min \operatorname{Re} k_{p,q}(e^{it}) = R_\pi$.

For $0<p<1, q=q_1(p)$, and $-1<p<0, q=q_2(p)$ the values at the critical points are R_0, and R_π, thus then $\max \operatorname{Re} k_{p,q}(e^{it}) = R_0$, and $\min \operatorname{Re} k_{p,q}(e^{it}) = R_\pi$. In conclusion the thesis follows. □

Proposition 2.7 *Let $-1<p,q<1$. Then, for $|z|=r<1$*

$$\frac{r}{(1+|p|r)(1+|q|r)} \le |k_{p,q}(z)| \le \frac{r}{(1-|p|r)(1-|q|r)},$$

and

$$\frac{1}{(1+|p|)(1+|q|)} \le |k_{p,q}(e^{it})| \le \frac{1}{(1-|p|)(1-|q|)}.$$

Proof. The proof is obvious by well known modulus properties. □

Remark 2.3 *By the above Proposition we see that $k_{p,q}(\mathbb{D})$ contains at least a disk of a radius $\dfrac{1}{(1+|p|)(1+|q|)}$.*

Proposition 2.8 *Let $-1<p,q<1$. Then we have*

$$\max_{0\le t\le 2\pi} \left|\operatorname{Im} k_{p,q}(e^{it})\right| = \begin{cases} \dfrac{1}{1-p^2} & for \quad q=0,\\ \dfrac{1}{1-q^2} & for \quad p=0,\\ \dfrac{1-p^2}{(1+p^2)^2} & for \quad p\in[1-\sqrt{2},\sqrt{2}-1]\setminus\{0\}, q=-p,\\ \dfrac{1}{4|p|} & for \quad p\in(-1,1-\sqrt{2})\cup(\sqrt{2}-1,1), q=-p,\\ \Phi(t_0) & for \quad (p,q)\in l_1\cup l_2,\\ \Phi(t_1) & q>-p,\\ \Phi(t_2) & q<-p,\end{cases}$$

where

$$l_1 = \left\{p \in (-1,\sqrt{3}-2],\ q(p) = \frac{-4p-\sqrt{-1+14p^2-p^4}}{1+p^2}\right\},$$

$$l_2 = \left\{p \in [2-\sqrt{3},1),\ q(p) = \frac{-4p+\sqrt{-1+14p^2-p^4}}{1+p^2}\right\},$$

$\Phi(t)$ is a function given by the equality

$$\Phi(t) = \frac{(1+pq)\sin t}{(1-2p\cos t+p^2)(1-2q\cos t+q^2)},$$

t_0 is a given by the equation

$$\cos t = -\sqrt[3]{\frac{(p+q)(1+pq)}{2pq}},$$

and t_1, t_2 are the solutions of the equation

$$-4pq\cos^3 t + [8pq + (1+p^2)(1+q^2)]\cos t - 2(p+q)(1+pq) = 0,$$

from the interval $(0, \pi/2)$, and $(\pi/2, \pi)$, respectively. Moreover

$$|\Phi(t)| \le \frac{1+pq}{(1-|p|)^2(1-|q|)^2} \quad (-1 < p, q < 1).$$

Proof. We note that $\operatorname{Im} k_{p,q}(e^{it})$ equals

$$\operatorname{Im} k_{p,q}(e^{it}) = \frac{(1+pq)\sin t}{(1-2p\cos t+p^2)(1-2q\cos t+q^2)} =: \Phi(t),$$

so that may be estimated by

$$|\Phi(t)| \le \frac{1+pq}{(1-|p|)^2(1-|q|)^2}.$$

However, in some special cases we can get better bounds. The maximum of $\Phi(t)$ is attained at the point in which

$$-4pq\cos^3 t + [8pq + (1+p^2)(1+q^2)]\cos t - 2(p+q)(1+pq) = 0.$$

Let the above polynomial be denoted by $W(t)$. Without loss of generality, we restrict our consideration to the interval $t \in [0, \pi]$. The roots of the polynomial W may be found directly by the well known Cardano method, but they have a very complicated form, therefore we describe its locations.

In the case, when $pq = 0$, say $q = 0$, we immediately have

$$\max_{0\le t\le 2\pi} \left|\operatorname{Im} k_{p,q}(e^{it})\right| = \frac{1}{1-p^2}.$$

Hence, in the sequel, we assume $pq \ne 0$.

We note that

$$W(0) = (1+pq-p-q)^2 > 0,$$

and

$$W(\pi) = -(1+pq+p+q)^2 < 0,$$

therefore exist one or three zeros of W in $(0, \pi)$. Set

$$q_3(p) = \frac{-4p - \sqrt{-1+14p^2-p^4}}{1+p^2}, \quad q_4(p) = \frac{-4p + \sqrt{-1+14p^2-p^4}}{1+p^2},$$

and

$$l_1 = \left\{p \in (-1, \sqrt{3}-2],\ q = q_3(p)\right\}, \quad l_2 = \left\{p \in [2-\sqrt{3}, 1),\ q = q_4(p)\right\}.$$

Also, let $D_1, D_2, \ldots, D_{10}$ denote the sets from the Fig. 2.4.

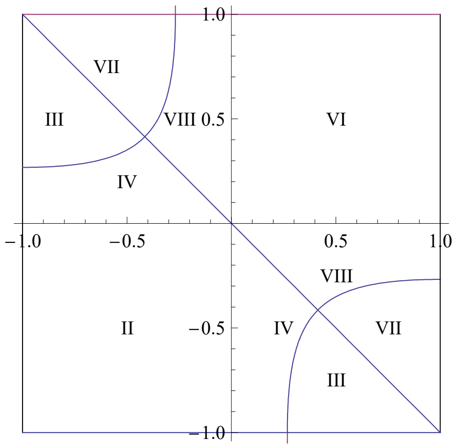

Fig. 2.4. The range of the parameters p, q for $\max |\operatorname{Im} k_{p,q}(e^{it})|$.

For the case $q = -p$ we have $W(\pi/2) = 0$ (this case correspond to a domain $k_{p,q}(\mathbb{D})$ which is symmetric with respect to both axes, cf. (2.9)). Moreover, when $p \in (-1, 1-\sqrt{2}) \cup (\sqrt{2}-1, 1)$, and $q = -p$ there are additional zeros t_1, t_2 of W, satisfying

$$\cos^2 t = 1 - \frac{(1+p)^2(1-p)^2}{4p^2}.$$

At such points Φ achieves its maximum that equals $\Phi(t_{1,2}) = 1/(4|p|)$. For $p \in [1-\sqrt{2}, \sqrt{2}-1] \setminus \{0\}$ and $q = -p$, the function $\Phi(t)$ attains its maximum at $t = \pi/2$, that equals $(1-p^2)/(1+p^2)^2$.

Assume now $q \neq -p$. We have

$$\begin{aligned}
8pq + (1+p^2)(1+q^2) &> 0 \quad \text{for} \quad (p,q) \in D_1 \cup D_2 \cup ... \cup D_6, \\
8pq + (1+p^2)(1+q^2) &< 0 \quad \text{for} \quad (p,q) \in D_7 \cup ... \cup D_{10}, \\
8pq + (1+p^2)(1+q^2) &= 0 \quad \text{for} \quad (p,q) \in l_1 \cup l_2.
\end{aligned}$$

In the case, when $(p,q) \in l_1 \cup l_2$ there exist t_0, such that $W(t_0) = 0$, given by the equation

$$-4pq\cos^3 t - 2(p+q)(1+pq) = 0.$$

When $p + q > 0$ then $t_0 \in (0, \pi/2)$, and when $p + q < 0$ then $t_0 \in (\pi/2, \pi)$, and $\Phi(t_0)$ attains its maximum.

Let now $(p,q) \in D_2 \cup ... \cup D_6$. The sign of the derivative

$$W'(t) = \sin t \left\{12pq\cos^2 t - [8pq + (1+p^2)(1+q^2)]\right\}$$

depends only on the second factor of the product, since $\sin t > 0$ on $(0, \pi)$. The equation

$$12pq\cos^2 t - [8pq + (1+p^2)(1+q^2)] = 0,$$

has no solution $t \in (0, \pi)$ for $(p,q) \in D_2 \cup ... \cup D_6$ since a left hand side of the above equation is negative in the entire interval $(0, \pi)$. Therefore $W' < 0$ on $(0, \pi)$, so that W decreases in $(0, \pi)$, and we conclude that there exists the only zero $t_0 \in (0, \pi)$ in which Φ attains its maximum. If $q > -p$, then $t_0 \in (0, \pi/2)$, and if $q < -p$

then $t_0 \in (\pi/2, \pi)$. Hence, for $(p,q) \in D_2 \cup D_6$ we have $t_0 \in (0, \pi/2)$, and when $(p,q) \in D_3 \cup D_5$ $t_0 \in (\pi/2, \pi)$.

Now, set $x = \cos t$, and $w(x) = W(\cos t)$. We discus the signs of coefficients of the polynomial w, in order to describe the location of its zeros. In the case, when $(p,q) \in D_1$ the sequence of the sign of the coefficients of the polynomial w is $(-,+,-)$, and for $w(-x)$ is $(+,-,-)$. Therefore, by the classical rule of Descartes-Harriot, there are at most two positive and one negative root x. Since $w(-1) < 0, w(0) < 0$ and $w(1) > 0$ the root $x \in (0,1)$, so that there exists single $t_0 \in (0, \pi/2)$ in which Φ attains its maximum.

Similar situation holds, when $(p,q) \in D_4$. Then the signs of the coefficients of the polynomial $w(x)$ are $(-,+,+)$ and $(-x)$ are $(+,-,+)$. By the fact that $w(-1) < 0, w(0) > 0$ and $w(1) > 0$ the root $x \in (-1, 0)$, thus there is $t_0 \in (\pi/2, \pi)$ in which Φ attains its maximum.

For $(p,q) \in D_7 \cup ... \cup D_{10}$ the an analysis of a behavior of a polynomial w (by using Descartes-Harriot method of signs of coefficients) indicates that there are three roots t_1, t_2, t_3 of a polynomial W. In $t_1 \in (0, \pi/2), t_3 \in (\pi/2, \pi)$ a function Φ attains its maxima and a minimum at t_2.

Reassuming, we obtain the thesis. $\square$

Chapter 3

Polynomials connected with $\mathcal{T}^{p,q}$ class of functions

The classical Chebyshev polynomials of the first and second kind have been known since the late 18th century, when was defined using de Moivre's formula by Chebyshev [31]. In the study of differential equations they arise as the solution to the Chebyshev differential equations

$$(1-x^2)y'' - xy' + n^2y = 0,$$

and

$$(1-x^2)y'' - 3xy' + n(n+2)y = 0,$$

for the polynomials of the first and second kind, respectively. They are the special cases of the Sturm–Liouville differential equation. Two properties of Chebyshev polynomials make them exceptionally suitable for approximations: monic Chebyshev polynomials minimize all norms among monic polynomials of a given degree, and satisfy discrete orthogonality relation. Based on these properties they are widely used in many areas of numerical analysis; uniform approximation, least-squares approximation, numerical solution of ordinary and partial differential equations, and so on. Therefore several its extensions occur, see Akhiezer [3], [4], and Akhiezer and Tomčuk [5], Tomčuk [152], Ismail [70], Peherstorfer [111] and many others. The Chebyshev polynomials are orthogonal not only as polynomials in real variable but also as polynomials in a complex variable z on elliptical contours and domains of the complex plane (the foci of the ellipses being ± 1). This property is exploited in fields that rely on complex variable techniques. Later on polynomials which are not fully in agreement with orthogonal polynomials, hence called the (generalized) Chebyshev type polynomials appeared, see for example Peherstorfer [111]. The Chebyshev type polynomials satisfy similar extremal properties to the classical Chebyshev polynomials on $[-1,1]$. The extremal polynomials also have the property that they are orthogonal with respect to some weight function. For other generalization the reader is referred to [30]. Some of generalized Chebyshev type polynomials are associated with generalized Koebe function, as was observed in [148].

In this chapter, we will explore the sequences of polynomials of the generalized Chebyshev polynomial of the second kind $U_n(p,q;e^{i\theta})$ and of the first kind $T_n(p,q;e^{i\theta})$. Each of these sequences is useful in applications for a particular reason. The Chebyshev polynomials of second kind are defined by the fact its connections with the generalized typically real functions; similarly as was in the classical case. The coefficient problem for generalized typically real functions provides one motivation to study properties of Chebyshev polynomials. Another motivation for the present body of work is discussed in great detail in Section 3.3. This chapter highlights the connections between this two types of orthogonal polynomials. Thus, we start with several explicit expressions for these polynomials. The obtained results for $p=q=1$ give the corresponding ones for Chebyschev polynomials of the first and second kind.

3.1 Basic properties of $U_n(p,q,e^{i\theta})$ and $T_n(p,q;e^{i\theta})$

First we concentrate on a clear presentation of some properties of the polynomials $U_n(p,q;e^{i\theta})$. The polynomials $U_n(p,q;e^{i\theta})$ appear by its connection with the class of generalized typically real functions in Chapter 4.

Definition 3.1 *Let $\theta \in [0,2\pi], -1 \le p,q \le 1$. The generalized Chebyshev polynomials of the second kind $U_n(p,q;e^{i\theta})$ are defined by*

$$\begin{aligned} \Psi^{(p,q)}(e^{i\theta};z) &= \frac{1}{(1-pze^{i\theta})(1-qze^{-i\theta})} \\ &= \sum_{n=0}^{\infty} U_n(p,q;e^{i\theta})z^n \quad (z\in\mathbb{D}), \end{aligned} \tag{3.1}$$

where $\theta \in [0,2\pi], -1 \le p,q \le 1$, or by an explicit formulas

$$\begin{aligned} U_0(p,q;e^{i\theta}) &= 1, \\ U_1(p,q;e^{i\theta}) &= pe^{i\theta}+qe^{-i\theta}, \\ U_n(p,q;e^{i\theta}) &= \frac{p^{n+1}e^{i(n+1)\theta}-q^{n+1}e^{-i(n+1)\theta}}{pe^{i\theta}-qe^{-i\theta}} \quad (n\ge 2). \end{aligned} \tag{3.2}$$

Remark 3.1 *We observe moreover, that the polynomials $U_n(p,q;e^{i\theta})$ can be expressed via classical Chebyshev polynomials of the second kind $U_n(x)$, where the variable x is now complex and have special form. Indeed, putting in the generating function* (3.1) *the value $\frac{z}{\sqrt{pq}}$, $pq\neq 0$ instead of z, and comparing the result with the generating function for $U_n(x)$ we conclude that*

$$U_n(p,q;e^{i\theta}) = (\sqrt{pq})^n U_n\left(\frac{pe^{i\theta}+qe^{-i\theta}}{2\sqrt{pq}}\right) \quad (pq\neq 0).$$

It can be easily seen that, if $\theta \in [0, 2\pi]$, then the function

$$\omega(\theta) = \frac{pe^{i\theta} + qe^{-i\theta}}{2\sqrt{pq}}$$

describes an ellipse E with semi-axes: $a = \left|\frac{(p+q)}{2\sqrt{pq}}\right|$ and $b = \left|\frac{(p-q)}{2\sqrt{pq}}\right|$.

Definition 3.2 *Let $\theta \in [0, 2\pi], -1 \le p, q \le 1$. The generalized Chebyshev polynomials of the first kind are defined by*

$$T_n(p,q;e^{i\theta}) = \frac{1}{2}(p^n e^{in\theta} + q^n e^{-in\theta}) \quad (n = 0, 1, ...), \tag{3.3}$$

where $\theta \in [0, 2\pi]$, $-1 \le p, q \le 1$.

Remark 3.2 *For $p = q = 1$ and $\theta \in [0, 2\pi]$ the polynomials $T_n(p,q;e^{i\theta})$ reduce to the Chebyshev polynomials of the first kind.*

Using the representation (3.2) we establish the fundamental properties of the polynomials $U_n(p,q;e^{i\theta})$ below.

Theorem 3.1

a) *The trigonometric polynomials $U_n(p,q;e^{i\theta})$ satisfy the three-term recurrence relation*

$$\begin{aligned} U_0(p,q;e^{i\theta}) &= 1, \\ U_1(p,q;e^{i\theta}) &= pe^{i\theta} + qe^{-i\theta}, \\ U_{n+2}(p,q;e^{i\theta}) &= (pe^{i\theta} + qe^{-i\theta})U_{n+1}(p,q;e^{i\theta}) - pqU_n(p,q;e^{i\theta}) \quad (n \ge 0). \end{aligned} \tag{3.4}$$

b) *The function $y(\theta) = U_n(p,q;e^{i\theta})$ satisfies the following differential equation of the second order*

$$y''(\theta)(pe^{i\theta} - qe^{-i\theta}) + 2i(pe^{i\theta} + qe^{-i\theta})y'(\theta) + n(n+2)(pe^{i\theta} - qe^{-i\theta})y(\theta) = 0.$$

c) *The polynomials $U_n(p,q;e^{i\theta})$, satisfy the following orthogonality relation*

$$\int_E U_n(p,q;e^{i\theta})\overline{U_m}(p,q;e^{i\theta})\rho(\theta)d\theta = \begin{cases} 0 & \text{if} \quad m \ne n, \\ \dfrac{\pi}{4pq}\left(p^{2(n+1)} + q^{2(n+1)}\right) & \text{if} \quad m = n, \end{cases}$$

where $\rho(\theta) = \dfrac{-1}{2i\sqrt{pq}}(pe^{-i\theta} - qe^{i\theta})$.

Proof.

a) Setting $n = 0$ and $n = 1$ in (3.2) we get the two first equalities. Now we put

$$U_{n+1}(p,q;e^{i\theta}) = \frac{p^{n+2}e^{i(n+2)\theta} - q^{n+2}e^{-i(n+2)\theta}}{pe^{i\theta} - qe^{-i\theta}}. \tag{3.5}$$

Next

$$U_{n+2}(p,q;e^{i\theta}) = \frac{p^{n+3}e^{i(n+3)\theta} - q^{n+3}e^{-i(n+3)\theta}}{pe^{i\theta} - qe^{-i\theta}}.$$

We multiply (3.5) by $pe^{i\theta}+qe^{-i\theta}$ and (3.2) by pq, then we get

$$\begin{aligned} & pe^{i\theta}+qe^{-i\theta}U_{n+1}(p,q;e^{i\theta}) \\ = & \frac{p^{n+3}e^{i(n+3)\theta}-pq^{n+2}e^{-i(n+1)\theta}-q^{n+3}e^{-i(n+3)\theta}+qp^{n+2}e^{i(n+1)\theta}}{pe^{i\theta}-qe^{-i\theta}} \end{aligned}$$

and

$$pqU_n(p,q;e^{i\theta})=\frac{qp^{n+2}e^{i(n+1)\theta}-pq^{n+2}e^{-i(n+1)\theta}}{pe^{i\theta}-qe^{-i\theta}}.$$

Afterwards we get the desired recurrence formula by substracting $pqU_n(p,q;e^{i\theta})$ from $pe^{i\theta}+qe^{-i\theta}U_{n+1}(p,q;e^{i\theta})$.

b) From (3.2) we have

$$(pe^{i\theta}-qe^{-i\theta})y(\theta)=p^{n+1}e^{i(n+1)\theta}-q^{n+1}e^{-i(n+1)\theta}. \tag{3.6}$$

After differentiation of (3.6) we first get

$$i(pe^{i\theta}+qe^{-i\theta})y(\theta)+(pe^{i\theta}-qe^{-i\theta})y'(\theta)=i(n+1)(p^{n+1}e^{i(n+1)\theta}+q^{n+1}e^{-i(n+1)\theta})$$

and next

$$\begin{aligned} - & (pe^{i\theta}-qe^{-i\theta})y(\theta)+2i(pe^{i\theta}+qe^{-i\theta})y'(\theta)+(pe^{i\theta}-qe^{-i\theta})y''(\theta) \\ = & -(n+1)^2(p^{n+1}e^{i(n+1)\theta}-q^{n+1}e^{-i(n+1)\theta}), \end{aligned}$$

from which we obtain the result.

c) In order to prove c), we compute

$$\begin{aligned} & \int\limits_E U_n(p,q;e^{i\theta})\overline{U_m}(p,q;e^{i\theta})\rho(\theta)d\omega \\ = & \int\limits_{-\pi}^{\pi} U_n(p,q;e^{i\theta})\overline{U_m}(p,q;e^{i\theta})\frac{1}{2i\sqrt{pq}}(pe^{-i\theta}-qe^{i\theta})\Big(\frac{pe^{i\theta}-qe^{-i\theta}}{2\sqrt{pq}}\Big)id\theta \\ = & \frac{1}{4pq}\int\limits_{-\pi}^{\pi}(p^{n+m+2}e^{i(n-m)\theta}-p^{n+1}q^{m+1}e^{i(n+m+2)\theta} \\ - & p^{m+1}q^{n+1}e^{-i(n+m+2)\theta}+q^{n+m+2}e^{i(m-n)\theta})d\theta \\ & = \begin{cases} 0 & \text{if} \quad m\neq n, \\ \dfrac{\pi}{4pq}\left(p^{2(n+1)}+q^{2(n+1)}\right) & \text{if} \quad m=n, \end{cases} \end{aligned}$$

and the thesis follows. □

Remark 3.3 *We observe that the trigonometric polynomials $U_n(p,q;e^{i\theta})$ can be considered as the boundary values for $z=e^{i\theta}$ of the following symmetric Laurent polynomials:*

$$U_n(p,q;z)=p^nz^n+p^{n-1}qz^{n-2}+p^{n-2}q^2z^{n-4}+...+pq^{n-1}\frac{1}{z^{n-2}}+\frac{q^n}{z^n} \quad (z\neq 0),$$

$$U_n(p,q;z)=U_n\left(p,q;\frac{q}{pz}\right).$$

Theorem 3.2 *Let $\theta \in [0, 2\pi], -1 \leq p, q \leq 1$. Then the following relations hold.*

a) The trigonometric polynomials $T_n(p,q;e^{i\theta})$ satisfy the three-term recurrence relation

$$\begin{aligned} T_0(p,q;e^{i\theta}) &= 1, \\ T_1(p,q;e^{i\theta}) &= \tfrac{1}{2}(pe^{i\theta} + qe^{-i\theta}), \\ T_{n+2}(p,q;e^{i\theta}) &= (pe^{i\theta} + qe^{-i\theta})T_{n+1}(p,q;e^{i\theta}) - pqT_n(p,q;e^{i\theta}) \quad (n = 0, 1, ...). \end{aligned} \tag{3.7}$$

b) The function $y(\theta) = T_n(p,q;e^{i\theta})$, satisfies the differential equation of the second order

$$y''(\theta) + n^2 y(\theta) = 0.$$

c) The trigonometric polynomials $T_n(p,q;e^{i\theta})$ satisfy the following orthogonality relation

$$\int_{-\pi}^{\pi} T_n(p,q;e^{i\theta})\overline{T_m}(p,q;e^{i\theta})d\theta = \begin{cases} 0 & \text{if } m \neq n, \\ \frac{\pi}{2}(p^{2n} + q^{2n}) & \text{if } m = n \neq 0, \\ 2\pi & \text{if } m = n = 0. \end{cases} \tag{3.8}$$

d) The generating function of $\{T_n(p,q;e^{i\theta})\}_{n\geq 0}$ has the form:

$$\sum_{n=0}^{\infty} T_n(p,q;e^{i\theta})z^n = \frac{1 - \frac{(pe^{i\theta}+qe^{-i\theta})}{2}z}{(1-pe^{i\theta}z)(1-qe^{-i\theta}z)},$$

where $z \in \mathbb{D}$, $\theta \in [0, 2\pi]$, $-1 \leq p, q \leq 1$.

e)

$$\max_{0\leq\theta\leq 2\pi} |T_n(p,q;e^{i\theta})| = \frac{1}{2}\sqrt{p^{2n} + q^{2n} + 2(pq)^n}.$$

Proof.

a) Setting $n = 0$ and $n = 1$ in (3.3) we get the first two equalities. Now we put

$$2T_n(p,q;e^{i\theta}) = p^n e^{in\theta} + q^n e^{-in\theta}. \tag{3.9}$$

Therefore

$$2T_{n+1}(p,q;e^{i\theta}) = p^{n+1}e^{i(n+1)\theta} + q^{n+1}e^{-i(n+1)\theta}, \tag{3.10}$$

and

$$2T_{n+2}(p,q;e^{i\theta}) = p^{n+2}e^{i(n+2)\theta} + q^{n+2}e^{-i(n+2)\theta}.$$

First we multiply (3.9) by pq and (3.10) by $pe^{i\theta} + qe^{-i\theta}$, then we get

$$2pqT_n(p,q;e^{i\theta}) = qp^{n+1}e^{in\theta} + pq^{n+1}e^{-in\theta},$$

$$2(pe^{i\theta}+qe^{-i\theta})T_{n+1}(p,q;e^{i\theta}) = p^{n+2}e^{i(n+2)\theta}+pq^{n+1}e^{-in\theta}+qp^{n+1}e^{in\theta}+q^{n+2}e^{-i(n+2)\theta}.$$

Afterwards we get the desired formula by substracting $pqT_n(p,q;e^{i\theta})$ from $(pe^{i\theta} + qe^{-i\theta})T_{n+1}(p,q;e^{i\theta})$.

b) Using the explicit formula (3.3), after double differentiation we obtain the desired result.

c) The scalar product of $T_n(p,q;e^{i\theta})$ and $T_m(p,q;e^{i\theta})$ equals

$$\begin{aligned}\langle T_n, T_m\rangle &= \int_0^{2\pi} T_n(p,q;e^{i\theta})\overline{T_m}(p,q;e^{i\theta})d\theta \\ &= \frac{1}{4}\int_0^{2\pi} (p^{n+m}e^{i(n-m)\theta} + p^n q^m e^{i(n+m)\theta} + q^n p^m e^{-i(n+m)\theta} \\ &+ \ q^{n+m}e^{-i(n-m)\theta})d\theta.\end{aligned}$$

Since

$$\int_0^{2\pi} e^{im\theta}d\theta = \begin{cases} 0 & \text{if } m \neq 0, \\ 2\pi & \text{if } m = 0, \end{cases}$$

we easily see that $\langle T_m, T_n\rangle$ is expressed by (3.8).

d) By (3.3) we have

$$\begin{aligned}\sum_{n=0}^{\infty} T_n(p,q;e^{i\theta})z^n &= \frac{1}{2}\sum_{n=0}^{\infty}(p^n e^{in\theta} + q^n e^{-in\theta})z^n \\ &= \frac{1}{2}\sum_{n=0}^{\infty}(pe^{i\theta}z)^n + \frac{1}{2}\sum_{n=0}^{\infty}(qe^{-i\theta}z)^n \\ &= \frac{1}{2}\frac{1}{1-pe^{i\theta}z} + \frac{1}{2}\frac{1}{1-qe^{-i\theta}z} \\ &= \frac{1-\frac{(pe^{i\theta}+qe^{-i\theta})}{2}z}{(1-pe^{i\theta}z)(1-qe^{-i\theta}z)}.\end{aligned}$$

e) Straightforward calculation of (3.3) gives

$$\begin{aligned}|T_n(p,q;e^{i\theta})| &= \sqrt{\tfrac{1}{4}(p^n+q^n)^2\cos^2 n\theta + \tfrac{1}{4}(p^n-q^n)^2\sin^2 n\theta} \\ &= \tfrac{1}{2}\sqrt{p^{2n}+q^{2n}+2p^nq^n\cos 2n\theta},\end{aligned}$$

from which we get the desired result.

□

Remark 3.4 *Note that the norm of $T_n(p,q;e^{i\theta}$ is given by*

$$||T_n(p,q;e^{i\theta})||^2 = \int_0^{2\pi} T_n(p,q;e^{i\theta})\overline{T_n}(p,q;e^{i\theta})d\theta = \frac{\pi(p^{2n}+q^{2n})}{2},$$

and

$$||T_0(p,q;e^{i\theta})||^2 = 2\pi.$$

The system $\{T_n(p,q;e^{i\theta})\}$ is therefore not orthonormal. By introducing the respective weight we find that the system

$$\left\{\frac{T_0(p,q;e^{i\theta})}{\sqrt{2\pi}}, \Big(\sqrt{\frac{2}{\pi(p^{2n}+q^{2n})}}T_n(p,q;e^{i\theta}), \quad n=1,2,...\Big)\right\}$$

is orthonormal.

It is well known that the classical Chebyshev polynomials $U_n(x)$ and $T_n(x)$ are connected by several relations (see [9]). We observe that similar relations between $U_n(p,q;e^{i\theta})$ and $T_n(p,q;e^{i\theta})$ also holds.

Proposition 3.3 *For polynomials $U_n(p,q;e^{i\theta})$ and $T_n(p,q;e^{i\theta})$, where $\theta \in [0,2\pi]$, $-1 \le p,q \le 1$, we have*

$$\begin{aligned} T_n(p,q;e^{i\theta})T_m(p,q;e^{i\theta}) &= \tfrac{1}{2}T_{n+m}(p,q;e^{i\theta}) + \tfrac{1}{2}(pq)^m T_{n-m}(p,q;e^{i\theta}) \\ &= \tfrac{1}{2}T_{n+m}(p,q;e^{i\theta}) + \tfrac{1}{2}(pq)^n T_{m-n}(p,q;e^{i\theta}) \end{aligned}$$

and

$$U_{m-1}(T_n) = \frac{2p^m T_{2n}(p,q;e^{i\theta}) + 2p^{n+m}q^n - q^m}{2T_n(p,q;e^{i\theta})(pe^{i\theta} + qe^{-i\theta})}.$$

Proposition 3.4 *The trigonometric polynomials $U_n(p,q;e^{i\theta})$ and $T_n(p,q;e^{i\theta})$, where $\theta \in [0,2\pi]$, $-1 \le p,q \le 1$, have the following representation:*

$$U_n(p,q;e^{i\theta}) = \sum_{k=0}^{[n/2]} (-pq)^k \frac{(n-k)!}{(n-2k)!k!}(pe^{i\theta} + qe^{-i\theta})^{n-2k},$$

$$T_n(p,q;e^{i\theta}) = \frac{n}{2}\sum_{k=0}^{[n/2]} (-pq)^k \frac{(n-k-1)!}{(n-2k)!k!}(pe^{i\theta} + qe^{-i\theta})^{n-2k}.$$

Remark 3.5 *We observe that*

$$T_n(p,q;-e^{i\theta}) = (-1)^n T_n(p,q;e^{i\theta}),$$

or in other words, the even degree Chebyshev polynomials are even functions and the odd Chebyshev polynomials are odd functions. This follows immediately from (3.3).

3.2 Products, integrals and derivatives of generalized Chebyshev polynomials

It is well known that the classical Chebyshev polynomials satisfy several equalities, that involve factor x or $1 - x^2$ (with $x = \cos\theta$) see, for instance [98]. Also the product of both polynomials $T_n(x)$ and $U_n(x)$ were considered [98]. Let us remain the well know relation for $T_n(x), U_n(x)$ below.

$$\begin{aligned}
xT_n(x) &= \frac{1}{2}\left(T_{n+1}(x)+T_{|n-1|}(x)\right),\\
xU_n(x) &= \frac{1}{2}\left(U_{n+1}(x)+U_{|n-1|}(x)\right),\\
(1-x^2)T_n(x) &= -\frac{1}{4}T_{n+2}(x)+\frac{1}{2}T_n(x)-\frac{1}{4}T_{|n-2|}(x),\\
(1-x^2)U_n(x) &= -\frac{1}{4}U_{n+2}(x)+\frac{1}{2}U_n(x)-\frac{1}{4}U_{|n-2|}(x),\\
\int T_n(x)dx &= \begin{cases}\frac{1}{2}\left(\frac{T_{n+1}(x)}{n+1}-\frac{T_{|n-1|}}{n-1}\right), & n\neq 1,\\ \frac{1}{4}T_2(x), & n=1,\end{cases}\\
\int U_n(x)dx &= \frac{1}{n+1}T_{n+1}(x)+c,\\
T_n'(x) &= \frac{n}{2}\frac{T_{n-1}(x)-T_{n+1}(x)}{1-x^2},\\
U_n'(x) &= \frac{(n+2)U_{n-1}(x)-nU_{n+1}(x)}{2(1-x^2)},
\end{aligned}$$

In this section we formulate similar properties for generalized Chebyshev polynomials $T_n(p,q;e^{i\theta})$ and $U_n(p,q;e^{i\theta})$, below.

Proposition 3.5 *Let* $T_n(p,q;e^{i\theta})$ *and* $U_n(p,q;e^{i\theta})$ *be defined by* (3.3) *and* (3.2). *Then*

$$\begin{aligned}
\cos\theta T_n(p,q;e^{i\theta}) &= T_n(p,q;1)\left(\frac{\mathrm{Re}T_{n+1}(p,q;e^{i\theta})}{T_{n+1}(p,q;1)}+\frac{\mathrm{Re}T_{|n-1|}(p,q;e^{i\theta})}{T_{|n-1|}(p,q;1)}\right)\\
&+ U_{|n-1|}(p,q;1)\left(\frac{i\mathrm{Im}T_{n+1}(p,q;e^{i\theta})}{U_n(p,q;1)}+\frac{i\mathrm{Im}T_{|n-1|}(p,q;e^{i\theta})}{U_{|n-2|}(p,q;1)}\right).
\end{aligned}\tag{3.11}$$

$$\begin{aligned}
(1-\cos^2\theta)T_n(p,q;e^{i\theta}) &= \frac{1}{2}T_n(p,q;e^{i\theta})\\
&- \frac{1}{4}T_n(p,q;1)\left(\frac{\mathrm{Re}T_{n+2}(p,q;e^{i\theta})}{T_{n+2}(p,q;1)}+\frac{\mathrm{Re}T_{|n-2|}(p,q;e^{i\theta})}{T_{|n-2|}(p,q;1)}\right)\\
&- \frac{1}{4}U_{|n-1|}(p,q;1)\Big(\frac{i\mathrm{Im}T_{n+2}(p,q;e^{i\theta})}{U_{n+1}(p,q;1)}\\
&+ \frac{i\mathrm{Im}T_{|n-2|}(p,q;e^{i\theta})}{U_{|n-3|}(p,q;1)}\Big).
\end{aligned}\tag{3.12}$$

Proof. In order to prove (3.11) we apply (3.3). Then we have

$$
\begin{aligned}
\cos\theta T_n(p,q;e^{i\theta}) &= \cos\theta\left(\frac{1}{2}(p^n+q^n)\cos n\theta+\frac{1}{2}i(p^n-q^n)\sin n\theta\right)\\
&= \frac{1}{2}(p^n+q^n)\cos(n+1)\theta+\frac{1}{2}(p^n+q^n)\cos(n-1)\theta\\
&+ \frac{1}{2}i(p^n-q^n)\sin(n+1)\theta+\frac{1}{2}i(p^n-q^n)\sin(n-1)\theta\\
&= (p^n+q^n)\left(\frac{\mathrm{Re}T_{n+1}(p,q;e^{i\theta})}{p^{n+1}+q^{n+1}}+\frac{\mathrm{Re}T_{|n-1|}(p,q;e^{i\theta})}{p^{n-1}+q^{n-1}}\right)\\
&+ (p^n-q^n)\left(\frac{i\mathrm{Im}T_{n+1}(p,q;e^{i\theta})}{p^{n+1}-q^{n+1}}+\frac{i\mathrm{Im}T_{|n-1|}(p,q;e^{i\theta})}{p^{n-1}-q^{n-1}}\right)\\
&= T_n(p,q;1)\left(\frac{\mathrm{Re}T_{n+1}(p,q;e^{i\theta})}{T_{n+1}(p,q;1)}+\frac{\mathrm{Re}T_{|n-1|}(p,q;e^{i\theta})}{T_{|n-1|}(p,q;1)}\right)\\
&+ U_{|n-1|}(p,q;1)\left(\frac{i\mathrm{Im}T_{n+1}(p,q;e^{i\theta})}{U_n(p,q;1)}+\frac{i\mathrm{Im}T_{|n-1|}(p,q;e^{i\theta})}{U_{|n-2|}(p,q;1)}\right).
\end{aligned}
$$

Similarly we prove the equality (3.12)

$$
\begin{aligned}
(1-\cos^2\theta)T_n(p,q;e^{i\theta}) &= \sin^2\theta\left(\tfrac{1}{2}(p^n+q^n)\cos n\theta+\tfrac{1}{2}i(p^n-q^n)\sin n\theta\right)\\
&= \frac{1}{2}(1-\cos 2\theta)T_n(p,q;e^{i\theta})\\
&= \frac{1}{2}T_n(p,q;e^{i\theta})\\
&- \frac{1}{4}(p^n+q^n)\left(\frac{\mathrm{Re}T_{n+2}(p,q;e^{i\theta})}{p^{n+2}+q^{n+2}}+\frac{\mathrm{Re}T_{|n-2|}(p,q;e^{i\theta})}{p^{n-2}+q^{n-2}}\right)\\
&- \frac{1}{4}(p^n-q^n)\left(\frac{i\mathrm{Im}T_{n+2}(p,q;e^{i\theta})}{p^{n+2}-q^{n+2}}+\frac{i\mathrm{Im}T_{|n-2|}(p,q;e^{i\theta})}{p^{n-2}-q^{n-2}}\right)\\
&= \frac{1}{2}T_n(p,q;e^{i\theta})\\
&- \frac{1}{4}T_n(p,q;1)\left(\frac{\mathrm{Re}T_{n+2}(p,q;e^{i\theta})}{T_{n+2}(p,q;1)}+\frac{\mathrm{Re}T_{|n-2|}(p,q;e^{i\theta})}{T_{|n-2|}(p,q;1)}\right)\\
&- \frac{1}{4}U_{|n-1|}(p,q;1)\Bigg(\frac{i\mathrm{Im}T_{n+2}(p,q;e^{i\theta})}{U_{n+1}(p,q;1)}\\
&+ \frac{i\mathrm{Im}T_{|n-2|}(p,q;e^{i\theta})}{U_{|n-3|}(p,q;1)}\Bigg).
\end{aligned}
$$

□

Proposition 3.6 *Let $T_n(p,q;e^{i\theta})$ be defined by (3.3). Then the indefinite integral of $T_n(x)$ can be expressed in terms of Chebyshev polynomials as follows.*

$$
\begin{aligned}
\int T_n(p,q;e^{i\theta})dx &= T_n(p,q;1)\left(\frac{\mathrm{Re}T_{n+1}(p,q;e^{i\theta})}{T_{n+1}(p,q;1)(n+1)}-\frac{\mathrm{Re}T_{|n-1|}(p,q;e^{i\theta})}{T_{|n-1|}(p,q;1)|n-1|}\right)\\
&- U_{n-1}(p,q;1)i\left(\frac{\mathrm{Im}T_{n+1}(p,q;e^{i\theta})}{U_n(p,q;1)(n+1)}-\frac{\mathrm{Im}T_{|n-1|}(p,q;e^{i\theta})}{U_{|n-2|}(p,q;1)|n-1|}\right),\\
\int T_1(p,q;e^{i\theta})dx &= -iT_1(p,q;e^{i\theta}),
\end{aligned}
$$

where $n \neq 1$.

Proof. Applying the formula (3.3), by integration we obtain:

$$\begin{aligned}
\int T_n(p,q;e^{i\theta})dx &= -\int \frac{1}{2}(p^n+q^n)\cos n\theta \sin\theta d\theta - i\int \frac{1}{2}(p^n-q^n)\sin n\theta\sin\theta d\theta \\
&= \frac{1}{2}(p^n+q^n)\frac{\cos(n+1)\theta}{n+1} - \frac{1}{2}(p^n+q^n)\frac{\cos|n-1|\theta}{|n-1|} \\
&- i\frac{1}{2}(p^n-q^n)\frac{\sin(n+1)\theta}{n+1} + \frac{1}{2}i(p^n-q^n)\frac{\sin|n-1|\theta}{|n-1|} \\
&= (p^n+q^n)\left(\frac{\mathrm{Re}T_{n+1}(p,q;e^{i\theta})}{(p^{n+1}+q^{n+1})(n+1)} - \frac{\mathrm{Re}T_{|n-1|}(p,q;e^{i\theta})}{(p^{n-1}+q^{n-1})|n-1|}\right) \\
&- (p^n-q^n)i\left(\frac{\mathrm{Im}T_{n+1}(p,q;e^{i\theta})}{(p^{n+1}-q^{n+1})(n+1)} - \frac{\mathrm{Im}T_{|n-1|}(p,q;e^{i\theta})}{(p^{n-1}-q^{n-1})|n-1|}\right) \\
&= T_n(p,q;1)\left(\frac{\mathrm{Re}T_{n+1}(p,q;e^{i\theta})}{T_{n+1}(p,q;1)(n+1)} - \frac{\mathrm{Re}T_{|n-1|}(p,q;e^{i\theta})}{T_{|n-1|}(p,q;1)|n-1|}\right) \\
&- U_{n-1}(p,q;1)i\left(\frac{\mathrm{Im}T_{n+1}(p,q;e^{i\theta})}{U_n(p,q;1)(n+1)} - \frac{\mathrm{Im}T_{|n-1|}(p,q;e^{i\theta})}{U_{|n-2|}(p,q;1)|n-1|}\right).
\end{aligned}$$

(where the first term in the bracket is to be omitted in the case $n=1$). □

Proposition 3.7 *Let $T_n(p,q;e^{i\theta})$ be defined by* (3.3)*, then we have*

$$\begin{aligned}
\frac{d}{d\theta}T_n(p,q;e^{i\theta}) &= \frac{T_n(p,q;1)}{\sin^2\theta}\left(\frac{n\mathrm{Re}T_{|n-1|}(p,q;e^{i\theta})}{2T_{|n-1|}(p,q;1)} - \frac{\mathrm{Re}T_{n+1}(p,q;e^{i\theta})}{T_{n+1}(p,q;1)}\right) \\
&- \frac{U_{|n-1|}(p,q;1)}{\sin^2\theta}i\left(\frac{n\mathrm{Im}T_{|n-1|}(p,q;e^{i\theta})}{2U_{|n-2|}(p,q;1)} - \frac{\mathrm{Im}T_{n+1}(p,q;e^{i\theta})}{U_n(p,q;1)}\right).
\end{aligned}$$

Proof. Differentiation of (3.3), we have

$$\begin{aligned}
\frac{d}{d\theta}T_n(p,q;e^{i\theta}) &= \frac{1}{2}(p^n+q^n)\frac{n\sin n\theta}{\sin\theta} - \frac{1}{2}i(p^n-q^n)\frac{n\cos n\theta}{\sin\theta} \\
&= \frac{1}{2}(p^n+q^n)\frac{\frac{1}{2}n(\cos(n-1)\theta - \cos(n+1)\theta)}{\sin^2\theta} \\
&- \frac{1}{2}i(p^n-q^n)\frac{\frac{1}{2}n(\sin(n-1)\theta - \sin(n+1)\theta)}{\sin^2\theta} = \\
&= \frac{p^n+q^n}{\sin^2\theta}\left(\frac{n\mathrm{Re}T_{|n-1|}(p,q;e^{i\theta})}{2(p^{n-1}+q^{n-1})} - \frac{\mathrm{Re}T_{n+1}(p,q;e^{i\theta})}{p^{n+1}+q^{n+1}}\right) \\
&- \frac{p^n-q^n}{\sin^2\theta}i\left(\frac{n\mathrm{Im}T_{|n-1|}(p,q;e^{i\theta})}{2(p^{n-1}-q^{n-1})} - \frac{\mathrm{Im}T_{n+1}(p,q;e^{i\theta})}{p^{n+1}-q^{n+1}}\right) \\
&= \frac{T_n(p,q;1)}{\sin^2\theta}\left(\frac{n\mathrm{Re}T_{|n-1|}(p,q;e^{i\theta})}{2T_{|n-1|}(p,q;1)} - \frac{\mathrm{Re}T_{n+1}(p,q;e^{i\theta})}{T_{n+1}(p,q;1)}\right) \\
&- \frac{U_{|n-1|}(p,q;1)}{\sin^2\theta}i\left(\frac{n\mathrm{Im}T_{|n-1|}(p,q;e^{i\theta})}{2U_{|n-2|}(p,q;1)} - \frac{\mathrm{Im}T_{n+1}(p,q;e^{i\theta})}{U_n(p,q;1)}\right).
\end{aligned}$$

□

Various similar formulas are readily obtained for $U_n(p,q;e^{i\theta})$. For instance, we get:

Proposition 3.8 *Let $U_n(p,q;e^{i\theta})$ be defined by* (3.2)*, then we have*

a)

$$\begin{aligned}
&\cos\theta U_{n-1}(p,q;e^{i\theta}) \\
&= \frac{1}{2}U_{n-1}(p,q;1)\left(\frac{\mathrm{Re}U_n(p,q;e^{i\theta})}{U_n(p,q;1)} + \frac{\mathrm{Re}U_{n-2}(p,q;e^{i\theta})}{U_{n-2}(p,q;1)}\right) \\
&+ \frac{1}{2}iT_n(p,q;1)\left(\frac{\mathrm{Im}U_n(p,q;e^{i\theta})}{T_{n+1}(p,q;1)} + \frac{\mathrm{Im}U_{n-2}(p,q;e^{i\theta})}{T_{n-1}(p,q;1)}\right),
\end{aligned}$$

b)

$$\begin{aligned}
& (1-\cos^2\theta)U_{n-1}(p,q;e^{i\theta}) \\
= \; & \frac{1}{2}U_{n-1}(p,q;e^{i\theta}) \\
- \; & \frac{1}{4}U_{n-1}(p,q;1)\left(\frac{\mathrm{Re}U_{n+1}(p,q;e^{i\theta})}{U_{n+1}(p,q;1)}+\frac{\mathrm{Re}U_{n-3}(p,q;e^{i\theta})}{U_{n-3}(p,q;1)}\right) \\
- \; & \frac{1}{4}iT_n(p,q;1)\left(\frac{\mathrm{Im}U_{n+1}(p,q;e^{i\theta})}{T_{n+2}(p,q;1)}+\frac{\mathrm{Im}U_{n-3}(p,q;e^{i\theta})}{T_{n-2}(p,q;1)}\right),
\end{aligned}$$

c)

$$\begin{aligned}
& \cos m\theta T_n(p,q;e^{i\theta}) \\
= \; & \frac{1}{2}T_n(p,q;1)\left(\frac{\mathrm{Re}T_{n+m}(p,q;e^{i\theta})}{T_{n+m}(p,q;1)}+\frac{\mathrm{Re}T_{|n-m|}(p,q;e^{i\theta})}{T_{|n-m|}(p,q;1)}\right) \\
+ \; & \frac{1}{2}iU_n(p,q;1)\left(\frac{\mathrm{Im}T_{n+m}(p,q;e^{i\theta})}{U_{n+m-1}(p,q;1)}+\frac{\mathrm{Im}T_{|n-m|}(p,q;e^{i\theta})}{U_{n-m-1}(p,q;1)}\right),
\end{aligned}$$

d)

$$\begin{aligned}
& \cos m\theta U_{n-1}(p,q;e^{i\theta}) \\
= \; & \frac{1}{2}U_{n-1}(p,q;1)\left(\frac{\mathrm{Re}U_{n+m-1}(p,q;e^{i\theta})}{U_{n+m-1}(p,q;1)}+\frac{\mathrm{Re}U_{n-m-1}(p,q;e^{i\theta})}{U_{n-m-1}(p,q;1)}\right) \\
+ \; & \frac{1}{2}iT_n(p,q;1)\left(\frac{\mathrm{Im}U_{n+m-1}(p,q;e^{i\theta})}{T_{n+m}(p,q;1)}+\frac{\mathrm{Im}U_{n-m-1}(p,q;e^{i\theta})}{T_{n-m}(p,q;1)}\right),
\end{aligned}$$

e)

$$\frac{dU_{n-1}(p,q;e^{i\theta})}{d\theta} = i\frac{(n-1)U_n(p,q;e^{i\theta})-pq(n+1)U_{n+1}(p,q;e^{i\theta})}{pe^{i\theta}-qe^{-i\theta}}.$$

The key fact on $U_n(p,q;e^{i\theta})$ polynomials we will apply in the next chapter when it will be relevant to our proof in Section 4.4.

3.3 Application

The generalized Chebyshev polynomials of first kind occurs first in [79], where it was proposed to study the polynomials with one parameter, namely

$$T_n(q;e^{i\theta}) = \frac{1}{2}(e^{in\theta}+q^n e^{-in\theta}) \quad (q\in[-1,1]).$$

Another example is work of Freund [69]. In this paper author is concerned with a classical inequality due to Bernstein which estimates the norm of polynomials on any given ellipse in terms of their norm on any smaller ellipse with the same foci. These Bernstein type inequalities are closely connected with certain constrained Chebyshev approximation problems on ellipses. The authors introduce an analogy to the Chebyshev polynomials

$$T_{k+\frac{1}{2}}(\varphi) = a_k\cos\left(k+\frac{1}{2}\right)\varphi + ib_k\cos\left(k+\frac{1}{2}\right)\varphi \quad (\varphi\in[-\pi,\pi]),$$

where

$$a_k = \frac{1}{2}\left(r^{k+\frac{1}{2}} + \frac{1}{r^{k+\frac{1}{2}}}\right), \ b_k = \frac{1}{2}\left(r^{k+\frac{1}{2}} - \frac{1}{r^{k+\frac{1}{2}}}\right) \quad (r \geq 1).$$

Finally Freund and Fischer [69] deals with constrained Chebyshev approximation problem of the type

$$\min_{p \in \prod_n : p(c)=1} \max_{z \in E} |P(z)|.$$

Here $\prod_n$ denotes the set of all complex polynomials of degree at most n, E is any ellipse in the complex plane, and $c \in \mathbb{C} \setminus E$. In this paper it is showed that the extremal points of $P(z)$ on E_r has a form of (3.3).

The above mentioned facts are motivation of studies the properties of the polynomials given by (3.2) and (3.3).

Chapter 4

Extension of typically real function - $\mathcal{T}^{p,q}$

In Chapter 2 we introduced the generalized Koebe function (2.7). Here, we apply this function to define a related class of functions.

Definition 4.1 *By $\mathcal{T}^{p,q}$ we denote the class of generalized typically real functions, defined as a class of functions of $f \in \mathcal{A}$, and having an integral representation*

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{z}{(1-e^{i\theta}pz)(1-e^{-i\theta}qz)} d\mu(\theta) \quad (z \in \mathbb{D}), \tag{4.1}$$

where $\mu(\theta)$ is the unique probability measure on the segment $[0, 2\pi]$.

This chapter is composed of two types of problems. The first type is concerned with coefficients problem for $\mathcal{T}^{p,q}$ and the second type with extremal problems for $\mathcal{T}^{p,q}$ class of functions.

First we turn our attention in Section 4.1 to $\mathcal{P}^{p,q}$ class of functions, which is connected with an extension of typically real functions. In Section 4.2 we present a Theorem 4.2 which establishes a relationship between this two class of functions. In Section 4.4 we consider an important coefficients problem for $\mathcal{P}^{p,q}$. While the nature of this problem may at first appear unremarkable, its solution in terms of coefficients for $\mathcal{T}^{p,q}$ is of substantial importance, because it provides a very useful tool for founding coefficients bounds in $\mathcal{T}^{p,q}$. Next in Section 4.4 we conclude with some coefficients bounds for $\mathcal{T}^{p,q}$ class of function, especially problem related to Zalcman conjecture is solved. Therefore, studying the extremal problems for $\mathcal{T}^{p,q}$, especially coefficient problems, is a natural problem associated to the general study of "the trigonometric polynomials" $U_n(p, q; e^{i\theta})$ defined by (3.1). For a detailed description of the above orthogonal polynomials, the state of the art and motivation we refer the reader to Chapter 3.

Our investigation related to extremal problems for $\mathcal{T}^{p,q}$ starts in Section 4.5. Here, we present Theorem 4.12 that immediately highlights the importance of propositions from Section 2.2.1. We conclude in Section 4.6 with some important results about set of the local univalence and radius of local univalence.

4.1 Class $\mathcal{P}^{p,q}$

Let $\mathcal{P}$ be the class of functions of the form $\varphi(z) = 1 + c_1 z + \cdots$, holomorphic, and with positive real part in the unit disk $\mathbb{D}$; a class of such functions is called the Carathéodory class. $\mathcal{P}$ has interesting properties and many useful applications, particularly in the study of special classes of univalent functions. Any function $\varphi \in \mathcal{P}$ has useful Herglotz representation

$$\varphi(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + ze^{-i\theta}}{1 - ze^{-i\theta}} d\mu(\theta) = \frac{1}{2\pi} \int_0^{2\pi} L(ze^{-i\theta}) d\mu(\theta),$$

where $\mu(\theta)$ is a probability measure on $[0, 2\pi]$ and $L(z) = (1+z)/(1-z)$ is the Möbius function. A significant subclass of the class $\mathcal{P}$ is a class of all functions which are real on $(-1, 1)$; we denote it here by $\mathcal{P}_{\mathbb{R}}$. Since $\varphi \in \mathcal{P}_{\mathbb{R}}$ has real coefficients then $\varphi(z) = \overline{\varphi(\bar{z})}$. Therefore [121]

$$\varphi(z) = \frac{\varphi(z) + \overline{\varphi(\bar{z})}}{2} = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - z^2}{1 - 2z\cos\theta + z^2} d\mu(\theta).$$

Successively, the new subclasses of $\mathcal{P}$ and $\mathcal{P}_{\mathbb{R}}$ consisting of functions φ such that $\varphi(\mathbb{D})$ is a proper subdomain of a right halfplane appeared in the literature (see, for example [71, 74, 122]).

Now, we construct a natural extension of the class $\mathcal{P}_{\mathbb{R}}$ as follows. For $-1 \le p, q \le 1$ let

$$\varphi_1(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + e^{i\theta} pz}{1 - e^{i\theta} pz} d\mu(\theta) = \frac{1}{2\pi} \int_0^{2\pi} L(pze^{-i\theta}) d\mu(\theta),$$

$$\varphi_2(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + e^{-i\theta} qz}{1 - e^{-i\theta} qz} d\mu(\theta) = \frac{1}{2\pi} \int_0^{2\pi} \overline{L(q\bar{z}e^{i\theta})} d\mu(\theta).$$

Then $\varphi_1, \varphi_2 \in \mathcal{P}$. Hence

$$\frac{1}{2}\left(\varphi_1(z) + \varphi_2(z)\right) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - pqz^2}{(1 - e^{i\theta} pz)(1 - e^{-i\theta} qz)} d\mu(\theta)$$

is also in $\mathcal{P}$, since $\mathcal{P}$ is convex.

Definition 4.2 *For $-1 \le p, q \le 1$ we define the class $\mathcal{P}^{p,q}$ as the class of functions $\varphi \in \mathcal{P}$ that are of the form*

$$\varphi(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - pqz^2}{(1 - e^{i\theta}pz)(1 - e^{-i\theta}qz)} d\mu(\theta), \tag{4.2}$$

where $\mu(\theta)$ is the unique probability measure on the interval $[0, 2\pi]$.

We note that setting $p = q = 1$ the class $\mathcal{P}^{p,q}$ becomes $\mathcal{P}_{\mathbb{R}}$.

Let now $p = 1, q = -1$ or $p = -1, q = 1$. Then (4.2) reduces to

$$\varphi(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + z^2}{(1 + 2zi \sin\theta - z^2)} d\mu(\theta),$$

and the classes $\mathcal{P}^{1,-1} = \mathcal{P}^{-1,1}$ consists of all functions φ, satisfying $\varphi(z) = \overline{\varphi(-\bar{z})}$, that is functions symmetric with respect to the imaginary axis. Moreover, φ has real even coefficients and purely imaginary odd coefficients.

We note that for $\varphi \in \mathcal{P}^{p,q}$ we immediately obtain

Proposition 4.1 *Let $-1 \le p, q \le 1$, and let $\varphi \in \mathcal{P}^{p,q}$. Then for $|z| = r$, $0 < r < 1$, it holds*

$$|\varphi(z)| \le \frac{1 + |p||q|r^2}{(1 - |p|r)(1 - |q|r)}.$$

Also, we have

$$\frac{1 - |p||q|}{(1 + |p|)(1 + |q|)} \le |\varphi(z)| \le \frac{1 + |p||q|}{(1 - |p|)(1 - |q|)}.$$

Remark 4.1 *There is close relation between $\mathcal{P}^{p,q}$ and the generalized Chebyshev polynomials of the second kind* (3.1), *namely if $\varphi \in \mathcal{P}^{p,q}$, then*

$$\varphi(z) = \frac{1}{2\pi} \int_0^{2\pi} (1 - pqz^2)\Psi_{p,q}(e^{i\theta}; z) d\mu(\theta).$$

4.2 Relation between the classes $\mathcal{P}^{p,q}$ and $\mathcal{T}^{p,q}$

The motivation for introduction a class $\mathcal{P}^{p,q}$ is that the integral representation have been found to have many connection with $\mathcal{T}^{p,q}$ class of function. By considering this relation we gain further insight the coefficient problem for $\mathcal{T}^{p,q}$ class of functions. With the above motivation we now look back to connection between $\mathcal{P}^{p,q}$ and $\mathcal{T}^{p,q}$, below.

Theorem 4.2 *Let $-1 \le p, q \le 1$. If $g \in \mathcal{P}^{p,q}$ then*

$$f(z) = \frac{z}{1 - pqz^2} g(z) \in \mathcal{T}^{p,q}.$$

Conversely, if $f \in \mathcal{T}^{p,q}$, then

$$g(z) = \frac{1 - pqz^2}{z} f(z) \in \mathcal{P}^{p,q}.$$

Proof. Let $f \in \mathcal{T}^{p,q}$. Then the function
$$g(z) = \frac{1-pqz^2}{z} f(z) = 1 + a_2 z + (a_3 - pq)z^2 + \cdots$$
is regular in $\mathbb{D}$ since the pole at $z=0$ is canceled by the zero of f at the same point, so that $g(0)=1$. Also, we note that
$$\begin{aligned} g(z) &= \frac{1-pqz^2}{z}\frac{1}{2\pi}\int_0^{2\pi} \frac{z d\mu(\theta)}{(1-e^{i\theta}pz)(1-e^{-i\theta}qz)} \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{1-pqz^2}{(1-e^{i\theta}pz)(1-e^{-i\theta}qz)} d\mu(\theta). \end{aligned}$$
Then $g \in \mathcal{P}^{p,q}$.

Conversely suppose $g \in \mathcal{P}^{p,q}$ and $f(z) = zg(z)/(1-pqz^2)$. The function
$$f(z) = \frac{z}{1-pqz^2} g(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{z}{(1-e^{i\theta}pz)(1-e^{-i\theta}qz)} d\mu(\theta)$$
has the standard normalization $f(0) = 0 = f'(0) - 1$ and is univalent in $\mathbb{D}$. □

Remark 4.2 *We observe that* $\mathcal{T}^{1,1} = \mathcal{T}_{\mathbb{R}}$ *and* $\mathcal{T}^{1,0} = \mathcal{T}^{0,1} = \overline{co}\,\mathcal{CV}$.

4.3 Geometric interpretation of $\mathcal{T}^{p,q}$

Let $z = e^{it}$, and $f \in \mathcal{T}^{p,q}$. Then, we have
$$\begin{aligned} \mathrm{Re}\frac{1-pqz^2}{z} f(z) &= \mathrm{Re}\,(e^{-it} - pqe^{it})\, f(e^{it}) \\ &= (1-pq)\cos t\, \mathrm{Re} f(e^{it}) + (1+pq)\sin t\, \mathrm{Im} f(e^{it}) \end{aligned}$$
so that $\mathrm{Re}\frac{1-pqz^2}{z} f(z) > 0$, if and only if (extending the inequality to the whole $\mathbb{D}$)
$$(1-pq)\mathrm{Re}\,\{z\}\mathrm{Re}\,\{f(z)\}) + (1+pq)\mathrm{Im}\,\{f(z)\}\,\mathrm{Im}\,\{z\} > 0. \tag{4.3}$$
We see that if $pq = 1$ the above condition reduces to the following
$$\mathrm{Im}\,\{f(z)\}\,\mathrm{Im}\,\{z\} \geq 0 \quad (z \in \mathbb{D}), \tag{4.4}$$
that is a geometric interpretation of a class $\mathcal{T}_{\mathbb{R}}$. Also, when $pq = -1$, we obtain
$$\mathrm{Re}\,\{z\}\mathrm{Re}\,\{f(z)\} > 0 \quad (z \in \mathbb{D}), \tag{4.5}$$
that means that the image of the unit disk under f is symmetric with respect to the imaginary axis.

In the remaining cases the condition (4.3) may be rewritten in the form
$$\frac{1-pq}{2}\mathrm{Re}\,\{z\}\mathrm{Re}\,\{f(z)\} + \frac{1+pq}{2}\mathrm{Im}\,\{f(z)\}\,\mathrm{Im}\,\{z\} > 0. \tag{4.6}$$
We observe that (4.6) constitute an arithmetic means of (4.4) and (4.5), thus the class $\mathcal{T}^{p,q}$ provides an arithmetic bridge between the classes of normalized functions symmetric with respect to the imaginary and real axis.

4.4 Coefficient problems

One of the most interesting questions in a geometric functions theory is to address the region of variability of the n-th Taylor coefficient for functions f that belongs to some class of analytic functions. Leading example is the Bieberbach conjecture settled by de Branges in 1985 for the class of normalized and univalent functions $\mathcal{S}$, although corresponding results for important subclasses of $\mathcal{S}$ were established positive much earlier. Also, sharp bounds for the coefficients in the class $\mathcal{P}$ are well known; they constitute one of the main tool in a geometric theory of analytic functions.
In this section we prove coefficients bounds of functions from the class $\mathcal{T}^{p,q}$. In the case when $p = q = 1$ these results become the well known estimates in the class of typically real functions $\mathcal{T}_{\mathbb{R}}$. Next we move on to discuss of a related class of $\mathcal{P}^{p,q}$. Using coefficients relations and estimates that hold in that class, we also obtain some sharp coefficients bounds for generalized typically real functions. Orthogonal polynomials $U_{n-1}(p, q; e^{i\theta})$ defined by (3.1) have been used to solve problems related to coefficients bounds in $\mathcal{T}^{p,q}$ class. Namely, from the integral representation (4.1) it follows that the coefficients $f \in \mathcal{T}^{p,q}$ can be represented as

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} U_{n-1}(p, q; a^{i\theta}) d\mu(\theta). \tag{4.7}$$

Therefore, studying the properties of the class $\mathcal{T}^{p,q}$ is a natural problem associated to the general study of orthogonal polynomials defined and considered in Chapter 3.

Proposition 4.3 *If $f \in \mathcal{T}^{p,q}$, then we have the following sharp bound*

$$|a_n| \le \begin{cases} \dfrac{|p|^n - |q|^n}{|p| - |q|} & \textit{if} \quad |p| \neq |q|, \\ n|p|^{n-1} & \textit{if} \quad |p| = |q|. \end{cases} \tag{4.8}$$

The extremal functions have the form $f(z) = \dfrac{z}{(1-pz)(1-qz)}$ for $pq > 0$ and $f(z) = \dfrac{z}{(1-ipz)(1+iqz)}$ for $pq < 0$.

Proof. Applying (4.7), we have

$$\begin{aligned} |a_n| &= \frac{1}{2\pi} \left| \int_0^{2\pi} U_{n-1}(p, q; e^{i\theta}) d\mu(\theta) \right| \le \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{p^n e^{in\theta} - q^n e^{-in\theta}}{pe^{i\theta} - qe^{-i\theta}} \right| d\mu(\theta) \\ &= \frac{1}{2\pi} \int_0^{2\pi} |p^{n-1} e^{i(n-1)\theta} + p^{n-2} q e^{i(n-2)\theta} + \ldots + pq^{n-2} e^{-i(n-2)\theta} + q^{n-1} e^{-i(n-1)\theta}| d\mu(\theta) \\ &\le \begin{cases} \dfrac{|p|^n - |q|^n}{|p| - |q|} & \text{if} \quad |p| \neq |q|, \\ n|p|^{n-1} & \text{if} \quad |p| = |q|. \end{cases} \end{aligned}$$

$\square$

Proposition 4.4 *Let $-1 \le p, q \le 1$, and let $\varphi \in \mathcal{P}^{p,q}$ be of the form*

$$\varphi(z) = 1 + c_1 z + \cdots , \tag{4.9}$$

then

$$|c_n| \le |p|^n + |q|^n. \tag{4.10}$$

Proof. By the integral representation (4.2) we have that if φ is of the form (4.9), then

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} (e^{in\theta} p^n + e^{-in\theta} q^n) d\mu(\theta).$$

Therefore, the immediate consequence follows. $\square$

Remark 4.3 *We observe that the above bounds becomes known estimates in the class $\mathcal{P}$ in the case, when $|p| = |q| = 1$.*

Theorem 4.5 *Let $-1 \le p, q \le 1$, and let $f \in \mathcal{T}^{p,q}$ be of the form (4.1). Then*

$$|a_{n+2} + pqa_n| \le \begin{cases} (|p| + |q|) \dfrac{|p|^{n+1} - |q|^{n+1}}{|p| - |q|} & for \quad |p| \ne |q|, \\ 2(n+1)|p|^{n+1} & for \quad |p| = |q|. \end{cases} \tag{4.11}$$

Also, it holds

$$|a_{n+2} - pqa_n| \le |p|^{n+1} + |q|^{n+1} \quad (n \in \mathbb{N} \cup \{0\}). \tag{4.12}$$

Here we set $a_0 = 0$ and $a_1 = 1$. The results are sharp with the equalities for $k_{p,q}(z) = z/[(1 - pz)(1 - qz)]$.

Proof. Using the recurrence relation (3.4) we get

$$\begin{aligned} \int_0^{2\pi} U_{n+2}(p, q; e^{i\theta}) d\mu(\theta) \quad &- \quad \int_0^{2\pi} (pe^{i\theta} + qe^{-i\theta}) U_{n+1}(p, q; e^{i\theta}) d\mu(\theta) \\ &+ \quad pq \int_0^{2\pi} U_n(p, q; e^{i\theta}) d\mu(\theta) = 0. \end{aligned}$$

The above and the formula (4.7) yield

$$a_{n+3} + pqa_{n+1} = \frac{1}{2\pi} \int_0^{2\pi} (pe^{i\theta} + qe^{-i\theta}) U_{n+1}(p, q; e^{i\theta}) d\mu(\theta),$$

and hence

$$|a_{n+3} + pqa_{n+1}| \le (|p| + |q|)|a_{n+2}|.$$

Applying the inequality (4.8), and renumerating, we obtain (4.11).

For the second bound, we observe that by Theorem 4.2

$$\frac{1 - pqz^2}{z} f(z) = 1 + \sum_{n=0}^{\infty} (a_{n+2} - pqa_n) z^{n+1}$$

is of the class $\mathcal{P}^{p,q}$. The above and (4.10) therefore yield

$$|a_{n+2} - pqa_n| \le |p|^{n+1} + |q|^{n+1} \quad (n \in \mathbb{N} \cup \{0\}).$$

$\square$

As an immediate consequences of (4.12) we obtain an estimates for the following integral.

Lemma 4.6 *Let $-1 \le p, q \le 1$ and $\mu(\theta)$ be a probability measure on $[0, 2\pi]$, then*

$$\left| \int_0^{2\pi} \frac{p^{n+2}e^{i(n+2)\theta} - q^{n+2}e^{-i(n+2)\theta} - pq\left(p^n e^{in\theta} - q^n e^{-in\theta}\right)}{pe^{i\theta} - qe^{-i\theta}} d\mu(\theta) \right| \le 2\pi \left(|p|^{n+1} + |q|^{n+1}\right).$$

Proof. Let $-1 \le p, q \le 1$, and let $f \in \mathcal{T}^{p,q}$ be of the form (4.1). Applying (3.2) and (4.7) we have

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} \frac{p^n e^{in\theta} - q^n e^{-in\theta}}{pe^{i\theta} - qe^{-i\theta}} d\mu(\theta).$$

Making use the above together with (4.12), we get

$$\begin{aligned} |a_{n+2} - pqa_n| &= \left| \frac{1}{2\pi} \int_0^{2\pi} \frac{p^{n+2}e^{i(n+2)\theta} - q^{n+2}e^{-i(n+2)} - pq\left(p^n e^{in\theta} - q^n e^{-in\theta}\right)}{pe^{i\theta} - qe^{-i\theta}} d\mu(\theta) \right| \\ &\le |p|^{n+1} + |q|^{n+1}, \end{aligned}$$

and the assertion follows. $\square$

It is well known that, for $\varphi \in \mathcal{P}$ of the form (4.9), $|c_n| \le 2$ and hence $|c_{n+1} - c_n| \le 4$, which is sharp. However, Robertson [125] proved the following result for successive coefficients in $\mathcal{P}$; if $\varphi \in \mathcal{P}$ is of the form (4.9), then

$$|c_{n+1} - c_n| \le (2n+1)|2 - c_1|,$$

and also

$$||c_{n+1}| - |c_n|| \le (2n+1)(2 - |c_1|).$$

The factor $(2n+1)$ cannot be improved as can be seen by considering the function

$$\varphi(z) = \frac{1 - z^2}{1 - 2z\cos\theta + z^2}.$$

In 1983, Goodman ([62], page 104) pointed out that the sharp bound on $|c_{n+1} - c_n|$ in class $\mathcal{P}$ with fixed c_1 is unknown. Livingston [91] proved that if $\varphi \in \mathcal{P}$ and $c_1 = 1$, then

$$|c_{n+1} - c_n| \le 2.$$

This inequality also holds in the class $\mathcal{P}_{\mathbb{R}}$. For a more recent result concerning successive coefficients in $\mathcal{P}$ see Lecko [88],

$$||c_{n+1} - c_n| - |c_n - c_{n-1}|| \le 2(2 - \mathrm{Re} c_1),$$

and the inequality is sharp.

The above results were the motivation for our considerations in this direction. In a next part of this section we consider certain successive coefficients for functions in $\mathcal{P}^{p,q}$. Particulary, we obtain the following sharp estimates.

Theorem 4.7 *If $-1 \leq p, q \leq 1$, and let $\varphi \in \mathcal{P}^{p,q}$ is of the form* (4.9)*. Then*

$$|c_n - c_m c_{n-m}| \leq |p^m q^{n-m} + q^m p^{n-m}| \quad (n, m \in \mathbb{N}). \tag{4.13}$$

The results are sharp wit the equality for $\varphi_e(z) = \dfrac{1 - pqz^2}{(1 - pz)(1 - qz)}$.

Proof. We begin by observing that $\mathcal{P}^{p,q}$ is a convex set, that can be easily checked using representation (4.2). Therefore, any function $\varphi \in \mathcal{P}^{p,q}$, may be considered as a sum

$$\varphi(z) = \sum_{k=1}^{s} \lambda_k \frac{1 - pqe^{2it_k} z^2}{(1 - pe^{it_k} z)(1 - qe^{it_k} z)}, \tag{4.14}$$

where $\lambda_k \geq 0, 1 \leq k \leq s$, and $\sum\limits_{k=1}^{s} \lambda_k = 1$, that is a special case of the representation (4.2). Hence, we need only prove the assertion for functions of the form (4.14). For such functions,

$$c_n = (p^n + q^n) \sum_{k=1}^{s} \lambda_k e^{int_k} \quad (n \geq 1),$$

so that

$$\begin{aligned} |c_n - c_m c_{n-m}| &= \Bigg| (p^n + q^n) \sum_{k=1}^{s} \lambda_k e^{int_k} \\ &- (p^m + q^m) \sum_{k=1}^{s} \lambda_k e^{imt_k} (p^{n-m} + q^{n-m}) \sum_{k=1}^{s} \lambda_k e^{i(n-m)t_k} \Bigg| \\ &= \left| \sum_{k=1}^{s} \lambda_k \left((p^n + q^n) e^{int_k} - (p^m + q^m) e^{imt_k} (p^{n-m} + q^{n-m}) A \right) \right| \\ &= \sum_{k=1}^{s} |\lambda_k B_k| \leq \sum_{k=1}^{s} \lambda_k |B_k|, \end{aligned}$$

where

$$A = \sum_{k=1}^{s} \lambda_k e^{i(n-m)t_k},$$

and

$$B_k = (p^n + q^n) e^{int_k} - (p^m + q^m)(p^{n-m} + q^{n-m}) e^{imt_k} A \quad (1 \leq k \leq s).$$

Moreover, we have

$$\begin{aligned}\sum_{k=1}^{s}\lambda_k|B_k|^2 &= \sum_{k=1}^{s}\lambda_k|B_k|^2\\ &= \sum_{k=1}^{s}\lambda_k\left((p^n+q^n)e^{int_k}-(p^m+q^m)(p^{n-m}+q^{n-m})e^{imt_k}A\right)\\ &\quad\times((p^n+q^n)e^{-int_k}(p^m+q^m)(p^{n-m}+q^{n-m})e^{-imt_k}\overline{A})\\ &= (p^n+q^n)^2+(p^m+q^m)^2(p^{n-m}+q^{n-m})^2|A|^2\\ &- 2(p^n+q^n)(p^m+q^m)(p^{n-m}+q^{n-m})\mathrm{Re}\sum_{k=1}^{s}\lambda_k e^{-i(n-m)t_k}A,\end{aligned}\tag{4.15}$$

and

$$\begin{aligned}\mathrm{Re}\sum_{k=1}^{s}\lambda_k e^{-i(n-m)t_k}A &= \mathrm{Re}\sum_{k=1}^{s}\lambda_k e^{-i(n-m)t_k}\left(\sum_{q=1}^{s}\lambda_q e^{i(n-m)t_q}\right)\\ &= \mathrm{Re}\left(\sum_{q=1}^{s}\lambda_q e^{i(n-m)t_q}\sum_{k=1}^{s}\lambda_k e^{-i(n-m)t_k}\right)\\ &= \mathrm{Re}\left(\left|\sum_{q=1}^{s}\lambda_q e^{i(n-m)t_q}\right|^2\right)\\ &= \left|\sum_{q=1}^{s}\lambda_q e^{i(n-m)t_q}\right|^2=|A|^2.\end{aligned}\tag{4.16}$$

Therefore (4.15) and (4.16) yield

$$\begin{aligned}\sum_{k=1}^{s}\lambda_k|B_k|^2 &= (p^n+q^n)^2+(p^m+q^m)(p^{n-m}+q^{n-m})|A|^2\\ &\quad\times\ ((p^n+q^n)(p^{n-m}+q^{n-m})-2(p^n+q^n))\\ &\le \left((p^m+q^m)(p^{n-m}+q^{n-m})-(p^n+q^n)\right)^2.\end{aligned}\tag{4.17}$$

By the Schwarz inequality

$$\left(\sum_{k=1}^{s}\lambda_k|B_k|\right)^2\le\sum_{k=1}^{s}\lambda_k\sum_{k=1}^{s}\lambda_k|B_k|^2=\sum_{k=1}^{s}\lambda_k|B_k|^2.$$

Hence, by (4.17) and the above, we have

$$\sum_{k=1}^{s}\lambda_k|B_k|\le|p^m p^{n-m}+q^m p^{n-m}|,$$

that completes the proof of the Theorem. □

Remark 4.4 *We note that inserting $n+1$ instead of n, and setting $m=1, p=q=1$ in Theorem 4.7 we obtain the result by Livingston* [91].

As a consequence of Theorem 4.7, we have the following result.

Corollary 4.8 *Let $\mu(t)$ be a probability measure on $[0, 2\pi]$, then*

$$\begin{aligned}&\Bigg|(p^n+q^n)(p^m+q^m)\int_0^{2\pi} e^{int}d\mu(t)\int_0^{2\pi} e^{imt}d\mu(t)\\ -\;& 2\pi(p^{n+m}+q^{n+m})\int_0^{2\pi} e^{i(n+m)t}d\mu(t)\Bigg|\\ \le\;& 4\pi^2|p^m q^{n-m}+q^m p^{n-m}| \quad (n,m=1,2,3,...).\end{aligned}$$

Proof. The coefficients of the function

$$\varphi(z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-pqe^{2it}z^2}{(1-pe^{it}z)(1-qe^{it}z)}d\mu(t),$$

can be represented as

$$c_n=\frac{p^n+q^n}{2\pi}\int_0^{2\pi} e^{int}d\mu(t). \tag{4.18}$$

By the previous theorem we have

$$|c_n c_m - c_{n+m}| \le |p^m q^{n-m}+q^m p^{n-m}| \quad (n,m\in\mathbb{N}),$$

or equivalently, using (4.18)

$$\begin{aligned}&\Bigg|\frac{(p^n+q^n)(p^m+q^m)}{4\pi^2}\int_0^{2\pi} e^{int}d\mu(t)\int_0^{2\pi} e^{imt}d\mu(t)\\ -\;& \frac{p^{n+m}+q^{n+m}}{2\pi}\int_0^{2\pi} e^{i(n+m)t}d\mu(t)\Bigg|\\ \le\;& |p^m q^{n-m}+q^m p^{n-m}| \quad (n,m=1,2,3,...),\end{aligned}$$

and the assertion follows. □

At the end of 1960's, Lawrence Zalcman posed a conjecture that the coefficients of univalent functions f on the unit disk of the form (1.1) satisfy the sharp inequality $|a_n^2 - a_{2n-1}| \le (n-1)^2$, with equality only for the Koebe function. This remarkable conjecture implies the Bieberbach conjecture. Brown and Tsao [25] proved the Zalcman conjecture for the starlike functions. They also obtained $|a_n^2 - a_{2n-1}| \le (n-1)^2$ for $f \in \mathcal{T}_{\mathbb{R}}$. Furthermore, Livingston [91] showed that the Zalcman conjecture holds for close-to-convex functions when $n \ge 4$.

In 1998 Ma [95] proposed and solved generalized Zalcman conjecture for starlike functions and univalent functions with real coefficients. Ma also gave sharp sharp bounds on $|a_n a_m - a_{n+m-1}|$ for coefficients of $f \in \mathcal{T}_{\mathbb{R}}$, namely if f is of the form (1.1), then

$$|a_n a_m - a_{n+m-1}| \le \begin{cases} n+1 & for \quad m=2, n=2,4,6,...\\ m+1 & for \quad n=2, m=2,4,6,...\\ (n-1)(m-1) & otherwise.\end{cases}$$

It turns out that the generalized Zalcman conjecture does not hold for all typically real functions when one of n and m equals 2 and the other equals $2, 4, \ldots$. This does not provide a counterexample for the generalized Zalcman conjecture since not all typically real functions are univalent.

Now we consider some coefficients problems for generalized typically real functions related to the Zalcman conjecture.

Proposition 4.9 *Let $-1 \le p, q \le 1$, and let $f \in \mathcal{T}^{p,q}$ be of the form* (1.1)*. Also, let $\varphi \in \mathcal{P}^{p,q}$ be of the form* (4.9)*. Then for $k = 1, 2, \ldots$ the following relations hold*

$$a_{2k} = (pq)^{k-1}c_1 + (pq)^{k-2}c_3 + \ldots + c_{2k-1}, \tag{4.19}$$

and

$$a_{2k+1} = (pq)^k + (pq)^{k-1}c_2 + (pq)^{k-2}c_4 + \ldots + c_{2k}. \tag{4.20}$$

In particular

$$a_2 = c_1, \quad a_3 = pq + c_2.$$

Proof. The proof is obvious by virtue of Theorem 4.2. $\square$

Theorem 4.10 *Let $-1 \le p, q \le 1$, and let $f \in \mathcal{T}^{p,q}$ be of the form* (1.1)*. Also, let $\varphi \in \mathcal{P}^{p,q}$ be of the form* (4.9)*. Then, for $m = 2, n = 2, 4, 6, \ldots$ the following estimate holds*

$$|a_n a_m - a_{n+m-1}| \le |pq|^{\frac{n}{2}} + \sum_{l=0}^{\frac{n}{2}-1} |pq|^{\frac{n}{2}-l} |p^{2l} + q^{2l}|.$$

For $\quad m = 2, n = 3, 5, 7, \ldots$ *it holds*

$$|a_n a_m - a_{n+m-1}| \le \sum_{l=0}^{\frac{n-3}{2}} |pq|^{\frac{n-1}{2}-l} |p^{2l+1} + q^{2l+1}|,$$

for $m = 3, n = 2, 4, 6, \ldots$ the following holds

$$\begin{aligned} &|a_n a_m - a_{n+m-1}| \\ &\le p^2q^2|p^{n-3} + q^{n-3}| + (p^2+q^2) \sum_{l=1}^{\frac{n}{2}-1} |pq|^{\frac{n}{2}-l} (|p|^{2l-1} + |q|^{2l-1}), \end{aligned}$$

and for $m = 3, n = 3, 5, \ldots$

$$\begin{aligned} &|a_n a_m - a_{n+m-1}| \\ &\le p^2q^2|p^{n-3} + q^{n-3}| + (p^2+q^2) \left(|pq|^{\frac{n-1}{2}} + \sum_{l=1}^{\frac{n-3}{2}} |pq|^{\frac{n-1}{2}-l} (p^{2l} + q^{2l}) \right). \end{aligned}$$

The cases $n = 2, 3$ and $m = 2, 3, 4, \ldots$ are satisfied by the symmetry. The results are sharp, in the first two cases the equalities hold for $f(z) = k^*_{p,q}(z) = z(1 - p^2q^2z^4)/[(1 - pqz^2)(1 - p^2z^2)(1 - q^2z^2)]$, and for the last two cases, if $f(z) = k_{p,q}(z) = z/[(1 - pz)(1 - qz)]$.

Proof. We set $b = pq$. First, we consider the case when $m = 2$ and $n = 2k$, $k = 1, 2, ., ...$ Using (4.19), (4.20) and (4.13), we get

$$\begin{aligned} & |a_2a_{2k} - a_{2k+1}| \\ = & \left|c_1(b^{k-1}c_1 + b^{k-2}c_3 + \cdots + c_{2k-1}) - (b^k + b^{k-1}c_2 + \cdots + c_{2k})\right| \\ = & \left|(b^{k-1}c_1^2 - b^{k-1}c_2) + (c_1b^{k-2}c_3 - b^{k-2}c_4) + \cdots + (c_1c_{2k-1} - c_{2k}) - b^k\right| \\ \leq & |b|^{k-1}|pq + pq| + |b|^{k-2}|pq^3 + qp^3| + \cdots + |pq^{2k-1} + qp^{2k-1}| + |b|^k \\ = & \sum_{l=0}^{k-1} |b|^{k-l}|p^{2l} + q^{2l}| + |b|^k. \end{aligned}$$

This proves the first inequality.

Now, we consider the case $m = 2$ and $n = 2k + 1$, $k = 1, 2, 3, ...$ Similarly as in the previous case we use (4.19), (4.20) and (4.13) Then

$$\begin{aligned} & |a_2a_{2k+1} - a_{2k+2}| \\ = & \left|c_1(b^k + b^{k-1}c_2 + \cdots + c_{2k}) - (b^kc_1 + b^{k-1}c_3 + \cdots + c_{2k+1})\right| \\ = & \left|(b^{k-1}c_1c_2 - b^{k-1}c_3) + (c_1b^{k-2}c_4 - b^{k-2}c_5) + ... + (c_1c_{2k} - c_{2k+1})\right| \\ \leq & |b|^{k-1}|pq^2 + qp^2| + |b|^{k-2}|pq^4 + qp^4| + \cdots + |pq^{2k} + qp^{2k}| \\ = & \sum_{l=0}^{k-1} |b|^{k-l}|p^{2l+1} + q^{2l+1}|. \end{aligned}$$

We note that for $\varphi^*(z) = \dfrac{1 - p^2q^2z^4}{(1 - p^2z^2)(1 - q^2z^2)}$ we have $c_1 = c_3 = ... = c_{2k-1} = 0$, and $c_{2k} = p^{2k} + q^{2k}$. Therefore, the inequalities become equalities for $k^*_{p,q}(z) = \dfrac{z(1 - p^2q^2z^4)}{(1 - pqz^2)(1 - p^2z^2)(1 - q^2z^2)}$.

For the case when $m = 3$ we apply as above (4.19), (4.20) and (4.13). Thus

$$\begin{aligned} & |a_3a_{2k} - a_{2k+2}| \\ = & \left|(b + c_2)(b^{k-1}c_1 + b^{k-2}c_3 + \cdots + c_{2k-1}) - (b^kc_1 + b^{k-1}c_3 + \cdots + c_{2k+1})\right| \\ = & \left|c_2(b^{k-1}c_1 + \cdots + bc_{2k-3}) + (c_2c_{2k-1} - c_{2k+1})\right| \\ \leq & \left|c_2 \sum_{l=1}^{k-1} b^{k-l}c_{2l-1}\right| + |p^2q^{2k-1} + q^2p^{2k-1}| \\ \leq & |p^2 + q^2| \sum_{l=1}^{k-1} |b|^{k-l}(|p|^{2l-1} + |q|^{2l-1}) + p^2q^2|p^{2k-3} + q^{2k-3}| \end{aligned}$$

and, similarly

$$\begin{aligned} & |a_3a_{2k+1} - a_{2k+3}| \\ = & \left|(b + c_2)(b^k + b^{k-1}c_2 + b^{k-2}c_4 + ... + c_{2k}) - (b^{k+1} + b^kc_2 + \cdots + c_{2k+2})\right| \\ = & \left|c_2(b^k + b^{k-1}c_2 + ... + bc_{2k-2}) + (c_2c_{2k} - c_{2k+2})\right| \\ \leq & \left|c_2(b^k + \sum_{l=1}^{k-l} b^{k-l}c_{2l})\right| + |p^2q^{2k} + q^2p^{2k}| \\ \leq & (p^2 + q^2)\left[|b|^k + \sum_{l=1}^{k-1} |b|^{k-l}(|p|^{2l} + |q|^{2l})\right] + p^2q^2|p^{2k-2} + q^{2k-2}|. \end{aligned}$$

The equality in the above inequalities hold, if $\varphi_e(z) = \dfrac{1-pqz^2}{(1-pz)(1-qz)}$, that is for $k_{p,q}(z) = z/((1-pz)(1-qz))$. □

4.5 Extremal problems for $\mathcal{T}^{p,q}$

Making use an integral representation (4.1) and Propositions 2.6, 2.7 and 2.8 from the Chapter 2 we may solve several extremal problems in the class $\mathcal{T}^{p,q}$, below. Here, we also use the following theorem.

Theorem 4.11 [65, p. 189] *Let the function in a class $\mathcal{M}$ have an integral representation of the form*

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} K(z,\alpha)d\mu(\alpha),$$

where $K(z,\alpha)$ is analytic for $z \in \mathbb{D}$ and continuous in $\alpha \in [0,2\pi]$. Then for each fixed $z \in \mathbb{D}$ the region of values for the set $\mathcal{M}$ is the convex cover of the curve $\Gamma:\ w = K(z,\alpha), 0 \le \alpha \le 2\pi$. This region is bounded closed convex set.

Theorem 4.12 *Let $-1 \le p,q \le 1,\ pq \ne 0$. If $f \in \mathcal{T}^{p,q}$, then for $z = re^{it}, 0 < r < 1$, we have*

$$|f(z)| \le \begin{cases} |k_{p,q}(z)| & for \quad \mathrm{Re}\dfrac{1+pqz^2}{z} \ge |p+q|, \\ \left[\left|\mathrm{Im}\dfrac{1+pqz^2}{z}\right| - |p-q|\right]^{-1} & for \quad \left|\mathrm{Re}\dfrac{1+pqz^2}{z}\right| \le |p+q|, \\ |k_{p,q}(-z)| & for \quad \mathrm{Re}\dfrac{1+pqz^2}{z} \le -|p+q|, \end{cases}.$$

The result in a first and third case is sharp; the equality is attained for $f(z) = K(z,0) = k_{p,q}(z), K(z,\pi) = k_{p,q}(-z)$, respectively. Moreover

$$|\mathrm{Im}\, k_{p,q}(-z)| \le |\mathrm{Im}\, f(z)| \le |\mathrm{Im}\, k_{p,q}(z)| \quad for \quad \mathrm{Re}\frac{1+pqz^2}{z} \ge |p+q|,$$

$$|\mathrm{Im}\, k_{p,q}(z)| \le |\mathrm{Im}\, f(z)| \le |\mathrm{Im}\, k_{p,q}(-z)| \quad for \quad \mathrm{Re}\frac{1+pqz^2}{z} \le -|p+q|,$$

$$|\mathrm{Im}\, f(z)| \le \left[\left|\mathrm{Im}\frac{1+pqz^2}{z}\right| - |p-q|\right]^{-1} \quad for \quad \left|\mathrm{Re}\frac{1+pqz^2}{z}\right| \le |p+q|.$$

The functions that realize the equalities in the last inequalities are the same as the estimates for $|f(z)|$.

Proof. We have

$$K(z,\theta) = \frac{z}{(1-e^{i\theta}pz)(1-e^{-i\theta}qz)}.$$

For fixed z we examine $K(z,\theta)$ and $\lambda K(z,\theta_1)+(1-\lambda)K(z,\theta_2)$, where $\lambda\in[0,1], 0\le \theta,\theta_1,\theta_2\le 2\pi$. We note that for a fixed z and $-1\le p,q\le 1,\ p\neq q$ the equation

$$w=\frac{1}{K(z,\theta)}=pqz+\frac{1}{z}-(pe^{i\theta}+qe^{-i\theta}).$$

describes the ellipse with the center $pqz+1/z$ and semi-axes $p+q$ and $p-q$, and a segment with endpoints $w_1=z/(1+pz)^2, w_2=z/(1-pz)^2$ when $p=q$. The inverse $1/w$ of the ellipse is a hippopede that does not pass through the origin, with convex hull being an oval with endpoints $w_1=z/(1+pz)(1+qz), w_2=z/(1-pz)(1-qz)$. Let first $\mathrm{Re}\dfrac{1+pqz^2}{z}\ge |p+q|$. In this case the majorant region lies in the right half-plane and the point $w_2=z/(1-pz)(1-qz)$ have the largest modulus with the equality for $f(z)=k_{p,q}(z)$. Symmetric situation is for $\mathrm{Re}\dfrac{1+pqz^2}{z}\le -|p+q|$. If $\left|\mathrm{Re}\dfrac{1+pqz^2}{z}\right|\le |p+q|$ the points having the greatest modulus are the ends of a segment joining the farthest points in the direction of the imaginary axis. Reasoning along the same line we obtain bounds of $|\mathrm{Im}\, f(z)|$. $\square$

4.6 Set and radius of local univalence for $\mathcal{T}^{p,q}$

The set of local univalence for $T_{\mathbb{R}}$ has been found in [61], and is of a lens-shape, bounded by two arcs of the symmetric circles. For the class $\mathcal{T}^{p,q}$ the boundary curve of the set of local univalence is more complicated. Moreover, the method of calculations is completely different, as we can see below. Here, we clarify and improve the description of the sets obtained in [104].

In the proof we use the following practical result of Koczan and Szapiel [80].

Lemma 4.13 *Denote*

$$\mathcal{B}=\left\{f(z)\in\ \mathcal{H}(\mathbb{D}):\ f(z)=\int_a^b K(z,\theta)d\mu(\theta),\ z\in\mathbb{D},\ \mu\in\mathcal{P}_{[a,b]}\right\},$$

where $S(z,\cdot)$ is holomorphic in $\mathbb{D}$ and $S(z,\cdot)$ if continuous in $[a,b]$, and $\mathcal{P}_{[a,b]}$ denote the set of probability measures on $[a,b]$. The set of local univalence of function of class $\mathcal{B}$ is given by the formula

$$D'(\mathcal{B})=\bigcap_{f\in\mathcal{B}}\{f'(z)\neq 0\}=\{z\in\mathbb{D}:|\Delta_{a\le\theta\le b}\arg K'(z,\theta)|<\pi\},$$

where $\Delta_{a\le\theta\le b}\arg F(z,\theta)$ denotes increase of argument of $F(z,\theta)$ as θ increases from a to b.

Theorem 4.14 *Let $-1\le p,q\le 1$ and $f\in\mathcal{T}^{p,q}$. For $z=re^{it}\in\mathbb{D}$ set*

$$A=\sqrt{2(p-q)^2+4pq\sin^2 t}.$$

The equation of the boundary of the set of the local univalence $D'(\mathcal{T}^{p,q})$ in polar coordinates $z = r(t)e^{it}$ is given by the formula:

$$r = r(t) = \begin{cases} 1 & \text{if} \quad A < 1 - pq, \\ \frac{2}{\sqrt{A^2+4pq}+A} & \text{if} \quad A \geq 1 - pq. \end{cases}$$

Proof. For $f \in \mathcal{T}^{p,q}$ we have

$$K(z,\theta) = \frac{z}{(1-pze^{i\theta})(1-qze^{-i\theta})},$$

then

$$K'(z,\theta) = \frac{\frac{1}{z^2} - pq}{\left((\frac{1}{z} + pqz) - (pe^{i\theta} + qe^{-i\theta})\right)^2},$$

and

$$\arg\, K'(z,\theta) = \arg\left(\frac{1}{z^2} - pq\right) - 2\arg\left[\left(\frac{1}{z} + pqz\right) - (pe^{i\theta} + qe^{-i\theta})\right].$$

Let us put $z = re^{it}$, $r \in (0,1)$, $t \in [0, 2\pi]$, and $z_0 = x_0 + iy_0 = \frac{1}{z} + pqz$. We observe that $w(\theta) = z_0 - (pe^{i\theta} + qe^{-i\theta})$, for $\theta \in [0, 2\pi]$, represent the ellipse

$$\mathcal{L} : w = w(\theta) = u + iv = x_0 - (p+q)\cos\theta + i[y_0 - (p-q)\sin\theta], \tag{4.21}$$

with

$$x_0 = \frac{1+pqr^2}{r}\cos t, \;\; y_0 = -\frac{(1-pqr^2)}{r}\sin t. \tag{4.22}$$

Denoting

$$\psi(\theta) := \arg\, w(\theta) = \arctan\frac{y_0 - (p-q)\sin\theta}{x_0 - (p+q)\cos\theta},$$

we see that the problem

$$\Delta_{0\leq\theta\leq 2\pi} \arg\, K'(z,\theta)$$

is equivalent to finding

$$\max_{0\leq\theta\leq 2\pi} \psi(\theta) - \min_{0\leq\theta\leq 2\pi} \psi(\theta).$$

From geometrical point of view this is nothing else that finding the biggest angle formed by the rays, with the vertex at the origin, tangent to the ellipse $\mathcal{L}$ given by (4.21). The requested equations of rays have the form

$$v = m_1 u, \;\; v = m_2 u, \;\; m_1 = \tan\alpha_1, \;\; m_2 = \tan\alpha_2,$$

where m_1 and m_2 are the roots of the equation:

$$[x_0^2 - (p+q)^2]m^2 - 2x_0y_0m + [y_0^2 - (p-q)^2] = 0. \tag{4.23}$$

Using the formula

$$\tan(\alpha_2 - \alpha_1) = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$$

and the fact that

$$|\Delta \arg\, K'(z,\theta)| = 2|\max\psi - \min\psi| = 2|\alpha_2 - \alpha_1|,$$

we see that the equation of the boundary of $D'(\mathcal{T}^{p,q})$ is given by the condition $m_1 m_2 = -1$, which is

$$\frac{y_0^2-(p-q)^2}{x_0^2-(p+q)^2}=-1$$

or equivalently

$$x_0^2+y_0^2=2(p^2+q^2). \tag{4.24}$$

The equation (4.24) with the notation (4.22) is equivalent to

$$p^2q^2r^4-2(p^2+q^2-pq\cos 2t)r^2+1=0$$

or

$$(pqr^2+Ar-1)(pqr^2-Ar-1)=0$$

where

$$A^2=2(p-q)^2+4pq\sin^2 t. \tag{4.25}$$

One can verify that the expression pqr^2-Ar-1 is negative for $r\in(0,1)$. Therefore, the equation $r=r(t)$ of the boundary of $D'(\mathcal{T}^{p,q})$ is a solution of

$$pqr^2+Ar-1=0 \quad \text{if} \quad r(t)\le 1, \quad \text{and} \quad r=1 \quad \text{elsewhere.} \tag{4.26}$$

Because of the symmetry we can consider only the case $t\in[0,\pi]$. A solution of (4.26) is given by the formula,

$$r=r(t)=\frac{\sqrt{A^2+4pq}-A}{2pq}, \tag{4.27}$$

and is less then 1, if $A\ge 1-pq$; if $A<1-pq$ we put $r(t)=1$. Inequality $A\ge 1-pq$ is equivalent to

$$4pq\sin^2 t\ge(1-pq)^2-2(p-q)^2. \tag{4.28}$$

Let us consider two cases: 1. $pq>0$ and 2. $pq<0$, and set

$$q_1'(p)=\frac{\sqrt{2}(1-p^2)-p}{2-p^2}, \quad q_2'(p)=\frac{\sqrt{2}(1-p^2)+p}{p^2-2},$$
$$q_3'(p)=\frac{p+\sqrt{2}(1-p^2)}{2-p^2}, \quad q_4'(p)=\frac{p-\sqrt{2}(1-p^2)}{2-p^2},$$

and let sets $C_1, C_2, C_3, C_4, C_1', C_2', C_3', C_4'$ be denoted as in the Fig. 4.1 (we note that the common points of the curves and axes are $p=\pm 1/\sqrt{2},\ q=\pm 1/\sqrt{2}$.

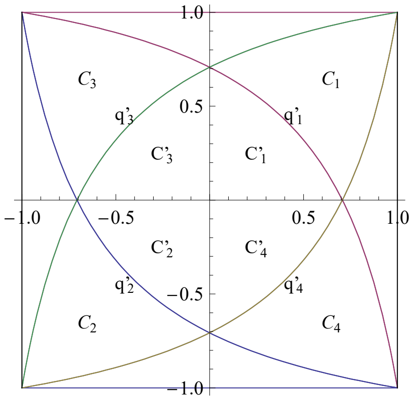

Fig. 4.1. The range of the parameters p, q for the set of local univalence.

Case 1. Inequality (4.28)

a) holds for any $t \in [0, \pi]$, if $(1-pq)^2-2(p-q)^2 \leq 0$, that is for $(p, q) \in [0, 1]^2 \setminus C_1 \cup C_1'$, and symmetrically $(p, q) \in [-1, 0]^2 \setminus C_2 \cup C_2'$. In this case we have $r = r(t)$, given by (4.27),

b) does not hold for any $t \in [0, \pi]$, if $(1 - pq)^2 - 2(p - q)^2 \geq 4pq$, so that for $(p, q) \in C_1' \cup C_2'$, and then $r = 1$,

c) holds, for $t \in [t_0, \pi - t_0]$, where

$$t_0 = \arcsin \sqrt{\frac{(1 - pq)^2 - 2(p - q)^2}{4pq}},$$

if $0 \leq (1 - pq)^2 - 2(p - q)^2 \leq 4pq$, or equivalently if $(p, q) \in C_1 \cup C_2$. Thus, the equation of the boundary of $D'(\mathcal{T}^{p,q})$ is

$$r = \begin{cases} r(t) & \text{if } t \in [0, t_0] \cup [\pi - t_0, \pi], \\ 1 & \text{if } t \in [t_0, \pi - t_0]. \end{cases} \tag{4.29}$$

Case 2. In analogous way one can prove that inequality (4.28)

a) holds for any $t \in [0, \pi]$, if $(1 - pq)^2 - 2(p - q)^2 \leq 4pq$, that is equivalent to $(p, q) \in [-1, 0] \times [0, 1] \setminus C_3 \cup C_3'$, and $(p, q) \in [0, 1] \times [-1, 0] \setminus C_4 \cup C_4'$. Then $r = r(t)$, given by (4.27),

b) does not hold for any $t \in [0, \pi]$, if $(1 - pq)^2 - 2(p - q)^2 \geq 0$, which reduces to $(p, q) \in C_3' \cup C_4'$. In this case $r = 1$,

c) holds, for $t \in [t_0, \pi - t_0]$, if $4pq \leq (1-pq)^2-2(p-q)^2 \leq 0$, holds, for $t \in [t_0, \pi-t_0]$, where

$$t_0 = \arcsin \left(\frac{(1 - pq)^2 - 2(p - q)^2}{4pq} \right)^{\frac{1}{2}},$$

if $0 \le (1-pq)^2 - 2(p-q)^2 \le 4pq$. Hence the equation of the boundary of $D'(\mathcal{T}^{p,q})$ is

$$r = \begin{cases} r(t) & \text{if } t \in [0,t_0] \cup [\pi - t_0, \pi], \\ 1 & \text{if } t \in [t_0, \pi - t_0]. \end{cases} \tag{4.30}$$

□

Let us describe the sets from the Fig. 4.1. analytically. We have

$$C_1 = \left\{0 \le p \le \frac{1}{\sqrt{2}}, \quad q_2'(p) \le q \le q_4'(p)\right\} \cup \left\{\frac{1}{\sqrt{2}} \le p \le 1, \quad q_3'(p) \le q \le q_4'(p)\right\},$$
$$C_1' = \left\{0 \le p \le \frac{1}{\sqrt{2}}, 0 \le q \le q_2'(p)\right\},$$
$$C_2 = \left\{-1 \le p \le -\frac{1}{\sqrt{2}}, \quad q_3'(p) \le q \le q_4'(p)\right\} \cup \left\{-\frac{1}{\sqrt{2}} \le p \le 0, \quad q_3'(p) \le q \le q_1'(p)\right\},$$
$$C_2' = \left\{-\frac{1}{\sqrt{2}} \le p \le 0, \quad q_1'(p) \le q \le 0\right\},$$
$$C_3 = \left\{-1 \le p \le -\frac{1}{\sqrt{2}}, \quad q_1'(p) \le q \le q_2'(p)\right\} \cup \left\{-\frac{1}{\sqrt{2}} \le p \le 0, \quad q_4'(p) \le q \le q_2'(p)\right\},$$
$$C_3' = \left\{-\frac{1}{\sqrt{2}} \le p \le 0, \quad 0 \le q \le q_4'(p)\right\},$$
$$C_4 = \left\{0 \le p \le \frac{1}{\sqrt{2}}, \quad q_1'(p) \le q \le q_3'(p)\right\} \cup \left\{\frac{1}{\sqrt{2}} \le p \le 1, \quad q_1'(p) \le q \le q_2'(p)\right\},$$
$$C_4' = \left\{0 \le p \le \frac{1}{\sqrt{2}}, \quad q_3'(p) \le q \le 0\right\}.$$

Applying Theorem 4.14 we come to the following conclusion about the radius of local univalence of $\mathcal{T}^{p,q}$.

Theorem 4.15 *The sharp value of the radius $r_0^{(p,q)}$ of local univalence of the class $\mathcal{T}^{p,q}$ is given by the formula:*

$$r_0^{(p,q)} = \begin{cases} \frac{\sqrt{2}}{\sqrt{p^2+q^2+p+q}} & \text{if } \ (p,q) \in \{[0,1]^2 \setminus C_1'\} \cup \{[-1,0]^2 \setminus C_2'\}, \\ \frac{\sqrt{2}}{\sqrt{p^2+q^2+|p-q|}} & \text{if } \ (p,q) \in \{[-1,0] \times [0,1] \setminus C_3'\} \cup \{[0,1] \times [-1,0] \setminus C_4'\}, \\ 1 & \text{if } \ (p,q) \in \bigcup_{k=1}^4 C_k', \end{cases} \tag{4.31}$$

where the set C_j, C_j' $j = 1, ..., 4$, were predefined. The equalities when $(p,q) \in \{[0,1]^2 \setminus C_1'\} \cup \{[-1,0]^2 \setminus C_2'\}$ are attained for

$$f_0(z) = \frac{1}{2}\left(\frac{z}{(1-pze^{i\theta})(1-qze^{-i\theta})} + \frac{z}{(1+pze^{-i\theta})(1+qze^{i\theta})}\right) \quad \textit{at} \quad z = \pm i r_0,$$
$$\textit{where} \quad \cos\theta = \frac{|p+q|}{\sqrt{2(p^2+q^2)}}, \quad \sin\theta = -\frac{(p-q)}{\sqrt{2(p^2+q^2)}},$$

and, if $(p,q) \in \{[-1,0] \times [0,1] \setminus C_3'\} \cup \{[0,1] \times [-1,0] \setminus C_4'\}$

$$f_0(z) = \frac{1}{2}\left(\frac{z}{(1-pze^{i\theta})(1-qze^{-i\theta})} + \frac{z}{(1-pze^{-i\theta})(1-qze^{i\theta})}\right) \quad at \quad z = \pm r_0,$$

$$where \quad \cos\theta = \frac{|p+q|}{\sqrt{2(p^2+q^2)}}, \quad \sin\theta = \frac{(p-q)}{\sqrt{2(p^2+q^2)}}.$$

Proof. The radius $r_0^{(p,q)}$ of the largest disc with the center at the origin which is contained in $D'(\mathcal{T}^{p,q})$ for any $t \in [0, 2\pi]$ is the radius of local univalence of the class $\mathcal{T}^{p,q}$. Finding the maximal value of $r(t)$ given by (4.27), which is attained for $t = 0$ if $pq < 0$ and for $t = \frac{\pi}{2}$ if $pq > 0$ we find (4.31). The form of the extremal functions follows from (4.22) and (4.23). □

Theorem 4.16 *The radius of univalence $r_u^{(p,q)}$ of the class $\mathcal{T}^{p,q}$ satisfy the inequality:*

$$r_0^{(p,q)} \geq r_u^{(p,q)} \geq \hat{r}^{(p,q)}$$

where $\hat{r}^{(p,q)}$ is the unique root of the equation

$$1 - |p||q|r^2 = 2r^2(|p|\sqrt{1-q^2r^2} + |q|\sqrt{1-p^2r^2})^2. \tag{4.32}$$

and $r_0^{(p,q)}$ is given by (4.31).

Proof. We will use the sufficient condition for univalence: $\mathrm{Re} f'(z) > 0$. From (4.1) we have for $f \in \mathcal{T}^{p,q}$

$$f'(z) = \int_0^{2\pi} \frac{1 - pqz^2}{(1-pze^{i\theta})^2(1-qze^{-i\theta})^2} d\mu(\theta) = \int_{-\pi}^{\pi} K'(z,\theta)d\mu(\theta).$$

We see that $\mathrm{Re} f'(z) > 0$ for $|\arg K'(z,\theta)| < \frac{\pi}{2}$. Putting $z = re^{it},\ r \in (0,1),\ t \in [0, 2\pi]$, we find that

$$\arg K'(z,\theta) = \Big\{ -\arctan\frac{pqr^2 \sin 2t}{1 - pqr^2 cos2t} + 2\arctan\frac{pr\sin(t+\theta)}{1 - pr\cos(t+\theta)}$$
$$+ 2\arctan\frac{qr\sin(t-\theta)}{1-qr\cos(t-\theta)}\Big\}.$$

Because

$$\max \min_{0 \leq \varphi \leq 2\pi} \arctan\frac{\tau\sin\varphi}{1-\tau\cos\varphi} = \pm\frac{|\tau|}{\sqrt{1-\tau^2}} \quad (|\tau| < 1),$$

and

$$\arctan\frac{\tau}{\sqrt{1-\tau^2}} = \arcsin\tau,$$

we conclude that

$$\begin{aligned} |\arg K'(z,\theta)| &< \arctan\frac{|p||q|r^2}{\sqrt{1-p^2q^2r^2}} \\ &+ 2\Big(\arctan\frac{|p|r}{\sqrt{1-p^2r^2}} + \arctan\frac{|q|r}{\sqrt{1-q^2r^2}}\Big) \\ &= \arcsin|p||q|r^2 + 2(\arcsin|p|r + \arcsin|q|r). \end{aligned}$$

Using the formula:

$$\arcsin x + \arcsin y = \eta \arcsin(x\sqrt{1-y^2} + y\sqrt{1-x^2}) + \varepsilon\pi,$$

where

$$\eta = 1,\ \varepsilon = 0, \quad \text{iff} \quad xy < 0 \text{ or } x^2 + y^2 \leq 1,$$

$$\eta = -1,\ \varepsilon = -1, \quad \text{iff} \quad x^2 + y^2 > 1,\ x < 0,\ y < 0,$$

$$\eta = -1,\ \varepsilon = 1, \quad \text{iff} \quad x^2 + y^2 > 1,\ x > 0,\ y > 0,$$

we come to the conclusion, that $|\arg K'(z,\theta)| < \frac{\pi}{2}$ if and only if

$$1 - |p||q|r^2 > 2r^2(|p|\sqrt{1-q^2r^2} + |q|\sqrt{1-p^2r^2})^2,$$

which ends the proof. □

Observe that by formula (4.32) we have $\hat{r}^{(1,1)} = \frac{\sqrt{2}}{4} = 0.35\ldots$; which is not sharp ($\hat{r}^{(1,1)} = \sqrt{2} - 1$) [45], however $\hat{r}^{(1,0)} = \frac{\sqrt{2}}{2}$ is the sharp value [160].

Chapter 5

Generalized Meixner - Pollaczek polynomials - $P_n^{\lambda}(x;\theta,\psi)$

In Section 2.1 we presented the motivation to introduce Generalized Meixner - Pollaczek (GMP) polynomials by the formula (2.6). Now in the Section 5.1, we explore more detailed the properties of GMP polynomials. Our discussion of GMP polynomials includes recurrence relation, explicite formula, hypergeometric representation and difference equation. One of the main result of this section is Theorem 5.6, where we prove the orthogonality.

In Section 5.2 we tackle the problem for a fixed value of ψ. In fact from now on we fix the value of the parameter ψ to be $\pi+\theta$. The resulting polynomials are called the Symmetric Generalized Meixner - Pollaczek polynomials and denote in this thesis by $S_n^{\lambda}(x;\theta)$. Next, we consider the density function of probability measure $w(x) = \dfrac{1}{2\cosh \pi x/2}$. Furthermore, it has interesting properties that make it useful as a weight function for orthogonal polynomials. The most useful property of the weight function $w(x)$ is that it can be interpreted as a Poisson kernel [142], namely we have the following.

Proposition 5.1 *Let the function f be continuous and harmonic in the strip $\mathcal{Z} = \{z \in \mathbb{C} : |\mathrm{Im} z| \le 1\}$, and suppose further that $|f(z)| < Ce^{a|z|}$, for some $a \in [a, \frac{\pi}{2}]$. Then*

$$f(0) = \int_{-\infty}^{+\infty} \frac{f(x+i)+f(x-i)}{2} \frac{dx}{2\cosh \frac{\pi}{2}x}.$$

Since the weight w is closely related to the strip $\mathcal{Z}$, we describe an orthogonal basis for the space $H^2(\mathcal{Z},\mathcal{M})$ where $\mathcal{M}$ is the Poisson measure for 0. This is summarized in Theorem 5.9. We designate the special case of $S_n^{\lambda}(x;\theta)$ and its important result of

Section 5.2 that $\sigma_n(2x)$ is the limiting case of the Symmetric Generalized Meixner - Pollaczek polynomials, as the parameter $\lambda \to 0^+$. This is the content of the Remark 5.3

Finally, in Section 5.3 we consider another special case of $P_n^\lambda(x;\theta,\psi)$ for $\psi = \pi - \theta$. We call this polynomials quasi-symmetric Meixner - Pollaczek polynomials and denote them by $Q_n^\lambda(x;\theta)$. We investigate a basic question of information theory, namely the evaluation of the Fisher information for $Q_n^\lambda(x;\theta)$. We present the explicit expression of this quantity in Theorem 5.12.
The Fisher information was first introduced in the framework of statistical estimation theory, where it plays a key role [48]. The Fisher information is a gradient functional of the polynomials, so that it is a local measure of the concentration of the polynomials. This quantity has been discussed in detail and explicitly calculated for the Laguerre, Hermite and Jacobi polynomials and other special functions [158]. In Section 5.3.2 we obtain the explicit expression of this quantity for quasi-symmetric Meixner - Pollaczek polynomials.

5.1 Generalized Meixner - Pollaczek polynomials

5.1.1 Fundamental properties

In this section we find the three-term recurrence relation, the explicit formula, the hypergeometric representation, the difference equation and the orthogonality relation for GMP polynomials $P_n^\lambda(x;\theta,\psi)$, defined by (2.6).

Theorem 5.2 *Let us set $P_{-1}^\lambda = 0$. The polynomials $P_n^\lambda = P_n^\lambda(x;\theta,\psi)$ have the following properties*

a) P_n^λ satisfy the three-term recurrence relation

$$\begin{aligned} P_0^\lambda &= 1, \\ nP_n^\lambda &= [(\lambda - ix)e^{i\theta} + (\lambda + ix)e^{i\psi} + (n-1)(e^{i\theta} + e^{i\psi})]P_{n-1}^\lambda \\ &- (2\lambda + n - 2)e^{i(\theta+\psi)}P_{n-2}^\lambda \quad (n \geq 1). \end{aligned}$$

b) P_n^λ are given by the formula

$$P_n^\lambda(x;\theta,\psi) = e^{in\theta}\sum_{j=0}^{n}\frac{(\lambda+ix)_j(\lambda-ix)_{n-j}}{j!(n-j)!}e^{ij(\psi-\theta)} \quad (n \in \mathbb{N}\cup\{0\}). \tag{5.1}$$

c) P_n^λ have the hypergeometric representation

$$n!P_n^\lambda(x;\theta,\psi) = (2\lambda)_n e^{in\theta}{}_2F_1(-n, \lambda + ix, 2\lambda; 1 - e^{i(\psi-\theta)}). \tag{5.2}$$

d) Let $y(x) = P_n^\lambda(x;\theta,\psi)$. The function $y(x)$ satisfies the following difference equation

$$\begin{aligned} e^{i\theta}(\lambda - ix)y(x+i) + [ix(e^{i\theta}+e^{i\psi}) - (n+\lambda)(e^{i\theta}-e^{i\psi})]y(x) \\ -e^{i\psi}(\lambda+ix)y(x-i) = 0. \end{aligned} \tag{5.3}$$

Proof.

a) We differentiate the formula (2.6) with respect to z, and after multiplication by $(1-ze^{i\theta})(1-ze^{i\psi})$ we compare the leading coefficients of z^{n-1}.

b) The Cauchy product of the power series

$$(1-ze^{i\theta})^{-(\lambda-ix)} = \sum_{n=0}^{\infty} \frac{(\lambda-ix)_n e^{in\theta}}{n!} z^n$$

and

$$(1-ze^{i\psi})^{-(\lambda+ix)} = \sum_{n=0}^{\infty} \frac{(\lambda+ix)_n e^{in\psi}}{n!} z^n$$

gives (5.1).

c) Applying the formula from ([45, vol.1, p.82]):

$$(1-s)^{a-c}(1-s+sz)^{-a} = \sum_{n=0}^{\infty} \frac{(c)_n}{n!} {}_2F_1(-n,a;c;z)s^n \quad (|s|<1,\ |s(1-z)|<1),$$

with $s = ze^{i\theta},\ a = \lambda + ix,\ c = 2\lambda,\ z = 1 - e^{i(\psi-\theta)}$, one obtains

$$(1-ze^{i\theta})^{-(\lambda-ix)}(1-ze^{i\psi})^{-(\lambda+ix)} = \sum_{n=0}^{\infty} \frac{e^{in\theta}(2\lambda)_n}{n!} {}_2F_1(-n,\lambda+ix,2\lambda;1-e^{i(\psi-\theta)})z^n.$$

Comparing the coefficients of the power series of both sides, we get (5.2).

d) Inserting $(x+i)$ and $(x-i)$ instead of x into the generating function (2.6) we find

$$y(x+i) = \sum_{k=0}^{n-1} P_k^\lambda(x;\theta,\psi)(e^{i(n-k)\theta} - e^{i[(n-k-1)\theta+\psi]}) + P_n^\lambda,$$

$$y(x-i) = \sum_{k=0}^{n-1} P_k^\lambda(x;\theta,\psi)(e^{i(n-k)\psi} - e^{i[(n-k-1)\psi+\theta]}) + P_n^\lambda,$$

which implies that

$$\begin{aligned} & e^{i\theta}(\lambda-ix)y(x+i) - e^{i\psi}(\lambda+ix)y(x-i) \\ = \ & (e^{i\theta}-e^{i\psi}) \sum_{k=0}^{n-1} P_k^\lambda(x;\theta,\psi)[(\lambda-ix)e^{i(n-k)\theta} + (\lambda+ix)e^{i(n-k)\psi}] \\ + \ & [e^{i\theta}(\lambda-ix) - e^{i\psi}(\lambda+ix)]P_n^\lambda. \end{aligned} \tag{5.4}$$

Differentiation of the generating function (2.6) with respect to z and equating the leading coefficient of z^{n-1} yields:

$$nP_n^\lambda(x;\theta,\psi) = \sum_{k=0}^{n-1} P_k^\lambda(x;\theta,\psi)[(\lambda-ix)e^{i(n-k)\theta} + (\lambda+ix)e^{i(n-k)\psi}]$$

which together with (5.4) gives (5.3).

□

The first four polynomials P_n^λ are given by the following formulas.

Corollary 5.3

$$
\begin{aligned}
P_0^\lambda &= 1, \\
P_1^\lambda &= ix(e^{i\psi}-e^{i\theta}) + \lambda(e^{i\theta}+e^{i\psi}), \\
2P_2^\lambda &= -x^2(e^{i\psi}-e^{i\theta})^2 + ix(2\lambda+1)(e^{2i\psi}-e^{2i\theta}) \\
&+ \lambda\left[(1+\lambda)e^{2i\psi} + 2\lambda e^{i(\psi+\theta)} + (1+\lambda)e^{2i\theta}\right], \\
6P_3^\lambda &= ix^3\left[3e^{i\theta}e^{i\psi}(e^{i\psi}-e^{i\theta}) - (e^{3i\psi}-e^{3i\theta})\right] \\
&+ 3(1+\lambda)x^2\left[e^{i\theta}e^{i\psi}(e^{i\psi}+e^{i\theta}) - (e^{3i\psi}+e^{3i\theta})\right] \\
&+ ix\left[3\lambda^2 e^{i\theta}e^{i\psi}(e^{i\psi}-e^{i\theta}) + (3\lambda^2+6\lambda+2)(e^{3i\psi}-e^{3i\theta})\right] \\
&+ \lambda(1+\lambda)\left[3\lambda e^{i\theta}e^{i\psi}(e^{i\psi}+e^{i\theta}) + (\lambda+2)(e^{3i\psi}+e^{3i\theta})\right], \\
24P_4^\lambda &= x^4\left[(e^{i\psi}-e^{i\theta})^4 + 4e^{2i\psi}e^{2i\theta}\right] + 2ix^3(2\lambda+3)(e^{2i\psi}-e^{2i\theta})(e^{i\psi}+e^{i\theta})^2 \\
&+ x^2\Big[-(6\lambda^2+18\lambda+11)(e^{4i\psi}+e^{4i\theta}) + 4(3\lambda+2)e^{i\psi}e^{i\theta}(e^{2i\psi}+e^{2i\theta}) \\
&+ 6(2\lambda^2+2\lambda+1)e^{2i\psi}e^{2i\theta}\Big] \\
&+ 2ix(e^{2i\psi}-e^{2i\theta})\Big[(4\lambda^3+9\lambda^2+11\lambda+3)(e^{2i\psi}+e^{2i\theta}) \\
&+ 2\lambda(2\lambda+3)e^{i\psi}e^{i\theta}\Big] + \lambda(1+\lambda)\Big[(\lambda+2)(\lambda+3)(e^{4i\psi}+e^{4i\theta}) \\
&+ 4\lambda(\lambda+2)e^{i\psi}e^{i\theta}(e^{2i\psi}+e^{2i\theta}) + 6\lambda(\lambda+1)e^{2i\psi}e^{2i\theta}\Big].
\end{aligned}
$$

Proposition 5.4 *For $x \in \mathbb{R}, \psi \in \mathbb{R}, \lambda > 0$, and $n \in \mathbb{N}$ the following explicit formula holds:*

$$P_n^\lambda(x;\theta,\psi) = e^{i\theta n}\sum_{k=0}^{[\frac{n}{2}]}\frac{(\lambda-ix)_{n-2k}(\lambda+ix)_k}{(n-2k)!k!}\left(\frac{e^{i\psi}}{e^{2i\theta}}\right)^k. \tag{5.5}$$

Proof. By (2.6) we have

$$\sum_{n=0}^{\infty} P_n^\lambda(x;\theta,\psi)z^n = (1-ze^{i\theta})^{-\lambda+ix}(1-ze^{i\psi})^{-\lambda-ix}. \tag{5.6}$$

Using the binomial power series we obtain

$$
\begin{aligned}
&(1-ze^{i\theta})^{-\lambda+ix}(1-ze^{i\psi})^{-\lambda-ix} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} z^n\frac{(\lambda-ix)_{n-k}}{(n-k)!}e^{i\theta(n-k)}\frac{(\lambda+ix)_k}{k!}e^{i\psi k)} \\
&= \sum_{n=0}^{\infty} z^n\sum_{k=0}^{[\frac{n}{2}]}\frac{(\lambda-ix)_{n-2k}}{(n-2k)!}e^{i\theta(n-2k)}\frac{(\lambda+ix)_k}{k!}e^{i\psi k}.
\end{aligned}
\tag{5.7}
$$

Comparing coefficients of (5.6) and (5.7), (5.5) follows. □

Proposition 5.5 *The family of generalized Meixner - Pollaczek polynomials $P_n^\lambda(x;\theta,\psi)$ can be extended to the case $\lambda = 0$ as follows:*

$$
\begin{aligned}
P_0^0(x;\theta,\psi) &= 1, \\
nP_n^0(x;\theta,\psi) &= \left(\frac{e^{i\theta}-e^{i\psi}}{i}\right)xP_{n-1}^1(x;\theta,\psi) \quad (n \geq 1).
\end{aligned}
\tag{5.8}
$$

Proof. Since from (5.2) we have

$$\begin{aligned}
&\lim_{\lambda\to 0} P_n^\lambda(x;\theta,\psi)\\
= &\lim_{\lambda\to 0}(2\lambda)_n e^{in\theta}{}_2F_1\left(-n,\lambda+ix,2\lambda;1-\frac{e^{i\psi}}{e^{i\theta}}\right)\\
= &\frac{1}{n!}e^{in\theta}\left(1-e^{i(\psi-\theta)}\right)\Gamma(n)(-n)(ix){}_2F_1\left(-n+1,ix+1,2;1-e^{i(\psi-\theta)}\right)\\
= &\frac{1}{n!}e^{in\theta}\left(1-e^{i(\psi-\theta)}\right)\Gamma(n)(-n)(ix)P_{n-1}^1(x;\theta,\psi)\frac{n-1}{(2)_{n-1}}e^{-i(n-1)\theta}\\
= &\frac{1}{n}\left(\frac{e^{i\theta}-e^{i\psi}}{i}\right)xP_{n-1}^1(x;\theta,\psi),
\end{aligned}$$

then (5.8) is a natural consequence. □

5.1.2 Orthogonality

Before we state the main theorem, we must introduce additional terminology.

Hjalmar Mellin (1854-1933) gave his name to the *Mellin transform* that associates to a function f defined over the positive reals, and is the complex function $f^*(s)$ where

$$\mathcal{M}[f(x);s]=f^*(s)=\int_0^{+\infty}f(x)x^{s-1}dx.$$

The change of variables $x=e^{-u}$ shows that the Mellin transform is closely related to the Laplace transform and the Fourier transform. However, despite this connection, there are numerous applications where it proves convenient to operate directly with the Mellin form rather than the Laplace-Fourier version. This is often the case in complex function theory, in number theory, in applied mathematics. Now we can state our theorem.

Theorem 5.6 *The polynomials $P_n^\lambda(x;\theta,\psi)$ are orthogonal on $(-\infty,+\infty)$ with the weight $w_{\theta,\psi}^\lambda(x)=\dfrac{1}{2\pi}e^{(\theta-\psi+\pi)x}|\Gamma(\lambda+ix)|^2$, where $\lambda>0$, $\theta\in(-\frac{\pi}{2},\frac{\pi}{2})$, and*

$$\int_{-\infty}^{+\infty}w_{\theta,\psi}^\lambda(x)P_n^\lambda(x;\theta,\psi)\overline{P_m^\lambda(x;\theta,\psi)}dx=\delta_{nm}\frac{\Gamma(n+2\lambda)}{n!\,(2\cos{(\theta-\psi+\pi)/2})^{2\lambda}}.$$

Proof. Let $F(s)$ and $H(s)$ be the Mellin transforms of $f(x)$ and $h(x)$, i.e.

$$\{\mathcal{M}f\}(s)=F(s)=\int_0^\infty f(x)x^{s-1}dx,\quad \{\mathcal{M}h\}(s)=H(s)=\int_0^\infty h(x)x^{s-1}dx.$$

Then the following formula (Parseval's identity) holds [115]:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}F(s)H(1-s)ds=\int_0^\infty f(x)h(x)dx, \tag{5.9}$$

and [45]

$$\int_0^{+\infty} u^{\alpha-1} e^{-pu} e^{-iqu} du = \Gamma(\alpha)(p^2+q^2)^{-\frac{\alpha}{2}} e^{-i\alpha \arctan{(p/q)}}. \tag{5.10}$$

For $f(x) = 2x^{2(\lambda+j)}e^{-x^2}$ and $h(x) = 2x^{2(\lambda+k)-1}e^{-x^2}$, we have

$$F(s) = \Gamma(\lambda + j + \frac{s}{2}), \quad H(s) = \Gamma(\lambda + k + \frac{s-1}{2}).$$

By the well know property

$$\{\mathcal{M}f\}(e^{i\theta}x) = e^{-i\theta s}F(s),$$

we have

$$\{\mathcal{M}f\}(e^{i(\theta-\psi+\pi)/2}x) = e^{-is(\theta-\psi+\pi)/2}F(s).$$

Consecutively, applying first the formula $(a)_j = \frac{\Gamma(a+j)}{\Gamma(a)}$ $(j = 1, 2, ...)$, and (5.9), next setting $\alpha = 2\lambda + k + j$, $p = \cos(\theta - \psi + \pi) + 1$, $q = \sin(\theta - \psi + \pi)$ in (5.10), we have

$$\begin{aligned}
& \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{(\theta-\psi+\pi)x}(\lambda+ix)_j(\lambda-ix)_k|\Gamma(\lambda+ix)|^2dx \\
= & \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{(\theta-\psi+\pi)x}\Gamma(\lambda+j+ix)\Gamma(\lambda+k-ix)dx \\
= & \frac{1}{4\pi i}\int_{-\infty}^{+\infty} e^{-ix(\theta-\psi+\pi)/2}\Gamma\left(\lambda+j+\frac{x}{2}\right)\Gamma\left(\lambda+k-\frac{x}{2}\right)dx \\
= & 2e^{i(\theta-\psi+\pi)(\lambda+j)}\int_0^{+\infty} x^{2(2\lambda+k+j)-1}\exp(-(e^{(\theta-\psi+\pi)i}+1)x^2)dx \\
= & e^{i(\theta-\psi+\pi)(\lambda+j)}\int_0^{+\infty} x^{2\lambda+k+j-1}\exp(-(e^{(\theta-\psi+\pi)i}+1)x)dx \\
= & \frac{e^{i(j-k)(\theta-\psi+\pi)/2}\Gamma(2\lambda+k+j)}{(2\cos((\theta-\psi+\pi)/2)))^{2\lambda+k+j}}.
\end{aligned} \tag{5.11}$$

Set

$$P_n^\lambda(x;\theta,\psi) = \sum_{k=0}^{n} A_k(\lambda+ix)_k,$$

where

$$A_k = \frac{e^{ik\theta}(2\lambda)_k(-k)_k(1-e^{i(\psi-\theta)})^k}{k!(2\lambda)_k k!}.$$

Then

$$\begin{aligned}
J := & \frac{1}{2\pi}\int_{-\infty}^{+\infty} P_n^{\lambda}(x;\theta,\psi)(\lambda-ix)_k e^{(\theta-\psi+\pi)x}|\Gamma(\lambda+ix)|^2dx \\
= & \frac{1}{2\pi}\int_{-\infty}^{+\infty}\sum_{j=0}^{n} A_j(\lambda+ix)_j(\lambda-ix)_k e^{(\theta-\psi+\pi)x}|\Gamma(\lambda+ix)|^2dx \\
= & \frac{1}{2\pi}\sum_{j=0}^{n} A_j\int_{-\infty}^{+\infty}(\lambda+ix)_j(\lambda-ix)_k e^{(\theta-\psi+\pi)x}|\Gamma(\lambda+ix)|^2dx \\
= & \frac{(2\lambda)_n e^{in\theta}}{n!}\sum_{j=0}^{n}\frac{(-n)_j(1-e^{i(\psi-\theta)})^j}{(2\lambda)_j j!} \\
& \times\frac{1}{2\pi}\int_{-\infty}^{+\infty}(\lambda+ix)_j(\lambda-ix)_k e^{(\theta-\psi+\pi)x}|\Gamma(\lambda+ix)|^2dx.
\end{aligned}$$

Using (5.11) and (1.5), we obtain

$$\begin{aligned}
J = & \frac{(2\lambda)_n e^{in\theta}}{n!}\sum_{j=0}^{n}\frac{(-n)_j(1-e^{i(\psi-\theta)})^j}{(2\lambda)_j j!}\frac{e^{i(j-k)(\theta-\psi+\pi)/2}\Gamma(2\lambda+k+j)}{(2\cos((\theta-\psi+\pi)/2)))^{2\lambda+k+j}} \\
= & \frac{(2\lambda)_n e^{in\theta}}{n!}\Gamma(2\lambda+k)\frac{e^{-ik(\theta-\psi+\pi)/2}}{(2\cos((\theta-\psi+\pi)/2))^{2\lambda+k}} \\
& \times\sum_{j=0}^{n}\frac{(-n)_j(2\lambda+k)_j}{(2\lambda)_j j!}\frac{(1-e^{i(\psi-\theta)})^j}{\left(e^{i(\theta-\psi+\pi)/2}+e^{-i(\theta-\psi+\pi)/2}\right)^j\left(e^{-i(\theta-\psi+\pi)/2}\right)^j} \\
= & \frac{(2\lambda)_n e^{in\theta}}{n!}\Gamma(2\lambda+k)\frac{e^{-ik(\theta-\psi+\pi)/2}}{(2\cos((\theta-\psi+\pi)/2))^{2\lambda+k}}\,{}_2F_1(-n,2\lambda+k;2\lambda;1).
\end{aligned}$$

By the formula (1.5) the above reduces to

$$\begin{aligned}
& \frac{1}{2\pi}\int_{-\infty}^{+\infty} P_n^{\lambda}(x;\theta,\psi)(\lambda-ix)_k e^{(\theta-\psi+\pi)x}|\Gamma(\lambda+ix)|^2dx \\
= & \frac{e^{i(n\theta-k(\theta-\psi+\pi)/2)}\Gamma(2\lambda+k)}{n!\,(2\cos((\theta-\psi+\pi)/2))^{2\lambda+k}}(-k)_n.
\end{aligned} \tag{5.12}$$

Since $(-k)_n = 0$ for $k < n$, then (5.12) is nonzero only for the case $k = n$. Then

$$\begin{aligned}
& \frac{1}{2\pi}\int_{-\infty}^{+\infty} P_n^{\lambda}(x;\theta,\psi)(\lambda-ix)_k e^{(\theta-\psi+\pi)x}|\Gamma(\lambda+ix)|^2dx \\
= & \frac{e^{i(n\theta-k(\theta-\psi+\pi)/2)}\Gamma(2\lambda+k)}{n!\,(2\cos((\theta-\psi+\pi)/2))^{2\lambda+k}}(-n)_n.
\end{aligned}$$

From this and relation (5.11), it follows that

$$\begin{aligned}
& \int_{-\infty}^{+\infty} P_n^{\lambda}(x;\theta,\psi)\overline{P_m^{\lambda}(x;\theta,\psi)}w_{\theta}^{\lambda}(x)dx \\
= & \delta_{nm}\frac{e^{-in\theta}(2\lambda)_n(-n)_n(1-e^{-i(\psi-\theta)})^n}{n!(2\lambda)_n n!}\frac{e^{i(n\theta-\frac{n}{2}(\theta-\psi+\pi))}\Gamma(2\lambda+n)}{n!\left(2\cos\frac{\theta-\psi+\pi}{2}\right)^{2\lambda+n}}(-n)_n \\
= & \delta_{nm}\frac{\Gamma(n+2\lambda)}{n!\left(2\cos\frac{\theta-\psi+\pi}{2}\right)^{2\lambda}}.
\end{aligned}$$

□

5.2 Symmetric Generalized Meixner - Pollaczek polynomials

Let us consider now the case $\psi = \pi + \theta$. We observe that such case leads to the very interesting family of symmetric polynomials. Some special cases of $P_n^\lambda(x;\theta,\pi+\theta;z)$ are known in the literature for $\theta = \frac{\pi}{2}$. These are the Symmetric Meixner - Pollaczek polynomials, denoted by $P_n^\lambda(x/2;\theta),\ \lambda > 0$. The importance of symmetric Meixner - Pollaczek polynomials was indicated by Bender, Mead and Pinsky [13], and Koornwinder [85] have shown that there is a connection between the Symmetric Meixner - Pollaczek polynomials $P_n^{\frac{1}{2}}(\frac{x}{2},\frac{\pi}{2})$ and the Heisenberg algebra. Another example is [10], where the Symmetric Meixner - Pollaczek polynomials are considered.

Definition 5.1 *We define the Symmetric Generalized Meixner - Pollaczek (SGMP) polynomials $S_n^\lambda(x;\theta)$ by the following generating function:*

$$\begin{aligned} G^\lambda(x;\theta,\pi+\theta;z) &= \frac{1}{(1-ze^{i\theta})^{\lambda-ix}(1+ze^{i\theta})^{\lambda+ix}} \\ &= \frac{e^{-2x\arctan(ze^{i(\theta+\pi/2)})}}{(1-z^2e^{2i\theta})^\lambda} \\ &= \sum_{n=0}^{\infty} S_n^\lambda(x;\theta)z^n \quad (z\in\mathbb{D}). \end{aligned}$$

This sequence of polynomials has a hypergeometric representation

$$S_n^\lambda(x;\theta) = e^{in\theta}\frac{(2\lambda)_n}{n!}\,{}_2F_1(-n,\lambda+ix,2\lambda;2), \tag{5.13}$$

and an integral representation

$$S_n^\lambda(x;\theta) = \frac{1}{2\pi i}\int \frac{e^{-2x\arctan(ze^{i(\theta+\pi/2)})}}{(1-z^2e^{2i\theta})^\lambda}\frac{dz}{z^{n+1}}.$$

Denote $\mathcal{Z} = \{z\in\mathbb{C} : |\mathrm{Im}z| < 1\}$ and $w(x) = \dfrac{1}{2\cosh(\pi x/2)}$.

Proposition 5.7 *The function $w(x)$ is a density function of a probability measure on $\mathcal{Z}$.*

Proof. This is the case because,

$$\int_{-\infty}^{+\infty} w(x)dx = \int_{-\infty}^{+\infty}\frac{dx}{2\cosh\frac{\pi x}{2}} = \left[\frac{1}{\pi}\arctan\sinh\frac{\pi}{2}x\right]_{-\infty}^{+\infty} = 1.$$

□

The function $w(x)$ has two interesting properties that make it useful as a weight function for orthogonal polynomials. The first is that it is up to a dilation its own

Fourier transform, or that it can be said, $w(x) = (1/\cosh x)(x)$. The second is that $w(x)$ is essentially the Poisson kernel for a strip of width two. The first property makes it possible to interpret its moments as values at 0 of successive derivatives, while the second can be used for direct computations of many integrals. Using these two properties we define orthogonal polynomial systems $\{\sigma_n(x)\}$ which turns out to be a limiting case of SGMP.
As we mentioned before the most useful property of the weight function $w(x)$ is that it can be interpreted as a Poisson kernel, namely we have the following;

Proposition 5.8 [142, Stein,] *Let the function f be continuous and harmonic in the strip $\mathcal{Z}$ and suppose further that $|f(z)| < C \exp a|z|$, for some a $(0 \le a \le \pi/2)$. Then*

$$f(0) = \int_{-\infty}^{\infty} \frac{f(x+i) + f(x-i)}{2} \frac{dx}{2\cosh \frac{\pi}{2} x}.$$

Since the weight $w(x)$ is so closely related to the strip $\mathcal{Z}$, we shall also describe an orthogonal basis for the Hilbert space $H^2(\mathcal{Z}, \mathcal{M})$, where $\mathcal{M}$ is the Poisson measure for 0. The inner product for any two functions $f, g \in H^2(\mathcal{Z}, \mathcal{M})$ is given by the formula:

$$(f,g)_{H^2(\mathcal{Z},\mathcal{M})} := \int_R \frac{f(x+i)\overline{g(x+i)} + f(x-i)\overline{g(x-i)}}{4\cosh \frac{\pi}{2} x} dx.$$

Definition 5.2 *We consider the system $\{\sigma_n(x)\}$ given by the recursion relation:*

$$\begin{aligned} &\sigma_{-1} = 0, \quad \sigma_0 = 1, \\ &(n+1)\sigma_{n+1}(z) + ie^{i\theta} z\sigma_n(z) - e^{2i\theta}(n-1)\sigma_{n-1}(z) = 0. \end{aligned} \tag{5.14}$$

Theorem 5.9 *Let the system $\{\sigma_n(x)\}_{n=0}^{\infty}$ be given by* (5.14)*, then:*

a) the system satisfies

$$G_\sigma(z,s) = \sum_{k=0}^{\infty} \sigma_k(z) s^k = e^{-z \arctan s e^{i(\theta + \frac{\pi}{2})}},$$

b) the sequence of polynomials $\{\sigma_n\}_0^\infty$ is an orthogonal basis in the Hilbert space $H^2(\mathcal{Z}, \mathcal{M})$,

c) the norm of polynomials σ_n is $\sqrt{2}$, if $k \ge 1$ and 1, if $k = 0$.

Proof.

a) By (5.14) we have

$$(k+1)\sigma_{k+1}(z) + ie^{i\theta} z\sigma_k(z) - e^{2i\theta}(k-1)\sigma_{k-1}(z) = 0.$$

Multiplying the above relations by s^k, summing over k and simplifying, we obtain:

$$\begin{aligned} 0 &= \sum_{k=0}^{\infty}\left[(k+1)\sigma_{k+1}(z)+ie^{i\theta}z\sigma_k(z)-e^{2i\theta}(k-1)\sigma_{k-1}(z)\right]s^k \\ &= \sum_{k=0}^{\infty}(k+1)\sigma_{k+1}(z)s^k+ie^{i\theta}z\sum_{k=0}^{\infty}\sigma_k(z)s^k-e^{2i\theta}\sum_{k=0}^{\infty}(k-1)\sigma_{k-1}(z)s^k \\ &= \frac{\partial G_\sigma(z,s)}{\partial s}+ie^{i\theta}zG_\sigma(z,s)-e^{2i\theta}s^2\frac{\partial G_\sigma(z,s)}{\partial s}. \end{aligned}$$

This implies that,

$$(1-e^{2i\theta}s^2)\frac{\partial G_\sigma(z,s)}{\partial s}=-ie^{i\theta}zG_\sigma(z,s),$$

which in turn implies

$$\frac{\partial G_\sigma(z,s)}{\partial s}=\frac{-ie^{i\theta}z}{1-e^{2i\theta}s^2}G_\sigma(z,s).$$

Integrating both sides with respect to s with the condition $G_\sigma(0,0)=1$, we obtain

$$G_\sigma(z,s)=\left(\frac{1-e^{i\theta}s}{1+e^{i\theta}s}\right)^{\frac{iz}{2}}=e^{-z\arctan se^{i(\theta+\frac{\pi}{2})}}.$$

b) In order to prove the orthogonality of $\sigma_n(x)$, and compute their norms it suffices to show that

$$\int_{\partial\mathcal{Z}}G_\sigma(z,s)\overline{G_\sigma(z,t)}d\mathcal{M}_z=\frac{1+s\bar{t}}{1-s\bar{t}}. \tag{5.15}$$

We note that for $\alpha=-\arctan se^{i(\theta+\frac{\pi}{2})}$, $\beta=-\arctan\bar{t}e^{-i(\theta+\frac{\pi}{2})}$ we obtain

$$\int_{-\infty}^{\infty}\frac{e^{(\alpha+\beta)x}}{2\cosh\frac{\pi}{2}x}dx=\frac{1}{\cos(\alpha+\beta)}.$$

Then

$$\begin{aligned} \int_{\partial\mathcal{Z}}G_\sigma(z,s)\overline{G_\sigma(z,t)}d\mathcal{M}_z &= \int_{-\infty}^{\infty}\frac{e^{(x+i)\alpha+(x-i)\beta}+e^{(x-i)\alpha+(x+i)\beta}}{4\cosh(\frac{\pi}{2}x)}dx \\ &= \frac{e^{i(\alpha-\beta)}+e^{-i(\alpha-\beta)}}{2}\int_{-\infty}^{\infty}\frac{e^{(\alpha+\beta)x}}{2\cosh(\frac{\pi}{2}x)}dx \\ &= \frac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)} \\ &= \frac{1+\tan\alpha\tan\beta}{1-\tan\alpha\tan\beta}=\frac{1+s\bar{t}}{1-s\bar{t}}. \end{aligned}$$

c) In the light of a) and equation (5.15) we have

$$\begin{aligned}
&\int_{-\infty}^{+\infty} G_\sigma(z,s)\overline{G_\sigma(z,t)}\frac{dx}{2\cosh\frac{\pi}{2}x}dx \\
=& \int_{-\infty}^{+\infty}\left(\sum_{k=0}^{\infty}\sigma_k(x)s^k\right)\left(\sum_{n=0}^{\infty}\sigma_n(x)\overline{t}^n\right)\frac{dx}{2\cosh\frac{\pi}{2}x}dx \\
=& \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}s^k\overline{t}^n\int_{-\infty}^{+\infty}\sigma_k(x)\sigma_n(x)\frac{dx}{2\cosh\frac{\pi}{2}x}dx \\
=& -1+2\sum_{k=0}^{\infty}(s\overline{t})^k.
\end{aligned}$$

Comparing the coefficients of the powers of s and $\overline{t}$, we obtain the desired result.

□

Remark 5.1 *Applying Cauchy's integral formula to the generating function of the system, one obtains the integral representation:*

$$\sigma_n(x) = \frac{1}{2\pi i}\int_K e^{-zarctan(te^{i(\theta+\frac{\pi}{2})})}\frac{dt}{t^{n+1}},$$

around a closed contour K about the origin with radius less than 1.

Remark 5.2 *Let $y(x) = \sigma_n(x)$. The function $y(x)$ satisfies the following difference equation:*

$$\frac{y(x+i)-y(x-i)}{2i} = \frac{ny(x)}{x}.$$

Proposition 5.10 *The system $\{\sigma_n\}$ satisfies the following relation:*

$$\sigma_n(2x) = \lim_{\lambda\to 0^+} S_n^\lambda(x;\theta).$$

Proof. By (5.13) and by the definition of ${}_2F_1(a,b,c;z)$ we have

$$\begin{aligned}
&\lim_{\lambda\to 0^+} S_n^\lambda(x;\theta) = \lim_{\lambda\to 0^+} e^{in\theta}\frac{(2\lambda)_n}{n!}{}_2F_1(-n,\lambda+ix,2\lambda;2) \\
=& \lim_{\lambda\to 0^+} e^{in\theta}\frac{(2\lambda)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k(\lambda+ix)_k}{(2\lambda)_k}\frac{2^k}{k!} \\
=& \frac{e^{in\theta}}{n!}\lim_{\lambda\to 0^+}\sum_{k=0}^{n}(-n)_k(\lambda+ix)_k(2\lambda+k)_{n-k}\frac{2^k}{k!} \\
=& \frac{e^{in\theta}}{n!}\sum_{k=0}^{n}(-n)_k(ix)_k(k)_{n-k}\frac{2^k}{k!} = \sigma_n(2x).
\end{aligned}$$

□

Remark 5.3 *From Remark 5.1 we get*

$$\sigma_n(x) = \frac{xe^{i\theta}}{in}S_{n-1}^1\left(\frac{x}{2};\theta\right).$$

5.3 Quasi-Symmetric Meixner - Pollaczek polynomials

Definition 5.3 *For $\theta \in \mathbb{R}, x \in \mathbb{R}, \lambda > 0$ let the quasi-symmetric Meixner - Pollaczek (QMP) polynomials $Q_n^\lambda(x;\theta)$ be defined by the generating function:*

$$G^\lambda(x;\theta,\pi-\theta;z) = \frac{1}{(1-ze^{i\theta})^{\lambda-ix}(1+ze^{-i\theta})^{\lambda+ix}} = \sum_{n=0}^{\infty} Q_n^\lambda(x;\theta)z^n \quad (z \in \mathbb{D}).$$

5.3.1 Basic properties

Setting $\psi = \pi - \theta$ we obtain from (2.6) the following:

Corollary 5.11

a) *The (QMP) polynomials $Q_n^\lambda = Q_n^\lambda(x;\theta)$ satisfy the three-term recurrence relation:*

$$\begin{aligned} Q_{-1}^\lambda &= 0, \\ Q_0^\lambda &= 1, \\ nQ_n^\lambda &= 2i[(\lambda+n-1)\sin\theta - x\cos\theta]Q_{n-1}^\lambda + (2\lambda+n-2)Q_{n-2}^\lambda \quad (n \geq 1). \end{aligned}$$

b) *The polynomials $Q_n^\lambda = Q_n^\lambda(x;\theta)$ are given by the formula:*

$$Q_n^\lambda(x;\theta) = e^{in\theta}\sum_{j=0}^{n}(-1)^j \frac{(\lambda+ix)_j(\lambda-ix)_{n-j}}{j!(n-j)!}e^{-2ij\theta} \quad (n \in \mathbb{N}\cup\{0\}).$$

c) *The polynomials $Q_n^\lambda = Q_n^\lambda(x;\theta)$ have the hypergeometric representation*

$$Q_n^\lambda(x;\theta) = e^{in\theta}\frac{(2\lambda)_n}{n!}{}_2F_1(-n,\lambda+ix,2\lambda;1+e^{-2i\theta}). \tag{5.16}$$

d) *The polynomials $y(x) = Q_n^\lambda(x;\theta)$ satisfy the following difference equation*

$$e^{i\theta}(\lambda-ix)y(x+i) - 2[x\sin\theta + (n+\lambda)\cos\theta]y(x) + e^{-i\theta}(\lambda+ix)y(x-i) = 0.$$

e) *The polynomials $Q_n^\lambda(x;\theta)$ are orthogonal on $(-\infty,+\infty)$ with the weight*

$$w_\theta^\lambda(x) = \frac{1}{2\pi}e^{2\theta x}|\Gamma(\lambda+ix)|^2$$

for $\lambda > 0$ and $\theta \in (-\frac{\pi}{2},\frac{\pi}{2})$ and

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{2\theta x}|\Gamma(\lambda+ix)|^2 Q_n^\lambda(x;\theta)\overline{Q_m^\lambda(x;\theta)}dx = \delta_{mn}\frac{\Gamma(n+2\lambda)}{(\cos\theta)^{2\lambda}n!}. \tag{5.17}$$

5.3.2 Fisher information for $P_n^\lambda(x;\theta,\pi-\theta)$

The aim of this part of thesis is to compute the Fisher information for quasi-symmetric Meixner - Pollaczek polynomials. First we define the Fisher information of a probability density and we point out some of the properties relevant for the purpose of this work.

Definition 5.4 [47, Fisher,] *Let X denote a continuous random variable with probability density function $\rho(x)$ (for the sake of simplicity, we confine ourselves to the one-dimensional case). The Fisher information corresponding to this probability distribution is defined as*

$$I(X) = \int \frac{(\rho'(x))^2}{\rho(x)} dx.$$

Importance of Fisher information as a measure of the information in a distribution is well known. It is named after R. A. Fisher, who invented the concept of maximum likehood estimator and discovered several its properties. It has many implications in estimation theory, as exemplified by the Cramér-Rao bound which is fundamental limit on the variance of an estimator (see e.g. [36, Chapter 12]),

$$(\Delta X)^2 I(X) \geq 1.$$

Recent applications for computing performance bounds can be found in [6, 156]. It is used as a method of inference and understanding in statistical physics and biology, as promoted Friden [52–54]. It is also used as a tool for characterizing complex signals or systems [96, 103, 154] with applications in geophysics [12, 129, 149], in biology [50] or in reconstruction [21, 22]. Other applications are in random censoring [153], hypothesis testing [101], classification [37]. Fisher information for orthogonal polynomials and special functions have been studied in [128, 158].

For brevity, let us mention that according Definition 5.4, the Fisher information is a measure of derivative (gradient) content of the probability density. So, when $\rho(x)$ has a discontinuity, the local slope value changes drastically and then the Fisher information strongly alerts. This indicates that it is a local quantity [36].

Moreover, the Fisher information has been shown to be a measure of disorder or smoothness of the probability density $\rho(x)$ and uncertainty of the associated random variable X. The disorder aspect has been discussed by Frieden [52, 53], and the uncertainty properties are clearly shown by the Stam inequality [141]. For a deeper understanding of the Fisher notion, let us point out that broad and smooth densities have low gradient contents and so their Fisher information is small. Conversely, if $\rho(x)$ shows an undue preference or bias towards particular x values, then it is steeply sloped about these x values, and so the value of the Fisher information becomes large. Then a largely uniform or unbiased density exhibits large uncertainty (high disorder) and conversely.

Now we find the Fisher information $I_\theta(Q_n^\lambda)$ for QMP polynomials using the ideas present in [41]. Dominici in [41] considered a sequence $P_n(x)$ of orthogonal polynomials with respect to the weight function $\rho(x)$ satisfying

$$\sum_{x=0}^{\infty} P_n(x)P_m(x)\rho(x) = h_n\delta_{nm} \quad (n, m = 0, 1, ...) \ .$$

Introducing the functions

$$\rho_n(x) = \frac{[P_n(x)]^2\rho(x)}{h_n} \quad (n \in \mathbb{N}_0), \tag{5.18}$$

the Fisher information corresponding to the functions (5.18) may be described as follows

$$I_\theta(P_n) = \sum_{x=0}^{\infty} \left[\frac{\partial}{\partial\theta}\rho_n(x)\right]^2 \frac{1}{\rho_n(x)} \quad (n \in \mathbb{N}_0).$$

We recall the result from [41] for the family $P_n(x)$ of polynomials defined by

$$P_n(x) = {}_2F_1[-n, -x, c; z(\theta)] \quad (n \in \mathbb{N}_0).$$

It was proved that

$$\frac{\partial P_n}{\partial\theta} = \frac{n}{z}\frac{\partial z}{\partial\theta}[P_n(x) - P_{n-1}(x)] \quad (n \in \mathbb{N}_0). \tag{5.19}$$

Now we can state our theorem.

Theorem 5.12 *The Fisher information of QMP polynomials is given by*

$$I_\theta(Q_n^\lambda) = \int_{-\infty}^{\infty} \left[\frac{\partial}{\partial\theta}\rho_n(x)\right]^2 \frac{1}{\rho_n(x)}dx = \frac{-2[n^2 + (2n+1)\lambda]}{\cos^2\theta} \quad (n \in \mathbb{N}_0),$$

with $\rho_n(x)$ defined as in (5.18).

Proof. We note that for QMP we have $\rho(x) = w_\theta^\lambda(x) = \frac{1}{2\pi}e^{2\theta x}|\Gamma(\lambda + ix)|^2$.
From (5.16) and using (5.19) we have

$$\frac{\partial Q_n^\lambda}{\partial\theta} = -n\tan(\theta)Q_n^\lambda + i\frac{2\lambda + n - 1}{\cos\theta}Q_{n-1}^\lambda,$$

while (5.18) and (5.17) give

$$\rho_n(x) = \frac{e^{2\theta x}|\Gamma(\lambda + ix)|^2(\cos\theta)^{2\lambda}n![Q_n(x)]^2}{2\pi\Gamma(n + 2\lambda)}. \tag{5.20}$$

Observe that

$$\int_{-\infty}^{\infty} \rho_n(x)dx = 1 \quad (n \in \mathbb{N}_0). \tag{5.21}$$

Differentiating (5.20) with respect to θ, we obtain

$$\frac{\partial\rho_n(x)}{\partial\theta} = \frac{i\rho_n(x)}{\cos\theta Q_n^\lambda}[(n+1)Q_{n+1}^\lambda - (2\lambda + n - 1)Q_{n-1}^\lambda].$$

Therefore

$$\begin{aligned}
&\left[\frac{\partial}{\partial\theta}\rho_n(x)\right]^2 \frac{1}{\rho_n(x)} = \frac{-\rho_n(x)}{\cos^2\theta (Q_n^\lambda)^2} \\
\times\ &\left[(n+1)^2(Q_{n+1}^\lambda)^2 - 2(n+1)(2\lambda+n-1)Q_{n+1}^\lambda Q_{n-1}^\lambda + (2\lambda+n-1)(Q_{n-1}^\lambda)^2\right] \\
=\ &\frac{1}{\cos^2\theta}\Big[(n+1)(n+2\lambda)\rho_{n+1}(x) + n(n+2\lambda-1)\rho_{n-1} \\
-\ &2(n+1)(2\lambda+n-1)\frac{(\cos\theta)^{2\lambda} n!}{\Gamma(n+2\lambda)\rho(x)Q_{n+1}^\lambda Q_{n-1}^\lambda}\Big].
\end{aligned} \tag{5.22}$$

Integrating (5.22) and using the orthogonality relation (5.17) and (5.21), we get

$$\begin{aligned}
I_\theta(Q_n^\lambda) &= \int_{-\infty}^{\infty} \left[\frac{\partial}{\partial\theta}\rho_n(x)\right]^2 \frac{1}{\rho_n(x)} dx \\
&= \frac{-1}{\cos^2\theta}\left[(n+1)(n+2\lambda) + n(n+2\lambda-1)\right]
\end{aligned}$$

and the result follows. □

5.3.3 Conclusions

The analytic determination of spreading quantifiers of special functions of applied mathematics and mathematical physic beyond the familiar variance is the goal of a mathematical programme which includes the calculation of the global (e.g. Shannon, Renyi, Tsallis) and local (Fisher) measures of the probability distributions associated with them. The performance of this programme has far reaching consequences for numerous scientific fields. However, except for the asymptotic determination of the Shannon entropy of Airy [127] and Bessel [38, 39] functions, most attempts have been done for orthogonal polynomials. It is based on the known algebraic properties of these functions other than the differential equation which they satisfy.

Contrary to the Shanon entropies, where explicit expressions are unreachable up until now, the Fisher information [128] have recently been determined for the classical orthogonal polynomials (Hermite, Laguerre, Jacobi) as a function of the degree and the parameters of the polynomials.

In this section we have extended these efforts by calculated the Fisher information for quasi-symmetric Meixner - Pollaczek polynomials via its hypergeometric representation. Let us mention that the study of asymptotic behavior of Fisher information for QMP is left as an open problem.

Bibliography

[1] L. V. Ahlfors, *Conformal invariants. Topics in Geometric Function Theory*, McGraw-Hill, New York, 1973.

[2] N. I. Akhiezer and I. M. Glazman, *Theory of linear operators in Hilbert space*, Vol. 1, Frederick Unar Publishing Company, New York, 1961.

[3] N. I. Akhiezer, *Orthogonal polynomials on several intervals*, translated in Soviet Math. 1 (1960), 989 - 992.

[4] ______, *The classical moment problem and some related questions in analysis*, Oliver and Boyed, Edinburgh, 1965. English translation.

[5] N. I. Akhiezer and Ju. Ja. Tomčuk, *On the theory of orthogonal polynomials over several intervals*, Dokl. Akad. Nauk USSR (1961), 743 - 746 (Russian).

[6] M. M. Ali and S. Nadarajah, *Information matrices for normal and Laplace mixtures*, Inform. Sci. **177** (2007), no. 3, 947 - 955.

[7] J. W. Alexander, *Functions which map the interior of the unit circle upon simple regions*, Ann. of Math. **17** (1915), 12 - 22.

[8] W. A. Al - Salam, W. Allaway, and R. Askey, *Sieved ultraspherical orthogonal polynomials*, Trans. Amer. Math. Soc. **284** (1984), 39 - 55.

[9] G. Andrews, R. Askey, and R. Roy, *Special functions*, Cambridge University Press, Cambridge, 1999.

[10] T. K. Araaya, *The symmetric Meixner - Pollaczek polynomials*, Uppsala University, 2003.

[11] R. Askey and J. Wilson, *Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials*, Memoirs Amer. Math. Soc., 1985.

[12] M. Balasco, V. Lapenna, M. Lovallo, G. Romano, A. Siniscalchi, and L. Telesca, *Fisher information measure analysis of earth's apparent resistivity*, Int. J. Nonlinear Sci. **5** (2008), 230 - 236.

[13] C. M. Bender, L. R. Mead, and S. Pinsky, *Continuous Hahn polynomials and the Heisenberg Algebra*, J. Math. Phys. **28** (1987), no. 3, 509 - 513.

[14] L. Bieberbach, *Einführung in die konforme Abbildung*, Sammlung Göschen, Band 768/786a, Berlin, 1967.

[15] L. Bieberbach, *Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln*, S. - B. Preuss. Akad. Wiss. **138** (1916), 940 - 955.

[16] M. Biernacki, *Sur la représentation conforme des domaines linéairement accesible*, Prace Mat.-Fiz. **44** (1936), 294 - 314.

[17] J. Booth, *Researches on the geometrical properties of elliptic integrals* **142** (1852), 311 - 416.

[18] ______, *Researches on the geometrical properties of elliptic integrals* **144** (1854), 53 - 69.

[19] J. Booth, *A treatise on some new geometrical methods*, Vol. I, Longmans, Green, Reader and Dyer, London, 1873.

[20] ———, *A treatise on some new geometrical methods*, Vol. II, Longmans, Green, Reader and Dyer, London, 1877.

[21] J. M. Borwein, A. S. Lewis, M. N. Limber, and D. Noll, *Maximum entropy reconstruction using derivative information part 2: computational results*, Numer. Math. **69** (1995), 243 - 256.

[22] J. M. Borwein, A. S. Lewis, and D. Noll, *Maximum entropy reconstruction using derivative information part 1: Fisher information and convex duality*, Math. Oper. Res. **21** (1996), 442 - 468.

[23] L. D. Branges, *A proof of the Bieberbach conjecture*, Acta Math. **154** (1985), 137 - 152.

[24] D. A. Brannan and W. E. Kirwan, *A covering theorem for typically real functions*, Glasg. Math. J. **10** (1969), 153 - 155.

[25] J. E. Brown and A. Tsao, *On the Zalcman conjecture for starlike and typically real functions*, Math. Z. **191** (1986), 467 - 474.

[26] D. M. Campbell, J. A. Cima, and J. A. Pfaltzgraff, *Linear space and linear-invariant families of locally univalent analytic functions*, Manuscripta Math. **4** (1971), 1 - 30.

[27] D. M. Campbell and J. A. Pfaltzgraff, *Mapping properties of* $\log g'(z)$, Colloq. Math. **32** (1974), 267 - 276.

[28] J. A. Charris, M. E. H. Ismail, and S. Monsalve, *On sieved orthogonal polynomials X : general blocks of recurrence relations*, Pacific J. Math. **163** (1994), 237 - 267.

[29] Y. Chen and M. E. H. Ismail, *Asymptotics of extreme zeros of the Meixner–Pollaczek polynomials*, J. Comput. Appl. Math. **82** (1997), 59 - 78.

[30] Y. Chen and N. Lawrence, *A generalization of the Chebyshev polynomials*, J. Phys. A **35** (2002), 4651 - 4699.

[31] P. L. Chebyshev, *Complete collected works*, Vol. II, Moscov-Leningrad, 1947 (Russian).

[32] Y. Chen and M. E. H. Ismail, *Asymptotics of extreme zeros of the Meixner–Pollaczek polynomials*, J. Comput. Appl. Math. **82** (1997), 59 - 78.

[33] Y. Chen and N. Lawrence, *A generalization of the Chebyshev polynomials*, J. Phys. A **35** (2002), 4651 - 4699.

[34] T. S. Chihara, *An introduction to orthogonal polynomials*, Gordon and Breach, New York, 1978.

[35] J. B. Conway, *Functions of one complex variable*, Springer-Verlag, New York, 1978.

[36] T. M. Cover and J. A. Thomas, *Elements of information theory*, Wiley - Interscience, New York, 1991.

[37] A. Dasgupta, *Stein's identity, Fisher information, and projection pursuit: a triangulation*, J. Statist. Plann. Inference **137** (2007), 3394 - 3409.

[38] J. S. Dehesa, A. Martínez-Finkelshtein, and V. N. Sorokin, *Asymptotics of information entropies of some Toda-like potentials*, J. Math. Phys. **44** (2003), 36 - 47.

[39] ———, *Short-wave asymptotics of the information entropy of a circular membrane*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **12** (2002), 2387 - 2392.

[40] L. E. Dickson, *History of the theory of numbers*, Vol. I, Chelsea Publishing Company, New York, 1919.

[41] D. Dominici, *Fisher information of orthogonal polynomials*, Int. J. Comput. Appl. Math. **233** (2010), no. 6, 1511 - 1518.

[42] M. Dorf and T. Suffridge, *The inner mapping radius of harmonic mappings of the unit disk*, Complex Variables Theory and Applications **33** (1997), 97 - 103.

[43] P. L. Duren, *Univalent functions*, Springer, New York, 1983.

[44] A. Eguchi and S. Owa, *On radius problems for analytic functions of Koebe type*, Vol. 1192, Sũrikaisekikenyũsho Kõkyũroku, 2001.

[45] A. Erdélyi et Al., *Higher transcendental functions*, Vol. I, Bateman Manuscript Project McGraw-Hill Book Company, New York, 1953.

[46] W. Fenchel, *Bemerkungen über die in Einheitskreis meromorphen schlichten Funktionen*, Preuss. Akad. Wiss. Phys. - Math. **22/23** (1931), 431 - 436.

[47] R. A. Fisher, *Statistical methods and scientific inference*, Hafner Press, New York, 1973.

[48] ———, *Theory of statistical estimation*, Math. Proc. Cambridge Philos. Soc. **22** (1925), 700 - 725.

[49] L. Fox and B. Parker, *Chebyshev polynomials in numerical analysis*, Oxford U.P., 1968.

[50] S. A. Frank, *Natural selection maximizes Fisher information*, Journal of Evolutionary Biology **22** (2009), 231 - 244.

[51] G. Freud, *Orthogonale polynome*, Birkhäuser, Akadémiai Kiadó Pergamon, Budapest Oxford, 1971.

[52] B. R. Friden, *Fisher information, disorder, and the equilibrium distributions of physics*, Phys. Rev. A **41** (1990), 42 - 65.

[53] ———, *Science from Fisher information*, Cambridge University Press, Cambridge, 2004.

[54] B. R. Friden and P. M. Binder, *Physics from Fisher information: a unification*, 1998.

[55] B. Friedman, *Two theorems on schlicht functions*, Duke Math. J. **13** (1946), 171 - 177.

[56] R. F. Gabriel, *The Schwarzian derivate and convex functions*, Proc. Amer. Math. Soc. **5** (1955), 58 - 66.

[57] A. Garcia and H. Stichtenoth, *On Chebyshev polynomials and maximal curves*, Acta Math. **XC** (1999), no. 4, 301 - 311.

[58] G. Gasper, *q-extension of Clausen's formula and of the inequalities used by de Branges in his proof of the Bieberbach, Robertson and Milin conjecture*, SIAM J. Math. Anal. **20** (1989), no. 4, 1019 - 1034.

[59] G. Gasper and M. Rahman, *Basic hypergeometric series*, Cambridge, 1997.

[60] G. M. Golusin, *Geometric theory of functions of a complex variable*, Izd. "Nauka", Moscow, 1966 (Russian).

[61] ———, *On typically real functions*, Mat. Sb. **27** (1950), 201 - 218 (Russian).

[62] A. W. Goodman, *On close-to-convex functions of higher order*, Ann. Univ. Sci. Budapest. Eötvös. Sect. Math. **15** (1972), 17 - 30.

[63] ______, *The critical points of a typically real function*, Proc. Amer. Math. Soc. **38** (1973), no. 1, 95 - 102.

[64] ______, *The domain covered by a typically real function*, Proc. Amer. Math. Soc. **64** (1977), no. 2, 233 - 237.

[65] ______, *Univalent functions*, Vol. I - II, Mariner Publishing Company, Inc., Tampa, Florida, U.S.A., 1983.

[66] H. Grunsky, *Zwei Bemerkungen zur konformen Abbildung*, Jahresbericht Deutsch. Math. - Verein. **43** (1934), 140 - 143.

[67] G. Herglotz, *Über Potenzreihen mit positivem, reelen Teil im Einheitskreis*, Ber. Verhandl. Sächs. Akad. Wiss. Leipzig Math - Nat.Kl. **63** (1911), 501 - 511.

[68] E. Hille, *Remarks on a paper by Zeev Nehari*, Bull. Amer. Math. Soc. **55** (1949), 552 - 553.

[69] R. Freund and B. Fisher, *On the constrained Chebyshev approximation problem on ellipses*, J. Approx. Theory **62** (1990), 297 - 315.

[70] M. E. Ismail, *Classical and quantum orthogonal polynomials in one variable*, Encylopedia of mathematics and its application's 98, Cambridge University Press, 2005.

[71] W. Janowski, *Extremal problem for a family of functions with positive real part and for some related families*, Ann. Polon. Math **23** (1970), 159-177.

[72] J. A. Jenkins, *On a conjecture of Goodman concerning meromorphic univalent functions*, Michigan Math. J. **9** (1962), 25 - 27.

[73] M. Kamali and H. M. Srivastava, *A sufficient condition for starlikeness of analytic functions of Koebe type*, Journal of Inequalities in Pure and Applied Mathematics **5** (2004), no. 3, 1 - 8.

[74] S. Kanas and T. Sugawa, *On conformal representation of the interior of an ellipse*, Ann. Acad. Sci. Fenn. Math. **31** (2006), 329-348.

[75] S. Kanas and A. Tatarczak, *Constrained coefficients problem for generalized typically real functions.* submitted.

[76] ______, *The generalized Meixner - Pollaczek polynomials*, Adv. Difference Equ. **131** (2013), 67 - 75.

[77] ______, *The generalized typically real functions.* submitted.

[78] W. Kaplan, *Close - to - convex schlicht functions*, Michigan Math. J. **1** (1952), 169 - 185.

[79] K. Kiepiela and D. Klimek, *An extension of Chebyshev polynomials*, J. Comput. Appl. Math. **17** (2005), 305 - 312.

[80] L. Koczan and W. Szapiel, *Sur certaines classes des fonctions holomorphes définies par une intègrale de Stieltjes*, Annales UMCS **28** (1974), 39 - 51.

[81] L. Koczan and P. Zaprawa, *Domains of univalence for typically real odd functions*, Complex Variables **48** (2003), no. 1, 1 - 17.

[82] P. Koebe, *Über die Uniformisierung beliebiger analytischer Kurven*, Nachr. Kgl. Ges. Wiss. Göttingen Math. Phys. (1907), 191 - 210.

[83] R. Koekoek and R. F. Swarttouw, *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue*, Delft University of Technology, 1998.

[84] H. T. Koelink and J. Van der Jeugt, *Convolutions for orthogonal polynomials from Lie and quantum algebra represatations*, SIAM J. Math. Anal. **29** (1998), 794 - 822.

[85] T. H. Koornwinder, *Meixner - Pollaczek polynomials and the Heisenberg Algebra*, J. Math. Phys. **30** (1989), no. 4, 767 - 769.

[86] I. V. Krasovsky, *Asymptotic distribution of zeros polynomials satisfying difference equations*, J. Comput. Appl. Math. **150** (2003), 57 - 70.

[87] M. G. Krein, *The ideas of P. L. Chebysheff and A. A. Markov in the theory of limiting values of integrals and their further development*, Amer. Math. Soc. Transl. Series 2 (1959), 1 - 122.

[88] A. Lecko, *On coefficient inequalities in the Caratheodory class of functions*, Ann. Polon. Math. **75** (2000), 59 - 67.

[89] Z. Lewandowski, *Sur la l'identité de certaines classes de fonctions univalentes*, Ann. Univ. Mariae Curie-Skłodowska **Sect. A** (1958), no. 12, 131 - 146.

[90] R. J. Libera, *Some classes of regular univalent functions*, Proc. Amer. Math. Soc. **16** (1965), 755 - 758.

[91] A. E. Livingston, *The coefficients of multivalent close-to-convex functions*, Proc. Amer. Math. Soc. **21** (1969), 545 - 550.

[92] X. Li and R. Wong, *On the asymptotics of the Meixner - Pollaczek polynomials and their zeros*, Const. Approx. **17** (2001), 59 - 90.

[93] G. Loria, *Spezielle algebraische und transzendente ebene kurven, tjeorie und geschichte*, Vol. I, F. Schütte, Teubner, Leipzig, 1910.

[94] K. Löwner, *Untersuchungen über die Verzerrung bei konformen Abbidungen des Einheitskreises $|z| < 1$, die durch Funktionen mit nichtverschwindender Ableitung geliefert werden*, S. B. Sächs. Akad. Wiss. Leipzig Berichte **69** (1917), 89 - 106.

[95] W. Ma, *Generalized Zalcman conjecture for starlike and typically real functions*, J. Math. Anal. Appl. **234** (1999), 328 - 339.

[96] M. T. Martin, F. Pennini, and A. Plastino, *Fisher's information and the analysis of complex signals*, Physics Letters A **256** (1999), no. 2 - 3, 173 - 180.

[97] A. Marks, *Untersuchunger uber schlichte ApplidgenUnter*, Mathematische Annalen **107** (33 1932), 40 - 67.

[98] J. C. Mason and D. C. Handscomb, *Chebyshev Polynomials*, Chapman and Hall/CRC, 2003.

[99] J. Meixner, *Orthogonale Polynomsysteme mit einer besonderen Gestalt der erzeugenden Funktion*, J. London Math. Soc. **9** (1934), 6 - 13.

[100] S. S. Miller, P. T. Mocanu, and M. O. Reade, *Subordination preserving integral operators*, Trans. Amer. Math. Soc. **283** (1984), 605 - 615.

[101] M. Montenegro, M. R. Casals, M. A. Lubiano, and M. Ángeles Gil, *Two-sample hypothesis tests of a fuzzy random variable*, Information Sciences **133** (2001), 89 - 100.

[102] A. G. Morris and T. S. Horner, *Chebyshev polynomials in the numerical solution of differential equations*, Math. Comp. **31** (1977), 881 - 891.

[103] A. Nagy, *Fisher information in density functional theory*, The Journal of Chemical Physics **119** (2003), no. 18, 9401 - 9405.

[104] I. Naraniecka, J. Szynal, and A. Tatarczak, *An extension of typically real functions and associated orthogonal polynomials*, Ann. UMCS, Mathematica **65** (2011), no. 2, 99 - 112.

[105] ______, *Linearly-invariant families and generalized Meixner-Pollaczek polynomials*, Ann. UMCS, Mathematica **67** (2013), no. 1, 45 - 56.

[106] I. Naraniecka, J. Szynal, and A. Tatarczak, *The generalized Koebe function*, Trudy Petrozawodskogo Universiteta, Matematika **17** (2010), 62 - 66.

[107] R. Nevanlinna, *Über die konforme Abbildung von Sterngebieten*, Översikt av Finska Vetenskaps - Soc. Förh. **63A** (Unknown Month 1920), 1 - 21.

[108] K. I. Noor, *On a generalizations of close-to-convexity*, Int. J. Math. Math. Sci. **6** (1983), 327 - 334.

[109] S. Nordebo, M. Gustafsson, and B. Nilsson, *Fisher information analysis for two-dimensional microwave tomography*, Inverse Problem **23** (2007), 859 - 877.

[110] Y. Okuyama, *On several properties of spiral-like functions*, Hokkaido Mathematical Journal **1112** (1999), 86 - 91.

[111] F. Peherstorfer, *On Bernstein-Szegö orthogonal polynomials on several intervals II: orthogonal polynomials with periodic recurrence coefficients*, J. Approx. Theo. **64** (1991), 123 - 161.

[112] F. Pollaczak, *Sur une famille de polynomes orthogonaux qui contient les polynomes d'Hermite et de Laguerre comme cas limites*, Ibid. **230** (1950), 1563 - 1565.

[113] Ch. Pommerenke, *Linear-invariant Familien analytischer Funktionen*, Mat. Ann. **155** (1964), 108 - 154.

[114] ______, *Univalent functions*, Vandenhoeck and Ruprecht, Göttingen, 1975.

[115] A. D. Poularikas, *The handbook of formulas and tables for signal processing*, Boca Raton: CRC Press LLC, 1999.

[116] M. Reade, *On close-to-convex univalent function*, Mich. Math. J. **3** (1955), 59 - 62.

[117] E. D. Rainville, *Special Functions*, MacMilan, 1960.

[118] W. D. Ray and A. E. N. T. Pitman, *Chebyshev Polynomial and other new approximations to Mills' ratio*, Ann. Math. Statist. **34** (1963), no. 3, 892 - 902.

[119] S. Richardson, *Some Hele-Shaw flows with time-dependent free boundaries*, J. Fluid Mech. **102** (1981), 263 - 278.

[120] T. J. Rivlin, *Chebyshev Polynomials: from approximation theory to algebra and number theory*, John Wiley and Sons, New York, 1990.

[121] M. S. Robertson, *On the coefficients of typically real functions*, Bull. Amer. Math. Soc. **41** (1935), 565 - 572.

[122] M. S. Robertson, *A coefficient problem for functions regular in an annulus*, Canad. J. Math. **4** (1952), 407-423.

[123] M. S. Robertson, *On the theory of univalent functions*, Annals of Mathematics **37** (1936), 374 - 408.

[124] ______, *The sum of univalent functions*, Duke Math. J. **37** (1970), 411 - 419.

[125] ______, *Univalent functions starlike with respect to a boundary point*, J. Math. Anal. Appl. **81** (1981), 327 - 345.

[126] W. W. Rogosinski, *Über positive harmonische Entwicklungen und typisch-reelle Potenzreihen*, Math. Z. **35** (1932), 93 - 121.

[127] P. Sánchez-Moreno, R. J. Yáñez, and V. S. Buyarov, *Asymptotics of the information entropy of the Airy function*, J. Phys. A: Math. Gen **38** (2005), 9969 - 9978.

[128] P. Sánchez-Ruiz and J. S. Dehesa, *Fisher information of orthogonal hypergeometric polynomials*, J. Comput. Appl. Math. **182** (2005), 150 - 164.

[129] K. D. Sen, J. Antolín, and J. C. Angulo, *Fisher-Shannon analysis of ionization processes and isoelectronic series*, Phys. Rev. A **76** (2007), 1 - 7.

[130] W. Shapiro, *A Combinatorial proof of a Chebyshev polynomial identity*, Discrete Math. **34** (1981), 203 - 206.

[131] H. Silverman, *Convex and starlike criteria*, Internat. J. Math. & Math. Sci. **22** (1999), no. 1, 75 - 79.

[132] B. Simon, *Orthogonal polynomials on the unit circle*, Classical theory, vol. I, AMS Colloquium Series, American Mathematical Society, Providence, RI, 2005.

[133] ______, *Orthogonal polynomials on the unit circle*, Spectral theory, vol. II, AMS Colloquium Series, American Mathematical Society, Providence, RI, 2005.

[134] S. Siregar and M. Darus, *Certain conditions for starlikeness of analytic functions of Koebe type*, Int. J. Math. Mathematical Sciences **2011** (2011).

[135] M. Sobczak - Kneć and K. Trąbka - Więcław, *On typically real functions which are generated by a fixed typically real function*, Czechoslovak Math. J. **61** (2011), no. 3, 733 - 742.

[136] I. E. Shparlinski, *Generalization of Chebyshev polynomials*, Sib. Math. J. **31** (1990), no. 1, 183 - 185.

[137] V. V. Starkov, *On some subclasses of linear-invariant families with integral representation*, Dep. VINITI, Moscow, 1981 (Russian).

[138] ______, *Linear-invariant families of functions*, Ekatirenburg, Dissertation, 1989 (Russian).

[139] ______, *Some linear-invariant families of functions having an integral representation*, Izv. Vyssh. Uchebn. Zaved. Mat. **5** (1983), 82 - 85.

[140] ______, *The estimates of coefficients in locally-univalent family* $U_\alpha^{'}$, Vestnik Lenin. Gosud. Univ. **13** (1984), 48 - 54 (Russian).

[141] A. Stam, *Some inequalities satisfied by the quantities of information of Fisher and Shannon*, Inform. and Control **2** (1959), 105 - 112.

[142] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, 1971.

[143] T. J. Stieltjes, *Sur quelques intégrals definies et leurs développement en fractions continues*, Quart. J. Math. **24** (1890), 370 - 382.

[144] ———, *Recherches sur les fractions continues*, Ann. Fac. Sci. Toulouse Math. **8** (1894), 1 - 122.

[145] E. Strohäcker, *Beitrage Zur Theorie Der Schlichten Funktionen*, Mathematische Zeitschrift **37** (1933), 356 - 380.

[146] D. Struik, *Lectures on Classical Differential Geometry*, Dover, New York, 1961.

[147] G. Szegö, *Orthogonal polynomials*, Vol. 23, American Mathematical Society, Colloquium Publications XXIII, Providence, RI, 1959.

[148] A. Tatarczak, *An extension of the Chebyshev polynomials of the first and second kind*. submitted.

[149] L. Telesca, V. Lapenna, and M. Lovallo, *Fisher information analysis of earthquake-related geoelectrical signals*, Nat. Hazards Earth Syst. Sci. **5** (2005), no. 4, 561 - 564.

[150] P. G. Todorov, *On the univalent functions starlike with respect to a boundary point*, Proc. Amer. Math. Soc. **97** (1986), no. 4, 602 - 604.

[151] ———, *The radii of starlikness and convexity of order alpha of typically real functions.*, Ann. Acad. Sci. Fenn. Ser. A I Math. **8** (1983), 93 - 106.

[152] Ju. Ja. Tomčuk, *Orthogonal polynomials over a system of intervals on the number line*, Zap. Mekh. - Mat. Fak. Kharkov Mat. Oshch. **29** (1964), 93-128 (Russian).

[153] C. Tsairidis, K. Ferentinos, and T. Papaioannou, *Information and random censoring*, Inform. Sci. **92** (1996), no. 1 - 4, 159 - 174.

[154] C. Vignat and J. F. Bercher, *On Fisher information inequalities and score functions in non-invertible linear systems*, Journal of Inequalities in Pure and Applied Mathematics **4** (2003), 1 - 9.

[155] R. Witula and D. Slota, *On modified Chebyshev polynomials*, J. Math. Anal. Appl. **324** (2006), no. 1, 321 - 343.

[156] B. Xu, Q. Chen, Z. Wu, and Z. Wang, *Analysis and approximation of performance bound for two-observer bearings-only tracking*, Inform. Sci. **178** (2008), no. 8, 2059 - 2078.

[157] S. Yamashita, *Nonunivalent generalized Koebe function*, Proc. Japan Acad. Ser. A Math. Sci. **79** (2003), no. 1, 9 - 10.

[158] R. J. Yáñez, P. Sánchez-Moreno, A. Zarzo, and J. S. Dehesa, *Fisher information of special functions and second-order differential equations*, J. Math. Phys. **49** (2008), 82 - 104.

[159] J. Zeng, *Weighted derangements and the linearization coefficients of orthogonal Sheffer polynomials*, Proc. Lond. Math. Soc. **65** (1992), 1 - 22.

[160] G. Zheng and J. L. Gastwirth, *Fisher information in randomly sampled sib pairs and extremely discordant sib pairs in genetic analysis for a quantitative trait locus*, J. Statist. Plann. Inference **130** (2005), no. 1 - 2, 299 - 315.

[161] C. Zwikker, *The advanced geometry of plane curves and their applications*, Dover, New York, 1963.