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Continuity results for parametric nonlinear singular Dirichlet problems

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Abstract: In this paper we study from a qualitative point of view the nonlinear singular Dirichlet problem depending on a parameter $\lambda > 0$ that was considered in [32]. Denoting by S_λ the set of positive solutions of the problem corresponding to the parameter λ, we establish the following essential properties of S_λ:

(i) there exists a smallest element u_λ^* in S_λ, and the mapping $\lambda \mapsto u_\lambda^*$ is (strictly) increasing and left continuous;

(ii) the set-valued mapping $\lambda \mapsto S_\lambda$ is sequentially continuous.

Keywords: Parametric singular elliptic equation, p-Laplacian, smallest solution, sequential continuity, monotonicity

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1 Introduction

Elliptic equations with singular terms represent a class of hot-point problems because they are mathematically significant and appear in applications to chemical catalysts processes, non-Newtonian fluids, and in models for the temperature of electrical conductors (see [3, 9]). An extensive literature is devoted to such problems, especially focusing on their theoretical analysis. For instance, Ghergu-Rădulescu [18] established several existence and nonexistence results for boundary value problems with singular terms and parameters; Gasínski-Papageorgiou [15] studied a nonlinear Dirichlet problem with a singular term, a $(p-1)$-sublinear term, and a Carathéodory perturbation; Hirano-Saccon-Shioji [21] proved Brezis-Nirenberg type theorems for a singular elliptic problem. Related topics and results can be found in Crandall-Rabinowitz-Tartar [7], Cîrstea-Ghergu-Rădulescu [6], Dupaigne-Ghergu-Rădulescu [10], Gasiński-Papageorgiou [17], Averna-Motreanu-Tornatore [2], Papageorgiou-Winkert [33], Carl [4], Faria-Miyagaki-Motreanu [11], Carl-Costa-Tehrani [5], Liu-Motreanu-Zeng [26] Papageorgiou-Rădulescu-Repovš [30], and the references therein.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$ and let $\gamma \in (0,1)$ and $1 < p < +\infty$. Recently, Papageorgiou-Vetro-Vetro [32] have considered the following parametric nonlinear singular Dirichlet problem

$$\begin{cases} -\triangle_p u(x) = \lambda u(x)^{-\gamma} + f(x, u(x)) & \text{in } \Omega \\ u(x) > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

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where the operator Δ_p stands for the p-Laplace differential operator

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \quad \text{for all} \quad u \in W_0^{1,p}(\Omega).$$

The nonlinear function f is assumed to satisfy the following conditions:

$\underline{H(f)}$: $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for a.e. $x \in \Omega$, $f(x,0) = 0$, $f(x,s) \geq 0$ for all $s \geq 0$, and

(i) for every $\rho > 0$, there exists $a_\rho \in L^\infty(\Omega)$ such that

$$|f(x,s)| \leq a_\rho(x) \quad \text{for a.e.} \quad x \in \Omega \quad \text{and for all} \quad |s| \leq \rho;$$

(ii) there exists an integer $m \geq 2$ such that

$$\lim_{s\to+\infty} \frac{f(x,s)}{s^{p-1}} = \widehat{\lambda}_m \;\text{ uniformly for a.e. }\; x \in \Omega,$$

where $\widehat{\lambda}_m$ is the m-th eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$, and denoting

$$F(x,t) = \int_0^s f(x,t)\,dt,$$

then

$$pF(x,s) - f(x,s)s \to +\infty \quad \text{as} \quad s \to +\infty, \quad \text{uniformly for a.e.} \quad x \in \Omega;$$

(iii) for some $r > p$, there exists $c_0 \geq 0$ such that

$$0 \leq \liminf_{s\to 0^+} \frac{f(x,s)}{s^{r-1}} \leq \limsup_{s\to 0^+} \frac{f(x,s)}{s^{r-1}} \leq c_0 \quad \text{uniformly for a.e.} \quad x \in \Omega;$$

(iv) for every $\rho > 0$, there exists $\widehat{\xi}_\rho > 0$ such that for a.e. $x \in \Omega$ the function

$$s \mapsto f(x,s) + \widehat{\xi}_\rho s^{p-1}$$

is nondecreasing on $[0,\rho]$.

The following bifurcation type result is proved in [32, Theorem 2].

Theorem 1. *If hypotheses $H(f)$ hold, then there exists a critical parameter value $\lambda^* > 0$ such that*

(a) *for all $\lambda \in (0,\lambda^*)$ problem (1) has at least two positive solutions $u_0, u_1 \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$;*
(b) *for $\lambda = \lambda^*$ problem (1) has at least one positive solution $u^* \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$;*
(c) *for all $\lambda > \lambda^*$ problem (1) has no positive solutions.*

In what follows, we denote

$$\mathcal{L} := \left\{\lambda > 0 \;:\; \text{problem (1) admits a (positive) solution}\right\} = (0,\lambda^*],$$

$$S_\lambda = \left\{u \in W_0^{1,p}(\Omega) \;:\; u \text{ is a (positive) solution of problem (1)}\right\}$$

for $\lambda \in \mathcal{L}$. In this respect, Theorem 1 asserts that the above hypotheses, in conjunction with the nonlinear regularity theory (see Liebermann [24, 25]) and the nonlinear strong maximum principle (see Pucci-Serrin [34]), ensure that there holds

$$S_\lambda \subset \operatorname{int}(C_0^1(\overline{\Omega})_+).$$

Also, we introduce the set-valued mapping $\Lambda\colon (0,\lambda^*] \to 2^{C_0^1(\overline{\Omega})}$ by

$$\Lambda(\lambda) = S_\lambda \quad \text{for all } \lambda \in (0,\lambda^*].$$

The following open questions need to be answered:

1. *Is there a smallest positive solution to problem (1) for each $\lambda \in (0, \lambda^\star]$?*
2. *If for each $\lambda \in (0, \lambda^\star]$ problem (1) has a smallest positive solution $u^\star_\lambda$, then the function $\Gamma\colon (0, \lambda^\star] \to C^1_0(\overline{\Omega})$ with $\Gamma(\lambda) = u^\star_\lambda$ is it monotone ?*
3. *If for each $\lambda \in (0, \lambda^\star]$ problem (1) has a smallest positive solution $u^\star_\lambda$, then is the function Γ continuous ?*
4. *Is the solution mapping Λ upper semicontinuous ?*
5. *Is the solution mapping Λ lower semicontinuous ?*

In this paper we answer in the affirmative the above open questions.

Theorem 2. *Assume that hypotheses $H(f)$ hold. Then there hold:*

(i) *the set-valued mapping $\Lambda\colon \mathcal{L} \to 2^{C^1_0(\overline{\Omega})}$ is sequentially continuous;*

(ii) *for each $\lambda \in \mathcal{L}$, problem (1) has a smallest positive solution $u^\star_\lambda \in \operatorname{int}(C^1_0(\overline{\Omega})_+)$, and the map Γ from $\mathcal{L}$ to $C^1_0(\overline{\Omega})$ given by $\Gamma(\lambda) = u^\star_\lambda$ is*

  (a) *(strictly) increasing, that is, if $0 < \mu < \lambda \le \lambda^\star$, then*

$$u^\star_\lambda - u^\star_\mu \in \operatorname{int}(C^1_0(\overline{\Omega})_+);$$

  (b) *left continuous.*

The rest of the paper is organized as follows. In Section 2 we set forth the preliminary material needed in the sequel. In Section 3 we prove our main results formulated as Theorem 2.

2 Preliminaries

In this section we gather the preliminary material that will be used to prove the main result in the paper. For more details we refer to [8, 13, 16, 19, 22, 28, 29, 35].

Let $1 < p < \infty$ and p' be its Hölder conjugate defined by $\frac{1}{p} + \frac{1}{p'} = 1$. In what follows, the Lebesgue space $L^p(\Omega)$ is endowed with the standard norm

$$\|u\|_p = \left(\int_\Omega |u(x)|^p\,dx\right)^{\frac{1}{p}} \quad \text{for all} \quad u \in L^p(\Omega).$$

The Sobolev space $W^{1,p}_0(\Omega)$ is equipped with the usual norm

$$\|u\| = \left(\int_\Omega |\nabla u(x)|^p\,dx\right)^{\frac{1}{p}} \quad \text{for all} \quad u \in W^{1,p}_0(\Omega).$$

In addition, we shall use the Banach space

$$C^1_0(\overline{\Omega}) = \left\{u \in C^1(\overline{\Omega}) \,:\, u = 0 \ \text{ on } \ \partial\Omega\right\}.$$

Its cone of nonnegative functions

$$C^1_0(\overline{\Omega})_+ = \left\{u \in C^1_0(\overline{\Omega}) \,:\, u \ge 0 \ \text{ in } \Omega\right\}$$

has a nonempty interior given by

$$\operatorname{int}(C^1_0(\overline{\Omega})_+) = \left\{u \in C^1_0(\overline{\Omega}) \,:\, u > 0 \text{ in } \Omega \ \text{ with } \ \frac{\partial u}{\partial n}\bigg|_{\partial\Omega} < 0\right\},$$

where $\frac{\partial u}{\partial n}$ is the normal derivative of u and $n(\cdot)$ is the outward unit normal to the boundary $\partial\Omega$.

Hereafter by $\langle\cdot,\cdot\rangle$ we denote the duality brackets for $(W^{1,p}(\Omega)^*, W^{1,p}(\Omega))$. Also, we define the nonlinear operator $A\colon W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ by

$$\langle A(u), v\rangle = \int_\Omega |\nabla u(x)|^{p-2}\left(\nabla u(x), \nabla v(x)\right)_{\mathbb{R}^N} dx \quad \text{for all } u, v \in W^{1,p}(\Omega). \tag{2}$$

The following statement is a special case of more general results (see Gasiński-Papageorgiou [14], Motreanu-Motreanu-Papageorgiou [29]).

Proposition 3. *The map $A\colon W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ introduced in* (2) *is continuous, bounded (that is, it maps bounded sets to bounded sets), monotone (hence maximal monotone) and of type (S_+), i.e., if $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$ and*

$$\limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \leqslant 0,$$

then $u_n \to u$ in $W^{1,p}(\Omega)$.

For the sake of clarity we recall the following notion regarding order.

Definition 4. *Let $(P, \leq)$ be a partially ordered set. A subset $E \subset P$ is called downward directed if for each pair $u, v \in E$ there exists $w \in E$ such that $w \leq u$ and $w \leq v$.*

For any $u, v \in W_0^{1,p}(\Omega)$ with $u(x) \leq v(x)$ for a.e. $x \in \Omega$, we set the ordered interval

$$[u, v] := \left\{w \in W_0^{1,p}(\Omega) : u(x) \leq w(x) \leq v(x) \quad \text{for a.e. } x \in \Omega\right\}.$$

For $s \in \mathbb{R}$, we denote $s^\pm = \max\{\pm s, 0\}$. It is clear that if $u \in W_0^{1,p}(\Omega)$ then it holds

$$u^\pm \in W_0^{1,p}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.$$

We recall a few things regarding upper and lower semicontinuous set-valued mappings.

Definition 5. *Let X and Y be topological spaces. A set-valued mapping $F\colon X \to 2^Y$ is called*

(i) *upper semicontinuous (u.s.c., for short) at $x \in X$ if for every open set $O \subset Y$ with $F(x) \subset O$ there exists a neighborhood $N(x)$ of x such that*

$$F(N(x)) := \bigcup_{y\in N(x)} F(y) \subset O;$$

if this holds for every $x \in X$, F is called upper semicontinuous;

(ii) *lower semicontinuous (l.s.c., for short) at $x \in X$ if for every open set $O \subset Y$ with $F(x) \cap O \neq \emptyset$ there exists a neighborhood $N(x)$ of x such that*

$$F(y) \cap O \neq \emptyset \text{ for all } y \in N(x);$$

if this holds for every $x \in X$, F is called lower semicontinuous;

(iii) *continuous at $x \in X$ if F is both upper semicontinuous and lower semicontinuous at $x \in X$; if this holds for every $x \in X$, F is called continuous.*

The propositions below provide criteria of upper and lower semicontinuity.

Proposition 6. *The following properties are equivalent:*

(i) *$F\colon X \to 2^Y$ is u.s.c.;*

(ii) *for every closed subset $C \subset Y$, the set*

$$F^{-}(C) := \{x \in X \mid F(x) \cap C \neq \emptyset\}$$

is closed in X.

Proposition 7. *The following properties are equivalent:*

(a) $F\colon X \to 2^Y$ *is l.s.c.;*

(b) *if $u \in X$, $\{u_\lambda\}_{\lambda \in J} \subset X$ is a net such that $u_\lambda \to u$, and $u^\star \in F(u)$, then for each $\lambda \in J$ there is $u_\lambda^\star \in F(u_\lambda)$ with $u_\lambda^\star \to u^\star$ in Y.*

3 Proof of the main result

In this section we prove Theorem 2. We start with the fact that, for each $\lambda \in \mathcal{L}$, problem (1) has a smallest solution. To this end, we will use the similar technique employed in [12, Lemma 4.1] to show that the solution set S_λ is downward directed (see Definition 4).

Lemma 8. *For each $\lambda \in \mathcal{L} = (0, \lambda^\star]$, the solution set S_λ of problem (1) is downward directed, i.e., if $u_1, u_2 \in S_\lambda$, then there exists $u \in S_\lambda$ such that*

$$u \le u_1 \quad \text{and} \quad u \le u_2.$$

Proof. Fix $\lambda \in (0, \lambda^\star]$ and $u_1, u_2 \in S_\lambda$. Corresponding to any $\varepsilon > 0$ we introduce the truncation $\eta_\varepsilon\colon \mathbb{R} \to \mathbb{R}$ as follows

$$\eta_\varepsilon(t) = \begin{cases} 0 & \text{if } t \le 0 \\ \frac{t}{\varepsilon} & \text{if } 0 < t < \varepsilon \\ 1 & \text{otherwise,} \end{cases}$$

which is Lipschitz continuous. It results from Marcus-Mizel [27] that

$$\eta_\varepsilon(u_2 - u_1) \in W_0^{1,p}(\Omega)$$

and

$$\nabla(\eta_\varepsilon(u_2 - u_1)) = \eta_\varepsilon'(u_2 - u_1)\nabla(u_2 - u_1).$$

Then for any function $v \in C_0^\infty(\Omega)$ with $v(x) \ge 0$ for a.e. $x \in \Omega$, we have

$$\eta_\varepsilon(u_2 - u_1)v \in W_0^{1,p}(\Omega)$$

and

$$\nabla\big(\eta_\varepsilon(u_2 - u_1)v\big) = v\nabla(\eta_\varepsilon(u_2 - u_1)) + \eta_\varepsilon(u_2 - u_1)\nabla v.$$

Since $u_1, u_2 \in S_\lambda$, there hold

$$\int_\Omega |\nabla u_i(x)|^{p-2}\big(\nabla u_i(x), \nabla\varphi(x)\big)_{\mathbb{R}^N}\, dx = \lambda \int_\Omega u_i(x)^{-\gamma}\varphi(x)\, dx + \int_\Omega f(x, u_i(x))\varphi(x)\, dx \quad \text{for all } \varphi \in W_0^{1,p}(\Omega),\ i = 1, 2.$$

Inserting $\varphi = \eta_\varepsilon(u_2 - u_1)v$ for $i = 1$ and $\varphi = (1 - \eta_\varepsilon(u_2 - u_1))v$ for $i = 2$, and summing the resulting inequalities yield

$$\begin{aligned}
&\int_\Omega |\nabla u_1(x)|^{p-2}(\nabla u_1(x), \nabla\big(\eta_\varepsilon(u_2 - u_1)v\big)(x))_{\mathbb{R}^N}\,dx \\
&\quad + \int_\Omega |\nabla u_2(x)|^{p-2}\big(\nabla u_2(x), \nabla\big((1 - \eta_\varepsilon(u_2 - u_1))v\big)(x)\big)_{\mathbb{R}^N}\,dx \\
&\quad = \int_\Omega \big[\lambda u_1(x)^{-\gamma} + f(x, u_1(x))\big]\big(\eta_\varepsilon(u_2 - u_1)v\big)(x)\,dx \\
&\qquad + \int_\Omega \big[\lambda u_2(x)^{-\gamma} + f(x, u_2(x))\big]\big(1 - \eta_\varepsilon(u_2 - u_1))v\big)(x)\,dx.
\end{aligned}$$

We note that

$$\begin{aligned}
&\int_\Omega |\nabla u_1(x)|^{p-2}(\nabla u_1(x), \nabla\big(\eta_\varepsilon(u_2 - u_1)v\big)(x))_{\mathbb{R}^N}\,dx \\
&\quad = \frac{1}{\varepsilon}\int_{\{0<u_2-u_1<\varepsilon\}} |\nabla u_1(x)|^{p-2}(\nabla u_1(x), \nabla(u_2 - u_1)(x))_{\mathbb{R}^N} v(x)\,dx \\
&\qquad + \int_\Omega |\nabla u_1(x)|^{p-2}(\nabla u_1(x), \nabla v(x))_{\mathbb{R}^N}\eta_\varepsilon\big(u_2(x) - u_1(x)\big)\,dx
\end{aligned}$$

and

$$\begin{aligned}
&\int_\Omega |\nabla u_2(x)|^{p-2}(\nabla u_2(x), \nabla\big((1 - \eta_\varepsilon(u_2 - u_1))v\big)(x))_{\mathbb{R}^N}\,dx \\
&\quad = -\frac{1}{\varepsilon}\int_{\{0<u_2-u_1<\varepsilon\}} |\nabla u_2(x)|^{p-2}(\nabla u_2(x), \nabla(u_2 - u_1)(x))_{\mathbb{R}^N} v(x)\,dx \\
&\qquad + \int_\Omega |\nabla u_2(x)|^{p-2}(\nabla u_2(x), \nabla v(x))_{\mathbb{R}^N}\big(1 - \eta_\varepsilon(u_2(x) - u_1(x))\big)\,dx.
\end{aligned}$$

Altogether, we obtain

$$\begin{aligned}
&\int_\Omega |\nabla u_1(x)|^{p-2}(\nabla u_1(x), \nabla v(x))_{\mathbb{R}^N}\eta_\varepsilon\big(u_2(x) - u_1(x)\big)\,dx \\
&\quad + \int_\Omega |\nabla u_2(x)|^{p-2}(\nabla u_2(x), \nabla v(x))_{\mathbb{R}^N}\big(1 - \eta_\varepsilon(u_2(x) - u_1(x))\big)\,dx \\
&\quad \geq \int_\Omega \big[\lambda u_1(x)^{-\gamma} + f(x, u_1(x))\big]\big(\eta_\varepsilon(u_2 - u_1)v\big)(x)\,dx \\
&\qquad + \int_\Omega \big[\lambda u_2(x)^{-\gamma} + f(x, u_2(x))\big]\big(1 - \eta_\varepsilon(u_2 - u_1))v\big)(x)\,dx.
\end{aligned}$$

Now we pass to the limit as $\varepsilon \to 0^+$. Using Lebesgue's Dominated Convergence Theorem and the fact that

$$\eta_\varepsilon\big((u_2 - u_1)(x)\big) \to \chi_{\{u_1<u_2\}}(x) \quad \text{for a.e. } x \in \Omega \quad \text{as } \varepsilon \to 0^+,$$

we find

$$\begin{aligned}&\int_{\{u_1<u_2\}} |\nabla u_1(x)|^{p-2}(\nabla u_1(x), \nabla v(x))_{\mathbb{R}^N}\, dx\\ &+ \int_{\{u_1\geq u_2\}} |\nabla u_2(x)|^{p-2}(\nabla u_2(x), \nabla v(x))_{\mathbb{R}^N}\, dx\\ &\geq \int_{\{u_1<u_2\}} \big[\lambda u_1(x)^{-\gamma} + f(x, u_1(x))\big] v(x)\, dx + \int_{\{u_1\geq u_2\}} \big[\lambda u_2(x)^{-\gamma} + f(x, u_2(x))\big] v(x)\, dx.\end{aligned} \tag{3}$$

Here the notation χ_D stands for the characteristic function of a set D, that is,

$$\chi_D(t) = \begin{cases} 1 & \text{if } t \in D\\ 0 & \text{otherwise.}\end{cases}$$

The gradient of $u := \min\{u_1, u_2\} \in W_0^{1,p}(\Omega)$ is equal to

$$\nabla u(x) = \begin{cases} \nabla u_1(x) & \text{for a.e. } x \in \{u_1 < u_2\}\\ \nabla u_2(x) & \text{for a.e. } x \in \{u_1 \geq u_2\}.\end{cases}$$

Consequently, we can express (3) in the form

$$\int_\Omega |\nabla u(x)|^{p-2}(\nabla u(x), \nabla v(x))_{\mathbb{R}^N}\, dx \geq \int_\Omega \big[\lambda u(x)^{-\gamma} + f(x, u(x))\big] v(x)\, dx \tag{4}$$

for all $v \in C_0^\infty(\Omega)$ with $v(x) \geq 0$ for a.e. $x \in \Omega$. Actually, the density of $C_0^\infty(\Omega)_+$ in $W_0^{1,p}(\Omega)_+$ ensures that (4) is valid for all $v \in W_0^{1,p}(\Omega)_+$.

Let $\widetilde{u}_\lambda$ be the unique solution of the purely singular elliptic problem

$$\begin{cases} -\Delta_p u(x) = \lambda u(x)^{-\gamma} & \text{in } \Omega\\ u > 0 & \text{in } \Omega\\ u = 0 & \text{on } \partial\Omega.\end{cases}$$

Proposition 5 of Papageorgiou-Smyrlis [31] guarantees that $\widetilde{u}_\lambda \in \operatorname{int}\big(C_0^1(\overline{\Omega})_+\big)$. We claim that

$$\widetilde{u}_\lambda \leq u \quad \text{for all} \quad u \in S_\lambda. \tag{5}$$

For every $u \in S_\lambda$, there holds

$$\int_\Omega |\nabla u(x)|^{p-2}(\nabla u(x), \nabla v(x))_{\mathbb{R}^N}\, dx = \int_\Omega \big[\lambda u(x)^{-\gamma} + f(x, u(x))\big] v(x)\, dx \tag{6}$$

whenever $v \in W_0^{1,p}(\Omega)$. Inserting $v = (\widetilde{u}_\lambda - u)^+ \in W_0^{1,p}(\Omega)$ in (6) and using the fact that $f(x, u(x)) \geq 0$, we derive

$$\begin{aligned}
&\int_\Omega |\nabla u(x)|^{p-2}(\nabla u(x), \nabla(\widetilde{u}_\lambda - u)^+(x))_{\mathbb{R}^N}\, dx \\
&\quad = \int_\Omega \big[\lambda u(x)^{-\gamma} + f(x, u(x))\big](\widetilde{u}_\lambda - u)^+(x)\, dx \\
&\quad \geq \int_\Omega \lambda u(x)^{-\gamma}(\widetilde{u}_\lambda - u)^+(x)\, dx \\
&\quad \geq \int_\Omega \lambda \widetilde{u}_\lambda(x)^{-\gamma}(\widetilde{u}_\lambda - u)^+(x)\, dx \\
&\quad = \int_\Omega |\nabla \widetilde{u}_\lambda|^{p-2}\big(\nabla \widetilde{u}_\lambda(x), \nabla(\widetilde{u}_\lambda - u)^+(x)\big)_{\mathbb{R}^N}\, dx.
\end{aligned}$$

Then the monotonicity of $-\Delta_p$ leads to (5).

Since $u_1, u_2 \in S_\lambda$ and $u := \min\{u_1, u_2\} \in W_0^{1,p}(\Omega)$, we conclude that $u \geq \widetilde{u}_\lambda$. Corresponding to the truncation

$$\widetilde{g}(x, s) = \begin{cases} \lambda \widetilde{u}_\lambda(x)^{-\gamma} + f(x, \widetilde{u}_\lambda(x)) & \text{if } s < \widetilde{u}_\lambda(x) \\ \lambda s^{-\gamma} + f(x, s) & \text{if } \widetilde{u}_\lambda(x) \leq s \leq u(x) \\ \lambda u(x)^{-\gamma} + f(x, u(x)) & \text{if } u(x) < s, \end{cases} \tag{7}$$

we consider the intermediate Dirichlet problem

$$\begin{cases} -\Delta_p w(x) = \widetilde{g}(x, w(x)) & \text{in } \Omega \\ w > 0 & \text{in } \Omega \\ w(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{8}$$

By [32, Proposition 7] there exists $\underline{u} \in W_0^{1,p}(\Omega)$ such that

$$\langle A(\underline{u}), h\rangle = \int_\Omega \widetilde{g}(x, \underline{u}(x))h(x)\, dx$$

for all $h \in W_0^{1,p}(\Omega)$. Inserting $h = (\underline{u} - u)^+$, through (4) and (7), we infer that

$$\begin{aligned}
\langle A(\underline{u}), (\underline{u} - u)^+\rangle &= \int_\Omega \big[\lambda u(x)^{-\gamma} + f(x, u(x))\big](\underline{u} - u)^+(x)\, dx \\
&\leq \langle A(u), (\underline{u} - u)^+\rangle.
\end{aligned}$$

It turns out that $\underline{u} \leq u$. Through the same argument, we also imply $\underline{u} \geq \widetilde{u}_\lambda$. So by virtue of (7) and (8) we arrive at $\underline{u} \in S_\lambda$ and $\underline{u} \leq \min\{u_1, u_2\}$. □

We are in a position to prove that problem (1) admits a smallest solution for every $\lambda \in \mathcal{L}$.

Lemma 9. *If hypotheses $H(f)$ hold and $\lambda \in \mathcal{L} = (0, \lambda^*]$, then problem (1) has a smallest (positive) solution $u_\lambda^\star \in S_\lambda$, that is,*

$$u_\lambda^\star \leq u \quad \textit{for all } u \in S_\lambda.$$

Proof. Fix $\lambda \in (0, \lambda^*]$. Invoking Hu-Papageorgiou [22, Lemma 3.10], we can find a decreasing sequence $\{u_n\} \subset S_\lambda$ such that

$$\inf S_\lambda = \inf_n u_n.$$

On the basis of (5) we note that

$$\widetilde{u}_\lambda \le u_n \quad \text{for all } n. \tag{9}$$

Next we verify that the sequence $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Arguing by contradiction, suppose that a relabeled subsequence of $\{u_n\}$ satisfies $\|u_n\| \to \infty$. Set $y_n = \frac{u_n}{\|u_n\|}$. This ensures

$$y_n \to y \quad \text{weakly in} \quad W_0^{1,p}(\Omega) \quad \text{and} \quad y_n \to y \quad \text{strongly in} \quad L^p(\Omega) \quad \text{with} \quad y \ge 0. \tag{10}$$

From (6) and $\{u_n\} \subset S_\lambda$ we have

$$\begin{aligned}\langle A(y_n), v\rangle &= \int_\Omega |\nabla y_n(x)|^{p-2}(\nabla y_n(x), \nabla v(x))_{\mathbb{R}^N}\, dx \\ &= \int_\Omega \left[\lambda \frac{u_n(x)^{-\gamma}}{\|u_n\|^{p-1}} + \frac{f(x, u_n(x))}{\|u_n\|^{p-1}}\right] v(x)\, dx\end{aligned} \tag{11}$$

for all $v \in W_0^{1,p}(\Omega)$. On the other hand, hypotheses $H(f)$(i) and (ii) entail

$$0 \le f(x, s) \le c_1(1 + |s|^{p-1}) \quad \text{for a.e.} \quad x \in \Omega \quad \text{and all} \quad s \ge 0, \tag{12}$$

with some $c_1 > 0$. By (10) and (12) we see that the sequence

$$\left\{\frac{f(\cdot, u_n(\cdot))}{\|u_n\|^{p-1}}\right\} \quad \text{is bounded in } L^{p'}(\Omega).$$

Due to hypothesis $H(f)$(ii) and Aizicovici-Papageorgiou-Staicu [1, Proposition 16], we find that

$$\left\{\frac{f(\cdot, u_n(\cdot))}{\|u_n\|^{p-1}}\right\} \to \widehat{\lambda}_m y^{p-1} \quad \text{weakly in} \quad L^{p'}(\Omega).$$

Then inserting $v = y_n - y$ in (11) and using (9) lead to

$$\lim_{n\to\infty} \langle A(y_n), y_n - y\rangle = 0.$$

We can apply Proposition 3 to obtain $y_n \to y$ in $W_0^{1,p}(\Omega)$. Letting $n \to \infty$ in (11) gives

$$\langle A(y), v\rangle = \widehat{\lambda}_m \int_\Omega y^{p-1} v\, dx \quad \text{for all } v \in W_0^{1,p}(\Omega),$$

so y is a nontrivial nonnegative solution of the eigenvalue problem

$$\begin{cases} -\Delta_p y(x) = \widehat{\lambda}_m y(x)^{p-1} & \text{in } \Omega \\ y = 0 & \text{on } \partial\Omega. \end{cases}$$

Consequently, y must be nodal because $m \ge 2$ and $y \ne 0$, which contradicts that $y \ge 0$ in Ω. This contradiction proves that the sequence $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$.

Along a relabeled subsequence, we may assume that

$$u_n \to u_\lambda^* \quad \text{weakly in} \quad W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to u_\lambda^* \quad \text{in} \quad L^p(\Omega), \tag{13}$$

for some $u_\lambda^* \in W_0^{1,p}(\Omega)$. In addition, we may suppose that

$$u_n(x)^{-\gamma} \to u_\lambda^*(x)^{-\gamma} \quad \text{for a.e. } x \in \Omega. \tag{14}$$

From $\widetilde{u}_\lambda \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ and (5), through the Lemma in Lazer-Mckenna [23], we obtain

$$0 \le u_n^{-\gamma} \le \widetilde{u}_\lambda^{-\gamma} \in L^{p'}(\Omega). \tag{15}$$

On account of (13)-(15) we have

$$u_n^{-\gamma} \to (u_\lambda^*)^{-\gamma} \quad \text{weakly in} \quad L^{p'}(\Omega) \tag{16}$$

(see also Gasiński-Papageorgiou [16, p. 38]).

Setting $u = u_n \in S_\lambda$ and $v = u_n - u_\lambda^* \in W^{1,p}(\Omega)$ in (6), in the limit as $n \to \infty$ we get

$$\lim_{n\to\infty} \langle Au_n, u_n - u_\lambda^* \rangle = 0.$$

The property of A to be of type (S_+) (according to Proposition 3) implies

$$u_n \to u_\lambda^* \quad \text{in } W_0^{1,p}(\Omega).$$

The above convergence and Sobolev embedding theorem enable us to deduce

$$\int_\Omega |\nabla u_\lambda^*(x)|^{p-2} \left(\nabla u_\lambda^*(x), \nabla v(x)\right)_{\mathbb{R}^N} dx = \int_\Omega \left[\lambda u_\lambda^*(x)^{-\gamma} + f(x, u_\lambda^*(x))\right] v(x)\, dx$$

for all $v \in W_0^{1,p}(\Omega)$. Consequently, we have

$$u_\lambda^* \in S_\lambda \subset \operatorname{int}(C_0^1(\overline{\Omega})_+) \quad \text{and} \quad u_\lambda^* = \inf S_\lambda,$$

which completes the proof. □

In the next lemma we examine monotonicity and continuity properties of the map $\lambda \mapsto u_\lambda^*$ from $\mathcal{L} = (0, \lambda^*]$ to $C_0^1(\overline{\Omega})$.

Lemma 10. *Suppose that hypotheses $H(f)$ hold. Then the map $\Gamma\colon \mathcal{L} = (0, \lambda^*] \to C_0^1(\overline{\Omega})$ given by $\Gamma(\lambda) = u_\lambda^*$ fulfills:*

(i) *Γ is strictly increasing, in the sense that*

$$0 < \mu < \lambda \le \lambda^* \quad \textit{implies} \quad u_\lambda^* - u_\mu^* \in \operatorname{int}\left(C_0^1(\overline{\Omega})_+\right);$$

(ii) *Γ is left continuous.*

Proof. (i) It follows from [32, Proposition 5] that there exists a solution $u_\mu \in S_\mu \subset \operatorname{int}(C_0^1(\overline{\Omega})_+)$ such that

$$u_\lambda^* - u_\mu \in \operatorname{int}(C_0^1(\overline{\Omega})_+).$$

The desired conclusion is the direct consequence of the inequality $u_\mu^* \le u_\mu$.

(ii) Let $\{\lambda_n\} \subset (0, \lambda^*]$ and $\lambda \in (0, \lambda^*]$ satisfy $\lambda_n \uparrow \lambda$. Denote for simplicity $u_n = u_{\lambda_n}^* = \Gamma(\lambda_n) \in S_{\lambda_n} \subset \operatorname{int}(C_0^1(\overline{\Omega})_+)$. It holds

$$\langle A(u_n), v \rangle = \int_\Omega \left[\lambda_n u_n(x)^{-\gamma} + f(x, u_n(x))\right] v(x)\, dx \tag{17}$$

for all $v \in W_0^{1,p}(\Omega)$. By assertion (i) we know that

$$0 \le u_1 \le u_n \le u_{\lambda^*}^*. \tag{18}$$

Choosing $v = u_n$ in (17) and proceeding as in the proof of Lemma 9, we verify that the sequence $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Given $r > N$, it is true that $(u_{\lambda_1}^*)^r \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$, so there is a constant $c_2 > 0$ such that

$$\widetilde{u}_1 \le c_2 (u_{\lambda_1}^*)^r = c_2 u_1^r,$$

or

$$\widetilde{u}_1^{\frac{-\gamma}{r}} \geq c_2^{\frac{-\gamma}{r}} u_1^{-\gamma}.$$

We can make use of the Lemma in Lazer-Mckenna [23] for having

$$0 \leq u_n^{-\gamma} \leq u_1^{-\gamma} \in L^r(\Omega) \quad \text{for all} \quad n.$$

Moreover, hypothesis $H(f)$(i) and (18) render that

$$\text{the sequence } \{f(\cdot, u_n(\cdot))\} \text{ is bounded in } \ L^r(\Omega).$$

Therefore, utilizing Guedda-Véron [20, Proposition 1.3] we obtain the uniform bound

$$\|u_n\|_{L^\infty(\Omega)} \leq c_3 \quad \text{for all } n, \tag{19}$$

with some $c_3 > 0$. Besides, the linear elliptic problem

$$\begin{cases} -\Delta v(x) = g_{\lambda_n}(x) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $g_{\lambda_n}(\cdot) = \lambda_n u_n(\cdot)^{-\gamma} + f(\cdot, u_n(\cdot)) \in L^r(\Omega)$, has a unique solution $v_{\lambda_n} \in W_0^{2,r}(\Omega)$ (see, e.g., [19, Theorem 9.15]). Owning to $r > N$, the Sobolev embedding theorem provides

$$v_{\lambda_n} \in C_0^{1,\alpha}(\overline{\Omega}),$$

with $\alpha = 1 - \frac{N}{r}$. For $w_n := \nabla v_{\lambda_n}$, we have $w_n \in C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^N)$ and

$$\begin{cases} -\operatorname{div}\left(|\nabla u_n(x)|^{p-2}\nabla u_n(x) - w_n(x)\right) = 0 & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

This allows us to apply the nonlinear regularity up to the boundary in Liebermann [24, 25] finding that $u_n \in C_0^{1,\beta}(\overline{\Omega})$ with some $\beta \in (0,1)$ for all n. Here the uniform estimate in (19) is essential. The compactness of the embedding of $C_0^{1,\beta}(\overline{\Omega})$ in $C_0^1(\overline{\Omega})$ and the monotonicity of the sequence $\{u_n\}$ guarantee

$$u_n \to \overline{u}_\lambda \quad \text{in } C_0^1(\overline{\Omega})$$

for some $\overline{u}_\lambda \in C_0^1(\overline{\Omega})$.

We claim that $\overline{u}_\lambda = u_\lambda^*$. Arguing by contradiction, suppose that there exists $x^* \in \Omega$ satisfying

$$\overline{u}_\lambda(x^*) < u_\lambda^*(x^*).$$

The known monotonicity property of $\{u_n\}$ entails

$$u_\lambda^*(x^*) < u_n(x^*) = u_{\lambda_n}^*(x^*) \quad \text{for all } n,$$

which contradicts assertion (i). It results that $\overline{u}_\lambda = u_\lambda^* = \Gamma(\lambda)$, thereby

$$\Gamma(\lambda_n) = u_n \to \overline{u}_\lambda = \Gamma(\lambda) \quad \text{as } n \to \infty,$$

completing the proof. □

Next we turn to the semicontinuity properties of the set-valued mapping Λ.

Lemma 11. *Assume that hypotheses $H(f)$ hold. Then the set-valued mapping $\Lambda\colon \mathcal{L} \to 2^{C_0^1(\overline{\Omega})}$ is sequentially upper semicontinuous.*

Proof. According to Proposition 6 we are going to show that for any closed set $D \subset C_0^1(\overline{\Omega})$, one has that

$$\Lambda^-(D) := \{\lambda \in \mathbb{R} : \Lambda(\lambda) \cap D \neq \emptyset\}$$

is closed in $\mathbb{R}$. Let $\{\lambda_n\} \subset \Lambda^-(D)$ verify $\lambda_n \to \lambda$ as $n \to \infty$. So,

$$\Lambda(\lambda_n) \cap D \neq \emptyset,$$

hence there exists a sequence $\{u_n\} \subset \operatorname{int}(C_0^1(\overline{\Omega})_+)$ satisfying

$$u_n \in \Lambda(\lambda_n) \cap D \quad \text{for all } n \in \mathbb{N},$$

in particular

$$\int_\Omega |\nabla u_n(x)|^{p-2} (\nabla u_n(x), \nabla v(x))_{\mathbb{R}^N}\, dx = \int_\Omega [\lambda_n u_n(x)^{-\gamma} + f(x, u_n(x))]\, v(x)\, dx \tag{20}$$

for all $v \in W_0^{1,p}(\Omega)$. As in the proof of Lemma 9, we can show that the sequence $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Therefore we may assume that

$$u_n \to u \quad \text{weakly in} \quad W_0^{1,p}(\Omega) \quad \text{and } u_n \to u \quad \text{in} \quad L^p(\Omega). \tag{21}$$

for some $u \in W_0^{1,p}(\Omega)$. Furthermore, the sequences $\{f(\cdot, u_n(\cdot))\}$ and $\{u_n^{-\gamma}\}$ are bounded in $L^{p'}(\Omega)$ as already demonstrated in the proofs of Lemmas 9 and 10. In (20), we choose $v = u_n - u \in W_0^{1,p}(\Omega)$ and then pass to the limit as $n \to \infty$. By means of (21) we are led to

$$\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0.$$

Since A is of type (S_+), we can conclude

$$u_n \to u \quad \text{in} \quad W_0^{1,p}(\Omega). \tag{22}$$

On account of (20), the strong convergence in (22) and Sobolev embedding theorem imply

$$\int_\Omega |\nabla u(x)|^{p-2} (\nabla u(x), \nabla v(x))_{\mathbb{R}^N}\, dx = \int_\Omega [\lambda u(x)^{-\gamma} + f(x, u(s))]\, v(x)\, dx$$

for all $v \in W_0^{1,p}(\Omega)$. This reads as $u \in S_\lambda = \Lambda(\lambda)$.

It remains to check that $u \in D$. Fix $\underline{\lambda} \in \mathcal{L}$ such that

$$\underline{\lambda} < \lambda_n \leq \lambda^* \quad \text{for all } n.$$

By Lemma 10 (i) we know that

$$u_{\underline{\lambda}}^* < u_{\lambda_n}^* \leq u_n \quad \text{for all } n.$$

The same argument as in the proof of Lemma 10 confirms that, for $r > N$ fixed, the function $x \mapsto \lambda_n u_n(x)^{-\gamma} + f(x, u_n(x))$ is bounded in $L^r(\Omega)$. Let $g_{\lambda_n}(x) = \lambda_n u_n(x)^{-\gamma} + f(x, u_n(x)) \in L^r(\Omega)$ and consider the linear Dirichlet problem

$$\begin{cases} -\Delta v(x) = g_{\lambda_n}(x) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{23}$$

The standard existence and regularity theory (see, e.g., Gilbarg-Trudinger [19, Theorem 9.15]) ensure that problem (23) has a unique solution

$$v_{\lambda_n} \in W^{2,r}(\Omega) \subset C_0^{1,\alpha}(\overline{\Omega}) \text{ with } \|v_{\lambda_n}\|_{C_0^{1,\alpha}(\overline{\Omega})} \leq c_4,$$

with a constant $c_4 > 0$ and $\alpha = 1 - \frac{N}{r}$. Denote $w_n(x) = \nabla v_{\lambda_n}(x)$ for all $x \in \Omega$. It holds $w_n \in C^{0,\alpha}(\overline{\Omega})$ thanks to $v_{\lambda_n} \in C_0^{1,\alpha}(\overline{\Omega})$. Notice that

$$\begin{cases} -\operatorname{div}\left(|\nabla u_n(x)|^{p-2}\nabla u_n(x) - w_n(x)\right) = 0 & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

The nonlinear regularity up to the boundary in Liebermann [24, 25] reveals that $u_n \in C_0^{1,\beta}(\overline{\Omega})$ for all $n \in \mathbb{N}$ with some $\beta \in (0,1)$. The compactness of the embedding of $C_0^{1,\beta}(\overline{\Omega})$ in $C_0^1(\overline{\Omega})$ and (22) yield the strong convergence

$$u_n \to u \quad \text{in } C_0^1(\overline{\Omega}).$$

Recalling that D is closed in $C_0^1(\overline{\Omega})$ it results that $u \in \Lambda(\lambda) \cap D$, i.e., $\lambda \in \Lambda^-(D)$. □

Lemma 12. *Suppose that hypotheses $H(f)$ hold. Then the set-valued mapping $\Lambda\colon \mathcal{L} \to 2^{C_0^1(\overline{\Omega})}$ is sequentially lower semicontinuous.*

Proof. In order to refer to Proposition 7, let $\{\lambda_n\} \subset \mathcal{L}$ satisfy $\lambda_n \to \lambda \neq 0$ as $n \to \infty$ and let $w \in S_\lambda \subset \operatorname{int}(C_0^1(\overline{\Omega})_+)$. For each $n \in \mathbb{N}$, we formulate the Dirichlet problem

$$\begin{cases} -\Delta_p u(x) = \lambda_n w(x)^{-\gamma} + f(x, w(x)) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{24}$$

In view of $w \geq \widetilde{u}_\lambda \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ (see (5)) and

$$\begin{cases} \lambda_n w(x)^{-\gamma} + f(x, w(x)) \geq 0 \quad \text{for all } x \in \Omega \\ \lambda_n w(x)^{-\gamma} + f(x, w(x)) \not\equiv 0, \end{cases}$$

it is obvious that problem (24) has a unique solution $u_n^0 \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$. Relying on the growth condition for f (see hypotheses $H(f)$(i) and (ii)), through the same argument as in the proof of Lemma 9 we show that the sequence $\{u_n^0\}$ is bounded in $W_0^{1,p}(\Omega)$. Then Proposition 1.3 of Guedda-Véron [20] implies the uniform boundedness

$$u_n^0 \in L^\infty(\Omega) \quad \text{and} \quad \|u_n^0\|_{L^\infty(\Omega)} \leq c_5 \quad \text{for all } n \in \mathbb{N},$$

with a constant $c_5 > 0$. As in the proof of Lemma 11, we set $g_{\lambda_n}(x) = \lambda_n w(x)^{-\gamma} + f(x, w(x))$ and consider the Dirichlet problem (23) to obtain that $\{u_n^0\}$ is contained in $C_0^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$. Due to the compactness of the embedding of $C_0^{1,\beta}(\overline{\Omega})$ in $C_0^1(\overline{\Omega})$, we may assume

$$u_n^0 \to u \quad \text{in} \quad C_0^1(\overline{\Omega}) \quad \text{as} \quad n \to \infty,$$

with some $u \in C_0^1(\overline{\Omega})$. Then (24) yields

$$\begin{cases} -\Delta_p u(x) = \lambda w(x)^{-\gamma} + f(x, w(x)) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Thanks to $w \in \Lambda(\lambda)$, a simple comparison justifies $u = w$. Since every convergent subsequence of $\{u_n\}$ converges to the same limit w, it is true that

$$\lim_{n\to\infty} u_n^0 = w.$$

Next, for each $n \in \mathbb{N}$, we consider the Dirichlet problem

$$\begin{cases} -\Delta_p u(x) = \lambda_n u_n^0(x)^{-\gamma} + f(x, u_n^0(x)) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Carrying on the same reasoning, we can show that this problem has a unique solution u_n^1 belonging to $\operatorname{int}(C_0^1(\overline{\Omega})_+)$ and that

$$\lim_{n\to\infty} u_n^1 = w.$$

Continuing the process, we generate a sequence $\{u_n^k\}_{n,k\geq 1}$ such that

$$\begin{cases} -\Delta_p u_n^k(x) = \lambda_n u_n^{k-1}(x)^{-\gamma} + f(x, u_n^{k-1}(x)) & \text{in } \Omega \\ u_n^k > 0 & \text{in } \Omega \\ u_n^k = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$\lim_{n\to\infty} u_n^k = w \quad \text{for all } k \in \mathbb{N}. \tag{25}$$

Fix $n \geq 1$. As before, based on the nonlinear regularity [24, 25], we notice that the sequence $\{u_n^k\}_{k\geq 1}$ is relatively compact in $C_0^1(\overline{\Omega})$, so we may suppose

$$u_n^k \to u_n \quad \text{in} \quad C_0^1(\overline{\Omega}) \quad \text{as} \quad k \to \infty,$$

for some $u_n \in C_0^1(\overline{\Omega})$. Then it appears that

$$\begin{cases} -\Delta_p u_n(x) = \lambda_n u_n(x)^{-\gamma} + f(x, u_n(x)) & \text{in } \Omega \\ u_n > 0 & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

which means that $u_n \in \Lambda(\lambda_n)$.

The convergence in (25) and the double limit lemma (see, e.g., [13, Proposition A.2.35]) result in

$$u_n \to w \quad \text{in} \quad C_0^1(\overline{\Omega}) \quad \text{as} \quad n \to \infty.$$

By Proposition 7 we conclude that Λ is lower semicontinuous. □

Proof of Theorem 2. (i) It suffices to apply Lemmas 11 and 12.

(ii) The stated conclusion is a direct consequence of Lemmas 9 and 10. □

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