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PLANAR NONAUTONOMOUS POLYNOMIAL EQUATIONS IV. NONHOLOMORPHIC CASE

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Abstract. We give a few sufficient conditions for the existence of periodic solutions of the equation $\dot{z} = \sum_{j=0}^{n} a_j(t)z^j - \sum_{k=1}^{r} c_k(t)\overline{z}^k$ where $n > r$ and a_j's, c_k's are complex valued. We prove the existence of one up to two periodic solutions.

Keywords: periodic orbits, polynomial equations.

Mathematics Subject Classification: 34C25, 34C37.

1. INTRODUCTION

The presented paper is a continuation of [21–23]. We study a planar nonautonomous differential equations of the form

$$\dot{z} = v(t,z) = \sum_{j=0}^{n} a_j(t)z^j - \sum_{k=1}^{r} c_k(t)\overline{z}^k, \tag{1.1}$$

where $n > r \geq 1$ and $a_j, c_k \in \mathcal{C}(\mathbb{R}, \mathbb{C})$ are T-periodic.

Equation (1.1) was investigated in many papers, e.g., [9, 13, 17–20]. In all of them $r > n$ holds. This allows to obtain the existence of periodic solutions and, in some cases, even some kind of chaos (for definition see [20]). All the papers use topological tools – method of isolating segments and isolating chains.

There is another subclass of (1.1) which was extensively studied

$$\dot{z} = \sum_{j=0}^{n} a_j(t)z^j. \tag{1.2}$$

The study was initiated in [14] and continued in many papers, e.g. [1, 2, 4, 5, 7, 8, 11, 12, 15]. One of the most important problems was to examine the structure of the set of periodic solutions. The second one is the investigation of a centre which is motivated by the Poincaré centre-focus problem. The third one is connected with the XVIth Hilbert problem for degree two equations in the plane which can be reduced to the problem of finding the maximal number of closed solutions of the equation (1.1) with $n = 3$ and special coefficients a_j. This leads to investigations of the maximal number of periodic solutions of (1.1). It is proved in [10] that in the general case there is no upper bound for this number provided that $n \geq 3$.

In some of these papers coefficients were only real. In others the holomorphicity of the vector field played an important role.

In the presented paper we deal with the case $n > r \geq 1$. Since the dominating term at infinity is of the form z^n, it is difficult to use the method of isolating segments. Moreover, the lack of holomorphicity makes us unable to use such tools like the Denjoy-Wolff fixed point theorem (see [3]).

We develop the idea from [21–23] and give a few sufficient conditions for the existence of one or two periodic solutions. We deal with the condition of geometric type which corresponds to the ones from [22, Subsection 3.1] and [23]. Namely, we try to include sets $a_j(\mathbb{R})$ and $c_k(\mathbb{R})$ in some sectors of the complex plane. The point is, we try to find one sector appropriate for all a_j's and c_k's.

The method we use may be easily applied to the more general equations

$$\dot{z} = \hat{v}(t,z) = \sum_{j=0}^{n} a_j(t)|z - b_j(t)|^{l_j} z^j - \sum_{k=1}^{r} c_k(t)|z - d_k(t)|^{s_k} \overline{z}^k, \tag{1.3}$$

where $l_j, s_k \in \mathbb{R}$ and $b_j, d_k \in \mathcal{C}(\mathbb{R}, \mathbb{C})$ are T-periodic.

By the existence of a greater number of coefficients a_j than in the case of the Riccati equation (i.e., equation (1.1) with $n = 2$), it is convenient to consider sectors where the axis of symmetry is the real axis. Other sectors need a change of variables of the form $w = e^{i\alpha} z$. We prove that there exists at least one T-periodic solution in every such sector. In many cases it is not unique in the sector.

The paper is organised as follows. In Section 2 we give definitions and introduce notion. In the next section we provide a few sufficient conditions for the existence of periodic solutions. Section 4 is devoted to the problem of uniqueness of the periodic solutions inside the sectors.

2. DEFINITIONS

2.1. PROCESSES

Let X be a topological space and $\Omega \subset \mathbb{R} \times X \times \mathbb{R}$ be an open set.

By a *local process* on X we mean a continuous map $\varphi : \Omega \longrightarrow X$ such that three conditions are satisfied:

(i) $I_{(\sigma,x)} = \{t \in \mathbb{R} : (\sigma, x, t) \in \Omega\}$ is an open interval containing 0 for every $\sigma \in \mathbb{R}$ and $x \in X$,

(ii) $\varphi(\sigma, \cdot, 0) = \mathrm{id}_X$ for every $\sigma \in \mathbb{R}$,

(iii) $\varphi(\sigma, x, s+t) = \varphi(\sigma+s, \varphi(\sigma, x, s), t)$ for every $x \in X$, $\sigma \in \mathbb{R}$ and $s, t \in \mathbb{R}$ such that $s \in I_{(\sigma,x)}$ and $t \in I_{(\sigma+s,\varphi(\sigma,x,s))}$.

For abbreviation, we write $\varphi_{(\sigma,t)}(x)$ instead of $\varphi(\sigma, x, t)$.

Let M be a smooth manifold and let $v : \mathbb{R} \times M \longrightarrow TM$ be a time-dependent vector field. We assume that v is so regular that for every $(t_0, x_0) \in \mathbb{R} \times M$ the Cauchy problem

$$\dot{x} = v(t, x), \tag{2.1}$$

$$x(t_0) = x_0 \tag{2.2}$$

has unique solution. Then the equation (2.1) generates a local process φ on X by $\varphi_{(t_0,t)}(x_0) = x(t_0, x_0, t + t_0)$, where $x(t_0, x_0, \cdot)$ is the solution of the Cauchy problem (2.1), (2.2).

Let T be a positive number. In the sequel T denotes the period. We assume that v is T-periodic in t. It follows that the local process φ is T-periodic, i.e.,

$$\varphi_{(\sigma+T,t)} = \varphi_{(\sigma,t)} \text{ for all } \sigma, t \in \mathbb{R},$$

hence there is a one-to-one correspondence between T-periodic solutions of (2.1) and fixed points of the Poincaré map $\varphi_{(0,T)}$.

2.2. TOPOLOGICAL TOOLS

The method presented in the paper comes from the method of isolating segments (for definition of isolating segment see [18]). In the paper we use only the simplest version of it so we omit the general definition. We investigate sets of the form $W = [0, T] \times B$ where $B \subset \mathbb{C}$ is homeomorphic to the unit disc. We deal with the behaviour of the vector field $(1, v)$ on the boundary of W. We need only the vector field $(1, v)$ point outwards W at every point of $[0, T] \times \partial B$ (it is also convenient if it points inwards at every point). In this situation we use the Brouwer fixed point theorem for the map $\varphi_{(0,-T)}|_B$ (or $\varphi_{(0,T)}|_B$, respectively).

We use the following lemma ([24, Lemma 1]) to strengthen the Brouwer fixed point theorem (we use it instead of the Denjoy-Wolff fixed point theorem).

Lemma 2.1. *Let $m \geq 1$ and X be a nonempty convex and closed subset of $\mathbb{R}^m$. Let $f \in \mathcal{C}(X, X)$ be such that for every $x, y \in X$, $x \neq y$, the inequality*

$$|f(x) - f(y)| < |x - y| \tag{2.3}$$

holds. If in addition the set $f(X)$ is bounded, then there exists exactly one fixed point $x_0 \in X$ of f. Moreover, x_0 is asymptotically stable and attracting in X.

2.3. BASIC NOTIONS

We make the general assumptions about the equation (1.1) that all its coefficients $a_j, c_k \in \mathcal{C}(\mathbb{R}, \mathbb{C})$ are T-periodic.

Let $g : M \longrightarrow M$ and $n \in \mathbb{N}$. We denote by g^n the n-th iterate of f, and by g^{-n} the n-th iterate of g^{-1} (if it exists).

We say that the point z_0 is *attracting* (*repelling*) *for* g *in the set* $W \subset M$ if the equality $\lim_{n\to\infty} g^n(w) = z_0$ ($\lim_{n\to\infty} g^{-n}(w) = z_0$) holds for every $w \in W$.

We call a T-periodic solution of (2.1) *attracting* (*repelling*) *in the set* $W \subset M$ if the corresponding fixed point of the Poincaré map $\varphi_{(0,T)}$ is attracting (repelling) in the set W.

Let $-\infty \leq \alpha < \omega \leq \infty$ and $s : (\alpha, \omega) \longrightarrow \mathbb{C}$ be a full solution of (1.1). We call s *forward blowing up* (shortly *f.b.*) or *backward blowing up* (*b.b.*) if $\omega < \infty$ or $\alpha > -\infty$, respectively.

We define the sector

$$\mathcal{S}(\alpha, \beta) = \{z \in \mathbb{C} : \alpha < \operatorname{Arg}(z) < \beta\},$$

where $-\pi \leq \alpha < \beta \leq \pi$. Moreover, for $0 < \alpha \leq \pi$ we define $\mathcal{S}(\alpha) = \mathcal{S}(-\alpha, \alpha)$ and $\widehat{\mathcal{S}}(\alpha)$ to be a set symmetric with respect to the origin to sector $\mathcal{S}(\alpha)$. Obviously, $0 \notin \mathcal{S}(\alpha, \beta)$.

Let us recall that the inner product of two vectors $a, b \in \mathbb{C}$ is given by the formula $\langle a, b\rangle = \mathfrak{Re}(a\bar{b}) = \mathfrak{Re}(\bar{a}b)$.

In the sequel we write $\operatorname{Arg}(0) = 0$. Let $a_j, c_k \in \mathcal{C}(\mathbb{R}, \mathbb{C})$ be the coefficients of the vector field v from equation (1.1).

We define $\tau_j \geq 0$ and $\widehat{\tau}_j \geq 0$ to be the smallest numbers such that $|\operatorname{Arg}[a_j(t)]| \leq \tau_j$ and $|\operatorname{Arg}[-a_j(t)]| \leq \widehat{\tau}_j$ hold for every $t \in \mathbb{R}$. We also define $\rho_k \geq 0$ to be the smallest numbers such that $|\operatorname{Arg}[c_k(t)]| \leq \rho_k$ holds for every $t \in \mathbb{R}$.

For a given two functions $f, g : \mathbb{C} \supset \Omega \longrightarrow \mathbb{C}$ we say that f is dominated by g at the point $p \in \operatorname{cl}\Omega$ iff $f = o(g)$ at p.

3. EXISTENCE OF PERIODIC SOLUTIONS

In the present section we assume that $a_1 \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ holds. It is possible to consider $a_1 \in \mathcal{C}(\mathbb{R}, \mathbb{C})$ (cf. [22, Subsection 3.1], [23, Section 3]), provided the equality

$$\int_0^T \mathfrak{Im}[a_1(s)]\, ds = 0 \tag{3.1}$$

is satisfied. In this case we make in (1.1) (or in (1.3)) the change of variables

$$w = B(t)z, \tag{3.2}$$

where

$$B(t) = e^{-i\int_{t_0}^{t} \mathfrak{Im}[a_1(s)]\, ds}$$

for some fixed $t_0 \in \mathbb{R}$ and get the equation

$$\dot{w} = B(t)a_0(t) + \mathfrak{Re}[a_1(t)]w + \sum_{j=2}^{n} B^{1-j}(t)a_j(t)w^j - \sum_{k=1}^{r} c_k(t)B^{k+1}\overline{w}^k$$

(or

$$\dot{w} = B(t)a_0(t)|w - B(t)a_0(t)|^{l_0} + \mathfrak{Re}[a_1(t)]w + \sum_{j=2}^{n} B^{1-j}(t)a_j(t)|w - B(t)a_j(t)|^{l_j} w^j$$
$$- \sum_{k=1}^{r} c_k(t)B^{k+1}|w - B(t)d_k(t)|^{s_k}\overline{w}^k,$$

provided that $l_1 = 0$, respectively).

3.1. NONZERO FREE TERM

We investigate the case $a_0 \not\equiv 0$. This allows us to control the vector field in the neighbourhood of the origin.

We state the main theorem of this section.

Theorem 3.1. *Let $n \geq 2$, $r \geq 1$, $a_1 \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ and $a_j, c_k \in \mathcal{C}(\mathbb{R}, \mathbb{C})$ for $j \in \{0, 2, 3, \ldots, n\}$, $k \in \{1, 2, \ldots, r\}$ be T-periodic. If*

$$n > r, \tag{3.3}$$

$$\sum_{j=r+1}^{n} |a_j| > 0 \tag{3.4}$$

and there exists number

$$M \geq \max\{3, n, r+2\} \tag{3.5}$$

such that

$$a_0 \not\equiv 0 \ \textit{and}\ \widehat{\tau}_0 < \frac{\pi}{M-1}, \tag{3.6}$$

$$\begin{cases} \tau_j < \frac{j-1}{M-1}\pi, & \textit{for } 2 \leq j \leq \frac{M+1}{2}, \\ \tau_j \leq \frac{M-j}{M-1}\pi, & \textit{for } \frac{M+1}{2} < j \leq n, \end{cases} \tag{3.7}$$

$$\begin{cases} \rho_k < \frac{k+1}{M-1}\pi, & \textit{for } 1 \leq k \leq \frac{M-3}{2}, \\ \rho_k \leq \frac{M-k-2}{M-1}\pi, & \textit{for } \frac{M-3}{2} < k \leq r, \end{cases} \tag{3.8}$$

hold, then the equation (1.1) *has in the sector $\mathcal{S}\left(\frac{\pi}{M-1}\right)$:*

- *at least one T-periodic solution ξ,*
- *infinitely many forward blowing up solutions.*

Moreover, the equation

$$\dot{z} = v_1(t, z) = \sum_{j=0}^{n} (-1)^j a_j(t) z^j - \sum_{k=1}^{r} (-1)^k c_k(t) \overline{z}^k \tag{3.9}$$

has in the sector $\widehat{\mathcal{S}}\left(\frac{\pi}{M-1}\right)$*:*

- *at least one* T*-periodic solution* χ*,*
- *infinitely many backward blowing up solutions.*

If in addition we assume that the equalities

$$a_j \equiv 0, \text{ for all odd numbers } j \geq 3, \tag{3.10}$$
$$c_k \equiv 0, \text{ for all odd numbers } k \geq 1 \tag{3.11}$$

hold, then the equation (1.1) *has:*

- *at least one* T*-periodic solution* χ *in the sector* $\widehat{\mathcal{S}}\left(\frac{\pi}{M-1}\right)$*,*
- *infinitely many backward blowing up solutions in the sector* $\widehat{\mathcal{S}}\left(\frac{\pi}{M-1}\right)$*.*

Proof. Our goal is to define a compact set $E \subset \mathbb{C}$ such that E is homeomorphic to the closed unit disc and there exists $t_o \in \mathbb{R}$ satisfying $\varphi^{-1}_{(t_0,t_0+T)}(E) \subset E$. It allows us to apply the Brouwer fixed point theorem and get the existence of a periodic solution inside the set E.

We set

$$0 < \varepsilon < \min\left\{\frac{\pi}{M-1} - \widehat{\tau}_0, \frac{\frac{j-1}{M-1}\pi - \tau_j}{j-1}, \frac{\frac{k+1}{M-1}\pi - \rho_k}{k+1} \text{ for } 2 \leq j \leq n, 1 \leq k \leq r\right\}$$

and

$$A = A(\varepsilon) = \left\{z \in \mathbb{C} : |\operatorname{Arg}(z)| \leq \frac{\pi}{M-1} - \varepsilon\right\}.$$

Let us recall that $0 \in A$. It is shown in the proof of [23, Theorem 4] that the vector field $\left(1, \sum_{j=0}^{n} a_j(t) z^j\right)$ points outwards or is tangent to the set $[0,T] \times A$ at every point of $[0,T] \times \partial A$. It comes mainly from (3.7).

Now we show the same for the $\left(1, -\sum_{k=1}^{r} c_k(t)\overline{z}^k\right)$. Let us consider the change of variables $w = g(z) = \frac{1}{z}$. We get $g\left(\mathcal{S}\left(\frac{\pi}{M-1} - \varepsilon\right)\right) = \mathcal{S}\left(\frac{\pi}{M-1} - \varepsilon\right)$. Applying this change to the term $-c_k(t)\overline{z}^k$ we get $c_k(t)|w|^{-2k}w^{2+k}$. But the condition (3.8) for the vector field $c_k(t)w^{k+2}$ is an analogue of (3.7) for $a_j(t)z^j$ (the term $|w|^{-2k}$ has no influence on the direction of $c_k(t)|w|^{-2k}w^{2+k}$ on $\mathbb{C}$). So for every $t \in [0,T]$ the vector field $-\sum_{k=1}^{r} c_k(t)\overline{z}^k$ points ouward or is tangent to the set A at every point from ∂A. Finally, $\left(1, -\sum_{k=1}^{r} c_k(t)\overline{z}^k\right)$ points outwards or is tangent to the set $[0,T] \times A$ at every point of $[0,T] \times \partial A$.

By (3.6) and (3.4), every solution starting from ∂A stays in ∂A not longer than T, so there is no periodic solutions in the set ∂A.

Let $m = \frac{M-1}{2} \geq 1$. We make the change of variables in the set $\{z \in \mathbb{C} : |\operatorname{Arg}(z)| < \frac{\pi}{M-1}\} \setminus \{0\}$ given by $w = f(z) = z^{-m} := e^{-m \operatorname{Log} z}$. It gives

$$\dot{w} = u(t,w) = -m\sum_{j=0}^{n} a_j(t) w^{\frac{m-j+1}{m}} + m\sum_{k=1}^{r} c_k(t)|w|^{\frac{-2k}{m}} w^{\frac{m+k+1}{m}}. \tag{3.12}$$

Let $\gamma > 0$. We set

$$C(\gamma) = f(A \setminus \{0\}) \cap \{w \in \mathbb{C} : \mathfrak{Re}(w) \geq \gamma\},$$
$$\widehat{C}(\gamma) = C(\gamma) \cap \{w \in \mathbb{C} : \mathfrak{Re}(w) = \gamma\}.$$

It is easy to see that $C(\gamma) \subset \{w \in \mathbb{C} : |\operatorname{Arg}(w)| \leq \frac{\pi}{2} - m\varepsilon_0\}$.

By (3.4), the dominating part of u on $\widehat{C}(\gamma)$ is $-m \sum_{j=0}^{n} a_j(t) w^{\frac{m-j+1}{m}}$, so, since this part points outwards $C(\gamma)$ (see proof of [23, Theorem 4]), the vector field $(1, u)$ points outwards $[0,T] \times C(\gamma)$ at every point of $[0,T] \times \widehat{C}(\gamma)$, provided that γ is sufficiently small.

Finally, we write $E = f^{-1}(C(\gamma)) \cup \{0\}$. Since $\varphi_{(0,T)}^{-1}(E) \subset E$ holds, we apply the Brouwer fixed point theorem and get the existence of periodic solution inside $E \subset \mathcal{S}\left(\frac{\pi}{M-1}\right)$.

The existence of forward blowing up solutions comes from the fact that, by (3.4), the dominating part of u on $\left\{w \in \mathbb{C} : |\operatorname{Arg}(w)| \leq \frac{\pi}{2} - m\varepsilon_0\right\} \setminus C(\gamma)$ is $-m \sum_{j=0}^{n} a_j(t) w^{\frac{m-j+1}{m}}$, so the qualitative behaviour of u is the same as this part (see proof of [23, Theorem 4]).

In the case (3.9) we investigate the set $[0,T] \times (-E)$. Here $\varphi_{(0,T)}(-E) \subset (-E)$ holds. The calculations are quite similar to the above ones. $\square$

Example 3.2. By Theorem 3.1, the equation

$$\dot{z} = -1 + \sin(t) + z^4 - e^{i\frac{\pi}{3}\sin(t)}\overline{z} - \overline{z}^3$$

has in the sector $\mathcal{S}\left(\frac{\pi}{4}\right)$ at least one 2π-periodic solution and infinitely many f.b. solutions. Here $M = 5$, $\widehat{\tau}_0 = \tau_4 = 0 < \frac{\pi}{4}$, $\rho_1 = \frac{\pi}{4} < \frac{\pi}{2}$, $\rho_3 = 0 \leq 0$.

Remark 3.3. Since the term $|z - b(t)|$ has no influence on the direction of the vector fields $|z - b(t)| z^j$ and $|z - b(t)| \overline{z}^k$, it is possible to formulate an analogue of Theorem 3.1 for the equation (1.3). We only need to make some additional assumptions:

1. the vector field $\hat{v}$ is defined and continuous at every point of the set $[0,T] \times \mathcal{S}\left(\frac{\pi}{M-1}\right)$ (or $[0,T] \times \widehat{\mathcal{S}}\left(\frac{\pi}{M-1}\right)$),
2. every $b_j, d_k \in \mathcal{C}(\mathbb{R}, \mathbb{C})$ are T-periodic and $l_j, s_k \in \mathbb{R}$,
3. $a_0(t)|z - b_0(t)|^{l_0}$ is the dominating term of $\hat{v}$ in the neighbourhood of the origin,
4. there exists a nonempty set $J \subset \{2, 3, \ldots, n\}$ such that for every $j \in J$ the vector field
$$\sum_{j=0}^{1} a_j(t)|z - b_j(t)|^{l_j} z^j - \sum_{k=1}^{r} c_k(t)|z - d_k(t)|^{s_k} \overline{z}^k,$$
$$\left(\text{or } \sum_{j=0}^{1} (-1)^j a_j(t)|z - b_j(t)|^{l_j} z^j - \sum_{k=1}^{r} (-1)^k c_k(t)|z - d_k(t)|^{s_k} \overline{z}^k, \text{ respectively}\right)$$
is dominated by the term $|z - b_j(t)|^{l_j} z^j$ in the neighbourhood of infinity and $\sum_{j \in J} |a_j| > 0$ (this condition replaces (3.3) and (3.4)).
5. $1 < j + l_j$ and $a_j \not\equiv 0$ for some $j \in J$.

Proof. Let E be like in the proof of Theorem 3.1. We set

$$\hat{E} = E \cap \{z \in \mathbb{C} : \mathfrak{Re}(z) \geq \nu\} \tag{3.13}$$

for some $\nu > 0$ sufficiently small. By assumption (3), the vector field points outwards $[0, T] \times \hat{E}$ at every point of the set $[0, T] \times \hat{E} \cap \{z \in \mathbb{C} : \mathfrak{Re}(z) = \nu\}$. By assumptions (2)–(5), the vector field points outwards at every other point of $[0, T] \times \partial\hat{E}$. Now we apply the Brouwer fixed point theorem to the set $\hat{E}$. □

Example 3.4. By Remark 3.3, the equation

$$\dot{z} = -\frac{2}{|z|^5} + e^{i\frac{\pi}{8}\cos(t)} z^2 + |z|^{-4} z^3 - |z|^{-10} \overline{z}^6$$

has at least one 2π-periodic solution in the sector $\mathcal{S}\left(\frac{\pi}{7}\right)$. Here $M = 8$, $\tau_2 = \frac{\pi}{8} < \frac{\pi}{7}$, $\rho_6 = 0 \leq 0$, $J = \{2, 3\}$.

Example 3.5. By Remark 3.3, the equation

$$\dot{z} = -|z|^{-1} + e^{i\frac{\pi}{3}\sin(t)} z^2$$

has at least one 2π-periodic solution in each of the sectors $\mathcal{S}\left(\frac{\pi}{2}\right)$, $\widehat{\mathcal{S}}\left(\frac{\pi}{2}\right)$. Here $M = 3$, $\widehat{\tau}_0 = 0$, $\tau_2 = \frac{\pi}{3} < \frac{\pi}{2}$, $J = \{2\}$.

Example 3.6. By Remark 3.3, the equation

$$\dot{z} = -|z - 1|^{-1} + e^{i\frac{\pi}{3}\sin(t)} z^2$$

has at least one 2π-periodic solution in the sector $\widehat{\mathcal{S}}\left(\frac{\pi}{2}\right)$. Here $M = 3$, $\widehat{\tau}_0 = 0$, $\tau_2 = \frac{\pi}{3} < \frac{\pi}{2}$, $J = \{2\}$. The remark says nothing about periodic solutions inside the sector $\mathcal{S}\left(\frac{\pi}{2}\right)$, because the vector field is not defined in the whole set $\mathbb{R} \times \mathcal{S}\left(\frac{\pi}{2}\right)$.

Example 3.7. By Remark 3.3, the equation

$$\dot{z} = -|z|^{-3} - \cos(t)|z|^3 z + e^{i\frac{\pi}{10}\sin(t)} z^2 + |z|^5 z^4 - \overline{z}^8$$

has at least one 2π-periodic solution in every of the sectors $\mathcal{S}\left(\frac{\pi}{9}\right)$, $\widehat{\mathcal{S}}\left(\frac{\pi}{9}\right)$. Here $M = 10$, $a_0 \equiv -1$, $\widehat{\tau}_0 = \tau_4 = 0$, $\tau_2 = \frac{\pi}{10} < \frac{\pi}{9}$, $\rho_8 = 0 \leq 0$, $J = \{4\}$.

Example 3.8. By Remark 3.3, the equation

$$\dot{z} = -|z + i\sin(t)|^3 + \left|z + e^{it}\right|^{\frac{1}{2}} z^5 - e^{i\frac{\pi}{4}\cos(t)} \overline{z}^2$$

has in the sector $\mathcal{S}\left(\frac{\pi}{4}\right)$ at least one 2π-periodic solution and infinitely many f.b. solutions. Here $M = 5$, $a_0 \equiv -1$, $\widehat{\tau}_0 = \tau_5 = 0$, $\rho_2 = \frac{\pi}{4} \leq \frac{\pi}{4}$, $J = \{5\}$.

Remark 3.9. In Theorem 3.1 and Remark 3.3, all monomials $a_j(t)z^j$, $c_k(t)\overline{z}^k$ coincide with respect to the sector in which the symmetry axis is the positive part of the real axis. If there is coincidence with respect to other sector one can use the change of variables

$$w = e^{i\mu} z. \tag{3.14}$$

Example 3.10. The equation

$$\dot{z} = -3i + e^{it} + iz^3 + z^4 - \overline{z}^2$$

does not fulfil the assumptions of Theorem 3.1, because $\tau_3 = \frac{\pi}{2}$ and it should be $\tau_3 \leq \frac{\pi}{3}$ for $M = 4$ and $\tau_3 < \frac{2}{M-1}\pi \leq \frac{\pi}{2}$ for $M \geq 5$. But, by the change of variables $w = e^{-\frac{2\pi i}{3}} z$, we get the equation

$$\dot{w} = (-3i + e^{it})e^{-\frac{2\pi i}{3}} + e^{-\frac{\pi i}{6}} w^3 + w^4 - \overline{w}^2.$$

Here $\tau_3 = \frac{\pi}{6} < \frac{\pi}{3}$ and $\widehat{\tau}_0 < \frac{\pi}{3}$ for $M = 4$. Then the main equation has in the sector $\mathcal{S}\big(\frac{\pi}{3}, \pi\big)$ at least one 2π-periodic solution and infinitely many f.b. ones.

3.2. TRIVIAL FREE TERM

In this section we assume that $a_0 \equiv 0$. The linear term $a_1(t)z$ may be the dominating one in the neighbourhood of the origin. If it satisfies the inequality

$$\int_0^T a_1(t)dt < 0, \tag{3.15}$$

then it is possible to adapt the method from the proof of Theorem 3.1. In this case $a_1 z$ plays similar role to a_0.

It is also possible that $c_1(t)\overline{z}$ is dominating in the neighbourhood of the origin, which is in some sense a dual situation to the one considered in [9].

We state the main theorem of the section.

Theorem 3.11. *Let $n \geq 2$, $r \geq 1$, $a_1 \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ and $a_j, c_k \in \mathcal{C}(\mathbb{R}, \mathbb{C})$ for $j \in \{2, 3, \ldots, n\}$, $k \in \{1, 2, \ldots, r\}$ be T-periodic. If the conditions* (3.3), (3.4) *hold and there exists M satisfying* (3.5), (3.7), (3.8) *and at least one of the following conditions* (3.15) *or*

$$a_1 \equiv 0 \text{ and } |c_1| > 0 \tag{3.16}$$

is satisfied, then the equation

$$\dot{z} = v(t, z) = \sum_{j=1}^{n} a_j(t) z^j - \sum_{k=1}^{r} c_k(t) \overline{z}^k \tag{3.17}$$

has in the sector $\mathcal{S}\big(\frac{\pi}{M-1}\big)$ at least one T-periodic solution and infinitely many f.b. solutions. Moreover, the equation

$$\dot{z} = v_1(t, z) = \sum_{j=1}^{n} (-1)^j a_j(t) z^j - \sum_{k=1}^{r} (-1)^k c_k(t) \overline{z}^k \tag{3.18}$$

has in the sector $\widehat{\mathcal{S}}\big(\frac{\pi}{M-1}\big)$ at least one T-periodic solution and infinitely many b.b. solutions.

Proof. Let E and ε be like in the proof of Theorem 3.1. Our goal is to modify the set E and apply the Brouwer fixed point theorem.

Write $m = \frac{M-1}{2}$. We make the change of varibles given by $w = h(z) = z^m$ and get

$$\dot{w} = u(t,w) = ma_1(t)w - mc_1(t)|w|^{\frac{2}{m}} w^{\frac{m-2}{m}} + f(t,w,\overline{w}), \tag{3.19}$$

where f is of order at least $\frac{m+1}{m}$ in w and $\overline{w}$, so may be omitted when dealing with u in the neighbourhood of the origin.

Write $H = h(E) \subset \mathcal{S}\left(\frac{\pi}{2} - m\varepsilon\right)$. The vector field $(1,u)$ points outwards at every point of the set $[0,T] \times (\partial H \setminus \{0\})$. For $\mu > 0$ small enough we write

$$\begin{aligned} H =& H(\mu) = H \cap \{w \in \mathbb{C} : \mathfrak{Re}[w] \geq \mu\}, \\ \hat{H} =& H \cap \{w \in \mathbb{C} : \mathfrak{Re}[w] = \mu\}. \end{aligned}$$

Similarly to the proof of Theorem 3.1, we can show that the vector field $(1,u)$ points outwards at every point of the set $[0,T] \times \left(\partial H \setminus \hat{H}\right)$. To deal with $[0,T] \times \hat{H}$, we fix $w \in \hat{H}$ and estimate

$$\begin{aligned} \left|\operatorname{Arg}\left[c_1(t)w^{\frac{m-2}{m}}\right]\right| <& \frac{\pi}{m} + \left(\frac{\pi}{2} - m\varepsilon\right)\frac{m-2}{m} = \\ =& \frac{\pi}{2} - (m-2)\varepsilon \end{aligned}$$

when $1 \leq m-1$ and

$$\begin{aligned} \left|\operatorname{Arg}\left[c_1(t)w^{\frac{m-2}{m}}\right]\right| \leq& \frac{m-1}{m}\pi + \left(\frac{\pi}{2} - m\varepsilon\right)\frac{2-m}{m} = \\ =& \frac{\pi}{2} - (2-m)\varepsilon \end{aligned}$$

when $m-1 < 1$, so

$$\begin{aligned} \mathfrak{Re}(\dot{w}) =& ma_1(t)\mathfrak{Re}(w) - m|w|^{\frac{2}{m}}\mathfrak{Re}\left[c_1(t)w^{\frac{m-2}{m}}\right] < \\ <& \begin{cases} ma_1(t)\mathfrak{Re}(w), & \text{when } c_1 \equiv 0, \\ 0, & \text{when } a_1 \equiv 0. \end{cases} \end{aligned}$$

Finally, $\psi^{-1}_{(0,T)}(H) \subset H$, where ψ is the local process generated by (3.19). Now, by the Brouwer fixed point theorem we get the existence of T-periodic solution of (3.17) inside $\mathcal{S}\left(\frac{\pi}{M-1}\right)$. □

Example 3.12. By Theorem 3.11, the equation

$$\dot{z} = -z + z^2 + e^{i\frac{\pi}{3}\sin(t)}z^4 - \overline{z}$$

has in the sector $\mathcal{S}\left(\frac{\pi}{6}\right)$ at least one 2π-periodic solution and infinitely many f.b. ones. Here $M = 7$ and $\tau_4 = \frac{\pi}{3} < \frac{\pi}{2}$.

Remark 3.13. It is possible to formulate the analogue of Theorem 3.11 for the equation

$$\dot{z} = \sum_{j=p+1}^{n} a_j(t)z^j - \sum_{k=p}^{r} c_k(t)\overline{z}^k, \tag{3.20}$$

provided that (3.3), (3.4), (3.5), (3.7), (3.8), $1 \le p \le r$ and $|c_p| > 0$ hold.

Proof. We follow the proof of Theorem 3.11 and point only differences out.

The equation (3.20) has now the form

$$\dot{w} = u(t,w) = m \sum_{j=p+1}^{n} a_j(t) w^{\frac{m-1+j}{m}} - m \sum_{k=p}^{r} c_k(t) |w|^{\frac{2k}{m}} \overline{w}^{\frac{m-1-k}{m}}.$$

The dominating term in the neighbourhood of the origin is $mc_p(t)|w|^{\frac{2p}{m}}\overline{w}^{\frac{m-1-p}{m}}$. It is enough to show that it points outwards at every point of $\hat{H}$.

We fix $w \in \hat{H}$ and estimate

$$\begin{aligned}\left|\operatorname{Arg}\left[c_p(t) w^{\frac{m-1-p}{m}}\right]\right| <& \frac{\pi}{m}\frac{p+1}{2} + \left(\frac{\pi}{2} - m\varepsilon\right)\frac{m-1-p}{2} = \\ =& \frac{\pi}{2} - (m-1-p)\varepsilon\end{aligned}$$

when $1 \le p \le m-1$ and

$$\begin{aligned}\left|\operatorname{Arg}\left[c_p(t) w^{\frac{m-1-p}{m}}\right]\right| \le& \frac{2m-p-1}{2m}\pi + \left(\frac{\pi}{2} - m\varepsilon\right)\left|\frac{m-1-p}{2}\right| = \\ =& \frac{\pi}{2} - (p-m+1)\varepsilon\end{aligned}$$

when $m-1 < p \le r$. Finally, $\left|\operatorname{Arg}\left[c_p(t) w^{\frac{m-1-p}{m}}\right]\right| < \frac{\pi}{2}$. □

Example 3.14. By Remark 3.13, the equation

$$\dot{z} = e^{i\frac{\pi}{3}\sin(t)} z^3 - \overline{z}^2$$

has in the sector $\mathcal{S}\left(\frac{\pi}{4}\right)$ at least one 2π-periodic solution and infinitely many f.b. ones. Here $M = 5$ and $\tau_3 = \frac{\pi}{3} < \frac{\pi}{2}$.

Example 3.15. By Remark 3.13, the equation

$$\dot{z} = z^5 + e^{i\frac{\pi}{4}\sin(t)} z^7 - \overline{z}^3$$

has in each of the sectors $\mathcal{S}\left(\frac{\pi}{8}\right)$, $\widehat{\mathcal{S}}\left(\frac{\pi}{8}\right)$ at least one 2π-periodic solution. Here $M = 9$ and $\tau_7 = \frac{\pi}{4}$.

Remark 3.16. Similarly to Remark 3.3, one can state version of Theorem 3.11 and Remark 3.13 for the equation

$$\dot{z} = \sum_{j=1}^{n} a_j(t)|z - b_j(t)|^{l_j} z^j - \sum_{k=1}^{r} c_k(t)|z - d_k(t)|^{s_k}\overline{z}^k.$$

It is possible when the dominating term in the neighbourhood of the origin is of the form $a_1(t)|z-b_1(t)|^{l_1}z$ (with (3.15)) or $c_k(t)|z-d_k(t)|^{s_k}\overline{z}^k$.

Example 3.17. By Remark 3.16, each of the equations

$$\begin{aligned}
\dot{z} &= -|z|z + z^3 + e^{i\frac{\pi}{4}\sin(t)}z^6 - \overline{z}^2,\\
\dot{z} &= |z|^2z^3 + e^{i\frac{\pi}{4}\sin(t)}z^6 - |z|\overline{z}^2,\\
\dot{z} &= -\frac{z}{|z|^2} + z^3 + e^{i\frac{\pi}{4}\sin(t)}z^6 - \overline{z}^2,\\
\dot{z} &= -\left|z+e^{it}\right|z + z^3 + e^{i\frac{\pi}{4}\sin(t)}z^6 - \overline{z}^2,\\
\dot{z} &= z^3 + e^{i\frac{\pi}{4}\sin(t)}z^6 - \frac{\overline{z}^2}{|z+2+e^{it}|},
\end{aligned}$$

has in the sector $\mathcal{S}\left(\frac{\pi}{19}\right)$ at least one 2π-periodic solution and infinitely many f.b. ones. Here $M=20$ and $\tau_6 = \frac{\pi}{4} < \frac{5}{19}\pi$.

4. UNIQUENESS OF PERIODIC SOLUTION IN SOME SECTORS

It is sometimes possible to find periodic solutions of equation (1.2) which are unique in sectors of the type $\mathcal{S}\left(\frac{\pi}{\alpha}\right)$ or $\widehat{\mathcal{S}}\left(\frac{\pi}{\alpha}\right)$ for some $\alpha>0$ (see [23]). It is due to the fact that they are asymptotically stable or asymptotically unstable (and were detected by the Denjoy–Wolff fixed point theorem).

In the case of equation (1.1) (and (1.3)) we can apply only the Brouwer fixed theorem which says nothing about the uniqueness of periodic solutions. In fact, equations which can be investigated by theorems from Section 3 may have more then one periodic solutions in the considered sectors as shown in the following example.

Example 4.1. By Theorem 3.1, the equation

$$\dot{z} = v(t,z) = -6 + re^{it} + 5z + 5z^2 + z^4 - 5\overline{z}^3$$

has in the sector $\mathcal{S}\left(\frac{\pi}{4}\right)$ at least one 2π-periodic solution, provided $r\geq 0$ is small enough (here $M=5$).

Taking $r=0$ and $z\in\mathbb{R}$, we get $v(t,z)=(z+1)(z-1)(z-2)(z-3)$, so for small $r>0$ we have at least three periodic solutions inside $\mathcal{S}\left(\frac{\pi}{4}\right)$ (and this effect cannot be removed by taking a thinner sector i.e. sectors of the form $\mathcal{S}\left(\frac{\pi}{\alpha}\right)$ for $\alpha>4$).

This example shows, that adding terms to (1.2) of the type $c_k(t)\overline{z}^k$ may destroy uniqueness. On the other hand, equations similar to (1.2) should not lose this property. It occurs that it holds for the equation

$$\dot{z} = v(t,z) = \sum_{j=0}^{n} a_j(t)|z-b_j(t)|^{l_j}z^j. \tag{4.1}$$

Unfortunately, we are able to obtain this only in sectors which are narrower than in the case of (1.2).

Now we state the main theorem of the present section. It is in some sense an improvement of Theorem 3.1.

Theorem 4.2. *Let $n \geq 2$, $a_1 \in \mathcal{C}(\mathbb{R}, [0, \infty))$ and $a_j \in \mathcal{C}(\mathbb{R}, \mathbb{C})$ for every $j \in \{0, 2, 3, \ldots, n\}$ be T-periodic. If*

$$l_0 = 0, \tag{4.2}$$

$$l_j + j > 1 \text{ and } a_j \not\equiv 0 \text{ for some } j \geq 1 \tag{4.3}$$

and there exists number

$$M \geq 2n - 1 \tag{4.4}$$

such that (3.6),

$$\tau_j < \frac{j-1}{M-1}\pi \text{ for } 2 \leq j \leq n, \tag{4.5}$$

$$j \cos\left(\tau_j + \frac{j-1}{M-1}\pi\right) > |l_j| \text{ for } 1 \leq j \leq n, \tag{4.6}$$

$$b_j(\mathbb{R}) \subset \operatorname{cl}\widehat{\mathcal{S}}\left(\frac{M-3}{2(M-1)}\pi\right) \text{ for every } j \geq 1 \tag{4.7}$$

and at least one of the following conditions

$$j + l_j \geq 1 \text{ for every } j \geq 1, \tag{4.8}$$

$$|a_0| > 0 \tag{4.9}$$

hold, then the equation (4.1) *has in the sector $\mathcal{S}\left(\frac{\pi}{M-1}\right)$:*

- *exactly one T-periodic solution ξ; it is asymptotically unstable and repelling in the whole sector;*
- *infinitely many forward blowing up solutions.*

Moreover, after replacing $\widehat{\mathcal{S}}\left(\frac{M-3}{2(M-1)}\pi\right)$ by $\mathcal{S}\left(\frac{M-3}{2(M-1)}\pi\right)$ in (4.7)*, the equation*

$$\dot{z} = \sum_{j=0}^{n} (-1)^j a_j(t) |z - b_j(t)|^{l_j} z^j \tag{4.10}$$

has in the sector $\widehat{\mathcal{S}}(\frac{\pi}{M-1})$:

- *exactly one T-periodic solution χ; it is asymptotically stable and attracting in the whole sector;*
- *infinitely many backward blowing up solutions.*

Remark 4.3. We write in (4.7)

$$\widehat{\mathcal{S}}(0) = \{z \in \mathbb{C} : \mathfrak{Re}[z] \leq 0, \mathfrak{Im}[z] = 0\}$$

and when dealing with (4.10) we write

$$\mathcal{S}(0) = \{z \in \mathbb{C} : \mathfrak{Re}[z] \geq 0, \mathfrak{Im}[z] = 0\}.$$

Proof of Theorem 4.2. By (4.6) and (4.7), the equation (4.1) in the sector $\mathcal{S}\left(\frac{\pi}{M-1}\right)$ is regular enough to generate a local process φ.

By (4.4) and (4.5) we get (3.7). Moreover, by (4.6) we get $|l_j| < j$ for all $j \geq 1$. Thus assumptions of Remark 3.3 are satisfied, so there exists at least one T-periodic solution inside $\hat{E} \subset \mathcal{S}\left(\frac{\pi}{M-1}\right)$, where $\hat{E}$ is given by (3.13). It is enough to show that it is the only one inside $\hat{E}$. To do that we use Lemma 2.1.

First of all, we show that $\hat{E}$ is a convex subset of $\mathbb{C}$. The real axis is its line of symmetry. By the construction of $\hat{E}$, it is enough to show that the part of its boundary given by $f^{-1}(C(\gamma)) \cap \{z \in \mathbb{C} : \mathfrak{Im}[z] > 0\}$ is a plot of a concave function $y : \mathbb{R} \supset (\mu, \nu) \ni x \mapsto y(x) \in \mathbb{R}$ (here we use notation from the proof of Theorem 3.1). We parameterise this part of boundary by

$$s(o) = x(o) + iy(o) = (\gamma + io)^{-\frac{1}{m}}, \text{ where } o \in \left(-\frac{\pi}{M-1} + \varepsilon, 0\right).$$

Thus

$$\frac{d^2y}{dx^2} = \frac{\mathfrak{Im}\left[s''\overline{s}'\right]}{\mathfrak{Re}^3\left[s'\right]} = \frac{(m+1)\left(\gamma^2 + o^2\right)^{\frac{1-m}{2m}}}{\sin^3\left[\frac{m+1}{m}\arctan\frac{o}{\gamma}\right]} < 0$$

and

$$\frac{dx}{do} = \frac{\sin\left[\frac{m+1}{m}\arctan\frac{o}{\gamma}\right]}{m\left(\gamma^2 + o^2\right)^{\frac{1+m}{2m}}} > 0,$$

which proves the convexity of $\hat{E}$.

Now we show that every solution is locally repelling as long as it stays in $\hat{E}$, i.e. for every solution η such that $\eta(t_0 + \tau) \in \hat{E}$ for every $\tau \in [0, \omega_{\hat{E}}]$, where $0 \leq \omega_{\hat{E}} \leq T$, there exists $\alpha > 0$ such that for every solution $\zeta \neq \eta$ satisfying $|\zeta(t) - \eta(t)| < \alpha$ for every $t \in [t_0, t_0 + \omega_{\hat{E}}]$ one gets

$$\frac{d}{dt}|\zeta(t) - \eta(t)| > 0 \text{ for every } t \in [t_0, t_0 + \omega_{\hat{E}}].$$

Since $\frac{d}{dt}|\zeta - \eta|^2 = 2\langle\zeta - \eta, \zeta' - \eta'\rangle$ and $\hat{E}$ is compact, it is enough to show that

$$\langle\zeta - \eta, \zeta' - \eta'\rangle > 0 \tag{4.11}$$

holds. To do this, we estimate the contribution to the inner product of every term of the vector field v. We fix η and write $\theta = \zeta - \eta$ and $A_j = a_j(t)|\zeta - b_j(t)|^{l_j}\zeta^j - a_j(t)|\eta - b_j(t)|^{l_j}\eta^j$.

By (4.2), we get $\langle \zeta-\eta, A_0\rangle = 0$. Now let $j \geq 1$. By (4.7), $\left|\frac{\theta}{\eta}\right|$ and $\left|\frac{\theta}{\eta-b_j}\right|$ may be arbitrarily small provided that η is small enough. Thus

$$\begin{aligned}
\langle \zeta-\eta, A_j\rangle = &\mathfrak{Re}\left[a_j(|\zeta-b_j|^{l_j}\zeta^j - |\eta-b_j|^{l_j}\eta^j)\overline{\theta}\right] = \\
&= \mathfrak{Re}\left[a_j\left(|\eta-b_j|^{l_j}\left|1+\frac{\theta}{\eta-b_j}\right|^{l_j}\zeta^j - |\eta-b_j|^{l_j}\eta^j\right)\overline{\theta}\right] = \\
&= \mathfrak{Re}\left[a_j|\eta-b_j|^{l_j}\left(\left|1+\frac{\theta}{\eta-b_j}\right|^{l_j}-1\right)\zeta^j\overline{\theta}\right] + \mathfrak{Re}\left[a_j|\eta-b_j|^{l_j}(\zeta^j-\eta^j)\overline{\theta}\right] = \\
&= \mathfrak{Re}\left[a_j|\eta-b_j|^{l_j}\left(l_j\mathfrak{Re}\left[\frac{\theta}{\eta-b_j}\right] + r_{l_j}\left(\frac{\theta}{\eta-b_j}\right)\right)\eta^j\left(1+\frac{\theta}{\eta}\right)^j\overline{\theta}\right] + \\
&\quad + \mathfrak{Re}\left[a_j|\eta-b_j|^{l_j}|\theta|^2\eta^{j-1}\sum_{k=0}^{j-1}\left(1+\frac{\theta}{\eta}\right)^k\right] = \\
&= B_j + C_j,
\end{aligned}$$

where

$$|1+z|^p = p\mathfrak{Re}[z] + r_p(z)$$

and $|r_p(z)| \leq |z|^2 D_p(z)$ for some continuous function $D_p : \mathbb{C} \longrightarrow \mathbb{R}$. Since $|\operatorname{Arg}(\eta)| < \frac{\pi}{M-1}$, we get

$$C_j \geq \cos\left[\tau_j + (j-1)\frac{\pi}{M-1} + F_j\left(\frac{\theta}{\eta}\right)\right] |a_j||\eta-b_j|^{l_j}|\theta|^2|\eta|^{j-1}\left(j + G_j\left(\frac{\theta}{\eta}\right)\right),$$

where $F_j(z) = \operatorname{Arg}\left[\sum_{k=0}^{j-1}(1+z)^k\right]$ and $G_j(z) = \left|\sum_{k=0}^{j-1}(1+z)^k\right| - j$. On the other hand,

$$|B_j| \leq |a_j||\eta-b_j|^{l_j}|\theta|^2|\eta|^{j-1}\left|1+\frac{\theta}{\eta}\right|^j |l_j|\left|\frac{\eta}{\eta-b_j}\right|\left[1+\frac{1}{|l_j|}\left|\frac{\theta}{\eta-b_j}\right| D_{l_j}\left(\frac{\theta}{\eta-b_j}\right)\right]$$

is satisfied. Thus

$$\begin{aligned}
&C_j - |B_j| \geq \\
&\geq |a_j||\eta-b_j|^{l_j}|\theta|^2|\eta|^{j-1}\Bigg\{\cos\left[\tau_j + (j-1)\frac{\pi}{M-1} + F_j\left(\frac{\theta}{\eta}\right)\right]\left(j+G_j\left(\frac{\theta}{\eta}\right)\right) - \\
&\quad - \left|1+\frac{\theta}{\eta}\right|^j |l_j|\left|\frac{\eta}{\eta-b_j}\right|\left[1+\frac{1}{|l_j|}\left|\frac{\theta}{\eta-b_j}\right| D_{l_j}\left(\frac{\theta}{\eta-b_j}\right)\right]\Bigg\}
\end{aligned}$$

holds and $C_j - |B_j| > 0$ for θ small enough provided that

$$\cos\left[\tau_j + (j-1)\frac{\pi}{M-1}\right] j > |l_j|\left|\frac{\eta}{\eta-b_j}\right|.$$

By (4.7), we get $\left|\frac{\eta}{\eta-b_j}\right| \leq 1$, so, by (4.6), $\langle \zeta-\eta, A_j\rangle > 0$ holds. Finally, (4.11) holds, which finishes the proof. □

The following examples are the straightforward applications of Theorem 4.2.

Example 4.4. The equation

$$\dot{z} = -1 + \cos(t) + \sqrt{|z|}z + [2 + i\sin(t)]z^2 + z^3$$

has in the sector $\mathcal{S}\left(\frac{\pi}{4}\right)$ exactly one 2π-periodic solution. It is asymptotically unstable and repelling in the whole sector. Here $M = 5$, $\tau_2 < \frac{\pi}{4}$ and the condition (4.8) is satisfied.

Example 4.5. The equation

$$\dot{z} = -1 + \frac{z}{\sqrt{|z|}} + e^{i\frac{\pi}{8}\sin(t)}|z|z^2 + \frac{z^4}{\sqrt{|z|}}$$

has in each of the sectors $\mathcal{S}\left(\frac{\pi}{7}\right)$, $\widehat{\mathcal{S}}\left(\frac{\pi}{7}\right)$ exactly one 2π-periodic solution, infinitely many solutions which are heteroclinic to them and infinitely many blowing up solutions. Here $M = 8$, $\tau_2 = \frac{\pi}{8} < \frac{\pi}{7}$, $\tau_4 = 0$, $4\cos\left(\frac{3\pi}{7}\right) > \frac{1}{2}$ and the condition (4.9) is satisfied.

The following facts show that the generalisation of Theorem 4.2 is rather hard.

Remark 4.6. It is impossible to use sector $\mathcal{S}\left(\frac{\pi}{M-1}\right)$ instead of $\mathcal{S}\left(\frac{M-3}{2(M-1)}\pi\right)$ in (4.7) as can be seen in the following example.

Example 4.7. The equation

$$\dot{z} = -900 + re^{it} + |z - 4 - i|^4\, z^5$$

has at least three 2π-periodic solutions inside $\mathcal{S}\left(\frac{\pi}{20}\right)$. Indeed, for $r = 0$ and $z \in \mathbb{R}$ we get three stationery solutions at $0 < z_1 < \frac{28-\sqrt{19}}{9} < z_2 < \frac{28+\sqrt{19}}{9} < z_3$. For $r > 0$ small enough this solutions continue to 2π-periodic ones.

Remark 4.8. It is impossible to allow $|l_j| > j$ instead of using condition (4.6) as can be seen in the following example.

Example 4.9. First of all, let us notice that for every $\beta > 0$ there exist $\gamma > 0$ and $\delta > 0$ such that the equation

$$1 = \gamma\left(\frac{z}{|z+1|^{1+\beta}} + \delta z^2\right)$$

has three positive solutions z_1, z_2 and z_3. Fixing such β, γ, δ one gets that the equation

$$\dot{z} = -1 + re^{it} + \gamma\left(\frac{z}{|z+1|^{1+\beta}} + \delta z^2\right)$$

has at least three 2π-periodic solutions inside $\mathcal{S}\left(\frac{\pi}{2}\right)$ provided that $r > 0$ is small enough.

Remark 4.10. It is impossible to prove an improvement of Theorem 3.11 as can be seen in the following example.

Example 4.11. The equation

$$\dot{z} = -z\sqrt{|z|} + 3\frac{z^2}{|z+2|} + \left(0.001 + re^{it}\right) z^3$$

has at least three 2π-periodic solutions inside $\mathcal{S}\left(\frac{\pi}{4}\right)$. Indeed, for $r = 0$ and $z \in \mathbb{R}$ we get three stationary solutions at $0 < z_1 < 1$ and $4 < z_2 < 76 < z_3$. For $r > 0$ small enough this solutions continue to 2π-periodic ones.

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