

Compressibility of dense nuclear matter in the ρ -meson variant of the Skyrme model

Miguel Huidobro ^a, Paul Leask ^{b,c}, Carlos Naya ^{d,e} and Andrzej Wereszczyński ^{e,f,g}

^a*Departamento de Física de Partículas, Universidad de Santiago de Compostela and Instituto Galego de Física de Altas Enerxías (IGFAE), Santiago de Compostela, E-15782, Spain*

^b*School of Mathematics, University of Leeds, Leeds, LS2 9JT, England, U.K.*

^c*Department of Physics, KTH Royal Institute of Technology, 10691 Stockholm, Sweden*

^d*Universidad de Alcalá, Departamento de Física y Matemáticas, 28805 Alcalá de Henares, Spain*

^e*Institute of Theoretical Physics, Jagiellonian University, Lojasiewiczza 11, 30-348 Kraków, Poland*

^f*Department of Applied Mathematics, University of Salamanca, Casas del Parque 2 and Institute of Fundamental Physics and Mathematics, University of Salamanca, Plaza de la Merced 1, 37008 — Salamanca, Spain*

^g*International Institute for Sustainability with Knotted Chiral Meta Matter (WPI-SKCM2), Hiroshima University, Higashi-Hiroshima, Hiroshima 739-8526, Japan*

E-mail: miguel.huidobro.garcia@usc.es, palea@kth.se, carlos.naya@uah.es, andrzej.wereszczynski@uj.edu.pl

ABSTRACT: We show that coupling the $SU(2)$ -valued Skyrme field to the ρ -meson solves the long-standing issue of (in)compressibility in the solitonic Skyrme model. Even by including only one $\rho\pi$ interaction term, motivated by a holographic-like reduction of Yang-Mills action by Sutcliffe, reduces the compression modulus from $K_0 \simeq 1080$ MeV, in the massive Skyrme model, to $K_0 \simeq 351$ MeV.

KEYWORDS: Solitons Monopoles and Instantons, Effective Field Theories, Sigma Models

ARXIV EPRINT: [2405.20757](https://arxiv.org/abs/2405.20757)

Contents

1	Introduction	1
2	Compression modulus problem	2
3	The ρ-Skyrme model	4
3.1	The $B = 1$ skyrmion: hedgehog ansatz	9
3.2	Multi-skyrmions with ρ -mesons	10
4	Nuclear matter in the ρ-Skyrme model	12
4.1	Varying the metric	13
4.2	Determining the energy minimizing period lattice	16
4.3	Crystalline configurations	16
5	Resolving the compression modulus problem	17
6	Concluding remarks	19
A	Compressibility from Derrick scaling of the cubic lattice of half-skyrmions	20

1 Introduction

Although the fundamental theory of strong interactions is known, a derivation of properties of baryons, atomic nuclei and (infinite) nuclear matter is still one of the main challenges of the contemporary theoretical physics. This is due to the well known fact that quantum chromodynamics (QCD) is a strongly interacting quantum theory in the low energy limit, where fundamental degrees of freedom, i.e., quarks and gluons, are confined into non-perturbative objects. Consequently, the power-full perturbative expansion does not apply in this regime and we are left with lattice QCD or with the effective field theory (EFT) approach.

The solitonic Skyrme model [1] (for a recent review see, e.g., [2]) is a very attractive example of an EFT which is not only well anchored in the underlying fundamental theory but, if contrasted with other EFT, has an extremely limited number of free parameters and therefore potentially possesses a huge predictive power. Skyrme's original theory consists of an effective mesonic Lagrangian involving only the lightest Goldstone bosons of QCD, the pions, with baryons emerging as stable topological solitons. The theory has $N_f = 2$ flavours of quarks (u,d) and the pion fields are encoded in the $SU(N_f = 2)$ -valued Skyrme field.

The predicative power of the model has recently been proven in many cases, elevating the Skyrme model from a qualitative framework to a theory which allows for very accurate computations. The main steps are the *vibrational quantization* [3], or in generality inclusion of vibrational, massive deformations of Skyrmions [4], and inclusion of additional terms, as e.g., the *sextic term* [5, 6] or additional degrees of freedom, i.e., *vector mesons* [7].

All that led to a very spectacular progress in the application of the Skyrme model to the description of atomic nuclei. Let us only mention the computation of vibrational-rotational bands of light nuclei, as ${}^7\text{Li}$ [3], ${}^{12}\text{C}$ [8, 9] and ${}^{16}\text{O}$ [10, 11], observation of α -particle clustering in atomic nuclei, computation of nuclear forces [12] and resolution of the too large binding energies problem (see also [6, 7, 13–15]). There has also been a significant progress in description of nuclear matter and neutron stars with very good agreement with astrophysical observables, such as the mass-radius curve [16–19]. In particular, taking advantage of the new approach to determining phases of nuclear matter within the Skyrme model [20], an equation of state covering the nuclear matter at all densities has been obtained [21].

However, these unquestionable successes all suffer from the last great issue of the Skyrme model which is the so-called compression modulus problem. This was recently remedied by Harland et al. [22] by investigating crystalline configurations in the Adkins-Nappi ω -meson variant of the Skyrme model [23]. Therein they combined the method for determining crystals in the massive Skyrme model [20] with the method developed to obtain energy minimizers in the ω -Skyrme model [15]. In this letter we follow a similar approach where we study light nuclei and crystalline configurations in the massive Skyrme model coupled to the vector ρ -meson. Using the ground state crystal in this theory, we are able to address the compression modulus problem.

2 Compression modulus problem

The (symmetric) energy $E(n_B)$ of isospin symmetric nuclear matter can be approximated about the nuclear saturation density n_0 by use of a power series expansion in the baryon density n_B , that is [24]

$$E(n_B) = E_0 + \frac{1}{2}K_0 \frac{(n_B - n_0)^2}{9n_0^2} + \mathcal{O}((n_B - n_0)^3), \quad (2.1)$$

The first multipole in this expansion is the energy per baryon E_0 of symmetric nuclear matter at the saturation density. There is no linear term since symmetric nuclear matter reaches a minimum of the energy at saturation. The next multipole, K_0 is the compression modulus. This is a fundamental quantity in nuclear physics which describes a resistance (stiffness) of nuclear matter under external force (pressure) at the saturation point.

In practice, the compression modulus is not directly measured, but it may be extracted from the Isoscalar Giant Monopole Resonance (ISGMR) frequency, ω_M [25, 26]. This resonance is a collective excitation of the nucleus, in which both protons and neutrons vibrate spherically in phase. It is measured through the low-momentum transfer in inelastic scattering collisions between isoscalar particles (like α particles or deuterons) and medium-heavy nuclei, $B \geq 90$. The frequency of this resonance for a given nucleus may be related to its compression modulus K ,

$$\omega_M^2 = \frac{K}{m_N R^2}, \quad (2.2)$$

where R is the radius of the nucleus and m_N is the mass of the nucleon. The value of this frequency was experimentally determined for different nuclei, such that the following

dependence on the baryon number was found, $\hbar\omega_M \sim 80B^{-1/3}$ MeV [27]. Then, from the well-known relation between the radius of a nucleus and its baryon number, $R \approx R_0B^{1/3}$ fm, where typically $R_0 = 1.18 - 1.25$ [28, 29], we find the B -independent value for the compression modulus [28],

$$K = K_0 \sim 215 - 240 \text{ MeV}. \quad (2.3)$$

Further numerical simulations from theoretical approaches yield a range of values for K which confirmed this value [28].

To compute the compression modulus in the Skyrme model we use an equivalent definition from the Taylor expansion

$$K_0 = 9n_0^2 \left. \frac{\partial^2 E}{\partial n_B^2} \right|_{n_0} = 9V^2 \left. \frac{\partial^2 E}{\partial V^2} \right|_{V_0} \quad (2.4)$$

applied to Skyrme crystal i.e., a periodic classical solution of the Skyrme model representing infinite symmetric nuclear matter. Here $V_0 = B/n_0$ is the volume of the unit crystal cell at saturation, i.e. the volume at which the energy is minimal. If the ground state crystalline configuration has cubic symmetry, then there is a corresponding energy minimizing length L_0 associated to $V_0 = L_0^3$.

It is a matter of fact that the compression modulus in the Skyrme theory is a few times larger than the experimental value. For different versions of the model and different crystal solutions its value ranges from ~ 900 MeV and ~ 1350 MeV for the massless and massive Skyrme [30] to even higher values if the sextic term is added (e.g., $K_0 \sim 1800$ MeV for $\lambda^2 = 3 \text{ MeV fm}^3$ and $K_0 \sim 2300$ MeV for $\lambda^2 = 7 \text{ MeV fm}^3$ [30]). This gets an improvement if one uses the recently found multi-wall solution which is the global crystal minimum provided the unit cell has $B_{\text{cell}} = 4$ baryon charge [21]. However, the value is still more than three times above the accepted value [21].

The appearance of a too large value of the compression modulus can be easily understood in the minimal massless \mathcal{L}_{24} -Skyrme model, where only the kinetic \mathcal{L}_2 and Skyrme term \mathcal{L}_4 are considered. Applying the scaling (Derrick) transformation $L \rightarrow \lambda L$, which for this version of the Skyrme model is an exact transformation relating solutions at different densities (volume), we can prove that $K_0/E_0 = 1$, see appendix A. For the full Skyrme model such a transformation leads to an approximate relation between the energies of solutions at different densities. However, for not too large M_π and λ (which are the coupling constants of the potential \mathcal{L}_0 and the sextic term \mathcal{L}_6), the approximation is quite accurate, especially at the saturation point. Thus, for these versions of the model the compression modulus must be even higher than that in the minimal Skyrme model.

When λ grows significantly the scaling transformation becomes useless. This is because of the fact that in the BPS Skyrme model, i.e., where only the potential and the sextic term are kept

$$\mathcal{L}_{\text{BPS}} = \mathcal{L}_6 + \mathcal{L}_0 \quad (2.5)$$

the Skyrme field is a perfect fluid matter [31]. Importantly, applying external pressure (squeezing skyrmionic matter) is not equivalent to the uniform rescaling of the solution.

On the contrary, regions with smaller densities are more squeezed than regions with higher density. As a result, the compression modulus is zero for any potential which provides a nonzero mass of pions [31]. Therefore, a near-BPS model, where the usual quadratic and quartic terms are added, may be a solution for the compression modulus problem. However, the values of the parameters to reach the near-BPS regime must be extremely large, rendering the numerical computations difficult and the physical interpretation cumbersome. Anyway, this opens a possible path for resolution of the compression modulus problem.

In the subsequent part of the paper we will follow another line and show that inclusion of further degrees of freedom, here ρ -mesons, reduces the stiffness of the pionic Skyrme model. The reason for this originates in an observation that the Yang-Mills theory in (4+0) dimensions is equivalent to the 3-dimensional massless Skyrme model with an infinite tower of vector mesons [7]. This identification is obviously kept on the level of solutions with self-dual instantons of charge N being “decomposed” into Skyrmions with baryon charge $B = N$ assisted by the vector mesons [7].

The first consequence of this equivalence is that skyrmions with infinitely many mesons have zero binding energy as the self-dual instantons saturate the pertinent linear topological bound on the energy. Following that small binding energies arise in a truncated model where only finite number of mesons are kept [32]. This provides a very natural path of resolution of the too large binding energy problem in the Skyrme model.

However, the instanton-skyrmion equivalence has another very important consequence. Since the Yang-Mills theory in four dimensions is conformally invariant, the energy of the self-dual instanton does not depend on the scale. Thus, it costs zero energy to change the size of the instantons. This finally results in zero compression modulus. Again, it is plausible to expect that a non-zero, but physically small, value of the compression modulus should be found if a truncated skyrmion-meson theory is considered. In fact, in this work, we provide the first evidence that this actually happens, although we start at the other end. Namely, we consider only one interaction term arising from the dimensional reduction of the instanton. Importantly, even this minimal extension of the pionic Skyrme model significantly reduces the compression modulus.

It is interesting to notice that, this ρ -pion interaction term effectively reduces the quartic Skyrme term. In the limit of maximal coupling it even kills the Skyrme term completely and the soliton does not exist any longer.

Finally, we remark that the ω -Skyrme model considered in [20] seems to be conceptually closer to a type Skyrme model with \mathcal{L}_6 included than to the ρ -meson extension. This is because of the fact that the ω -meson coupling effectively introduces the sextic term B_μ^2 term.

3 The ρ -Skyrme model

The general Skyrme model consists of a single scalar field $\varphi : \Sigma \rightarrow \text{SU}(2)$ where spacetime is given by the (3+1)-dimensional Lorentzian manifold $\Sigma = \mathbb{R} \times M$ with the pseudo-Riemannian product metric $\eta = -dt^2 + g$, and (M, g) is an oriented 3-dimensional Riemannian manifold with Riemannian metric g . We equip $(\text{SU}(2), h)$ with the canonical bi-invariant metric $h(X, Y) = \frac{1}{2} \text{Tr}(X^\dagger Y)$ and denote the Lie algebra of $\text{SU}(2)$ by $\mathfrak{su}(2)$. Let us introduce oriented local coordinates $x = (t, x^1, x^2, x^3)$ on the domain Σ and let $\{\partial_0, \partial_1, \partial_2, \partial_3\}$ be

a local basis for the tangent space $T_x\Sigma$ at $x \in \Sigma$, where we have denoted $\partial_\mu \equiv \partial/\partial x^\mu$. Let $\omega \in \Omega^1(\text{SU}(2)) \otimes \mathfrak{su}(2)$ be the left Maurer-Cartan form. Then, for any left invariant vector fields $X, Y \in T_{\varphi(x)}\text{SU}(2)$, where $x \in \Sigma$, we define $\Omega \in \Omega^2(\text{SU}(2)) \otimes \mathfrak{su}(2)$ to be an $\mathfrak{su}(2)$ -valued two-form on $\text{SU}(2)$ given by

$$\Omega(X, Y) = [\omega(X), \omega(Y)], \quad (3.1)$$

where $[\cdot, \cdot] : \mathfrak{su}(2) \times \mathfrak{su}(2) \rightarrow \mathfrak{su}(2)$ is the usual Lie bracket. The pullback of the left Maurer-Cartan form ω defines the $\mathfrak{su}(2)$ -valued *left current*

$$\varphi^*\omega =: L_\mu dx^\mu, \quad L_\mu = \omega_\varphi(\partial_\mu\varphi) = \varphi^\dagger \partial_\mu\varphi. \quad (3.2)$$

Let us write the pullback of the curvature as $\Omega_{\mu\nu} = \varphi^*\Omega(\partial_\mu, \partial_\nu)$. Then the curvature can be expressed in terms of the $\mathfrak{su}(2)$ -valued left current as

$$\Omega_{\mu\nu} = \Omega(d\varphi(\partial_\mu), d\varphi(\partial_\nu)) = [\omega_\varphi(\partial_\mu\varphi), \omega_\varphi(\partial_\nu\varphi)] = [L_\mu, L_\nu]. \quad (3.3)$$

For completeness, we define the natural three-form $\Xi \in \Omega^3(\text{SU}(2))$ by

$$\Xi(X, Y, Z) = \frac{1}{24\pi^2} h(\omega(X), \Omega(Y, Z)). \quad (3.4)$$

Note that this three-form Ξ coincides with the normalized volume form on $\text{SU}(2)$, where

$$\text{vol}_{\text{SU}(2)} = \frac{1}{24\pi^2} \text{Tr}(\varphi^{-1}d\varphi \wedge \varphi^{-1}d\varphi \wedge \varphi^{-1}d\varphi). \quad (3.5)$$

The baryon degrees of freedom are not explicitly visible in this model, they appear via topology. That is, there exists an associated homotopy invariant that can be identified with the physical baryon number, which is given by

$$B = \int_M \varphi^*\Xi \in \mathbb{Z}. \quad (3.6)$$

In general, we are interested in topological solitons in a $(3+1)$ -dimensional Minkowski spacetime $\Sigma = \mathbb{R} \times \mathbb{R}^3$ with constant coefficient metric η and metric signature $(-+++)$. Then, the Skyrme field is a single scalar field $\varphi : \mathbb{R} \times \mathbb{R}^3 \rightarrow \text{SU}(2)$, where the pion fields $\vec{\pi} = (\pi_1, \pi_2, \pi_3)$ are disguised into the Skyrme field φ via

$$\varphi = \begin{pmatrix} \sigma + i\pi_3 & i\pi_1 + \pi_2 \\ i\pi_1 - \pi_2 & \sigma - i\pi_3 \end{pmatrix} \in \text{SU}(2), \quad (3.7)$$

with the σ field imposing the constraint $\sigma + \pi_i\pi_i = 1$. We will consider the ρ -meson in its standard version, i.e. being a massive non-Abelian field defined by $R = R_\mu dx^\mu \in \Omega^1(\Sigma) \otimes \mathfrak{su}(2)$ where $R_\mu = i\rho_\mu^a \tau^a$ with $\rho_\mu^a \in \mathbb{R}$ and τ^a are the Pauli spin matrices. In the literature there is another *inequivalent* way to introduce the ρ -meson within the pionic effective Lagrangian. Namely, the Rho meson field is treated as the 2×2 four-vector $R_\mu = \rho_\mu^0 + i\tau^a \rho_\mu^a$. Such a construction was considered by Adkins [33], wherein he introduced the following Lagrangian

$$\begin{aligned} \mathcal{L}_{\text{Adkins}} = & \frac{F_\pi^2}{16\hbar} \text{Tr}(\partial_\mu\varphi^\dagger \partial^\mu\varphi) + \frac{\hbar}{32e^2} \text{Tr}\left([\partial_\mu\varphi\varphi^\dagger, \partial_\nu\varphi\varphi^\dagger]^2\right) + \frac{1}{8\hbar^3} F_\pi^2 m_\pi^2 (\text{Tr}(\varphi) - 2) \\ & - \frac{1}{8\hbar} \text{Tr}(R_{\mu\nu}^\dagger R^{\mu\nu}) + \frac{1}{4\hbar^3} m_\rho^2 \text{Tr}(R_\mu^\dagger R^\mu) + \alpha \text{Tr}(R_{\mu\nu} \partial^\mu\varphi^\dagger \varphi \partial^\nu\varphi^\dagger). \end{aligned} \quad (3.8)$$

The chirally invariant constraint $\text{Tr}(R_\mu^\dagger \varphi) = 0$ was required to reduce the number of degrees of freedom to the necessary amount for unit isospin. An alternative model was considered by Meissner where he replaced the Skyrme term with the sextic term [34]

$$\frac{\hbar}{32e^2} \text{Tr} \left(\left[(\partial_\mu \varphi) \varphi^\dagger, (\partial_\nu \varphi) \varphi^\dagger \right]^2 \right) \rightarrow \frac{1}{4} c_6^2 \text{Tr} (\mathcal{B}_\mu \mathcal{B}^\mu). \quad (3.9)$$

We, however, choose a non-constrained, Lie algebra valued formulation since it is computationally (and in particular numerically) simpler as it does not contain non-physical degrees of freedom.

The model we are interested in is the massive Skyrme model coupled to the ρ -meson, described by the Lagrangian

$$\begin{aligned} \mathcal{L} = & -\frac{1}{8\hbar^3} F_\pi^2 m_\pi^2 \text{Tr} (\text{Id} - \varphi) + \frac{F_\pi^2}{16\hbar} \eta^{\mu\nu} \text{Tr} (L_\mu L_\nu) + \frac{\hbar}{32e^2} \eta^{\mu\alpha} \eta^{\nu\beta} \text{Tr} ([L_\mu, L_\nu][L_\alpha, L_\beta]) \\ & - \frac{m_\rho^2}{4\hbar^3} \eta^{\mu\nu} \text{Tr} (R_\mu^\dagger R_\nu) - \frac{1}{8\hbar} \eta^{\mu\alpha} \eta^{\nu\beta} \text{Tr} (R_{\mu\nu}^\dagger R_{\alpha\beta}) + \frac{1}{2} \eta^{\mu\beta} \eta^{\nu\gamma} \alpha \text{Tr} (R_{\mu\nu} [L_\beta, L_\gamma]), \end{aligned} \quad (3.10)$$

where $L_\mu = \varphi^{-1} \partial_\mu \varphi$ is an $\mathfrak{su}(2)$ -valued left current and $R_{\mu\nu} = \partial_\mu R_\nu - \partial_\nu R_\mu$. The parameters of the model are the pion decay constant $F_\pi = 129$ MeV, the pion mass $m_\pi = 138$ MeV, the ρ -meson mass $m_\rho = 775$ MeV, the dimensionless Skyrme parameter e , and the $\rho\pi\pi$ coupling constant α . Also, $\hbar = 197.33$ MeV fm is the reduced Planck constant.

The above Skyrme- ρ theory can be motivated for two main reasons. First of all, both the field content as well as the interaction term, are consistent with the dimensional reduction of the four dimensional static Yang-Mills theory with $SU(2)$ gauge group [7]. Indeed this term is included in the Sutcliffe's effective action. However, the instanton derivation introduces six different interaction terms and also dictates their coupling constants. Importantly, the considered here interaction comes together, that is multiplied by same coupling constant, with another term, $\sim [L_\mu, L_\nu][L^\mu, R^\nu]$. In the current paper, due to the considerable complexity of the resulting model, we consider a simplified version of this construction and take into account only one of them. We also assume that its coupling constant is a free parameter rather than an instanton related fixed constant. One might argue that, to some extent, this takes into account the absence of all other interaction terms.

This is related to the second, physical source of the motivation for the considered model. Namely, this is the unique term in the instanton derivation which in the low energy limit provides the $\rho\pi\pi$ vertex

$$\mathcal{L}_{\rho\pi\pi} = 2\alpha m_\rho^2 \epsilon_{abc} \rho_c^\nu \pi_a \partial_\nu \pi_b. \quad (3.11)$$

This is by far the most important channel of ρ -meson decay with a branching rate of 99.9%.

Therefore, one could conclude that our proposal is rather a minimal ρ -meson extension of the Skyrme theory that a systematic limit of the Sutcliffe instanton approach.

For our numerical computations we will use a rescaled theory, with the energy scale $\tilde{E} = F_\pi/(4e)$ (MeV) and the length scale $\tilde{L} = 2\hbar/(eF_\pi)$ (fm). We also further rescale the ρ -meson field by $R_\mu \rightarrow F_\pi R_\mu$. Then the rescaled ρ -Skyrme Lagrangian in dimensionless

Skyrme units is given by

$$\begin{aligned} \mathcal{L} = & -M_\pi^2 \text{Tr}(\text{Id} - \varphi) + \frac{1}{2} \eta^{\mu\nu} \text{Tr}(L_\mu L_\nu) + \frac{1}{16} \eta^{\mu\alpha} \eta^{\nu\beta} \text{Tr}([L_\mu, L_\nu][L_\alpha, L_\beta]) \\ & + 2M_\rho^2 \eta^{\mu\nu} \text{Tr}(R_\mu R_\nu) + \eta^{\mu\alpha} \eta^{\nu\beta} \text{Tr}(R_{\mu\nu} R_{\alpha\beta}) + 2c_\alpha \eta^{\mu\alpha} \eta^{\nu\beta} \text{Tr}(R_{\mu\nu}[L_\alpha, L_\beta]), \end{aligned} \quad (3.12)$$

where $c_\alpha \equiv \alpha e$ and the rescaled pion and ρ -meson masses are, respectively,

$$M_\pi = \frac{2m_\pi}{eF_\pi}, \quad M_\rho = \frac{2m_\rho}{eF_\pi}. \quad (3.13)$$

The associated energy-momentum tensor (in dimensionless Skyrme units) can be obtained via the Hilbert prescription, which yields

$$\begin{aligned} T_{\mu\nu} = & -\text{Tr}(L_\mu L_\nu) - 4M_\rho^2 \text{Tr}(R_\mu R_\nu) - \frac{1}{4} \eta^{\alpha\beta} \text{Tr}([L_\mu, L_\alpha][L_\nu, L_\beta]) - 4\eta^{\alpha\beta} \text{Tr}(R_{\mu\alpha} R_{\nu\beta}) \\ & - 8c_\alpha \eta^{\alpha\beta} \text{Tr}(R_{\mu\alpha}[L_\nu, L_\beta]) + \eta_{\mu\nu} \mathcal{L}. \end{aligned} \quad (3.14)$$

From the time-like part of the energy-momentum tensor we obtain the static energy functional of theory, that is

$$\begin{aligned} E = \int d^3x \sqrt{-\eta} \left\{ M_\pi^2 \text{Tr}(\text{Id} - \varphi) - \frac{1}{2} \eta^{ij} \text{Tr}(L_i L_j) - \frac{1}{16} \eta^{ia} \eta^{jb} \text{Tr}([L_i, L_j][L_a, L_b]) \right. \\ \left. - 2M_\rho^2 \eta^{ij} \text{Tr}(R_i R_j) - \eta^{ia} \eta^{jb} \text{Tr}(R_{ij} R_{ab}) - 2c_\alpha \eta^{ia} \eta^{jb} \text{Tr}(R_{ij}[L_a, L_b]) \right\}. \end{aligned} \quad (3.15)$$

The first term in the energy (3.15) is the standard pion mass potential, the second is the kinetic term and the third is the Skyrme term, which corresponds to the four pion interaction. These three terms make up the standard massive \mathcal{L}_{024} -Skyrme model. There exists a generalization of the massive \mathcal{L}_{024} -model that includes the addition of the sixth order sextic term [5]

$$\mathcal{L}_6 = -\pi^4 \lambda^2 \eta^{\mu\nu} \mathcal{B}_\mu \mathcal{B}_\nu, \quad (3.16)$$

which yields an ω -meson like repulsion on short distances, while also allowing the fourth order Skyrme term to describe scalar meson effects. The constant λ is related to the ω -meson mass m_ω and the coupling constant β_ω of the ω -meson via $\lambda^2 = \beta_\omega^2 \hbar^3 / (2\pi^4 m_\omega^2)$ [35].

Inclusion of the sextic term, in general, stiffens the nuclear matter equation of state, especially at higher densities, and increases the maximal obtainable masses of neutron stars within the theory to be more in line with observed data [17]. However, the caveat is that it also may increase the compression modulus. Furthermore, this term is numerically quite challenging. For this reason, we will only be concerned with the massive \mathcal{L}_{024} -model as we are interested in the effects of the ρ -meson on the Skyrme model and, in particular, on the compression modulus (i.e., compressibility at the saturation point where the role of sextic is expected to be subleading).

Before discussing skyrmion solutions, let us observe that the model possesses a topological energy bound. Firstly, we note that we can express the ρ -Skyrme Lagrangian in adimensional units as

$$L = \frac{1}{2} \|d\varphi\|^2 - \int_M V \circ \varphi \text{vol}_g - \frac{1}{4} \|\varphi^* \Omega\|^2 + 8 \left(M_\rho^2 \|R\|^2 + \frac{1}{2} \|dR\|^2 + c_\alpha \langle dR, \varphi^* \Omega \rangle_{L^2} \right), \quad (3.17)$$

where $\|\cdot\|$ denotes the L^2 norm. Then, the usual Bogomolny trick (of completing the square) gives

$$\frac{1}{2} \langle dR, dR \rangle_{L^2} + c_\alpha \langle dR, \varphi^* \Omega \rangle_{L^2} = \frac{1}{2} \langle dR + c_\alpha \varphi^* \Omega, dR + c_\alpha \varphi^* \Omega \rangle_{L^2} - \frac{1}{2} c_\alpha^2 \langle \varphi^* \Omega, \varphi^* \Omega \rangle_{L^2}. \tag{3.18}$$

Hence, the energy can be expressed as

$$\begin{aligned} E &= \frac{1}{2} \|d\varphi\|^2 + \int_M V \circ \varphi \operatorname{vol}_g + \left(\frac{1}{4} - 4c_\alpha^2\right) \|\varphi^* \Omega\|^2 + 8M_\rho^2 \|R\|^2 + 4\|dR + c_\alpha \varphi^* \Omega\|^2 \\ &\geq \frac{1}{2} \|d\varphi\|^2 + \left(\frac{1}{4} - 4c_\alpha^2\right) \|\varphi^* \Omega\|^2 \\ &\geq 24\pi^2 \sqrt{\left(\frac{1}{4} - 4c_\alpha^2\right)} |B|, \end{aligned} \tag{3.19}$$

where in the last step we used the usual Faddeev-Skyrme bound. This computation makes sense only if the new coupling constant in front of quartic term is positive $\tilde{c}_4 \equiv \frac{1}{4} - 4e^2\alpha^2 \geq 0$. Thus we arrive at an upper bound for the coupling constant α

$$c_\alpha = \alpha e \leq \frac{1}{4}. \tag{3.20}$$

Larger values would destabilize skyrmions as they lead to non-positive definite energy integral. This analysis shows also the role of ρ -mesons. They effectively reduce the coupling constant of the quartic term and, in an extremal case, can make it arbitrary small.

In fact, using results of [36, 37] one can find a more restrictive bound. It reads

$$E \geq 4\pi^2 \sqrt{\frac{\tilde{c}_4}{2\pi^2}} \left[4 \frac{\sqrt{2\pi^2}}{a} \left(1 - \frac{a^2}{2} \left(\sqrt{1 + \frac{4}{a^2}} - 1 \right) \right)^{3/4} + 6\pi a \left(\sqrt{1 + \frac{4}{a^2}} - 1 \right) \right],$$

where

$$a = \frac{1}{2M_\pi \sqrt{\tilde{c}_4} (\langle V^{1/4} \rangle)^2}$$

and $\langle V^{1/4} \rangle$ is the average value of the potential when integrated over the whole target space. Although it is a tighter bound it does not give any additional constraint on \tilde{c}_4 , or equivalently on c_α .

Now we turn our attention to topological solitons in the ρ -Skyrme model, that is, skyrmions. These are solutions of the full non-linear Euler-Lagrange equations associated to the static energy functional (3.15). If we consider the model on a 3-torus, and also allow the fundamental period lattice of the torus to vary, we obtain crystalline configurations of skyrmions-skyrmion crystals. This is essential for understanding phases and phase transitions of nuclear matter within the Skyrme model. However, before we turn our attention to skyrmion crystals, we first detail the first few skyrmions, within the ρ -Skyrme model. They describe the nucleon ($B = 1$ hedgehog solution), and the lightest nuclei up to α particle ($B = 2, 3, 4$).

3.1 The $B = 1$ skyrmion: hedgehog ansatz

The $B = 1$ skyrmion is necessarily spherically symmetric and the pion fields $\vec{\pi}$ point radially outwards, earning the moniker the “hedgehog” skyrmion. This can be obtained by use of a rational map ansatz (RMA) [38]. The RMA is carried out in a local spherical coordinate system (r, θ, ϕ) , and the Skyrme field ansatz is defined by [38]

$$\varphi(r, \theta, \phi) = (\cos f(r), \sin f(r)\vec{n}_W(\theta, \phi)), \quad (3.21)$$

where the unit vector \vec{n}_W in terms of the rational map $W : \mathbb{C} \rightarrow \mathbb{C}$ is given explicitly by

$$\vec{n}_W = \left(\frac{2\Re(W)}{1 + |W|^2}, \frac{2\Im(W)}{1 + |W|^2}, \frac{1 - |W|^2}{1 + |W|^2} \right). \quad (3.22)$$

Here $f : [0, \infty) \rightarrow \mathbb{R}$ is a radial profile function and we identify the rational map domain $S^2 \cong \mathbb{C} \cup \{\infty\}$ via the Riemann sphere coordinate $z = \tan(\theta/2) \exp(i\phi)$. For suitable choices of profile functions and rational maps, the ansatz produces excellent approximations to the true skyrmions, with the correct symmetries as well. In terms of the RMA, the hedgehog ansatz is defined by the rational map $W(z) = z$, which gives the unit normal

$$\vec{n}_H = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta). \quad (3.23)$$

The corresponding standard hedgehog Skyrme field is given by

$$\varphi_H = \begin{bmatrix} \cos f(r) + i \cos \theta \sin f(r) & \sin \theta \sin \phi \sin f(r) + i \sin \theta \cos \phi \sin f(r) \\ -\sin \theta \sin \phi \sin f(r) + i \sin \theta \cos \phi \sin f(r) & \cos f(r) - i \cos \theta \sin f(r) \end{bmatrix}. \quad (3.24)$$

As a result of the hedgehog symmetry and the intrinsic parity of the ρ -mesons, the most general form for the spin-1 ρ -field is [39]

$$R_i^H = i\rho_i^a \tau^a, \quad \rho_i^a = \epsilon_{aij} n_H^j \xi(r). \quad (3.25)$$

The unknown profile functions $f(r)$ and $\xi(r)$ are obtained by substituting the ansatz (3.24) and (3.25) into the static energy (3.15), and minimizing the static nucleon energy

$$E_1 = 4\pi \int_0^\infty dr \mathcal{E}_H(r), \quad (3.26)$$

where the hedgehog energy density is

$$\begin{aligned} \mathcal{E}_H(r) = & 2M_\pi^2 r^2 [1 - \cos f(r)] + r^2 f'(r)^2 + 2 \sin^2 f(r) + \sin^2 f(r) \left[2f'(r)^2 + \frac{\sin^2 f(r)}{r^2} \right] \\ & + 8r^2 M_\rho^2 \xi(r)^2 + 8 \left[3\xi(r)^2 + 2r\xi(r)\xi'(r) + r^2 \xi'(r)^2 \right] \\ & + \frac{1}{r} 32c_\alpha \sin f(r) [rf'(r) \cos f(r) \{r\xi'(r) + \xi(r)\} + \xi(r) \sin f(r)]. \end{aligned} \quad (3.27)$$

That is, the profile functions are determined by solving the coupled pair of ODEs,

$$\begin{aligned} 2M_\pi^2 r^2 \sin f(r) + 2 \sin 2f(r) - 2r^2 f''(r) - 4 \sin f(r) \left\{ f''(r) \sin f(r) + f'(r)^2 \cos f(r) \right\} \\ - 4rf'(r) + \frac{4 \sin^3 f(r) \cos f(r)}{r^2} + 16c_\alpha \sin 2f(r) \left[\frac{2\xi(r)}{r} - \{r\xi''(r) + 2\xi'(r)\} \right] = 0 \end{aligned} \quad (3.28)$$

and

$$16c_\alpha r \left[-f''(r) \sin 2f(r) - 2f'(r)^2 \cos 2f(r) + \frac{2 \sin^2 f(r)}{r^2} \right] + 16 \left[\xi(r) \{ r^2 M_\rho^2 + 2 \} - r^2 \xi''(r) - 2r \xi'(r) \right] = 0. \quad (3.29)$$

To ensure the soliton is bounded in space, the appropriate boundary conditions are [33]

$$f(0) = \pi, \quad f(\infty) = 0, \quad \xi(0) = 0, \quad \xi(\infty) = 0. \quad (3.30)$$

The resulting profile functions $f(r)$ and $\xi(r)$ are obtained by applying a multiple shooting method to the coupled system (3.28)–(3.29) with the fixed boundary conditions (3.30). For increasing values of the coupling constant α , the energy minimizing profile functions are shown in figure 1 alongside the energy density. We remark that the profile function of the Skyrme field is very weakly affected by the ρ -meson, enjoying a sort of universality as recently observed in [40].

In addition, we also investigate the effect of the coupling constant c_α on the size of the hedgehog skyrmion. For simplicity, and due to difficulty quantizing the theory, we consider the nucleon to have zero isospin. Then, the mean charge radius of a nucleus with zero isospin is approximately the root-mean-square (RMS) radius [41]

$$\langle r^2 \rangle^{1/2} = \left(\frac{1}{B} \int_{\mathbb{R}^3} d^3x r^2 \mathcal{B}_0(\vec{x}) \right)^{\frac{1}{2}}. \quad (3.31)$$

We observe that the charge radius of the nucleon decreases as the coupling constant c_α increases. For c_α close to the minimal value the unit charge Skyrmion shrinks quite significantly. For example, for $c_\alpha = 0.22$ we find that $\langle r^2 \rangle^{1/2} = 0.353$ fm. As c_α tends to the critical value, then the RMS charge radius of the single soliton tends to about 1/3 of its physical value. This suggests that one would also need to couple to additional vector mesons, such as the ω -meson, or include further interaction terms if we want to consider this model in the low-density regime.

3.2 Multi-skyrmions with ρ -mesons

For higher charge skyrmions $B > 1$, there is no simplification to a one dimensional problem as there is for the hedgehog solution. We must solve the full non-linear Euler-Lagrange field equations. To solve the field equations we use arrested Newton flow, an accelerated gradient descent method with flow arresting, with the RMA as an initial configuration. The well-known rational maps of high symmetry which we use as initial configurations for the numerical relaxation are detailed in [38].

In general, the initial configuration is not a minimizer and so it swaps its potential energy for kinetic energy as it evolves. During the evolution we check to see if the energy is increasing. If the energy is indeed increasing, we take out all the kinetic energy in the system by setting the velocity to zero and restart the flow (this is the arresting criteria). Naturally the field will relax to a local, or global, minimum in some potential well. The evolution then terminates when every component of the energy gradient is zero within some specified tolerance, $\text{tol} = 10^{-5}$.

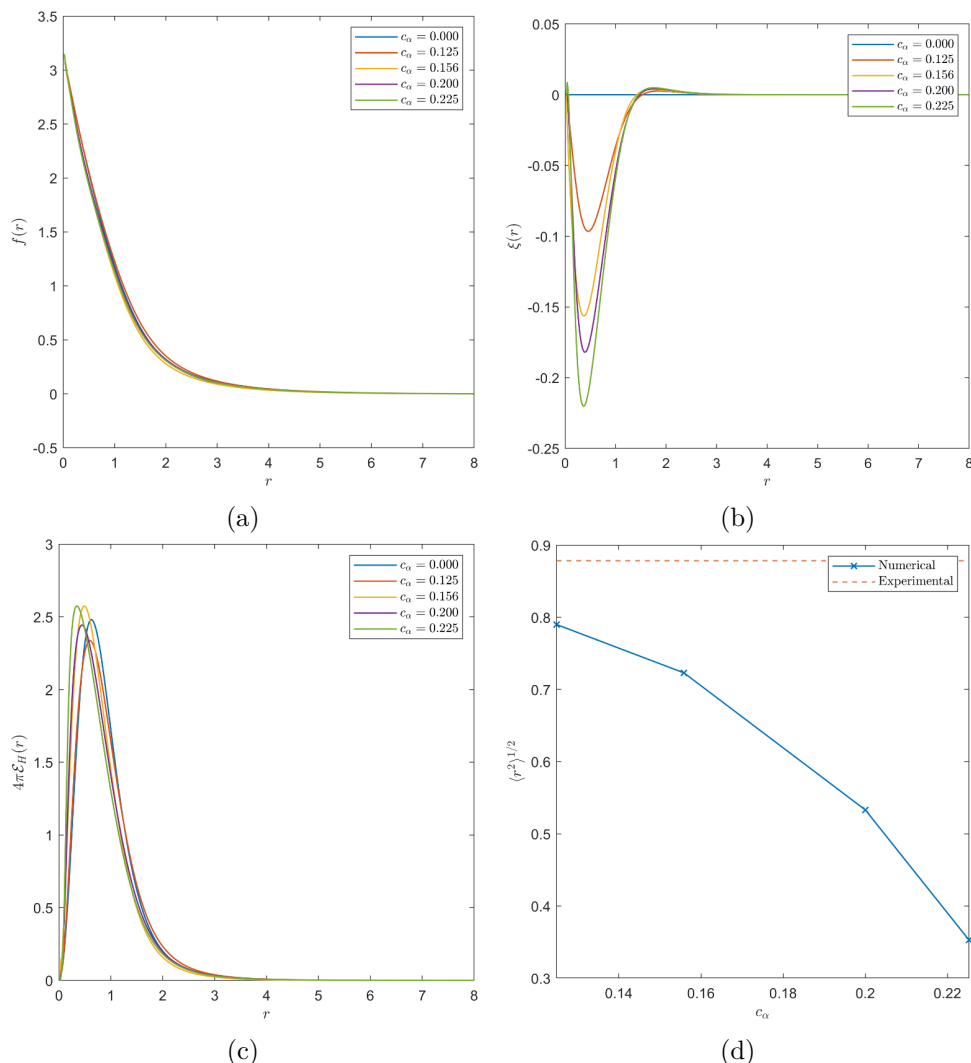


Figure 1. Plots for various coupling constants c_α of (a) the Skyrme radial profile functions $f(r)$, (b) the ρ -meson radial profile $\xi(r)$, (c) the energy density profile $\mathcal{E}_H(r)$ and (d) the rms charge radius obtained by minimizing the static nucleon mass E_1 via a multiple shooting method. The chiral angle $f(r)$ is related to the pion field and $\xi(r)$ is related to the spatial components of the ρ -meson.

Explicitly, we are solving Newton’s equations of motion for a particle on the discretised configuration space \mathcal{C} with potential energy E_{dis} . That is, we are solving the coupled system of 1st order ODEs

$$\dot{\psi}^\mu = -\frac{\delta\mathcal{E}}{\delta\varphi^\mu}, \quad \psi^\mu = \dot{\varphi}^\mu, \quad (3.32)$$

with initial configuration $\varphi(0) = \varphi_{\text{RMA}}$ and initial velocity $\psi(0) = \dot{\varphi}(0) = 0$. We implement a fourth order Runge-Kutta method to solve this coupled system. In conjunction, we do the exact same for the ρ -meson fields:

$$\dot{P}_b^a = -\frac{\delta\mathcal{E}}{\delta\rho_b^a}, \quad P_b^a = \dot{\rho}_b^a, \quad (3.33)$$

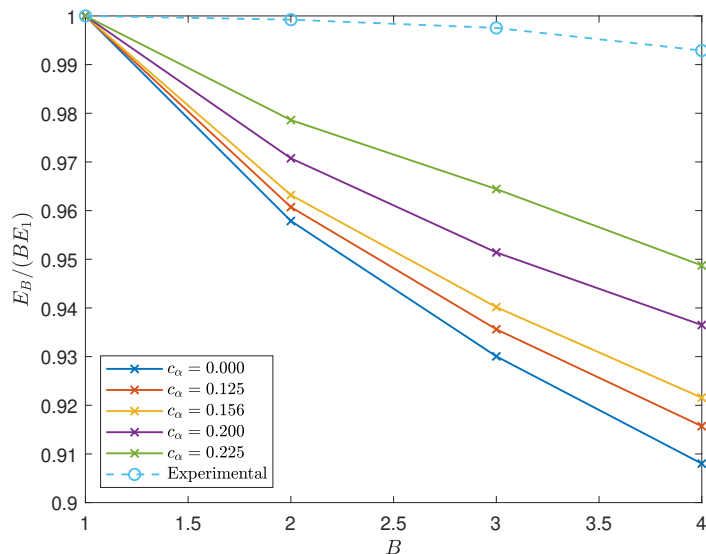


Figure 2. Comparison of the normalized energies per baryon of the multi-skyrmion solutions up to $B = 4$. It can clearly be seen that the binding energies decrease as the coupling is increased.

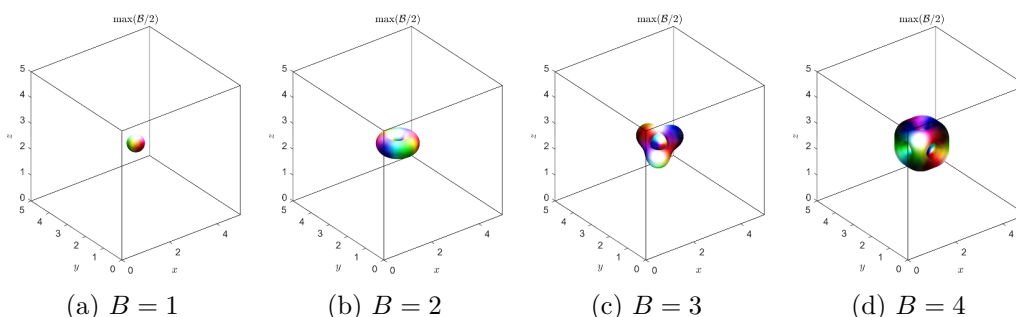


Figure 3. Level set plots of the baryon density of multi-skyrmions coupled to ρ -mesons for $B = 1, 2, 3, 4$ for the choice of coupling $c_\alpha = 0.200$.

with initial configuration $\rho_b^a = 0$ and initial velocity $P_b^a(0) = \dot{\rho}_b^a(0) = 0$. We also implement a fourth order Runge-Kutta method to solve this coupled system. The arresting criteria is implemented as follows. If $E(t + \delta t) > E(t)$, we take out all the kinetic energy in the Skyrme system by setting $\psi(t + \delta t) = 0$ and $P(t + \delta t) = 0$, and restart the flow for the Skyrme field and the ρ -mesons.

The resulting binding energies as a function the coupling constant c_α are plotted in figure 2. Alongside this, baryon density plots for representative $c_\alpha = 0.200$ are shown in figure 3. It is apparent that the binding energies decrease as the coupling c_α is increased.

4 Nuclear matter in the ρ -Skyrme model

The application of the skyrmion crystals to the analysis of nuclear matter was first studied by Klebanov [42] in the context of the massless \mathcal{L}_{24} -model. This was carried out for the simple cubic crystal of hedgehog skyrmions (SC_1) with unit cell charge $B_{\text{cell}} = 8$, where it was observed that this crystal undergoes a phase transition to a body centered cubic crystal

of half skyrmions ($BCC_{1/2}$) at higher densities [43]. It was later found that a crystal with lower energy than the $BCC_{1/2}$ exists, a simple cubic crystal of half skyrmions ($SC_{1/2}$) with unit cell charge $B_{\text{cell}} = 4$ [44, 45]. Baskerville took Klebanov’s method further and applied it to the $SC_{1/2}$ crystal, and the state corresponding to a neutron crystal was identified [46].

The phase structure of the \mathcal{L}_{24} -Skyrme model has been studied by Jackson and Verbaarschot [47]. This has been further investigated in the generalized \mathcal{L}_{0246} -Skyrme model [48]. Two candidates have been previously proposed as the minimal E/B crystal for \mathcal{L}_{024} -skyrmions, these are the $SC_{1/2}$ crystal and the α -particle crystal [49]. Many skyrmions can be constructed as chunks of the infinite crystal [50, 51] and many have been built from α -particles [52]. The phase transition between the α -particle and the α -particle lattice has been investigated in both the \mathcal{L}_{24} -model [53] and \mathcal{L}_{0246} -model [18]. This $B = 4$, or α -particle, cluster picture supports the α -particle model of nuclei in which medium to large skyrmions are composed of $B = 4$ skyrmions in many arrangements [51, 52]. This α -clustering model has been pretty successful in the predicting the energy spectrum of states of Carbon-12 [8, 54] and Oxygen-16 [10, 11]. For a recent review of the application of skyrmion crystals to the study of dense nuclear matter see, e.g., [55].

Recently, the crystalline structure of skyrmion matter was studied more rigorously in the massive \mathcal{L}_{024} -model [20] and the generalized \mathcal{L}_{0246} -model [21]. In both studies, four crystal solutions were found with unit cell charge $B_{\text{cell}} = 4$. Two of these solutions were known before and have cubic lattice geometry, these are the $SC_{1/2}$ and α -particle crystals. The two new solutions have non-cubic period lattices and take the form of a chain and a multiwall configuration. It was observed that the ground state crystalline configuration is the multiwall solution in the Skyrme model without vector mesons. This was found to be true at all densities and for various choices of the coupling constants.

However, when the theory is coupled to the ω -meson [22], the ground state crystal configuration is dependent upon the underlying parameters of the theory. A switching in the energy ordering of the four crystal solutions was observed, with the multiwall crystal not always being the ground state configuration. Before we can compute the compression modulus, we first need to determine the ground state crystal in this theory.

4.1 Varying the metric

In order to be able to address the compression modulus problem, we first need to understand the crystalline nature of dense nuclear matter within the ρ -Skyrme framework. To do this we follow the general methodology first proposed by Speight [56]. That is, we study static Skyrme fields $\varphi : \mathbb{R}^3 \rightarrow \text{SU}(2)$ and ρ -meson fields $R_\mu : \mathbb{R}^3 \rightarrow \mathfrak{su}(2)$ that are periodic with respect to some 3-dimensional period lattice

$$\Lambda = \left\{ n_1 \vec{X}_1 + n_2 \vec{X}_2 + n_3 \vec{X}_3 : n_i \in \mathbb{Z} \right\}. \tag{4.1}$$

We can equivalently interpret the domain of the fields as \mathbb{R}^3/Λ , and identify this with the unit 3-torus (\mathbb{T}^3, g) via the diffeomorphism

$$F : \mathbb{T}^3 \rightarrow \mathbb{R}^3/\Lambda, \quad (x^1, x^2, x^3) \mapsto x^1 \vec{X}_1 + x^2 \vec{X}_2 + x^3 \vec{X}_3. \tag{4.2}$$

Then the resulting metric g on \mathbb{T}^3 is the pullback of the metric d by F , i.e.

$$g = F^*d = g_{ij}dx^i dx^j, \quad g_{ij} = \vec{X}_i \cdot \vec{X}_j. \quad (4.3)$$

Now, varying the period lattice Λ_s with $\Lambda_0 = \Lambda$ is equivalent to considering variations of the metric g_s on \mathbb{T}^3 with $g_0 = F^*d$. Therefore, the energy minimized over variations g_s of the domain metric is equivalent to determining the energy minimizing period lattice Λ .

Throughout, it will be convenient to use the non-linear σ model (NL σ M) formulation of the model, where we identify $SU(2) \cong S^3$ and write the Skyrme field as the unit 4-vector $\varphi = (\sigma, \vec{\pi})$. In this notation we identify $\varphi_0 \equiv \sigma$ and $\varphi_i \equiv \pi_i$. So, now we can regard the Skyrme field as the map $\varphi : \mathbb{T}^3 \rightarrow S^3$ and the ρ -meson as the map $R_\mu : \mathbb{T}^3 \rightarrow \mathfrak{su}(2)$. Then, using the NL σ M formulation, the energy density can be written in index notation as

$$\begin{aligned} \mathcal{E} = & 2M_\pi^2(1 - \varphi_0) + g^{ij} \partial_i \varphi_\mu \partial_j \varphi_\mu + \frac{1}{2} g^{ik} g^{jl} (\partial_i \varphi_\mu \partial_k \varphi_\mu \partial_j \varphi_\nu \partial_l \varphi_\nu - \partial_i \varphi_\mu \partial_l \varphi_\mu \partial_j \varphi_\nu \partial_k \varphi_\nu) \\ & + 4M_\rho^2 g^{ij} \rho_i^a \rho_j^a + 2g^{ik} g^{jl} (\partial_i \rho_j^a - \partial_j \rho_i^a) (\partial_k \rho_l^a - \partial_l \rho_k^a) \\ & - 8\alpha e g^{ik} g^{jl} (\partial_i \rho_j^a - \partial_j \rho_i^a) (\epsilon^{abc} \partial_k \varphi_b \partial_l \varphi_c + \partial_k \varphi_0 \partial_l \varphi_a - \partial_k \varphi_a \partial_l \varphi_0). \end{aligned} \quad (4.4)$$

For numerical simulations involving the minimization of the energy functional with respect to variations of the metric, it will be convenient to follow the methodology laid out in [20, 21]. That is, we define the metric independent integrals

$$V = 2M_\pi^2 \int_{\mathbb{T}^3} d^3x (1 - \varphi_0), \quad (4.5)$$

$$L_{ij} = \int_{\mathbb{T}^3} d^3x \left\{ \partial_i \varphi_\mu \partial_j \varphi_\mu + 4M_\rho^2 \rho_i^a \rho_j^a \right\}, \quad (4.6)$$

$$\begin{aligned} \Omega_{ijkl} = & \int_{\mathbb{T}^3} d^3x \left\{ \frac{1}{2} (\partial_i \varphi_\mu \partial_k \varphi_\mu \partial_j \varphi_\nu \partial_l \varphi_\nu - \partial_i \varphi_\mu \partial_l \varphi_\mu \partial_j \varphi_\nu \partial_k \varphi_\nu) + 2 (\partial_i \rho_j^a - \partial_j \rho_i^a) (\partial_k \rho_l^a - \partial_l \rho_k^a) \right. \\ & \left. - 8\alpha e (\partial_i \rho_j^a - \partial_j \rho_i^a) (\epsilon^{abc} \partial_k \varphi_b \partial_l \varphi_c + \partial_k \varphi_0 \partial_l \varphi_a - \partial_k \varphi_a \partial_l \varphi_0) \right\}. \end{aligned} \quad (4.7)$$

In terms of these metric independent integrals, the energy is simply

$$E = E_0 + E_2 + E_4, \quad (4.8)$$

where

$$E_0 = \sqrt{g} V, \quad E_2 = \sqrt{g} g^{ij} L_{ij}, \quad E_4 = \sqrt{g} g^{ik} g^{jl} \Omega_{ijkl}. \quad (4.9)$$

We note that this can only be done for constant coefficient metrics.

To understand the role the metric plays in determining crystalline configurations, let us consider the rate of change of the energy with respect to varying the domain metric g on \mathbb{T}^3 . We will do this using the compact notation introduced above. The first variation of the energy with respect to the variation $g(s)$ of the metric on \mathbb{T}^3 is given by

$$\left. \frac{dE(\varphi, \rho, g_s)}{ds} \right|_{s=0} = \int_{\mathbb{T}^3} d^3x \sqrt{g} \langle S(\varphi, \rho, g), \delta g \rangle_g, \quad (4.10)$$

where $S(\varphi, \rho, g) \in \Gamma(\odot^2 T^* \mathbb{T}^3)$ is a symmetric 2-covariant tensor field on \mathbb{T}^3 , known as the *stress-energy tensor*, defined by

$$\int_{\mathbb{T}^3} d^3x \sqrt{g} S^{ij} = \frac{1}{2} g^{ij} \sqrt{g} V + \sqrt{g} \left(\frac{1}{2} g^{ij} g^{kl} - g^{ik} g^{jl} \right) L_{kl} + \sqrt{g} g^{km} \left(\frac{1}{2} g^{ln} g^{ij} - 2g^{il} g^{jn} \right) \Omega_{klmn}. \quad (4.11)$$

The space of allowed variations \mathcal{E} is a 6-dimensional subspace of the space of sections of the rank 6 vector bundle $\odot^2 T^* \mathbb{T}^3$,

$$\mathcal{E} = \left\{ \delta g_{ij} dx^i dx^j \in \Gamma(\odot^2 T^* \mathbb{T}^3) : \delta g_{ij} \text{ constant}, \delta g_{ji} = \delta g_{ij} \right\}. \quad (4.12)$$

By definition, the energy E is critical with respect to variations g_s of the metric if and only if

$$\left. \frac{dE(\varphi, \rho, g_s)}{ds} \right|_{s=0} = \int_{\mathbb{T}^3} d^3x \sqrt{g} \langle S(\varphi, \rho, g), \delta g \rangle_g = 0, \quad (4.13)$$

that is, if and only if $S \perp_{L^2} \mathcal{E}$. Now let the orthogonal complement of g in \mathcal{E} , the space of traceless parallel symmetric bilinear forms, given by

$$\mathcal{E}_0 = \left\{ \theta \in \Gamma(\odot^2 T^* \mathbb{T}^3) : \text{Tr}_g(\theta) = \langle \theta, g \rangle_g = 0 \right\}. \quad (4.14)$$

Then the criticality condition $S \perp_{L^2} \mathcal{E}$ can be reformulated as

$$\int_{\mathbb{T}^3} d^3x \sqrt{g} \langle S(\varphi, \rho, g), g \rangle_g = 0 \quad \text{and} \quad S \perp_{L^2} \mathcal{E}_0. \quad (4.15)$$

The first condition $S \perp_{L^2} g$ is analogous to a virial constraint and the second condition $S \perp_{L^2} \mathcal{E}_0$ coincides with the extended virial constraints derived by Manton [57]. One finds that the condition $S \perp_{L^2} g$ establishes the “usual” virial constraint

$$E_2 - E_4 + 3E_0 = 0. \quad (4.16)$$

To determine the extended virial constraint corresponding to the condition $S \perp_{L^2} \mathcal{E}_0$, we define a symmetric bilinear form $\Delta : T_x \mathbb{T}^3 \times T_x \mathbb{T}^3 \rightarrow \mathbb{R}$ corresponding to the trace-free part of the stress-energy tensor:

$$\Delta_{ij} = \sqrt{g} L_{ij} + 2\sqrt{g} g^{kl} \Omega_{ikjl}. \quad (4.17)$$

Then $S \perp_{L^2} \mathcal{E}_0$ if and only if Δ is orthogonal to \mathcal{E}_0 with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathcal{E}}$. Therefore, for $\lambda \in \mathbb{R}$ we must have

$$\Delta = \lambda g. \quad (4.18)$$

Taking the trace of both sides yields

$$3\lambda = \sqrt{g} g^{ij} L_{ij} + 2\sqrt{g} g^{ij} g^{kl} \Omega_{ikjl} = E_2 + 2E_4. \quad (4.19)$$

Thus, the condition $S \perp_{L^2} \mathcal{E}_0$ produces the extended virial constraint

$$\Delta = \frac{1}{3} (E_2 + 2E_4) g. \quad (4.20)$$

These extended virial constraints act as consistency check on our numerical algorithm, which we detail below.

4.2 Determining the energy minimizing period lattice

The general methodology for determining the period lattice that minimizes the static energy was laid out in [20]. The method is identical under redefinition of the metric independent integrals to include the ρ -mesons. We summarize the method here.

Let us fix the Skyrme field $\varphi : \mathbb{T}^3 \rightarrow \text{SU}(2)$ and ρ -meson fields $\rho_i^a : \mathbb{T}^3 \rightarrow \mathbb{R}$ and think of the energy as a function of the metric g on \mathbb{T}^3 , $E_{\varphi,\rho} : \text{SPD}_3 \rightarrow \mathbb{R}$, where SPD_3 is the space of symmetric positive-definite 3×3 -matrices. To minimize the energy functional with respect to variations of the metric, we use arrested Newton flow on SPD_3 . The essence of the algorithm is as follows: we solve Newton's equations of motion for a particle on SPD_3 with potential energy $E_{\varphi,\rho}$. Let g_s be a smooth one-parameter curve in SPD_3 with $g_0 = F^*d$. Explicitly, we are solving the system of 2nd order ODEs

$$\left. \frac{d^2}{ds^2} \right|_{s=0} (g_{ij})_s = - \frac{\partial E_{\varphi,\rho}}{\partial g_{ij}} = - \int_{\mathbb{T}^3} d^3x \sqrt{g} S_{\varphi,\rho}^{ij}, \quad (4.21)$$

with initial metric $(g_{ij})_0 = \vec{X}_i \cdot \vec{X}_j$ and initial velocity $\partial_s g_s|_{s=0} = 0$, where $S_{\varphi,\rho} = S(g)$ is the stress-energy tensor for fixed field configurations φ, ρ . Setting $\delta g_s = \partial_s g_s$ as the velocity with $\delta g = \delta g_0$ reduces the problem to a coupled system of first order ODEs. We implement a fourth order Runge-Kutta method to solve this coupled system. After each time step $t \mapsto t + \delta t$, we check to see if the energy is increasing. If $E(t + \delta t) > E(t)$, we take out all the kinetic energy in the system by setting $\delta g(t + \delta t) = 0$ and restart the flow. The flow then terminates when every component of the stress-energy tensor $S_{\varphi,\rho}$ is zero to within a given tolerance (we have used 10^{-6}).

4.3 Crystalline configurations

With the numerical method in place, we now turn our focus to constructing ρ -skyrmion crystals. The starting point in the construction of ρ -skyrmion crystals is the Fourier series-like expansion of the Skyrme fields as an initial configuration [58],

$$\sigma = \sum_{a,b,c}^{\infty} \beta_{abc} \cos\left(\frac{2a\pi x}{L}\right) \cos\left(\frac{2b\pi y}{L}\right) \cos\left(\frac{2c\pi z}{L}\right), \quad (4.22a)$$

$$\pi_1 = \sum_{h,k,l}^{\infty} \alpha_{hkl} \sin\left(\frac{2h\pi x}{L}\right) \cos\left(\frac{2k\pi y}{L}\right) \cos\left(\frac{2l\pi z}{L}\right), \quad (4.22b)$$

$$\pi_2 = \sum_{h,k,l}^{\infty} \alpha_{hkl} \cos\left(\frac{2l\pi x}{L}\right) \sin\left(\frac{2h\pi y}{L}\right) \cos\left(\frac{2k\pi z}{L}\right), \quad (4.22c)$$

$$\pi_3 = \sum_{h,k,l}^{\infty} \alpha_{hkl} \cos\left(\frac{2k\pi x}{L}\right) \cos\left(\frac{2l\pi y}{L}\right) \sin\left(\frac{2h\pi z}{L}\right), \quad (4.22d)$$

with initial metric $g = L^3 \text{Id}_3$ on \mathbb{T}^3 . For the $\text{SC}_{1/2}$ crystal, the Fourier coefficients β_{abc} and α_{hkl} must satisfy the conditions: h and k are odd, and l is even, and a, b, c are all odd [55]. Then, truncating the Fourier series (4.22) to only include the first terms in the expansions yields the approximation of Castillejo et al. [45].

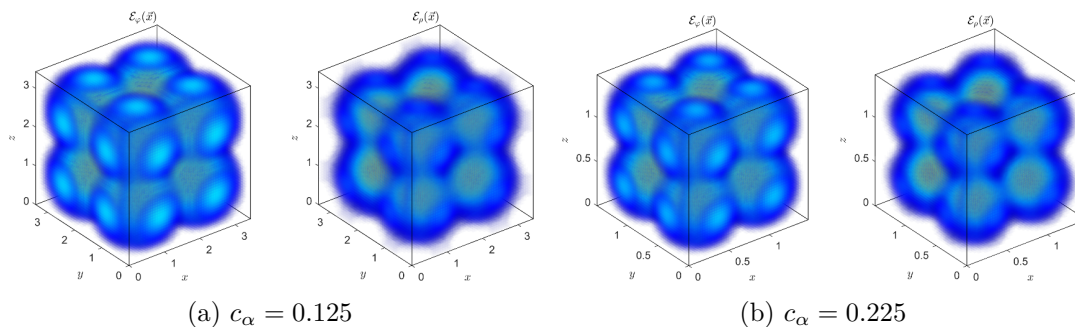


Figure 4. Energy density plots of the Skyrme \mathcal{E}_φ and ρ -meson \mathcal{E}_ρ contributions to the α -particle crystal for (a) low and (b) high coupling constants. At higher couplings, the crystals are more lightly bound and support α -particle clustering.

From the $SC_{1/2}$ crystal, the other three crystals can be constructed by applying a chiral $SO(4)$ transformation $Q \in SO(4)$, such that $\varphi = Q\varphi_{1/2}$, and minimizing the energy with respect to variations of the field and the lattice. These chiral transformations $Q \in SO(4)$ can be determined by considering a deformed energy functional on the moduli space of critical points of the Skyrme energy functional, and are found to be

$$Q \in \left\{ \text{Id}_4, \underbrace{\begin{pmatrix} (0, 1, 1, 1)/\sqrt{3} \\ * \end{pmatrix}}_{Q_\alpha}, \underbrace{\begin{pmatrix} (0, 0, 0, 1) \\ * \end{pmatrix}}_{Q_{\text{multiwall}}}, \underbrace{\begin{pmatrix} (0, 0, 1, 1)/\sqrt{2} \\ * \end{pmatrix}}_{Q_{\text{chain}}} \right\}. \quad (4.23)$$

The other three rows of the chiral transformations Q_α , $Q_{\text{multiwall}}$ and Q_{chain} , labeled by the asterisk, can be obtained by using the Gram-Schmidt process.

Following the methodology laid out in this paper for determining crystals in the ρ -Skyrme model, with the initial configuration as detailed above, we find that *the ground state configuration in this theory is the α -particle crystal with cubic lattice geometry*. The second lowest energy configuration is found to be the multiwall crystal. As the value of the coupling constant increases, the energy difference between these two solutions becomes greater and more apparent. Therefore, we will use the α -particle crystal to determine the compression modulus in this theory. Energy density plots of the Skyrme field φ and ρ -meson contributions to the α -particle crystal, for low and high c_α couplings, are shown in figure 4.

5 Resolving the compression modulus problem

To compute the compression modulus within the ρ -Skyrme model, we need to construct skyrmion crystals at various nuclear densities n_B . This is achieved by constraining the fundamental unit cell of the period lattice Λ to have fixed volume V . This amounts to setting $\det(g)$ constant and, in turn, fixes the baryon density $n_B = B/V$. In practice, we replace the stress-energy tensor S_{ij} by its projection $S_{ij} - \frac{1}{3}S_{kl}g^{kl}g_{ij}$. As the ground state is observed to be the α -particle crystal, we use this crystal to compute the compression modulus. This is carried out at various values of the coupling constant c_α and the resulting equations of state about saturation are shown in figure 5.

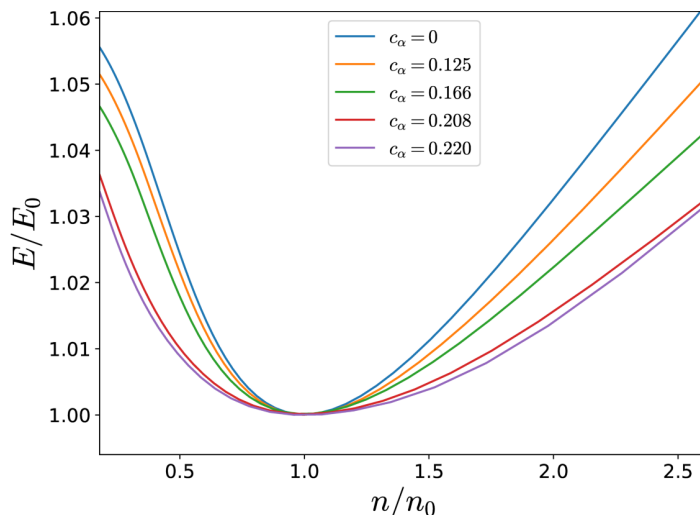


Figure 5. Comparison of the equations of state of the α -particle crystal about saturation for various choices of the ρ -meson coupling constant $c_\alpha = \alpha e$. It can be clearly seen that the compression modulus decreases as the coupling is increased.

c_α	K_0/E_0	K_0 (MeV)	BE (%)
0	1.170	1080	5.54
0.125	0.985	909	5.36
0.166	0.778	718	5.00
0.208	0.461	425	4.25
0.220	0.381	351	3.85

Table 1. Values of the adimensional coupling constant between ρ -mesons and pions with the corresponding values of the compression modulus and energy ratio. In the last two columns we compute the difference between the minimum and the $n_B \rightarrow 0$ asymptotic energy and also the corresponding value of the physical coupling constant, respectively.

It can be clearly seen that the equations of state about saturation get more shallow as the coupling constant c_α is increased. Consequently, this means that the compression modulus is getting smaller in value. Furthermore, the binding energies also decrease as the coupling c_α increases. This is summarized in table 1.

We now provide a simple argument that shows why a theory with low binding energies will also produce a lower value of the compression modulus. If the binding energies decrease then the asymptotic energy (as $n_B \rightarrow 0$) of the crystalline solution must also decrease. This reduces the energy difference between the asymptotic crystal (in our case, the isolated α -particle) and the (α -particle) crystal at saturation, which naturally reduces the curvature of the equation of state at saturation. Hence, a lower value of the compression modulus.

6 Concluding remarks

In the present paper we studied skyrmion crystals in an extended version of the Skyrme model where, the mesonic degrees of freedom are assisted by the ρ vector meson. This is a natural extension of the original Skyrme's idea, both from a phenomenological as well as a theoretical point of view. The ρ -meson is the next lightest mesonic particle. It is also naturally connected with the SU(2) pionic field via the dimensional reduction of (4+0) Yang-Mills theory à la Sutcliffe.

Specifically, we considered a minimal extension with only one interaction term, which gives rise to the usual two-pion vertex. Already at this level we obtained quite non-trivial and physically very encouraging results.

First of all, we found that the ground state is the α -particle crystal. This solution becomes more and more favored as the ρ -meson coupling constant grows. This should be contrasted with the generalized Skyrme model where the multiwall solution forms the ground state for a large range of parameters. It also differs from the ω -Skyrme crystal where the geometry of the ground state strongly depends on the values of the coupling constants.

Secondly, which is the most important result of the current work, the inclusion of ρ -meson significantly reduces the value of the compression modulus. For $c_\alpha = 0.220$, which is the ρ -meson coupling constant close to the maximal value, K_0 decreases from original 1080 MeV to 351 MeV. This is a striking improvement. Apparently, the ρ -meson makes the equation of state much softer at the saturation point which is a very welcome property of the considered model.

In comparison to Naya and Sutcliffe [59], the equivalent coupling constant corresponding to c_6 in their work is $c_\alpha \approx 0.248$, which is very close to the limit where the soliton is no longer stable, making this value not suitable for our approach. This suggests that it is pragmatic to consider the interaction coupling constant c_α as a free parameter, which is further motivated by the $\rho\pi\pi$ -vertex and the corresponding branching ratio.

An obvious question is whether the equation of state is stiff enough at higher densities to support neutron stars with masses in agreement with the current astrophysical data. This may require an addition of the sextic term and/or inclusion of the ω -meson. In fact, this meson is only 12 MeV heavier than ρ -meson. A variant of the Skyrme model with ω -meson has been very recently considered. Similarly, a significant improvement of the compression modulus has been reported [22]. Inclusion of ω -meson may potentially improve the unwanted small radii found for $c_\alpha \rightarrow 1/4$. Indeed, this meson contributes to a short distance repulsive force which should penalize too tight solutions.

Analytic methods have also been employed to study vector meson extensions of the Skyrme model [60, 61]. It has also been natural to consider generalizations of the non-linear pion theory by replacing the *ad hoc* Skyrme term with explicit interactions with vector mesons. One such approach is based on the so-called hidden local symmetry (HLS) method, where a hidden symmetry of the NL σ M is gauged and the corresponding gauge particle acquires mass through the Higgs mechanism [62]. This allows for the incorporation of ρ -mesons, as well as the ω -meson.

The HLS approach has been investigated extensively, especially in the context of dense nuclear matter [63–67]. Within the HLS framework, Lee et al. [68] incorporated a dilaton field to the massive Skyrme model, which is associated with the scale anomaly of QCD.

In their model the dilaton is crucial in realizing the phase transition from the Goldstone mode (spontaneously broken chiral symmetry FCC₁ phase) to the Wigner mode (unbroken chiral symmetry SC_{1/2} phase) consistently with the vector manifestation (VM) fixed point. This fixed point is characterized by the vanishing of both $\langle\sigma\rangle$ and the in-medium pion decay constant F_π^* , corresponding to the restoration of the spontaneously broken chiral symmetry. Interestingly, the addition of the ω -meson prevents the scale-anomaly dilaton field, and thus the in-medium pion decay constant, from developing a vanishing vacuum expectation value at the VM fixed point [69], resulting in a non-restoration of chiral symmetry. This is reminiscent of pseudo-gap phenomena in condensed matter physics.

In all of the above HLS crystal investigations, the Kugler and Shtrikmann Fourier series method was generalized to incorporate vector mesons. However, as was observed in this paper and in [22], the SC_{1/2} crystal is in fact not the ground state configuration in the Skyrme model with vector mesons. Therefore, generalizing the method presented in [20] to the Skyrme model with both, ρ and ω , mesons included seems to be a natural next step of the investigation.

Acknowledgments

A. W. and C. N. were supported by the Polish National Science center (NCN 2020/39/B/ST2/01553). P. L. acknowledges funding from UKRI, Grant No. EP/V520081/1.

A Compressibility from Derrick scaling of the cubic lattice of half-skyrmions

Consider a variation $\varphi_\lambda : \mathbb{R}^3 \times \mathbb{R} \rightarrow S^3$ of the Skyrme field φ such that $\varphi_{\lambda=0} = \varphi$. This has infinitesimal generator $\partial_\lambda \varphi_\lambda|_{\lambda=0} \in \Gamma(\varphi^{-1}TS^3)$, where $\varphi^{-1}TS^3$ is the vector bundle over \mathbb{R}^3 with fibre $T_{\varphi(x)}S^3$ over $x \in \mathbb{R}^3$. Explicitly, if we consider the spatial rescaling $x \mapsto e^\lambda x$, then we have a one-parameter family of maps $\varphi_\lambda = \varphi(e^\lambda x)$ such that $\varphi_{\lambda=0} = \varphi$. The rescaled massless \mathcal{L}_{24} static energy functional is then

$$E_\lambda = E_{24}[\varphi_\lambda] = e^\lambda E_2 + e^{-\lambda} E_4. \tag{A.1}$$

If the Skyrme field configuration φ is a minimiser of the \mathcal{L}_{24} -energy E , then we require

$$\left. \frac{d}{d\lambda} \right|_{\lambda=0} E_{24}[\varphi_\lambda] = E_2 - E_4 = 0, \tag{A.2}$$

which yields the familiar virial constraint $E_2 = E_4$.

The true \mathcal{L}_{24} -energy minimizing crystal is the cubic lattice of half-skyrmions found independently by Kugler & Shtrikmann [44] and Castillejo et al. [45]. Let us denote this minimal energy crystalline solution of the \mathcal{L}_{24} -Skyrme model by $\varphi_{1/2} \equiv \varphi(L_{1/2})$, where $L_{1/2}$ corresponds to the side length of the energy minimizing cubic lattice. Under the one-parameter variation $L_{1/2} \mapsto e^\lambda L_{1/2}$, the rescaled half-crystal configuration $\varphi(e^\lambda L_{1/2})$ approximates the true minimizer at volume $V = e^{3\lambda} L_{1/2}^3$ well for small λ . Then, the compression modulus may be related to the second derivative of the energy with respect to the scaling factor λ ,

$$K_0 = \frac{1}{B} \left. \frac{d^2}{d\lambda^2} \right|_{\lambda=0} E_{24}[\varphi(e^\lambda L_{1/2})] = \frac{1}{B} (E_2[\varphi_{1/2}] + E_4[\varphi_{1/2}]) = E_0. \tag{A.3}$$

Thus, one finds that $K_0/E_0 = 1$ and, hence, this leads to a value for the compression modulus that is roughly four times too large. If we consider the generalized \mathcal{L}_{0246} -model then the situation is worse [31]

$$K_0 = \frac{1}{B} \left(9E_0[\varphi_{1/2}] + E_2[\varphi_{1/2}] + E_4[\varphi_{1/2}] + 9E_6[\varphi_{1/2}] \right) = E_0 + \frac{8}{B} (E_0[\varphi_{1/2}] + E_6[\varphi_{1/2}]). \quad (\text{A.4})$$

Data Availability Statement. This article has no associated data or the data will not be deposited.

Code Availability Statement. This article has no associated code or the code will not be deposited.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] T.H.R. Skyrme, *A nonlinear field theory*, *Proc. Roy. Soc. Lond. A* **260** (1961) 127 [[INSPIRE](#)].
- [2] N.S. Manton, *Skyrmions — a theory of nuclei*, World Scientific, Singapore (2022) [[DOI:10.1142/q0368](#)] [[INSPIRE](#)].
- [3] C.J. Halcrow, *Vibrational quantisation of the B=7 Skyrmion*, *Nucl. Phys. B* **904** (2016) 106 [[arXiv:1511.00682](#)] [[INSPIRE](#)].
- [4] S. Bjarke Gudnason and C. Halcrow, *Vibrational modes of Skyrmions*, *Phys. Rev. D* **98** (2018) 125010 [[arXiv:1811.00562](#)] [[INSPIRE](#)].
- [5] A. Jackson et al., *A modified Skyrmion*, *Phys. Lett. B* **154** (1985) 101 [[INSPIRE](#)].
- [6] C. Adam, J. Sánchez-Guillén and A. Wereszczyński, *A Skyrme-type proposal for baryonic matter*, *Phys. Lett. B* **691** (2010) 105 [[arXiv:1001.4544](#)] [[INSPIRE](#)].
- [7] P. Sutcliffe, *Skyrmions, instantons and holography*, *JHEP* **08** (2010) 019 [[arXiv:1003.0023](#)] [[INSPIRE](#)].
- [8] P.H.C. Lau and N.S. Manton, *States of carbon-12 in the Skyrme model*, *Phys. Rev. Lett.* **113** (2014) 232503 [[arXiv:1408.6680](#)] [[INSPIRE](#)].
- [9] C. Adam, C. Naya and A. Wereszczyński, *Carbon-12 in the generalized Skyrme model*, [arXiv:2401.08778](#) [[INSPIRE](#)].
- [10] C.J. Halcrow, C. King and N.S. Manton, *A dynamical α -cluster model of ^{16}O* , *Phys. Rev. C* **95** (2017) 031303 [[arXiv:1608.05048](#)] [[INSPIRE](#)].
- [11] C.J. Halcrow, C. King and N.S. Manton, *Oxygen-16 spectrum from tetrahedral vibrations and their rotational excitations*, *Int. J. Mod. Phys. E* **28** (2019) 1950026 [[arXiv:1902.09424](#)] [[INSPIRE](#)].
- [12] C. Halcrow and D. Harland, *An attractive spin-orbit potential from the Skyrme model*, *Phys. Rev. Lett.* **125** (2020) 042501 [*Erratum ibid.* **129** (2022) 139901] [[arXiv:2007.01304](#)] [[INSPIRE](#)].
- [13] M. Gillard, D. Harland and M. Speight, *Skyrmions with low binding energies*, *Nucl. Phys. B* **895** (2015) 272 [[arXiv:1501.05455](#)] [[INSPIRE](#)].

- [14] C. Naya and P. Sutcliffe, *Skyrmions and clustering in light nuclei*, *Phys. Rev. Lett.* **121** (2018) 232002 [[arXiv:1811.02064](#)] [[INSPIRE](#)].
- [15] S.B. Gudnason and J.M. Speight, *Realistic classical binding energies in the ω -Skyrme model*, *JHEP* **07** (2020) 184 [[arXiv:2004.12862](#)] [[INSPIRE](#)].
- [16] C. Naya, *Neutron stars within the Skyrme model*, *Int. J. Mod. Phys. E* **28** (2019) 1930006 [[arXiv:1910.01145](#)] [[INSPIRE](#)].
- [17] C. Adam et al., *A new consistent neutron star equation of state from a generalized Skyrme model*, *Phys. Lett. B* **811** (2020) 135928 [[arXiv:2006.07983](#)] [[INSPIRE](#)].
- [18] C. Adam et al., *Dense matter equation of state and phase transitions from a generalized Skyrme model*, *Phys. Rev. D* **105** (2022) 074019 [[arXiv:2109.13946](#)] [[INSPIRE](#)].
- [19] C. Adam et al., *Quantum skyrmion crystals and the symmetry energy of dense matter*, *Phys. Rev. D* **106** (2022) 114031 [[arXiv:2202.00953](#)] [[INSPIRE](#)].
- [20] D. Harland, P. Leask and M. Speight, *Skyrme crystals with massive pions*, *J. Math. Phys.* **64** (2023) 103503 [[arXiv:2305.14005](#)] [[INSPIRE](#)].
- [21] P. Leask, M. Huidobro and A. Wereszczynski, *Generalized skyrmion crystals with applications to neutron stars*, *Phys. Rev. D* **109** (2024) 056013 [[arXiv:2306.04533](#)] [[INSPIRE](#)].
- [22] D. Harland, P. Leask and M. Speight, *Skyrmion crystals stabilized by ω -mesons*, *JHEP* **06** (2024) 116 [[arXiv:2404.11287](#)] [[INSPIRE](#)].
- [23] G.S. Adkins and C.R. Nappi, *Stabilization of chiral solitons via vector mesons*, *Phys. Lett. B* **137** (1984) 251 [[INSPIRE](#)].
- [24] U. Garg and G. Colò, *The compression-mode giant resonances and nuclear incompressibility*, *Prog. Part. Nucl. Phys.* **101** (2018) 55 [[arXiv:1801.03672](#)] [[INSPIRE](#)].
- [25] M.N. Harakeh et al., *Direct evidence for a new giant resonance at $A=80$ - 13 MeV in the lead region*, *Phys. Rev. Lett.* **38** (1977) 676 [[INSPIRE](#)].
- [26] U. Garg and G. Colò, *The compression-mode giant resonances and nuclear incompressibility*, *Prog. Part. Nucl. Phys.* **101** (2018) 55 [[arXiv:1801.03672](#)] [[INSPIRE](#)].
- [27] G. Colò et al., *Microscopic determination of the nuclear incompressibility within the nonrelativistic framework*, *Phys. Rev. C* **70** (2004) 024307 [[nucl-th/0403086](#)] [[INSPIRE](#)].
- [28] J.P. Blaizot, *Nuclear compressibilities*, *Phys. Rept.* **64** (1980) 171 [[INSPIRE](#)].
- [29] F.D. Mackie and G. Baym, *Compressible liquid drop nuclear model and mass formula*, *Nucl. Phys. A* **285** (1977) 332 [[INSPIRE](#)].
- [30] M. Huidobro, *Skyrme neutron stars*, Ph.D. thesis, University of Santiago de Compostela, Santiago de Compostela, Spain (2024) [[INSPIRE](#)].
- [31] C. Adam et al., *Thermodynamics of the BPS Skyrme model*, *Phys. Rev. D* **90** (2014) 045003 [[arXiv:1405.2927](#)] [[INSPIRE](#)].
- [32] P. Sutcliffe, *Skyrmions in a truncated BPS theory*, *JHEP* **04** (2011) 045 [[arXiv:1101.2402](#)] [[INSPIRE](#)].
- [33] G.S. Adkins, *ρ mesons in the Skyrme model*, *Phys. Rev. D* **33** (1986) 193 [[INSPIRE](#)].
- [34] U.G. Meissner, *ρ mesons in the Skyrme model: an alternative approach*, *Phys. Lett. B* **185** (1987) 399 [[INSPIRE](#)].

- [35] C. Adam et al., *Baryon chemical potential and in-medium properties of BPS skyrmions*, *Phys. Rev. D* **91** (2015) 125037 [[arXiv:1504.05185](#)] [[INSPIRE](#)].
- [36] D. Harland, *Topological energy bounds for the Skyrme and Faddeev models with massive pions*, *Phys. Lett. B* **728** (2014) 518 [[arXiv:1311.2403](#)] [[INSPIRE](#)].
- [37] C. Adam and A. Wereszczynski, *Topological energy bounds in generalized Skyrme models*, *Phys. Rev. D* **89** (2014) 065010 [[arXiv:1311.2939](#)] [[INSPIRE](#)].
- [38] C.J. Houghton, N.S. Manton and P.M. Sutcliffe, *Rational maps, monopoles and Skyrmions*, *Nucl. Phys. B* **510** (1998) 507 [[hep-th/9705151](#)] [[INSPIRE](#)].
- [39] U.G. Meissner and I. Zahed, *Skyrmions in the presence of vector mesons*, *Phys. Rev. Lett.* **56** (1986) 1035 [[INSPIRE](#)].
- [40] N.S. Manton, *Robustness of the hedgehog Skyrmion*, *JHEP* **08** (2024) 015 [[arXiv:2405.05731](#)] [[INSPIRE](#)].
- [41] C. Adam, M. Haberichter and A. Wereszczynski, *The volume of a soliton*, *Phys. Lett. B* **754** (2016) 18 [[arXiv:1511.01104](#)] [[INSPIRE](#)].
- [42] I.R. Klebanov, *Nuclear matter in the Skyrme model*, *Nucl. Phys. B* **262** (1985) 133 [[INSPIRE](#)].
- [43] A.S. Goldhaber and N.S. Manton, *Maximal symmetry of the Skyrme crystal*, *Phys. Lett. B* **198** (1987) 231 [[INSPIRE](#)].
- [44] M. Kugler and S. Shtrikman, *A new Skyrmion crystal*, *Phys. Lett. B* **208** (1988) 491 [[INSPIRE](#)].
- [45] L. Castillejo et al., *Dense Skyrmion systems*, *Nucl. Phys. A* **501** (1989) 801 [[INSPIRE](#)].
- [46] W.K. Baskerville, *Quantization of global isospin in the Skyrme crystal*, *Phys. Lett. B* **380** (1996) 106 [[hep-ph/9603216](#)] [[INSPIRE](#)].
- [47] A.D. Jackson and J.J.M. Verbaarschot, *Phase structure of the Skyrme model*, *Nucl. Phys. A* **484** (1988) 419 [[INSPIRE](#)].
- [48] I. Perapechka and Y. Shnir, *Crystal structures in generalized Skyrme model*, *Phys. Rev. D* **96** (2017) 045013 [[arXiv:1703.10673](#)] [[INSPIRE](#)].
- [49] D.T.J. Feist, *Interactions of $B=4$ Skyrmions*, *JHEP* **02** (2012) 100 [[arXiv:1112.2119](#)] [[INSPIRE](#)].
- [50] W.K. Baskerville, *Making nuclei out of the Skyrme crystal*, *Nucl. Phys. A* **596** (1996) 611 [[nucl-th/9510047](#)] [[INSPIRE](#)].
- [51] D.T.J. Feist, P.H.C. Lau and N.S. Manton, *Skyrmions up to baryon number 108*, *Phys. Rev. D* **87** (2013) 085034 [[arXiv:1210.1712](#)] [[INSPIRE](#)].
- [52] R. Battye, N.S. Manton and P. Sutcliffe, *Skyrmions and the alpha-particle model of nuclei*, *Proc. Roy. Soc. Lond. A* **463** (2007) 261 [[hep-th/0605284](#)] [[INSPIRE](#)].
- [53] J. Silva Lobo, *Deformed Skyrme crystals*, *JHEP* **10** (2010) 029 [[arXiv:1010.0619](#)] [[INSPIRE](#)].
- [54] J.I. Rawlinson, *An alpha particle model for Carbon-12*, *Nucl. Phys. A* **975** (2018) 122 [[arXiv:1712.05658](#)] [[INSPIRE](#)].
- [55] C. Adam, A.G. Martín-Caro, M. Huidobro and A. Wereszczynski, *Skyrme crystals, nuclear matter and compact stars*, *Symmetry* **15** (2023) 899 [[arXiv:2305.06639](#)] [[INSPIRE](#)].
- [56] J.M. Speight, *Solitons on tori and soliton crystals*, *Commun. Math. Phys.* **332** (2014) 355 [[arXiv:1307.3063](#)] [[INSPIRE](#)].
- [57] N.S. Manton, *Scaling identities for solitons beyond Derrick's theorem*, *J. Math. Phys.* **50** (2009) 032901 [[arXiv:0809.2891](#)] [[INSPIRE](#)].

- [58] M. Kugler and S. Shtrikman, *Skyrmion crystals and their symmetries*, *Phys. Rev. D* **40** (1989) 3421 [INSPIRE].
- [59] C. Naya and P. Sutcliffe, *Skyrmions in models with pions and rho mesons*, *JHEP* **05** (2018) 174 [arXiv:1803.06098] [INSPIRE].
- [60] G. Barriga, M. Torres and A. Vera, *Exact modulated hadronic tubes and layers at finite volume in a cloud of π and ω mesons*, *Nucl. Phys. B* **1001** (2024) 116501 [arXiv:2312.16131] [INSPIRE].
- [61] G. Barriga et al., *On the robustness of solitons crystals in the Skyrme model*, *Nucl. Phys. B* **983** (2022) 115913 [arXiv:2207.08712] [INSPIRE].
- [62] H. Forkel, A.D. Jackson and C. Weiss, *Skyrmions with vector mesons: stability and the vector limit*, *Nucl. Phys. A* **526** (1991) 453 [INSPIRE].
- [63] Y.-L. Ma et al., *Hidden local symmetry and infinite tower of vector mesons for baryons*, *Phys. Rev. D* **86** (2012) 074025 [arXiv:1206.5460] [INSPIRE].
- [64] Y.-L. Ma, G.-S. Yang, Y. Oh and M. Harada, *Skyrmions with vector mesons in the hidden local symmetry approach*, *Phys. Rev. D* **87** (2013) 034023 [arXiv:1209.3554] [INSPIRE].
- [65] Y.-L. Ma et al., *Dense baryonic matter in the hidden local symmetry approach: half-skyrmions and nucleon mass*, *Phys. Rev. D* **88** (2013) 014016 [Erratum *ibid.* **88** (2013) 079904] [arXiv:1304.5638] [INSPIRE].
- [66] Y.-L. Ma et al., *Dense baryonic matter in conformally-compensated hidden local symmetry: vector manifestation and chiral symmetry restoration*, *Phys. Rev. D* **90** (2014) 034015 [arXiv:1308.6476] [INSPIRE].
- [67] W.-G. Paeng, T.T.S. Kuo, H.K. Lee and M. Rho, *Scale-invariant hidden local symmetry, topology change and dense baryonic matter*, *Phys. Rev. C* **93** (2016) 055203 [arXiv:1508.05210] [INSPIRE].
- [68] H.-J. Lee, B.-Y. Park, M. Rho and V. Vento, *Sliding vacua in dense Skyrmion matter*, *Nucl. Phys. A* **726** (2003) 69 [hep-ph/0304066] [INSPIRE].
- [69] B.-Y. Park, M. Rho and V. Vento, *Vector mesons and dense Skyrmion matter*, *Nucl. Phys. A* **736** (2004) 129 [hep-ph/0310087] [INSPIRE].