

Chebyshev Polynomials and Continued Fractions Related

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Abstract

Let p, q be complex polynomials, $\deg p > \deg q \geq 0$. We consider the family of polynomials defined by the recurrence $P_{n+1} = 2pP_n - qP_{n-1}$ for $n = 1, 2, 3, \dots$ with arbitrary P_1 and P_0 as well as the domain of the convergence of the infinite continued fraction

$$f(z) = 2p(z) - \frac{q(z)}{2p(z) - \frac{q(z)}{2p(z) - \dots}}$$

Key words: Chebyshev polynomials, continued fractions, Binet formula, Cassini identity

1 Some polynomials of the Chebyshev type

Let P_0 and P_1 be polynomials of one complex variable, $\deg P_1 > \deg P_0 \geq 0$. Let p, q be polynomials of one complex variable, $\deg p > \deg q \geq 0$, $q \neq 0$. Define the family of polynomials P_n by the recurrence formula

$$P_{n+1}(z) = 2p(z)P_n(z) - q(z)P_{n-1}(z), \quad n = 1, 2, 3, \dots \quad (1)$$

Note that (1) gives the Chebyshev polynomials of

- the first kind T_n for $P_0(z) = 1$, $P_1(z) = z$, $p(z) = z$ and $q(z) = 1$
- the second kind U_n for $P_0(z) = 1$, $P_1(z) = 2z$, $p(z) = z$ and $q(z) = 1$
- the third kind V_n for $P_0(z) = 1$, $P_1(z) = 2z - 1$, $p(z) = z$ and $q(z) = 1$
- the fourth kind W_n for $P_0(z) = 1$, $P_1(z) = 2z + 1$, $p(z) = z$ and $q(z) = 1$.

(See [2], Appendix B, Table B.2).

We write the recurrence (1) in the matrix form

$$\begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix} = \begin{bmatrix} 2p & -q \\ 1 & 0 \end{bmatrix} \begin{bmatrix} P_n & P_{n-1} \\ P_{n-1} & P_{n-2} \end{bmatrix} \quad (2)$$

proceeding as in [1], p.80, where the Fibonacci sequence was considered, defined by the similar recurrence $\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix}$ with $F_0 = 0$ and $F_1 = 1$.

Note that the characteristic polynomial

$$w(\lambda) = \det \begin{bmatrix} 2p - \lambda & -q \\ 1 & -\lambda \end{bmatrix} = (\lambda - p)^2 - p^2 + q$$

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of the matrix

$$\begin{bmatrix} 2p & -q \\ 1 & 0 \end{bmatrix} \quad (3)$$

admits two different roots

$$\lambda_1 = p + \sqrt{p^2 - q} \text{ and } \lambda_2 = p - \sqrt{p^2 - q}, \quad (4)$$

as the polynomial q is assumed to be nonzero.

Theorem 1.1 For the polynomial P_n defined by (1) we get the following formula

$$P_n = \frac{1}{\lambda_1 - \lambda_2} \left[(\lambda_1^n - \lambda_2^n) P_1 - \lambda_1 \lambda_2 (\lambda_1^{n-1} - \lambda_2^{n-1}) P_0 \right] \quad (5)$$

where λ_1 and λ_2 are the eigenvalues (4) of the matrix (3).

Proof. By the Jordan decomposition of the matrix (3) we get

$$\begin{bmatrix} 2p & -q \\ 1 & 0 \end{bmatrix} = \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}.$$

The n -th power of the matrix (3) equals

$$\begin{bmatrix} 2p & -q \\ 1 & 0 \end{bmatrix}^n = \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}.$$

Hence, by the recurrence

$$\begin{aligned} \begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix} &= \begin{bmatrix} 2p & -q \\ 1 & 0 \end{bmatrix} \begin{bmatrix} P_n & P_{n-1} \\ P_{n-1} & P_{n-2} \end{bmatrix} = \begin{bmatrix} 2p & -q \\ 1 & 0 \end{bmatrix}^2 \begin{bmatrix} P_{n-1} & P_{n-2} \\ P_{n-2} & P_{n-3} \end{bmatrix} = \dots \\ &= \begin{bmatrix} 2p & -q \\ 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} P_2 & P_1 \\ P_1 & P_0 \end{bmatrix} \end{aligned}$$

we obtain

$$\begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix} = \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^{n-1} & 0 \\ 0 & \lambda_2^{n-1} \end{bmatrix} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} \begin{bmatrix} P_2 & P_1 \\ P_1 & P_0 \end{bmatrix}.$$

Multiplying the above matrices we get

$$P_n = \frac{1}{\lambda_1 - \lambda_2} \left[(\lambda_1^n - \lambda_2^n) P_1 - \lambda_1 \lambda_2 (\lambda_1^{n-1} - \lambda_2^{n-1}) P_0 \right]$$

□

Note that (5) corresponds to the well known *Binet formula* for the Fibonacci sequence

$$F_n = \frac{\mu_1^n - \mu_2^n}{\mu_1 - \mu_2} = \frac{1}{\sqrt{5}} \left[\left(\frac{\sqrt{5} + 1}{2} \right)^n - \left(\frac{-\sqrt{5} + 1}{2} \right)^n \right]$$

where $\mu_1 = \frac{\sqrt{5}+1}{2}$ and $\mu_2 = \frac{-\sqrt{5}+1}{2}$ are eigenvalues of the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ that defines the Fibonacci sequence $\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix}$ with $F_0 = 0$ and $F_1 = 1$.

Remark 1.2 The formula (5) works well with two known formulae (see [2] 1.49 and 1.52) for the Chebyshev polynomials of the first kind $T_0(x) = 1$, $T_1(x) = x$, $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ if we put $p(x) = x$, $q(x) = 1$, $\lambda_1(x) = x + \sqrt{x^2 - 1}$, $\lambda_2(x) = x - \sqrt{x^2 - 1}$, $P_0(x) = 1$ and $P_1(x) = x$:

$$\begin{aligned} T_n(x) &= \frac{1}{\lambda_1 - \lambda_2} \left[(\lambda_1^n - \lambda_2^n)x - (\lambda_1^{n-1} - \lambda_2^{n-1}) \right] \\ &= \frac{1}{\lambda_1 - \lambda_2} \left[\lambda_1^n \left(x - \frac{1}{\lambda_1} \right) - \lambda_2^n \left(x - \frac{1}{\lambda_2} \right) \right] \\ &= \frac{1}{2} (\lambda_1^n + \lambda_2^n) \\ &= \frac{1}{2} \left((x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n \right), \quad |x| \geq 1, \end{aligned}$$

and for the Chebyshev polynomials of the second kind $U_0(x) = 1$, $U_1(x) = 2x$, $U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$ if we put $p(x) = x$, $q(x) = 1$, $\lambda_1(x) = x + \sqrt{x^2 - 1}$, $\lambda_2(x) = x - \sqrt{x^2 - 1}$, $P_0(x) = 1$ and $P_1(x) = 2x$:

$$\begin{aligned} U_n(x) &= \frac{1}{\lambda_1 - \lambda_2} \left[(\lambda_1^n - \lambda_2^n)2x - (\lambda_1^{n-1} - \lambda_2^{n-1}) \right] \\ &= \frac{1}{\lambda_1 - \lambda_2} \left[\lambda_1^n \left(2x - \frac{1}{\lambda_1} \right) - \lambda_2^n \left(2x - \frac{1}{\lambda_2} \right) \right] \\ &= \frac{1}{\lambda_1 - \lambda_2} (\lambda_1^{n+1} - \lambda_2^{n+1}) \\ &= \frac{1}{2\sqrt{x^2 - 1}} \left((x + \sqrt{x^2 - 1})^{n+1} - (x - \sqrt{x^2 - 1})^{n+1} \right), \quad |x| \geq 1. \end{aligned}$$

Proceeding as above we get the next two formulae for the Chebyshev polynomials of the third and the fourth kind V_n , W_n , respectively:

$$\begin{aligned} V_n(x) &= \frac{1}{\lambda_1 - \lambda_2} \left[(\lambda_1^n - \lambda_2^n)(2x - 1) - (\lambda_1^{n-1} - \lambda_2^{n-1}) \right] \\ &= \frac{1}{\lambda_1 - \lambda_2} \left[\lambda_1^n \left(2x - 1 - \frac{1}{\lambda_1} \right) - \lambda_2^n \left(2x - 1 - \frac{1}{\lambda_2} \right) \right] \\ &= \frac{1}{\lambda_1 - \lambda_2} (\lambda_1^{n+1} - \lambda_2^{n+1}) - \frac{1}{\lambda_1 - \lambda_2} (\lambda_1^n - \lambda_2^n) \\ &= U_n(x) - U_{n-1}(x) \end{aligned}$$

and

$$\begin{aligned} W_n(x) &= \frac{1}{\lambda_1 - \lambda_2} \left[(\lambda_1^n - \lambda_2^n)(2x + 1) - (\lambda_1^{n-1} - \lambda_2^{n-1}) \right] \\ &= \frac{1}{\lambda_1 - \lambda_2} \left[\lambda_1^n \left(2x + 1 - \frac{1}{\lambda_1} \right) - \lambda_2^n \left(2x + 1 - \frac{1}{\lambda_2} \right) \right] \\ &= \frac{1}{\lambda_1 - \lambda_2} (\lambda_1^{n+1} - \lambda_2^{n+1}) + \frac{1}{\lambda_1 - \lambda_2} (\lambda_1^n - \lambda_2^n) \\ &= U_n(x) + U_{n-1}(x) \end{aligned}$$

Remark 1.3 If $\deg q = 0$, i.e. q is a nonzero constant, one may continue defining polynomials P_n for negative integers putting initial polynomials P_0, P_1 arbitrarily and the recurrence formula $P_{n-1} = -\frac{1}{q}P_{n+1} + \frac{2p}{q}P_n$ equivalent to the relation $P_{n+1} = 2pP_n(z) - qP_{n-1}$. Same as before we have

$$\begin{bmatrix} P_n & P_{n-1} \\ P_{n-1} & P_{n-2} \end{bmatrix} = \begin{bmatrix} 2p & -q \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix}$$

as

$$\begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix} = \begin{bmatrix} 2p & -q \\ 1 & 0 \end{bmatrix} \begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix}$$

We obtain

$$\begin{bmatrix} P_0 & P_{-1} \\ P_{-1} & P_{-2} \end{bmatrix} = \begin{bmatrix} 2p & -q \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} P_1 & P_0 \\ P_0 & P_{-1} \end{bmatrix}$$

and

$$\begin{aligned} \begin{bmatrix} P_{-n+1} & P_{-n} \\ P_{-n} & P_{-n-1} \end{bmatrix} &= \begin{bmatrix} 2p & -q \\ 1 & 0 \end{bmatrix}^{-n} \begin{bmatrix} P_1 & P_0 \\ P_0 & P_{-1} \end{bmatrix} \\ &= \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^{-n} & 0 \\ 0 & \lambda_2^{-n} \end{bmatrix} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} \begin{bmatrix} P_1 & P_0 \\ P_0 & P_{-1} \end{bmatrix} \end{aligned}$$

Multiplying the above matrices we get an analogous formula as (5) in Theorem 1.1:

$$P_{-n} = \frac{1}{\lambda_1 - \lambda_2} \left[(\lambda_1^{-n} - \lambda_2^{-n})P_1 - \lambda_1\lambda_2(\lambda_1^{-n-1} - \lambda_2^{-n-1})P_0 \right]$$

□

Calculating the determinant of the matrix

$$\begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix} = \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^{n-1} & 0 \\ 0 & \lambda_2^{n-1} \end{bmatrix} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} \begin{bmatrix} P_2 & P_1 \\ P_1 & P_0 \end{bmatrix}$$

we get the *Cassini type identity* for the polynomials P_n , corresponding to the *Cassini identity* for the Fibonacci sequence $F_{n+1}F_{n-1} - F_n^2 = \det \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = (-1)^n$:

$$\begin{aligned} (\lambda_1 - \lambda_2)^2 \det \begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix} &= \det \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \det \begin{bmatrix} \lambda_1^{n-1} & 0 \\ 0 & \lambda_2^{n-1} \end{bmatrix} \det \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} \det \begin{bmatrix} P_2 & P_1 \\ P_1 & P_0 \end{bmatrix} \\ \det \begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix} &= \det \begin{bmatrix} \lambda_1^{n-1} & 0 \\ 0 & \lambda_2^{n-1} \end{bmatrix} \det \begin{bmatrix} P_2 & P_1 \\ P_1 & P_0 \end{bmatrix} \end{aligned}$$

Since $\lambda_1 \lambda_2 = q$ we get the following remark.

Remark 1.4 The Cassini type identity for the polynomials P_n defined by (1) holds:

$$P_{n+1}P_{n-1} - P_n^2 = q^{n-1}(P_2P_0 - P_1^2)$$

which implies the four known formulae for the Chebyshev polynomials of the first, second, third and fourth kind, respectively:

$$\begin{aligned} T_{n+1}(x)T_{n-1}(x) - T_n^2(x) &= x^2 - 1 \\ U_{n+1}(x)U_{n-1}(x) - U_n^2(x) &= -1 \\ V_{n+1}(x)V_{n-1}(x) - V_n^2(x) &= -2x - 2 \\ W_{n+1}(x)W_{n-1}(x) - W_n^2(x) &= 2x - 2. \end{aligned}$$

Theorem 1.5 Let P_n be the sequence of polynomials defined by (1). The quotient P_{n+1}/P_n converges uniformly on compact subsets of the set

$$\{z \in \mathbb{C} : \left| \frac{\lambda_1(z)}{\lambda_2(z)} \right| > 1\}$$

to the limit λ_1 . The limit does not depend on the initial polynomials P_0 and P_1 .

Proof. By (5) the quotient of polynomials P_{n+1} and P_n equals

$$\begin{aligned} \frac{P_{n+1}}{P_n} &= \frac{(\lambda_1^{n+1} - \lambda_2^{n+1})P_1 - \lambda_1\lambda_2(\lambda_1^n - \lambda_2^n)P_0}{(\lambda_1^n - \lambda_2^n)P_1 - \lambda_1\lambda_2(\lambda_1^{n-1} - \lambda_2^{n-1})P_0} \\ &= \frac{(\lambda_1 - \lambda_2(\lambda_2/\lambda_1)^n)P_1 - \lambda_1\lambda_2(1 - (\lambda_2/\lambda_1)^n)P_0}{(1 - (\lambda_2/\lambda_1)^n)P_1 - \lambda_2(1 - (\lambda_2/\lambda_1)^{n-1})P_0} \end{aligned}$$

It converges on compact subsets the set $\{z \in \mathbb{C} : \left| \frac{\lambda_1(z)}{\lambda_2(z)} \right| > 1\}$ uniformly to the limit

$$\frac{\lambda_1 P_1 - \lambda_1 \lambda_2 P_0}{P_1 - \lambda_2 P_0} = \lambda_1$$

that is independent of P_0 and P_1 . □

2 Continued fractions related to polynomials P_n

Consider the infinite continued fraction

$$f(z) = 2p(z) - \frac{q(z)}{2p(z) - \frac{q(z)}{2p(z) - \dots}}$$

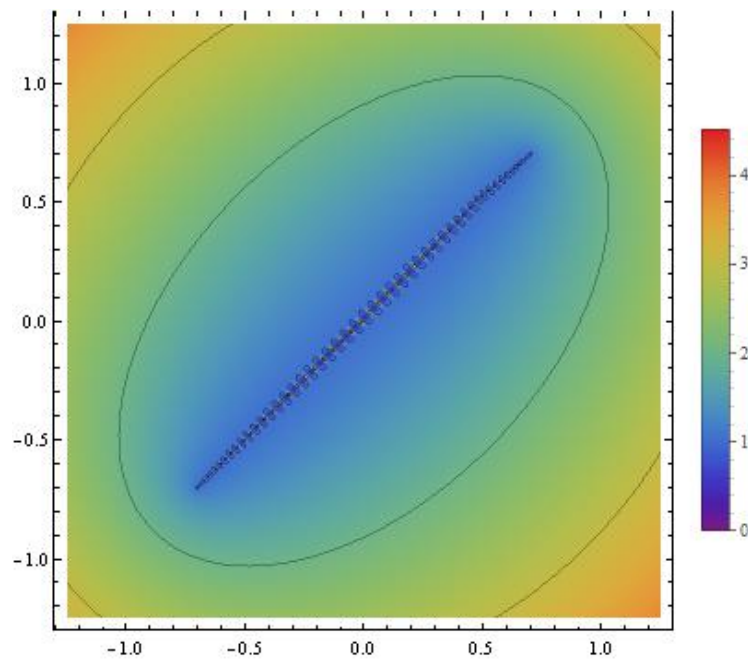


Figure 1: Density plot of $|r_{60}|$ for $p(z) = 2z$ and $q(z) = i$.

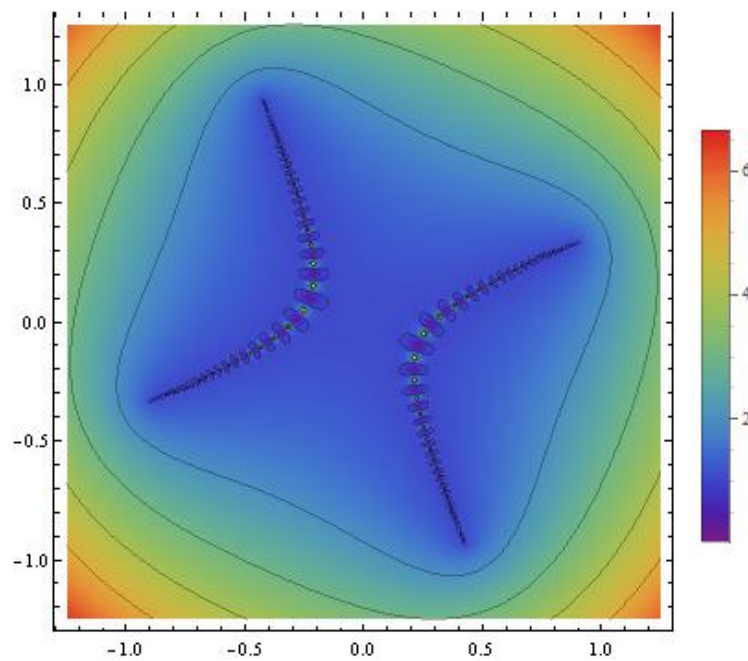


Figure 2: Density plot of $|r_{60}|$ for $p(z) = z^2 + \frac{i}{10}$, $q(z) = i$.

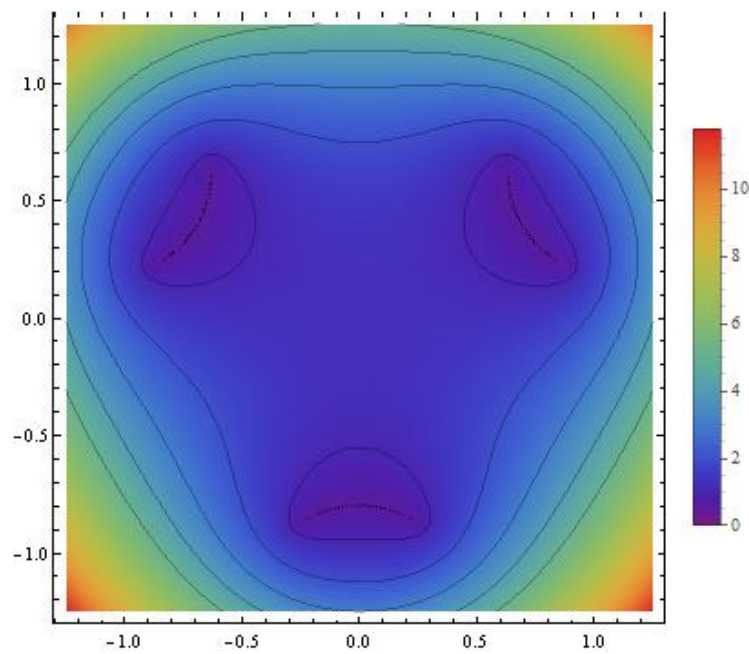


Figure 3: Density plot of $|r_{60}|$ for $p(z) = iz^3 + \frac{1}{2}$, $q(z) = -\frac{1}{3}$.

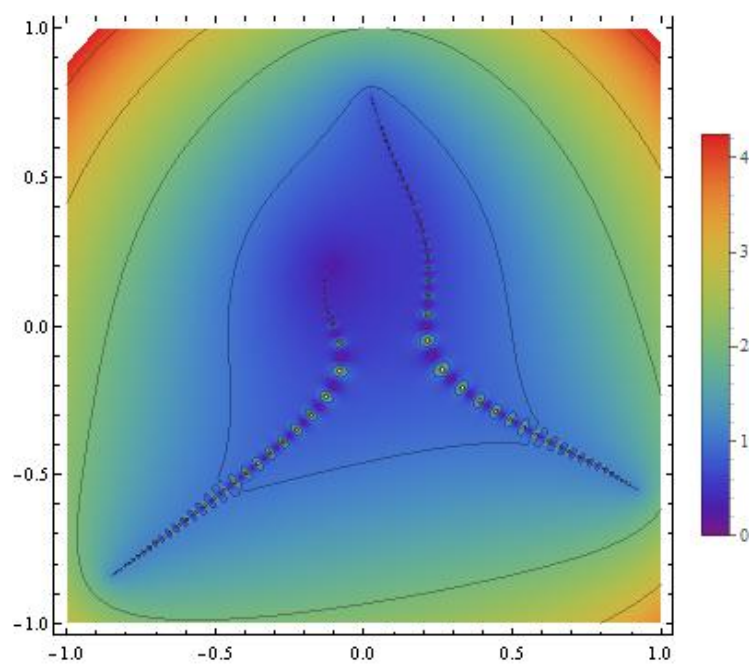


Figure 4: Density plot of $|r_{60}|$ for $p(z) = iz^2 - \frac{1}{5}z + \frac{1}{10}$, $q(z) = (i - \frac{1}{3})z + \frac{i}{7} + \frac{1}{5}$.