

## CARATHÉODORY COMPLETENESS ON THE PLANE

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*Dedicated to the memory of Professor Józef Siciak*

**Abstract.** M. A. Selby [8–10] and, independently, N. Sibony [11] proved that on the complex plane  $c$ -completeness is equivalent to  $c$ -finitely compactness. Their proofs are quite similar and are based on [4]. We give more refined equivalent conditions and, along the way, simplify the proofs.

**1. Introduction.** Let  $D \subset \mathbb{C}^n$  be a domain and let  $\zeta \in \partial D$  be its boundary point. We denote by  $A(D \cup \{\zeta\})$  the set of all bounded holomorphic functions on  $D$  which extend continuously to  $D \cup \{\zeta\}$ . Following [7], we say that  $\zeta$  is a *weak peak point* for  $D$  if there exists a function  $f \in A(D \cup \{\zeta\})$  such that  $|f| < 1$  on  $D$  and  $f(\zeta) = 1$ .

**THEOREM 1.** *Let  $D \subset \mathbb{C}$  be a domain and let  $\zeta \in \partial D$  be its boundary point. Then the following conditions are equivalent:*

- (1)  $\zeta$  is a weak peak point for  $D$ ;
- (2) there exists no finite Borel measure  $\mu$  on  $D$  such that

$$|f(\zeta)| \leq \int |f| d\mu \quad \text{for any } f \in A(D \cup \{\zeta\});$$

- (3) we have

$$\sum_{n=1}^{\infty} 2^n \gamma(A_n(\zeta, a) \setminus D) = +\infty,$$

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where  $A_n(\zeta) = \{z \in \mathbb{C} : \frac{1}{2^{n+1}} \leq |z - \zeta| \leq \frac{1}{2^n}\}$  and  $\gamma$  is the analytic capacity (see the definition below).

The equivalency of (1) and (3) in Theorem 1 was proved by M. A. Selby (see [9]). Note that the implication (1)  $\implies$  (2) is straightforward (also in a higher dimension). The implication (2)  $\implies$  (1) in any dimension is claimed in [2]. However, the proof is based on a false version of Hahn–Banach theorem, claimed in [5]. So, we give a new proof on the complex plane. In a higher dimension, it is still an open problem whether (2)  $\implies$  (1).

Let  $\mathbb{D}(\zeta, r) = \{z \in \mathbb{C} : |z - \zeta| < r\}$  denote the disk on the complex plane and let  $\mathbb{D} = \mathbb{D}(0, 1)$  denote the unit disk. We define the Poincaré function  $p$  on  $\mathbb{D}$  as

$$p(\lambda_1, \lambda_2) = \frac{1}{2} \log \frac{1 + m(\lambda_1, \lambda_2)}{1 - m(\lambda_1, \lambda_2)}, \quad \lambda_1, \lambda_2 \in \mathbb{D},$$

where  $m(\lambda_1, \lambda_2) = \left| \frac{\lambda_1 - \lambda_2}{1 - \overline{\lambda_1} \lambda_2} \right|$  is the Möbius function.

Let  $D \subset \mathbb{C}^n$ ,  $n \geq 1$ , be a domain. For  $z_1, z_2 \in D$  put

$$(1) \quad c_D(z_1, z_2) = \sup\{p(f(z_1), f(z_2)) : f \in \mathcal{O}(D; \mathbb{D})\},$$

$$(2) \quad c_D^*(z_1, z_2) = \sup\{m(f(z_1), f(z_2)) : f \in \mathcal{O}(D; \mathbb{D})\},$$

where  $\mathcal{O}(D; \mathbb{D})$  denotes the set of all holomorphic mappings  $D \rightarrow \mathbb{D}$ .  $c_D$  is called the Carathéodory pseudodistance for  $D$  (see e.g. [6]). In case when  $c_D$  is indeed a distance we say that  $D$  is  $c$ -hyperbolic. A  $c$ -hyperbolic domain  $D$  is called  $c$ -complete if any  $c_D$ -Cauchy sequence  $\{z_\nu\}_{\nu \geq 1} \subset D$  converges to a point  $z_0 \in D$  (w.r.t. Euclidean topology).

The aim of this paper is to study more carefully the completeness on the complex plane. Along the way we simplify the proofs by M. A. Selby [8–10] and by N. Sibony [11].

We say that a measurable set  $F \subset \mathbb{C}$  is of positive density at a point  $\zeta \in \mathbb{C}$  if

$$\limsup_{r \rightarrow 0+} \frac{\mathcal{L}(\overline{\mathbb{D}(\zeta; r)} \cap F)}{r^2} > 0.$$

First we show the following result.

**THEOREM 2.** *Let  $D \subset \mathbb{C}$  be a domain and let  $\zeta \in \partial D$  be its boundary point. If  $\zeta$  is not a weak peak point for  $D$  then*

$$\lim_{r \rightarrow 0+} \frac{\mathcal{L}(\mathbb{D}(\zeta; r) \cap D)}{\pi r^2} = 1.$$

We have the following inverse of Theorem 2.

**THEOREM 3.** *Let  $D \subset \mathbb{C}$  be a domain and let  $\zeta \in \partial D$  be its boundary point. Assume that*

$$\lim_{r \rightarrow 0+} \frac{\mathcal{L}(\mathbb{D}(\zeta; r) \cap D)}{\pi r^2} = 1.$$

*Then the following conditions are equivalent to conditions (1), (2), (3) in Theorem 1:*

- (4) *there exists a set  $A \subset D$  of positive density at  $\zeta$  such that for any sequence  $\{z_\nu\}_{\nu \geq 1} \subset A$  with  $z_\nu \rightarrow \zeta$  we have  $c_D(z_0, z_\nu) \rightarrow \infty$ ;*
- (5) *there exists a set  $A \subset D$  of positive density at  $\zeta$  such that for any sequence  $\{z_\nu\}_{\nu \geq 1} \subset A$  such that  $z_\nu \rightarrow \zeta$  there follows that  $\{z_\nu\}$  is not a  $c_D$ -Cauchy sequence.*

Note that the implications (1)  $\implies$  (4)  $\implies$  (5) are straightforward. Essentially, the main result of the paper is showing that (5)  $\implies$  (2). In case  $A = \Omega$  in Theorem 3, the result is proved in [8] and [11].

**2. Proof of Theorem 1.** Recall the definition of the analytic capacity (see e.g. Chapter VIII in [3]). Let  $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$  denote the Riemann sphere. The analytic capacity of a compact set  $K$  is defined by

$$\gamma(K) = \sup\{|f'(\infty)| : f \in \mathcal{O}(\Omega), \|f\| \leq 1, f(\infty) = 0\},$$

where  $\Omega$  is the unbounded component of  $\widehat{\mathbb{C}} \setminus K$  and

$$f'(\infty) = \lim_{z \rightarrow \infty} z(f(z) - f(\infty)).$$

For any set  $F \subset \mathbb{C}$  we put

$$\gamma(F) = \sup\{\gamma(K) : K \subset F \text{ compact}\}.$$

Recall also the following characterization (see Theorem VIII.4.5 in [3]).

**THEOREM 4** (Melnikov's criterion). *Let  $K \subset \mathbb{C}$  be a compact set and let  $\zeta \in K$ . Then  $\zeta$  is a peak point for  $R(K)$  if and only if*

$$\sum_{n=1}^{\infty} 2^n \gamma(A_n(\zeta) \setminus K) = +\infty.$$

Note that the implication (1)  $\implies$  (2) in Theorem 1 is immediate. Let us show the implication (2)  $\implies$  (3).

**PROOF OF (2)  $\implies$  (3) IN THEOREM 1.** Assume that

$$\sum_{n=1}^{\infty} 2^n \gamma(A_n(\zeta) \setminus D) < +\infty.$$

Using the continuity property of the analytic capacity (see the proof of Theorem 3.1 in [4]) one can show that there exists a compact set  $K \subset D \cup \{\zeta\}$  such that

$$\sum_{n=1}^{\infty} 2^n \gamma(A_n(\zeta) \setminus K) < +\infty.$$

By Melnikov's criterion  $\zeta$  is a not peak point for  $R(K)$ . Hence, by Bishop's characterization of peak points (see e.g. [3]) there exists a Borel probability measure  $\mu$  on  $K$  such that  $\mu(\{\zeta\}) = 0$  and

$$f(\zeta) = \int f d\mu \quad \text{for any } f \in R(K).$$

Note that  $A(D \cup \{\zeta\}) \subset R(K)$  (see Corollary 8 below). Hence,

$$f(\zeta) = \int f d\mu \quad \text{for any } f \in A(D \cup \{\zeta\}).$$

A contradiction. □

**3. Proof of Theorem 2.** Let  $\mathcal{L}$  denote the Lebesgue measure in  $\mathbb{C}$ . Recall the following well-known result (see e.g. [1], Lemma 1.5).

**PROPOSITION 5.** *Let  $K \subset \mathbb{C}$  be a compact set. Then the function*

$$f(z) = \int_K \frac{d\mathcal{L}(\eta)}{z - \eta}$$

*is holomorphic on  $\widehat{\mathbb{C}} \setminus K$ , continuous on  $\widehat{\mathbb{C}}$  and  $f(\infty) = 0$ . Moreover,*

$$(3) \quad |f(z)| \leq \int_K \frac{1}{|z - \eta|} d\mathcal{L}(\eta) \leq 2\sqrt{\pi\mathcal{L}(K)}.$$

As a corollary of Proposition 5 we get Theorem 2 (cf. Corollary VIII.4.2 in [3]).

**PROOF OF THEOREM 2.** Assume that

$$\limsup_{r \rightarrow 0+} \frac{\mathcal{L}(\overline{\mathbb{D}(\zeta; r)} \setminus D)}{r^2} > 0.$$

Choose  $r_n \rightarrow 0+$  and  $b > 0$  such that  $\mathcal{L}(K_n) > br_n^2$ , where  $K_n = \overline{\mathbb{D}(\zeta; r_n)} \setminus D$ . Put

$$g_n(z) = \frac{1}{\mathcal{L}(K_n)} \cdot (z - \zeta) \int_{K_n} \frac{d\mathcal{L}(\eta)}{z - \eta}.$$

From Proposition 5 there follows that  $g_n$  is a continuous function on  $\widehat{\mathbb{C}}$ , holomorphic on  $\widehat{\mathbb{C}} \setminus K_n$ ,  $g_n(\infty) = 1$ .

Note that for any  $z \in \mathbb{C}$  such that  $|z - \zeta| \leq r_n$  we have

$$|g_n(z)| \leq \frac{2r_n \sqrt{\pi \mathcal{L}(K_n)}}{\mathcal{L}(K_n)} \leq 2\sqrt{\frac{\pi}{b}}.$$

From the maximum principle we see that the above inequality holds on the whole  $\widehat{\mathbb{C}}$ . Now we proceed as in the proof of Theorem VIII.4.1 in [3] and get a weak peak function for  $D$ .  $\square$

**4. Proof of Theorem 3.** We denote by  $\mathcal{M}$  the set of all positive finite Borel measures in  $\mathbb{C}$ . For  $\mu \in \mathcal{M}$  we define its Newton potential as

$$M(z) = M_\mu(z) = \int \frac{1}{|z - \eta|} d\mu(\eta).$$

From the inequality (3) we have

$$(4) \quad \frac{1}{\pi r^2} \int_{\mathbb{D}(\eta, r)} |z - \eta| \cdot M(z) d\mathcal{L}(z) \leq 2\mu(\mathbb{C}),$$

and, therefore,  $M < \infty$  a.e. on  $\mathbb{C}$ . The following result, which essentially is a corollary of Fubini's theorem, shows the behaviour of the left side of (4) when  $r \rightarrow 0$  (see e.g. [12], Lemma 26.16).

**PROPOSITION 6.** *Let  $\mu \in \mathcal{M}$ . For any  $\eta \in \mathbb{C}$  we have*

$$\lim_{r \rightarrow 0} \frac{1}{\pi r^2} \int_{\mathbb{D}(\eta, r)} |z - \eta| \cdot M(z) d\mathcal{L}(z) = \mu(\{\eta\}).$$

*In particular, if  $\mu(\{\eta\}) = 0$ , then for any  $\epsilon > 0$  the set*

$$\Pi(\epsilon) = \{z \in \mathbb{C} : |z - \eta| \cdot M(z) > \epsilon\}$$

*is of zero density at  $\eta$ , i.e.,*

$$\lim_{r \rightarrow 0} \frac{\mathcal{L}(\Pi(\epsilon) \cap \mathbb{D}(\eta, r))}{r^2} = 0.$$

Recall the following approximation result (see e.g., Theorem 10.8 in Chapter VIII in [3]).

**THEOREM 7.** *Let  $D \subset \mathbb{C}$  be a domain and let  $\zeta \in \partial D$  be its boundary point. For any  $f \in H^\infty(D)$  there exists a sequence  $\{f_n\}_{n \geq 1} \subset H^\infty(D)$  with  $\|f_n\|_D \leq 17\|f\|_D$  such that  $f_n \rightarrow f$  locally uniformly on  $D$  and each  $f_n$  extends holomorphically to a neighborhood of  $\zeta$ . Moreover, if  $f$  extends continuously to  $\zeta$ , then  $f_n$  tends to  $f$  uniformly on  $D$ .*

From this we get.

**COROLLARY 8.** *Let  $D \subset \mathbb{C}$  be a domain and let  $\zeta \in \partial D$  be its boundary point. Then for any compact set  $K \subset D \cup \{\zeta\}$  we have  $A(D \cup \{\zeta\}) \subset R(K)$ .*

The following simple observation holds true.

PROPOSITION 9. *Let  $D \subset \mathbb{C}$  be a domain and let  $\zeta \in \partial D$ . Assume that  $\mu$  is a finite Borel measure in  $D$  such that*

$$|f(\zeta)| \leq \int |f| d\mu$$

*for any  $f \in A(D \cup \{\zeta\})$ . Then for any  $\eta \in D$  we have*

$$|f(\eta) - f(\zeta)| \leq 2\|f\|_\infty M(\eta)|\eta - \zeta|.$$

*In particular, for any  $\eta_1, \eta_2 \in D$  we have*

$$(5) \quad c_D^*(\eta_1, \eta_2) \leq 34 \left( |\zeta - \eta_1| M(\eta_1) + |\zeta - \eta_2| M(\eta_2) \right).$$

PROOF. Fix  $\eta \in D$ . Then for any  $f \in A(D \cup \{\zeta\})$  we have  $\tilde{f}(z) = \frac{f(z) - f(\eta)}{z - \eta} \in A(D \cup \{\zeta\})$ . Then

$$|\tilde{f}(\zeta)| \leq \int |\tilde{f}| d\mu.$$

Hence,

$$|f(\zeta) - f(\eta)| \leq |\zeta - \eta| \int \left| \frac{f(z) - f(\eta)}{z - \eta} \right| d\mu(z) \leq 2\|f\|_\infty M(\eta)|\eta - \zeta|.$$

Inequality (5) follows from Theorem 7.  $\square$

We have the following corollary, which proofs the implication (5)  $\implies$  (2).

COROLLARY 10. *Let  $D \subset \mathbb{C}$  be a domain and let  $\zeta \in \partial D$ . Assume that  $\mu$  is a finite Borel measure in  $D$  such that*

$$|f(\zeta)| \leq \int |f| d\mu$$

*for any  $f \in A(D \cup \{\zeta\})$ . Then for any measurable set  $A \subset D$  of positive density at  $\zeta$  there exists a  $c$ -Cauchy sequence  $\{\eta_n\}_{n \geq 1} \subset A$  such that  $\eta_n \rightarrow \zeta$ .*

PROOF. If

$$\liminf_{r \rightarrow 0} \frac{\mathcal{L}(\mathbb{D}(\zeta; r) \cap D)}{\pi r^2} < 1$$

then by Theorem 2  $\zeta$  is a weak peak point, which contradicts the existence of the measure  $\mu$ . So,

$$\lim_{r \rightarrow 0} \frac{\mathcal{L}(\mathbb{D}(\zeta; r) \cap D)}{\pi r^2} = 1.$$

Hence,

$$\limsup_{r \rightarrow 0} \frac{\mathcal{L}(\mathbb{D}(\zeta; r) \cap A)}{r^2} > 0.$$

Then by Proposition 6 there exists a sequence  $\{\eta_n\}_{n \geq 1} \subset D$  with  $\eta_n \rightarrow \zeta$  such that  $|\zeta - \eta_n|M(\eta_n) \leq \frac{1}{2^n}$ . From Theorem 9 we get the result.  $\square$

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