



Burgers-like equation for spontaneous breakdown of the chiral symmetry in QCD



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ABSTRACT

We link the spontaneous breakdown of chiral symmetry in Euclidean QCD to the collision of spectral shock waves in the vicinity of zero eigenvalue of the Dirac operator. The mechanism, originating from complex Burger's-like equation for viscid, pressureless, one-dimensional flows of eigenvalues, is similar to the recently observed weak-strong coupling phase transition in large N_c Yang–Mills theory. The spectral viscosity is proportional to the inverse size of the random matrix that replaces the Dirac operator in the universal (ergodic) regime. We obtain the exact scaling function and critical exponents of the chiral phase transition for the averaged characteristic polynomial for $N_c \geq 3$ QCD. We reinterpret our results in terms of known properties of chiral random matrix models and lattice data.

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1. Introduction

It was a recently emphasized that the Burgers equation could be used to understand universal features of the weak to strong coupling transition in two-dimensional Yang–Mills theory with a large number of colors N_c [1–3]. This transition, first studied by Durhuus and Olesen [4], can be pictured, in the language of the Burgers equation, as resulting from the collision of two spectral shock waves at the closure of the gap. The emergence of the Burgers equation in such kind of problems, and related questions in random matrix theory, seems generic, and can be simply understood by exploiting Dyson's original idea of matrix random walks. Then, after a proper rescaling of time that separates the fast motion of the eigenvalues caused by their mutual repulsion, from the diffusion that results from the random walks of the matrix elements, one can easily show, by using standard tools of statistical mechanics, that the average resolvent obeys indeed a Burgers equation (or simple generalizations) in the limit of large size matrices. Here we shall obtain the Burgers equation from an exact equation, akin to a diffusion equation, satisfied by the average characteristic polynomial. The average resolvent and average characteristic polynomial are simply related in the large N limit. However, the equation for the characteristic polynomial is exact for any

matrix size, while no such equation seems to exist for the average resolvent.

Returning to two-dimensional Yang–Mills theory, we note that the average characteristic determinant for a Wilson loop of a given area was indeed shown to obey an exact diffusion equation for any number of colors N_c (which yields a Burgers equation in the infinite N_c limit). For finite N_c , the flow of the eigenvalues is viscid, with a negative spectral viscosity $\nu_s = -\frac{1}{2N_c}$. It is the negative sign of the viscosity, or of the diffusion constant, that causes the rapid spectral oscillations at the closure of the gap. This picture was confirmed by extensive numerical studies of full Yang–Mills theory in three dimensions and recently, in four dimensions [3,5].

In this Letter, we demonstrate that a similar mechanism, governed by similar viscous spectral Burgers-like equation, is responsible for the universal spectral oscillations of the spectrum of the Dirac operator in QCD that accompany the spontaneous breakdown of chiral symmetry [6,7]. In the case of the Durhuus–Olesen transition, where the role of the time is played by the area of the Wilson loop, an explicit construction of a random matrix model was provided by Janik and Wieczorek [8], matrices attached to large loops being formed by multiplying random matrices representing small loops. Here, as already mentioned, following Dyson [9], we add a fictitious time (somewhat analogous to Schwinger's proper time) in order to describe the diffusion of the random matrices in four-dimensional Euclidean space. Similar diffusive approach turned out in the past to lead to several exact results in random matrix theory [10–12].

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2. Random matrix theory and QCD

A cornerstone for the microscopic understanding of the spontaneous breakdown of chiral symmetry is the Banks–Casher [13] relation

$$|\langle \bar{q}q \rangle| \equiv \Sigma = \pi \frac{\rho(0)}{V_4}, \quad (1)$$

where the quark condensate $\langle \bar{q}q \rangle$ is an order parameter for chiral symmetry, $\rho(0)$ is the averaged (over the gauge field configurations) level density of the Euclidean Dirac operator near vanishing eigenvalue, and $V_4 \equiv L^4$ is the Euclidean volume. This relation shows that chiral symmetry breaking requires a strong accumulation of eigenvalues near zero, i.e. a level spacing $\Delta \sim 1/L^4$, much larger than the level spacing $\Delta \sim 1/L$ of a free system [14]. This accumulation of eigenvalues leads to universal properties, that are well captured by random matrix theory: For eigenvalues smaller than a characteristic energy scale, referred to as the Thouless scale E_{Th} , the fluctuations of eigenvalues are described by chiral random matrix models respecting the global symmetries of the Dirac Hamiltonian. In QCD, the condition $E_{Th}^{QCD}/\Delta = F_\pi^2 L^2 \gg 1$, where F_π is the pion decay constant, determines the regime of applicability of random matrix theory [15,16].

In Euclidean QCD, all four Dirac matrices can be chosen to be anti-hermitian, hence the spectrum of the massless Dirac operator $D \equiv i\gamma_\mu(\partial_\mu - igA_\mu)$ is purely imaginary. The partition function, for a fixed topological sector, reads

$$Z_v^{QCD} = \left\langle \prod_f m_f^\nu \prod_l (\lambda_l^2 + m_f^2) \right\rangle_v \quad (2)$$

where the averaging is done with respect to gluonic configurations of a given topological charge ν , $\pm i\lambda_k$ are the eigenvalues of D , and m_f is the mass of a quark with flavor f . Due to chiral symmetry non-zero eigenvalues of D come in pairs, and the number of fermionic zero modes is related to the topological charge. In the chiral Gaussian random matrix model (hereafter χ GUE), corresponding to QCD with $N_c \geq 3$, the role of the massless Dirac operator is played by a random matrix $W (= -iD)$ of the following form:

$$W = \begin{pmatrix} 0 & K^\dagger \\ K & 0 \end{pmatrix}. \quad (3)$$

Here K is a rectangular $M \times N$ ($M > N$) matrix with complex entries, $K_{ij} \equiv x_{ij} + iy_{ij}$, where x_{ij} and y_{ij} are drawn from a Gaussian distribution. Note that W is hermitian, so that its eigenvalues κ_i 's are real. The block-diagonal structure of (3) reflects the chiral symmetry of the Dirac operator: W anticommutes with the analogue of the Dirac matrix γ_5 , defined here as $\gamma_5 = \text{diag}(\mathbf{1}_N, -\mathbf{1}_M)$. This implies in particular that the eigenvalues come in pairs of opposite values, $(\kappa, -\kappa)$. By construction, W has in addition $\nu \equiv M - N$ zero eigenvalues. These mimic the zero modes of quarks propagating in gauge fields of non-trivial topology.

General spectral properties of the random matrices can be obtained from correlation functions containing both products and ratios of the characteristic polynomial [17]

$$C(w_1, \dots, w_f; w'_1, \dots, w'_b) = \left\langle \frac{\prod_{i=1}^f Z(w_i)}{\prod_{j=1}^b Z(w'_j)} \right\rangle \quad (4)$$

where $\langle \dots \rangle$ denote averaging with respect to the χ GUE measure, and the characteristic polynomial is $Z(w) = \det(w - W)$. Directly related to (4) is the resolvent, $R(z) = \partial_w C(z; w)|_{z=w}$, whose imaginary part yields the spectral density. In the vicinity of zero, the microscopic (unfolded) resolvent predicts for QCD [18]

$$\frac{R(z)}{V_4 \Sigma} = x[I_a(x)K_a(x) + I_{a+1}(x)K_{a-1}(x)] \quad (5)$$

where $a = N_f + |\nu|$, with N_f the number of quark flavors, and $x = zV_4 \Sigma$. The appearance of Bessel functions I_a , K_a in the correlation functions, such as in Eq. (5), is generic. They encode the universal behaviors that show up already in the simplest objects, the average characteristic polynomial and/or the average of the inverse of the characteristic polynomial. As the first result of the Letter, we shall obtain an exact differential equation for the averaged characteristic polynomial. This equation is akin to a diffusion equation, from which a Burgers equation can be derived from a simple transformation.

3. Burgers equation and the characteristic polynomial for Dirac operator

We assume now that the entries of the matrix K follow independent random walks. Let us denote by $P(X, Y, t)$ the joint probability that the entries of K take the values $X = \{x_{ij}\}$ and $Y = \{y_{ij}\}$ at time t . The random walks of the matrix elements translate into the following diffusion equation for $P(X, Y, t)$:

$$\partial_t P(X, Y, t) = \frac{1}{4} \sum_{i=1}^M \sum_{j=1}^N (\partial_X^2 + \partial_Y^2) P(X, Y, t). \quad (6)$$

We shall be interested in this Letter with the time evolution of the averaged characteristic polynomial

$$\begin{aligned} Q_n^\nu(w, t) &\equiv \langle \det[w - W] \rangle \\ &= \int dX dY P(X, Y, t) \det[w - W(X, Y)], \end{aligned} \quad (7)$$

where $n = N + M$ is the rank of the polynomial, w is an arbitrary complex number, and $dX = \prod_{ij} dx_{ij}$, and similarly for dY . In order to get the equation obeyed by $Q_n^\nu(w, t)$, we consider first the equation satisfied by the average characteristic polynomial of the associated $N \times N$ Wishart matrix $K^\dagger K$, $M_N^\nu \equiv \langle \det[z - K^\dagger K] \rangle$. This, we write as an integral over N Grassmann variables $\eta_i, \bar{\eta}_i$:

$$\begin{aligned} M_N^\nu(z, t) &= \int d\eta d\bar{\eta} e^{z \sum_i \bar{\eta}_i \eta_i} \int dX dY P(X, Y, t) \\ &\quad \times \exp(-\bar{\eta} K^\dagger K \eta), \end{aligned} \quad (8)$$

where $\bar{\eta} K^\dagger K \eta = \sum_{i,k=1}^N \sum_{j=1}^M \bar{\eta}_i (x_{ij} - iy_{ij})(x_{jk} + iy_{jk}) \eta_k$. To derive the equation for M , we take a time derivative of the expression above. This acts on $P(X, Y, t)$, which, using Eq. (6), we transform into derivatives with respect to x_{ij} and y_{ij} . Then, we integrate by parts, and use standard Grassmann calculus to obtain (after a somewhat tedious but straightforward calculation) the following differential equation

$$\partial_t M_N^\nu(z, t) = -z \partial_z^2 M_N^\nu(z, t) - (\nu + 1) \partial_z M_N^\nu(z, t). \quad (9)$$

This equation is valid for any N and M , and arbitrary initial conditions. Note that for the trivial initial condition $K_{ij}(t=0) = 0$, its solution is given by time dependent associated Laguerre polynomial [19].

From the equation for $M_N^\nu(z, t)$, Eq. (9) above, one easily obtains the equation for $Q_N^\nu(w, t) = w^\nu M_N^\nu(z = w^2, t)$. This reads

$$\begin{aligned} \partial_t Q_n^\nu(w, t) &= -\frac{1}{4} \partial_w^2 Q_n^\nu(w, t) - \frac{1}{4w} \partial_w Q_n^\nu(w, t) \\ &\quad + \frac{\nu^2}{4w^2} Q_n^\nu(w, t). \end{aligned} \quad (10)$$

This is the announced before, first new result of the Letter. It will be also useful to consider the equation for the Cole–Hopf transform of $Q_n^\nu(w, t)$, $f_n^\nu \equiv \frac{1}{n} \partial_w \ln(Q_n^\nu(w, t))$. This object identifies with the average resolvent in the large n limit. After a rescaling of the time, $\tau = Mt$, one gets from Eq. (10)

$$\begin{aligned} \frac{n+\nu}{n} \partial_\tau f_n^\nu + f_n^\nu \partial_w f_n^\nu \\ = -\frac{1}{2n} \left[\partial_w^2 f_n^\nu + \frac{1}{w} \partial_w f_n^\nu - \frac{1}{w^2} f_n^\nu \right] - \left(\frac{\nu}{n} \right)^2 \frac{1}{w^3}, \end{aligned} \quad (11)$$

where we have separated on the left-hand side the terms that survive the large n limit, and on the right-hand side the terms that are explicitly suppressed by powers of $1/n$. Note the crucial role played by the rescaling of time in arriving at this equation. The motivation behind this rescaling is that the diffusion associated with the random walks is taking place over a time scale that is larger, typically by a factor n , than the time scale corresponding to the local rearrangements of the eigenvalues due to their mutual repulsion [9,19]. After rescaling, the diffusion terms are dwarfed by a factor $1/n$, and the large n dynamics is dominated by repulsion. The last term, of order $1/n^2$ finds its origin in the kinematical zero modes present when $\nu \neq 0$.

4. Large n limit

We consider now the limit $n \rightarrow \infty$, with ν constant. We set $g(w, \tau) = \lim_{n \rightarrow \infty} f_n^\nu(w, \tau)$. Eq. (11) reduces then to the inviscid Burgers equation [1,20,21], independent of ν :

$$\partial_\tau g(w, \tau) + g(w, \tau) \partial_w g(w, \tau) = 0. \quad (12)$$

It can be solved using complex characteristics. We choose the system to be initially in a chiral symmetric state, with eigenvalues localized at $\pm a$. The characteristic lines are given by $w = \xi + \tau g(\xi)$, with

$$g_0(w) \equiv g(w, \tau = 0) = \frac{1}{2} \left(\frac{1}{w-a} + \frac{1}{w+a} \right) = \frac{w}{w^2 - a^2}. \quad (13)$$

The solution $g(w, \tau)$ is constant along the characteristics, meaning $g(w, \tau) = g_0[\xi(w, \tau)]$. By eliminating ξ , one obtains an implicit equation for g :

$$\tau^2 g^3 - 2\tau w g^2 + (w^2 - a^2 + \tau)g - w = 0. \quad (14)$$

This equation can be solved by elementary means, and well-known results are recovered. In fact, the change of variables $\omega = w/\sqrt{\tau}$, $d = a/\sqrt{\tau}$, and $G_{st}(\omega) = \sqrt{\tau} g(\omega\sqrt{\tau}, \tau)$ transforms this equation into $G_{st}^3 - 2\omega G_{st}^2 + (\omega^2 - d^2 + 1)G_{st} = \omega$, an equation for a time-independent resolvent G_{st} that has been obtained in this context using different techniques [22–24]. In previous studies, the parameter d was introduced as the “deterministic” part of the chiral matrix (with K in Eq. (3) replaced by $K + d$, d being fixed and K random), in order to control the approach to the chiral transition. The aforementioned change of variables renders transparent the dynamics captured by the Burgers equation: For small time, i.e. $\tau < a^2$, the spectral density remains localized in humps centered around the values $\pm a$. As time reaches the critical value $\tau_c = a^2$ (corresponding to $d = 1$ in the static approach), the two domains of the spectrum merge at the origin, which we picture as the collision of two spectral shock waves. A finite “condensate” then develops, and chiral symmetry is spontaneously broken.

We now recover these features by studying the singularities of the characteristics, and the behavior of the solution in the vicinity of these singularities. This brief discussion will pave the way

for the scaling analysis to be performed in the next section. Singularities appear when characteristics start to cross (appearance of a spectral shock wave). This occurs for values of ξ that obey the equation,

$$\left. \frac{dw}{d\xi} \right|_{\xi=\xi_c} = 0 = 1 + \tau g'_0(\xi_c), \quad (15)$$

that is

$$\tau(a^2 + \xi_c^2) = (\xi_c^2 - a^2)^2. \quad (16)$$

The values of ξ_c correspond to the edges of the spectrum when $\tau < \tau_c = a^2$. At the critical time $\tau_c = a^2$, where the two humps of the spectrum start to merge, the equation for ξ_c admits a double solution at $\xi_c = 0$, which splits into two purely imaginary and opposite solutions when $\tau > a^2$.

In order to study the behavior of g at the edge of the spectrum, we expand g_0 in the vicinity of a singular point

$$\begin{aligned} g_0(\xi) = g_0(\xi_c) + (\xi - \xi_c)g'_0(\xi_c) + \frac{1}{2}(\xi - \xi_c)^2 g''_0(\xi_c) \\ + \frac{1}{6}(\xi - \xi_c)^3 g'''_0(\xi_c) + \dots \end{aligned} \quad (17)$$

When $\tau < a^2$, $g'_0(\xi_c) = -1/\tau$ and, using $g_0 = (w - \xi)/\tau$, we get

$$w - w_c = \frac{\tau}{2} (\xi - \xi_c)^2 g''_0(\xi_c) + \dots \quad (18)$$

One can then easily invert the relation between w and ξ , and get (for the rightmost edge)

$$\xi - \xi_c = \pm \sqrt{\frac{2}{\tau g''_0(\xi_c)}} \sqrt{w - w_c}, \quad (19)$$

so that

$$\begin{aligned} g(w, \tau) = g_0(\xi) \simeq g_0(\xi_c) + (\xi - \xi_c)g'_0(\xi_c) \\ = g_0(\xi_c) \mp \frac{1}{\tau} \sqrt{\frac{2}{\tau g''_0(\xi_c)}} \sqrt{w - w_c}, \end{aligned} \quad (20)$$

which exhibits the familiar square root behavior of the spectrum near its (right) edge.

For $\tau = a^2$, a similar analysis taking into account that $g_0(\xi_c) = 0 = g''_0(\xi_c)$, so that the cubic term must be kept, yields (for $w_c = 0$)

$$g(w, \tau) = \left(-\frac{w}{a^2} \right)^{1/3}. \quad (21)$$

For time $\tau > a^2$, we have $g(w = 0, \tau) = g_0(\xi_c(w = 0, \tau)) = -\frac{\sqrt{a^2 - \tau}}{\tau}$, which is imaginary and hence directly proportional to the spectral density $\rho(0)$.

The behavior of $g(w)$ at the edge of the spectrum determines the average eigenvalue spacing in the limit of large matrices. A singularity $\sim |w - w_c|^\alpha$ yields a level spacing $\sim n^{-\delta}$ with $\delta = 1/(1 + \alpha)$. We have therefore $\delta = 2/3$ for $\tau < a^2$, $\delta = 3/4$ for $\tau = a^2$ and $\delta = 1$ for $\tau > a^2$. We shall exploit these properties in the next section.

5. Critical properties of the characteristic polynomial

In this section we carry out a scaling analysis of the average characteristic polynomial $Q(w, \tau)$, or its Cole–Hopf transform $f(w, \tau)$, in the vicinity of the singular points. To that aim, we set

$$w = w_c + n^{-\delta} s, \quad f_N^\nu \rightarrow g_0(\xi_c) + n^{-\gamma} \chi(s, \tau), \quad (22)$$

with $\gamma = 1 - \delta$, and s and $\chi(s, \tau)$ remain finite as $n \rightarrow \infty$.

5.1. Airy edge

Let us focus first on the left edge of the positive part of the spectrum, for $\tau < a^2$, where we expect the solution to be of the form (see Eq. (20))

$$f_n^\nu(w, \tau) \approx g_0(\xi_c) + g'_0(\xi_c) \sqrt{\frac{2(w - w_c)}{\tau g''_0(\xi_c)}}. \quad (23)$$

This exhibits a square root singularity ($\delta = 2/3$), and in line with the discussion above we set $w = w_c + n^{-\frac{2}{3}}s$ and $f_n^\nu \rightarrow g_0(\xi_c) + n^{-\frac{1}{3}}\chi(s, \tau)$. (Notice that $g_0(\xi_c)$ is a function of τ only, $g_0(\xi_c) = \partial_\tau w_c$.) Substituting this ansatz into Eq. (11), and keeping only the dominant terms as $n \rightarrow \infty$, we get (with $\dot{g}_0(\xi_c) = \partial_\tau g_0(\xi_c)$)

$$\dot{g}_0(\xi_c) + \chi \partial_s \chi + \frac{1}{2} \partial_s^2 \chi = 0 = \partial_p (2p \dot{g}_0(\xi_c) + \chi^2 + \partial_s \chi). \quad (24)$$

This equation is easily integrated, $\chi^2 + \partial_s \chi + 2s \dot{g}_0(\xi_c) + u(\tau) = 0$, with $u(\tau)$ an arbitrary function of τ . Defining $\chi(s, \tau) = \partial_s \ln \phi(s, \tau)$ and shifting $s = \tilde{s} - u(\tau)/(2\dot{g}_0(\xi_c))$ (with $\phi(s, \tau) = \psi(\tilde{s}, \tau)$) one transforms the equation above into the equation for the Airy function, $\partial_p^2 \psi + 2\dot{g}_0(\xi_c) \tilde{p} \psi = 0$. The looked for solution is therefore

$$\phi(s, \tau) = Ai \left[-(2\dot{g}_0)^{\frac{1}{3}} \left(s + \frac{u}{2\dot{g}_0} \right) \right]. \quad (25)$$

The arbitrary function $u(\tau)$ can be determined by a careful matching of the asymptotic form as $s \rightarrow \infty$ of the solution to its large N limit. One then finds that u actually vanishes [19].

5.2. Bessel universality

For $\tau > a^2$, there is no singular behavior ($\delta = 1$). So, we set $w = n^{-1}s$ and $f_n^\nu = \chi$. In the large n limit (at ν constant) we obtain, following the same manipulations as above, the following partial differential equation:

$$0 = \partial_s \left(\chi^2 + \partial_s \chi + \frac{\chi}{s} - \frac{\nu^2}{s^2} \right), \quad (26)$$

which integrates to $\chi^2 + \partial_s \chi + \frac{\chi}{s} - \frac{\nu^2}{s^2} + u(\tau) = 0$. Then, setting $\chi(s, \tau) = \partial_s \ln \phi(s, \tau)$ we obtain

$$s^2 \partial_s^2 \phi + s \partial_s \phi + \phi [s^2 u(\tau) - \nu^2] = 0, \quad (27)$$

whose solution is

$$\phi(s, \tau) = J_\nu [s \sqrt{u(\tau)}]. \quad (28)$$

The determination of the arbitrary function $u(\tau)$ proceeds as in the previous case, by matching the asymptotic $\chi(s, \tau) \sim -i\sqrt{u(\tau)}$ with the large N solution. This gives $\sqrt{u(\tau)} = \sqrt{\tau - a^2}/\tau$. We recover the scaling of the ratio of spectral densities discussed for instance in [25–27].

5.3. Bessoid (axially symmetric Pearcey) universality

Finally, we move to the case of $\tau = \tau_c = a^2$, which will lead to the second new result of this Letter. As we have shown above, at $\tau = a^2$, the two pre-shocks collide. Before the collision, these pre-shocks are accompanied by oscillations of the Airy type. Our purpose now is to describe the modification of the pattern of oscillations for τ close to τ_c . To perform this analysis, it is most convenient to start from the diffusion equation obeyed by the characteristic polynomial, i.e., Eq. (10) which, after changing t into $\tau = Mt$, we rewrite as

$$\begin{aligned} \partial_\tau Q_n^\nu(w, \tau) = & -\frac{1}{4M} \partial_w^2 Q_n^\nu(w, \tau) - \frac{1}{4Mw} \partial_w Q_n^\nu(w, \tau) \\ & + \frac{\nu^2}{4Mw^2} Q_n^\nu(w, \tau). \end{aligned} \quad (29)$$

This equation is to be solved with the initial condition $Q_n^\nu(w, \tau = 0) = w^\nu (w^2 - a^2)^N$ (where the function of w , $(w^2 - a^2)^N$ is defined with a cut between $-a$ and a). It can be verified by a direct calculation (that does not require the explicit calculation of the integral below), that

$$\begin{aligned} Q_n^\nu(w, \tau) = & C \tau^{-1} \int_0^{\exp(i\phi_y)\infty} y^{\nu+1} \exp\left(M \frac{w^2 + y^2}{\tau}\right) \\ & \times I_\nu\left(\frac{2Myw}{\tau}\right) (y^2 - a^2)^N dy, \end{aligned} \quad (30)$$

where $C = (-1)^{\nu+1/2} 2M$, is a solution with the proper initial condition. (For $\nu = 0$, it agrees with a known solution [28].) The y -integral runs over a half-line that starts at the origin and goes to infinity, making a constant angle $\phi_y = \arg(y)$ with the real axis, with $-\pi \leq \arg(y) < \pi$. For the integral to be convergent, we require $\frac{\pi}{4} < |\phi_y| < \frac{3\pi}{4}$. The modified Bessel function $I_\nu(x)$ has the following asymptotic expansion $\lim_{|x| \rightarrow \infty} I_\nu(x) \simeq \frac{1}{\sqrt{2\pi x}} e^x$, valid for $|\arg(x)| < \frac{\pi}{2}$ (see [29]; here $x = \frac{2Myw}{\tau}$). This is useful in particular to verify the initial behavior. Indeed, as $\tau \rightarrow 0$, one may estimate the integral using the saddle point method. The saddle point equation yields $y + w = 0$, which fixes in particular $\arg(y) = \arg(-w)$. This new condition for ϕ_y , together with the convergence condition noted above, are easily seen to be compatible with the condition of validity of the asymptotic expansion of the Bessel function. In turn, these conditions limit the allowed arguments of w to $\frac{\pi}{4} < |\arg(w)| < \frac{3\pi}{4}$.

The integral representation (30) of the characteristic polynomial allows us to study the vicinity of the critical point. We note that the saddle point equation reads (in the large n limit)

$$\frac{y}{\tau} + \frac{w}{\tau} + \frac{y}{y^2 - a^2} = 0. \quad (31)$$

Identifying $y = -\xi$, we recognize the equation for the characteristic lines. This indicates how the large n dynamics is coded in this integral. We shall focus more specifically at the critical point, $w = 0, y = 0, \tau = a^2$. In this regime, we may expand $\ln(a^2 - y^2) \approx \ln(a^2) - \frac{y^2}{a^2} - \frac{y^4}{2a^4}$, and obtain

$$\begin{aligned} a^{2N} C' \tau^{-1} \exp\left(\frac{Mw^2}{\tau}\right) \int_0^{\exp(i\phi_y)\infty} y^{\nu+1} \\ \times \exp\left[-\frac{Ny^4}{2a^4} - \frac{Ny^2}{a^2} + \frac{My^2}{\tau}\right] I_\nu\left(\frac{2Myw}{\tau}\right) dy, \end{aligned} \quad (32)$$

with $C' = C(-1)^N$. To capture the critical behavior also as a function of time, we set $\tau = a^2 + \theta$. Then Eq. (32) becomes

$$\begin{aligned} a^{2(N-1)} C' \exp\left(\frac{Mw^2}{a^2}\right) \int_0^{\exp(i\phi_y)\infty} y^{\nu+1} \\ \times \exp\left[-\frac{Ny^4}{2a^4} + \nu \frac{y^2}{a^2} - \frac{My^2 \theta}{a^4}\right] I_\nu\left(\frac{2Myw}{a^2}\right) dy. \end{aligned} \quad (33)$$

This expression suggests the following change of variables that will ensure a smooth large n limit:

$$y = n^{-\frac{1}{4}} au, \quad w = n^{-\frac{3}{4}} aq, \quad \theta = n^{-\frac{1}{2}} a^2 \varphi. \quad (34)$$

We then define

$$B^\nu(q, \varphi) \equiv \lim_{N, M \rightarrow \infty} C'^{-1} a^{-n} n^{\frac{\nu+2}{4}} (-1)^{\nu+1} \times Q_n^\nu(n^{-\frac{3}{4}} a q, a^2 + n^{-\frac{1}{2}} a^2 \varphi), \quad (35)$$

where the limit is taken with ν constant. Finally

$$B^\nu(q, \varphi) = (-1)^{\nu+1} \int_0^{\exp(i\phi_q \pm i\pi)\infty} u^{\nu+1} \times \exp\left(-\frac{1}{4}u^4 - \frac{1}{2}u^2\varphi\right) I_\nu(qu) du. \quad (36)$$

This exact scaling function for the characteristic polynomial is the second important result of this Letter. We note that in the special case $\varphi = 0$, $\nu = 0$, $q = im$, where m is real, the above result reduces to the microscopic QCD partition function at the critical point obtained using other methods (Eq. (8) in [30]). This agreement provides another test for the validity of our approach. Furthermore, (36) has a form very similar to the Pearcey function, which gives the asymptotic behavior of the characteristic polynomial in the case when gap closes for GUE [31] or in the case of unitary diffusion on the circle [5]

$$P(q, \varphi) = \int_{-\infty}^{\infty} dy \exp(-y^4 - y^2\varphi + qy). \quad (37)$$

Here q is the rescaled angle representing the position of the eigenvalue on the unitary circle and φ parameterizes the fluctuations around the critical area. The critical indices that determine the scaling with n in Eq. (34) are identical to those in the Pearcey integral, but the form of the integral is different. The reason is the chiral symmetry, which imposes an additional polar symmetry of the spectrum, trading the exponential function of q in the Pearcey integral (37) for a Bessel function in Eq. (36).

6. Conclusions

In this Letter we have obtained an exact differential equation for the average characteristic polynomial of a chiral random matrix, and its Cole–Hopf transform. For the latter, the equation takes the form of a generalized viscous Burgers equation, where the viscosity is proportional to the inverse of the size of the matrix, but with a negative sign. This allowed us to provide a complete description of the full critical behavior of the average characteristic polynomial in chiral QCD, based on a single equation. In particular, we considered the case of chiral Gaussian Unitary Ensembles and we have identified the exact universal scaling function (Bessoid $B^\nu(q, \varphi)$) in the vicinity of the chiral critical point for the average characteristic polynomial. We did not analyze in this Letter the properties of the average of the inverse characteristic polynomial, but we have checked that it fulfills similar equations, albeit with non-trivial initial conditions (singular at $w = 0$), alike in the cases of unitary and GUE diffusions.

Since F_π scales like $\sqrt{N_c}$, more and more eigenvalues of the Dirac operator fall into the universal window when the number of colors tends to infinity, the volume of the lattice being kept finite. This suggests, that the present study is relevant also for analyzing the spontaneous breakdown of chiral symmetry at finite volume and large N_c QCD, which was observed, and explained by Neuberger and Narayanan [32]. For small lattice sizes, chiral symmetry is unbroken, while at some critical scale L_c a condensate is formed. The same authors [33] have also observed, that the N_c -dependence of the level spacing closest to zero goes from $1/N_c$ in the broken

chiral symmetry phase to $1/N_c^{2/3}$ in the symmetric (gapped) phase. At $L = L_c$, the critical scaling changes to $1/N_c^{3/4}$ behavior and the condensate vanishes at the critical size L_c as $\sqrt{L - L_c}$. These results are in agreement with our analysis.

The critical universal scaling function (Bessoid) for large N_c Dirac operator resembles closely the critical universal scaling function (Pearcey's cuspid) for the weak to strong coupling transition in Yang–Mills theory at large N_c . It would be interesting to study numerically both transitions simultaneously (at least in some simple model like [32]) to see the interplay between the cuspid and the Bessoid, or, in other words, the relation between the critical size L_c for chiral symmetry breakdown and the critical area $\sim L_c^2$ for the weak to strong coupling transition in Yang–Mills theory. Of particular interest would be to measure on the lattice the microscopic spectral density exactly at the point of the transition. As far as we know, the microscopic spectral density at criticality was constructed explicitly only for $a = N_f + |\nu| = 0$ case [30], and was never checked by the lattice simulation. Taking into account the still ongoing discussion on the nature of chiral phase transition, its relation to confinement and Anderson localization [34,35], lattice verification of analytic predictions for microscopic densities at the critical point may be a powerful tool to shed more light on this aspect of strong interactions.

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