Separator Theorem and Algorithms for Planar Hyperbolic Graphs

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Abstract

The hyperbolicity of a graph, informally, measures how close a graph is (metrically) to a tree. Hence, it is intuitively similar to treewidth, but the measures are formally incomparable. Motivated by the broad study of algorithms and separators on planar graphs and their relation to treewidth, we initiate the study of planar graphs of bounded hyperbolicity.

Our main technical contribution is a novel balanced separator theorem for planar δ-hyperbolic graphs that is substantially stronger than the classic planar separator theorem. For any fixed $\delta \geqslant 0$, we can find a small balanced separator that induces either a single geodesic (shortest) path or a single geodesic cycle in the graph.

An important advantage of our separator is that the union of our separator (vertex set Z) with any subset of the connected components of $G - Z$ induces again a planar δ-hyperbolic graph, which would not be guaranteed with an arbitrary separator. Our construction runs in near-linear time and guarantees that the size of the separator is $\mathrm{poly}(\delta) \cdot \log n$.

As an application of our separator theorem and its strong properties, we obtain two novel approximation schemes on planar δ-hyperbolic graphs. We prove that both Maximum Independent Set and the Traveling Salesperson problem have a near-linear time FPTAS for any constant δ, running in $n\,\mathrm{polylog}(n) \cdot 2^{\mathcal{O}(\delta^2)} \cdot \varepsilon^{-\mathcal{O}(\delta)}$ time.

We also show that our approximation scheme for Maximum Independent Set has essentially the best possible running time under the Exponential Time Hypothesis (ETH). This immediately follows from our third contribution: we prove that Maximum Independent Set has no $n^{o(\delta)}$-time algorithm on planar δ-hyperbolic graphs, unless ETH fails.

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1 Introduction

Many graph problems are known to be efficiently solvable on trees. A substantial research effort has long been underway to transfer this simple insight to more complex graphs that are somehow "tree-like". While many measures have been proposed (see e.g. [48, 71]), one of the most successful measures of tree-likeness has arguably been treewidth. We refer to the surveys of Bodlaender [9, 11, 12] or the recent book by Fomin et al. [39] for an overview of treewidth. The study of graph structure and algorithms on graphs of bounded treewidth has led to many celebrated results (see, e.g., [4, 10, 28, 29, 32, 68, 69]) that are important in their own way or as a subroutine in other algorithms. Treewidth has also been highly useful in practice, for example, for probabilistic inference in Bayesian networks [59]. For many other real-world networks and standard random models of them, treewidth is unfortunately very high [2, 43, 64, 31]. This makes algorithms for graphs of bounded treewidth not very useful in this context and seems to cast doubt on the tree-likeness of such networks.

It has been shown, however, that many real-world networks are *metrically* close to a tree. This idea can be formally cast to Gromov's notion of hyperbolicity. The graph G is *δ-hyperbolic* if for all $x, y, z, w \in V(G)$ we have that the greater two among the sums

$$\mathrm{dist}(x, y) + \mathrm{dist}(z, w),\ \mathrm{dist}(x, z) + \mathrm{dist}(y, w),\ \mathrm{dist}(x, w) + \mathrm{dist}(y, z)$$

differ by at most 2δ, where $\mathrm{dist}(.,.)$ denotes the shortest path distance in G. The *hyperbolicity* of a graph G is the smallest δ such that G is δ-hyperbolic. It is difficult to build good intuition for this definition without diving into hyperbolic geometry, but the alternative notion of δ'-slimness can be used instead and is easier to picture. Consider a triplet (x, y, z) of vertices and take any three shortest paths connecting the pairs (x, y), (y, z), and (z, x). We call the vertices of these three paths the sides of the triangle xyz. We then say that the graph is *δ'-slim* if for each triangle the side xy is within distance δ' from the union of the sides yz and zx. The *slimness* of a graph G is then the smallest δ' such that G is δ'-slim. It is known that the hyperbolicity and slimness of a graph differ by a constant factor from each other [45, 19], so they can almost be considered equivalent for the purposes of this section.

Since the notions of hyperbolicity and slimness are central to our work, let us develop some intuition for them through some simple examples. We can observe that the hyperbolicity and the slimness of a tree are both 0, which is in line with the idea that hyperbolicity measures tree-likeness in a metric sense. On the other hand, cycles are not metrically close to trees. Indeed, the cycle C_n is (roughly) $(n/4)$-hyperbolic and $(n/6)$-slim, which can be checked by placing the quadruple or the triangle vertices at equal distances along the cycle. In general, the hyperbolicity of a graph is bounded by its diameter or even its treelength[1] [21]. This is in a stark contrast with treewidth – note that C_n has treewidth 2. As another instructive example, observe that both the treewidth and the hyperbolicity of the $n \times n$ grid are n [31]. However, treewidth and hyperbolicity are not comparable, as the complete graph K_n has treewidth $n - 1$ but is 0-hyperbolic and 0-slim. Finally, it has been observed that many real-world networks are δ-hyperbolic for some small constant δ (see, e.g., [1, 2, 20, 31, 66] and the discussion in [1] for other relevant measures). There is evidence that the reason behind such a phenomenon is that real-world networks often have a large core of high-degree vertices through which most shortest paths pass [61, 66, 70]. Hence, hyperbolicity seems to be a useful measure by which to study networks (see also, e.g., [15]).

[1] For readers familiar with tree decompositions, the *treelength* of a tree decomposition of G is the maximum diameter of any bag, where distances are measured in G (and not in the metric of the subgraph induced by the bag). The *treelength* of a graph is the minimum treelength of any of its tree decompositions.

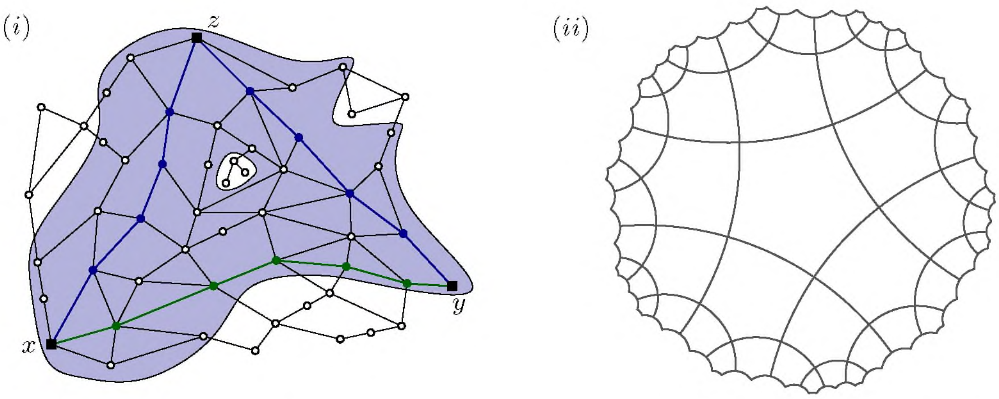

Figure 1 (i) The triangle xyz is 2-slim as the 2-neighborhood of the sides yz and zx (vertices in the blue shaded region) covers the side xy (green). (ii) Example of a planar constant-hyperbolic graph that does not directly resemble a tree. This graph is a part of the pentagonal tiling of the hyperbolic plane.

While a substantial research effort has focused on algorithms for treewidth and many other measures of tree-likeness, hyperbolicity has received comparatively limited attention. Chepoi et al. [21] studied spanners and the computation of the center and diameter of δ-hyperbolic graphs, while Chepoi and Estellon [22] considered packing and covering problems for balls. There have also been several investigations on how to compute the hyperbolicity of a graph and improve on the naive $\mathcal{O}(n^4)$-time algorithm [41, 16, 24, 25, 27, 38]. Some research has gone into studying algorithms for objects in hyperbolic geometry, e.g., point sets or graphs embedded in hyperbolic space. For example, Krauthgamer and Lee [58] and Kisfaludi-Bak [53] studied the Traveling Salesperson problem in this context. Using hyperbolic space, one can also define a graph where embedded vertices are adjacent if they are "close" in hyperbolic space, which gives rise to hyperbolic ball graphs. Hyperbolic random graphs, where the vertices are embedded randomly, are a particularly well-studied case of hyperbolic ball graphs (see, e.g., the survey of Friedrich [42] and the works of Bläsius et al. [6, 7, 8]). Kisfaludi-Bak [52] studied the complexity of several NP-hard problems on hyperbolic ball graphs. These results based on hyperbolic geometry are related but they have a different flavor. In particular, the geometric graphs are often but not always δ-hyperbolic graphs. Moreover, in the geometric setting one can always rely on the homogeneity of the underlying hyperbolic space. In contrast, δ-hyperbolic graphs (for small δ) have no underlying space that can be utilized. We note however that embedding δ-hyperbolic graphs into high-dimensional hyperbolic space with low distortion is possible [14].

To advance research on algorithms for graphs of bounded hyperbolicity, we consider it in the context of planar graphs. Treewidth is already well-studied in this context. Indeed, some problems have more efficient algorithms on bounded treewidth graphs if the graph is planar see, e.g., Dorn et al. [36]. (Recent work suggests that similar improvements may even be possible on general graphs [13, 29].) Treewidth is also an important tool in known approximation schemes for planar graphs [4, 33, 40]. For hyperbolicity, Cohen et al. [24] developed an algorithm to compute the hyperbolicity of an outerplanar graph in linear time. We are unaware, however, of any studies on general planar graphs in relation to hyperbolicity. Motivated by this gap in our knowledge, this paper initiates research on planar δ-hyperbolic graphs.

1.1 Main Contribution: A Novel Separator Theorem

A crucial tool for the algorithmic study of planar graphs has always been a balanced separator. Recall that for a graph class $\mathcal{G}$ and a function $\alpha : \mathbb{N} \to \mathbb{R}^+$ (possibly depending on $\mathcal{G}$), we say that an n-vertex graph $G \in \mathcal{G}$ has a *separator* of *balance* $\alpha(n)$ if there is a vertex set $Z \subset V(G)$ such that the number of vertices in any connected component of $G - Z$ is at most $(1 - \alpha(n)) \cdot n$. Typically, the balance is a constant independent of n. We say that a subgraph H of G is a geodesic path (cycle) if H is a path (cycle) where, for any $u, v \in V(H)$, we have $d_H(u, v) = d_G(u, v)$.

Famously, Lipton and Tarjan [62] proved that planar graphs have a $\frac{1}{2}$-balanced separator of size $\mathcal{O}(\sqrt{n})$. In fact, Lipton and Tarjan [62] showed these there exists such a separator that consists of *two* geodesic paths.

In this paper, we develop a balanced separator theorem for planar δ-hyperbolic graphs. Importantly, our separator consists of single geodesic path or geodesic cycle. This shows that hyperbolic planar graphs offer significantly more structure than general planar graphs.

▶ **Theorem 1** (Separator for planar δ-hyperbolic graphs). *Let G be a connected planar δ-hyperbolic graph on n vertices. Then G has a geodesic path separator X and a constant balance or G has a geodesic cycle separator Y and balance $2^{-\mathcal{O}(\delta)}/\log n$. Given G, such a separator X or Y can be computed in $\mathcal{O}(\delta^2 n \log^4 n)$ time.*

Additionally, $|X| = \mathcal{O}(\delta^2 \log n)$ and $|Y| = \mathcal{O}(\delta)$.

The proof of our separator theorem leans heavily on several novel techniques that we propose for planar δ-hyperbolic graphs. We first present a new variant of the well-known isoperimetric inequality for δ-hyperbolic graphs [19]. Then, we develop an iterative procedure that tries to construct a partition of a plane δ-hyperbolic graph into regions that are bounded by cycles of length $\mathcal{O}(\delta)$ and that contain a small number of vertices. We either obtain a balanced geodesic cycle separator during the execution of this procedure or, if the procedure finishes, we show how we can exploit the imbalance of the cycles to find a geodesic path separator by cutting "straight through" the embedding. In the latter case, it is highly non-trivial to argue that this separator is both short and balanced; this argument crucially relies on our new isoperimetric inequality. If the procedure cannot do anything, because all faces have length $\mathcal{O}(\delta)$, then we show how any separator can be turned into a balanced cycle separator using deep insights into the structure of such separators. Finally, we prove that we can shorten certain balanced cycle separators into geodesic cycle separators without losing our balance. As such, we develop new tools that combine insights into both planar and δ-hyperbolic graphs.

For a more detailed overview of the proof of Theorem 1, we refer to Section 2. The full proof is available in the full version of the paper [54].

Usefulness of single geodesic-cycle/geodesic-path separator

For a graph class $\mathcal{G}$, we say that a separator Z of a graph $G \in \mathcal{G}$ is *in-class* if $G[Z \cup \bigcup_{C \in \mathcal{C}} V(C)] \in \mathcal{G}$ for every subset $\mathcal{C}$ of the set of connected components of $G - Z$. Note, that any separator for a *hereditary* graph class is automatically in-class; for example, any separator for planar graphs. However, (planar) δ-hyperbolic graphs are *not hereditary*, as the deletion of any vertex can substantially alter distances. Indeed, a wheel graph W_{n+1} (consisting of C_n plus a central universal vertex) has hyperbolicity at most 2 (as hyperbolicity is upper bounded by diameter [21]), but removing its central vertex increases the hyperbolicity to (roughly) $n/4$. Hence, we need a different separator theorem that does guarantee the in-class property.

We now observe (and later prove formally) that if we separate a planar δ-hyperbolic graph along a geodesic path or cycle Z, then a shortest path P in G between two vertices in a component C of $G - Z$ such that P is fully not contained in C can be "rerouted" along Z and remain shortest in $G[V(C) \cup Z]$. Therefore, as a consequence of Theorem 1, we obtain the desired in-class separator:

▶ **Corollary 2.** *For any $\delta \geqslant 0$, the class of connected planar δ-hyperbolic graphs has a $1/2$-balanced in-class separator of size $2^{\mathcal{O}(\delta)} \log n$ that can be computed in $2^{\mathcal{O}(\delta)} \cdot n \log^5 n$ time.*

Discussion about the size of the separator

In the applications of Theorem 1 and Corollary 2, discussed later, we will mainly use the property that our separator is *in-class* and has *sublinear size.* Note that Theorem 1 even guarantees that the returned separator has size $\mathcal{O}(\delta^2 \log n)$. The separator of Theorem 1 thus is significantly smaller (for small values of δ) than the general planar separator of size $\mathcal{O}(\sqrt{n})$ [62] (which is not even in-class in our case). On the other hand, the size bound is reminiscent of known separators for hyperbolic ball graphs and random graphs, which also have size $\mathcal{O}(\log n)$ in certain regimes [7, 52, 57]. Recall, however, that hyperbolic ball graphs differ substantially in nature from the (planar) δ-hyperbolic graphs we study in this paper.

We observe that a separator of size $\mathcal{O}_\delta(\log n)$ that is *not in-class* can be easily obtained by combining two existing results. Chepoi et al. [21, Proposition 13] bounded the treelength of δ-hyperbolic graphs, and Dieng and Gavoille [35] (see also [34]) bounded the treewidth of a planar graph in terms of its treelength, which gives the following bound on the treewidth of planar δ-hyperbolic graphs:

▶ **Proposition 3** ([21] and [35]). *For any $\delta \geqslant 0$, the treewidth of any n-vertex planar δ-hyperbolic graph is $\mathcal{O}(\delta \log n)$.*

Observe that a constant-factor approximation of the treewidth tw of a planar graph can be computed in $\mathcal{O}(n \cdot \mathrm{tw}^2 \log \mathrm{tw})$ time [50, 46]. Using standard arguments (see, e.g., [68, (2.5)]), Proposition 3 and the fact that $\delta < n$ immediately implies the existence of a balanced separator:

▶ **Corollary 4.** *For any $\delta \geqslant 0$, the class of planar δ-hyperbolic graphs has a $1/2$-balanced separator of size $\mathcal{O}(\delta \log n)$ that can be computed in $\mathcal{O}(\delta^2\, n \log^3 n)$ time.*

We stress again that the effectiveness of Corollary 4 alone is somewhat doubtful. When attempting to employ it in recursive algorithms (a common approach for utilizing separators), the separator fails to guarantee that its components are again δ-hyperbolic. Theorem 1 guarantees that the separator consists of a single geodesic path or a single geodesic cycle, which allows us to develop novel algorithmic applications, which we discuss now.

1.2 Applications of our Separator Theorem

We present two applications of our separator theorem to well-known graph problems. Recall that an *independent set* of a graph G is a set $I \subseteq V(G)$ such that $uv \notin E(G)$ for any $u, v \in I$. Then the Maximum Independent Set problem asks to find an independent set of maximum size in a given graph G. We give a near-linear time FPTAS for Maximum Independent Set on planar δ-hyperbolic graphs (for any fixed δ).

▶ **Theorem 5.** *For any $\delta \geqslant 0$ and any $\varepsilon > 0$, the class of planar δ-hyperbolic graphs has a $(1-\varepsilon)$-approximation algorithm for* Maximum Independent Set *running in $2^{\mathcal{O}(\delta)} n \log^6 n + 2^{\mathcal{O}(\delta^2)} n/\varepsilon^{\mathcal{O}(\delta)}$ time.*

It is important to compare our approximation scheme to the known EPTAS for Maximum Independent Set on planar graphs, which runs in time $2^{\mathcal{O}(1/\varepsilon)}n$ [4] and is asymptotically optimal [65]. Our algorithm will be substantially faster for small values of δ. We also observe that the usual approach to planar approximation schemes that uses a treewidth bound (e.g., Proposition 3), as pioneered by Baker [4], is likely not possible here. Indeed, recall that the class of δ-hyperbolic graphs is not hereditary and thus removing BFS-layers does not necessarily preserve hyperbolicity. Hence, our algorithm uses the separator of Theorem 1 in the same way Lipton and Tarjan [63] did in their pioneering work. In particular, we show that we can compute a (weak) r-division of which each part induces a planar δ-hyperbolic graph. Our algorithmic approach is actually more general (using ideas of Chiba et al. [23]) and allows us to prove approximation schemes for several other problems (including e.g. the Maximum Induced Forest problem).

We next consider the Traveling Salesperson problem. We only consider the variant on undirected, unweighted graphs. We define a *tour* in a graph G to be a closed walk in G that visits every vertex of G at least once. Then the Traveling Salesperson problem (also known as the Traveling Salesman problem or Graph Metric TSP) asks, given an unweighted, undirected graph G, to find a shortest tour in G. We give a near-linear time FPTAS for the Traveling Salesperson problem on planar δ-hyperbolic graphs (for any fixed δ).

▶ **Theorem 6.** *For any $\delta \geqslant 0$ and any $\varepsilon > 0$, the class of planar δ-hyperbolic graphs has a $(1+\varepsilon)$-approximation algorithm for the* Traveling Salesperson *problem running in $2^{\mathcal{O}(\delta)} \cdot n \log^6 n + 2^{\mathcal{O}(\delta^2)} n/\varepsilon^{\mathcal{O}(\delta)}$ time.*

We again compare our approximation scheme to the known approximation schemes for the Traveling Salesperson problem on planar graphs. A first PTAS for this problem, running in time $n^{\mathcal{O}(1/\varepsilon)}$, was designed by Grigni et al. [44]. This later improved to an EPTAS running in time $2^{\mathcal{O}(1/\varepsilon)}n$ by Klein [55]. (For later generalizations, to the weighted case and H-minor-free graphs, see e.g. [55, 60, 26] and references therein.) Our scheme will be substantially faster for small values of δ. A crucial element in all these schemes is the definition of appropriate subproblems and the patching of partial solutions to form a general solution. Grigni et al. and Klein use different approaches to address these challenges: the former uses a recursive separator approach whereas the latter combines a spanner with a Baker-style shifting technique. Like for Maximum Independent Set, we must be careful that planar δ-hyperbolic graphs are not hereditary. Therefore, our approach again relies on the recursive separator approach of Lipton and Tarjan [63], although some of its ideas feel reminiscent of those underlying the previous schemes [44, 55].

Finally, we note that an FPTAS for Maximum Independent Set or the Traveling Salesperson problem is generally not possible, unless P=NP. However, the dependence on δ in Theorem 5 means that our schemes do not disprove the standard complexity theory assumptions. We note that our approximation schemes can also be seen as parameterized approximation schemes (see e.g. [37]), in particular as EPASes, with parameter δ.

1.3 Connection to Exact Algorithms

To build a connection to exact algorithms, we first observe that the following results are immediate from Proposition 3 combined with known algorithms on graphs of bounded treewidth for Maximum Independent Set [3] and the Traveling Salesperson problem [60, Appendix D] respectively.

▶ **Corollary 7.** *For any $\delta \geqslant 0$, the class of planar δ-hyperbolic graphs has an algorithm for* Maximum Independent Set *running in time $n^{\mathcal{O}(\delta)}$.*

▶ **Corollary 8.** *For any $\delta \geqslant 0$, the class of planar δ-hyperbolic graphs has an algorithm for the* Traveling Salesperson *problem running in time $n^{\mathcal{O}(\delta)}$.*

Note that, alternatively, these results follow from our approximation schemes (with an extra factor $2^{\mathcal{O}(\delta^2)}$ in the running time) by setting $\varepsilon = 1/\Omega(n)$.

For Maximum Independent Set, we prove a lower bound matching Corollary 7, conditional on the Exponential Time Hypothesis (ETH) [49], which asserts that there is no $2^{o(n)}$-time algorithm for the 3-Satisfiability problem. We prove:

▶ **Theorem 9.** *There is no $n^{o(\delta)}$-time algorithm for* Maximum Independent Set *in planar δ-hyperbolic graphs, unless ETH fails.*

This result immediately implies that the running time of Theorem 5 is also essentially optimal, in the sense that there is no $(1-\varepsilon)$-approximation scheme running in time $\text{poly}(n)/\varepsilon^{o(\delta)}$, unless ETH fails.

The lower bound of Theorem 9 also stands in contrast to our knowledge of graphs of bounded treewidth. It is known that Maximum Independent Set can be solved in $2^{\mathcal{O}(\text{tw})}n$ time on n-vertex graphs of treewidth tw, but such a result (fixed-parameter tractability) will be unlikely by Theorem 9.

1.4 Organization

We first give an overview of the main ideas of our paper in Section 2, particularly those behind Theorem 1 and Theorem 9. We discuss our results and ask open questions in Section 3. The full version of this paper [54] contains detailed proofs.

2 Overview of Main Ideas and Techniques

In this section, we discuss the combinatorial observations and ideas behind the proofs of Theorem 1 and Theorem 9.

2.1 Main Ideas and Techniques for the Separator Theorem

For the sake of convenience, we briefly restate Theorem 1.

▶ **Theorem 1** (Separator for planar δ-hyperbolic graphs). *Let G be a connected planar δ-hyperbolic graph on n vertices. Then G has a geodesic path separator X and a constant balance or G has a geodesic cycle separator Y and balance $2^{-\mathcal{O}(\delta)}/\log n$. Given G, such a separator X or Y can be computed in $\mathcal{O}(\delta^2 n \log^4 n)$ time.*

Additionally, $|X| = \mathcal{O}(\delta^2 \log n)$ and $|Y| = \mathcal{O}(\delta)$.

To grasp this theorem and how we prove it, it is important to understand why the dichotomy of the two types of separators in this theorem is unavoidable, and why the separator requires size $\Omega(\log n)$ and $\Omega(\delta)$, respectively. To this end, we give two illustrative examples.

First, there exist planar hyperbolic graphs of treewidth $\Omega(\log n)$. For example, Kisfaludi-Bak [51, Lemma 28] showed that a size-n patch of the pentagonal tiling of the hyperbolic plane (see Figure 2) contains a plane subgraph that is a subdivision of a $\log n \times \log n$ grid. Thus, a path separator of a large balance must be of length $\Omega(\log n)$. (See the full version

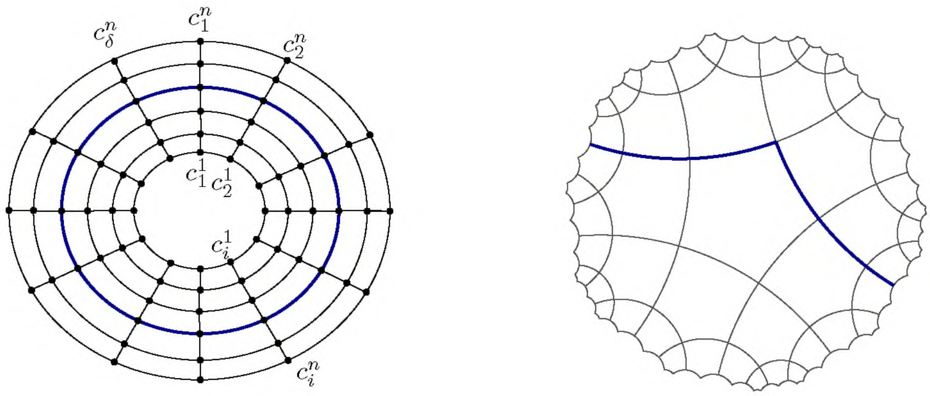

Figure 2 Left: the δ-cylinder with hyperbolicity $\Theta(\delta)$, and a typical geodesic cycle separator (in blue). Right: the pentagonal tiling graph and a shortest path separator (in blue).

of this paper [54] for a stronger lower bound of $\Omega(\delta \log n)$.) The pentagonal grid example (combined with the isoperimetric inequality [45, 19] discussed later) demonstrates that a geodesic cycle separator alone cannot always lead to a balanced separator, as any cycle of length ℓ in this graph can cut away only $\mathcal{O}(\ell)$ vertices.

Second, we consider the simple example of a δ-cylinder: a graph consisting of δ copies of a cycle $C = \{c_1, \ldots, c_\delta\}$ and for each i, a path through the vertices c_i of each copy (see Figure 2). The δ-cylinder has hyperbolicity $\Theta(\delta)$. In the δ-cylinder, any geodesic cycle that would be a balanced separator has at least δ vertices. This example also demonstrates that a geodesic path alone cannot lead to a balanced separator.

By these examples, our algorithm needs to output either a geodesic path or a geodesic cycle as a separator. Moreover, they need to be of the size as stated in the theorem, apart from a possible factor δ overhead in the size of the geodesic path separator. While the examples served as the starting point for our thinking in the proof of Theorem 1, we note that these examples are from being able to represent the full generality of planar δ-hyperbolic graphs.

We now give a high-level overview of the proof of Theorem 1. Let G be a planar δ-hyperbolic graph embedded on a sphere $\mathbb{S}^2$. For simplicity, we assume first that G is 2-connected; we later argue how we can reduce to this case.

Our algorithm heavily relies on a known procedure to create $\mathcal{O}(\delta)$-fillings in δ-hyperbolic graphs (see [19]). In the context of planar graphs, an *$\mathcal{O}(\delta)$-filling* of a cycle C of G is a 2-connected planar subgraph H of G that has C as a face, and where each face of H (except C) has length $\mathcal{O}(\delta)$. Intuitively, the minimum number of faces over all fillings serves as a discrete notion of *area*. Formally, we prove the following statement:

▶ **Lemma 10** (Greedy-Filling Procedure). *Let G be a 2-connected planar δ-slim graph on n vertices, and let C be a face of G. There exists a data structure that for a cycle C inside G given to it as a query, can construct a greedy filling H_C of C inside G. The returned graph H_C is 21δ-filling of C and has area $\mathcal{O}(\delta|C|)$. The initialization time of the data structure is $\mathcal{O}(\delta^2 n \log^4 n)$. The query time is $\mathcal{O}(\delta|C| \log\log n)$ if a cycle C is given as a query.*

The data structure can also answer shortest-path queries about distances of length at most 10δ in $\mathcal{O}(\delta + \log\log(n))$ time.

The so-called *isoperimetric inequality* [45, 19] in hyperbolic metric spaces asserts that the minimum number of faces in a filling of C is $\mathcal{O}(|C|)$. The procedure essentially "chops off" parts of the region enclosed by C in a greedy manner, where each part is bounded by a cycle

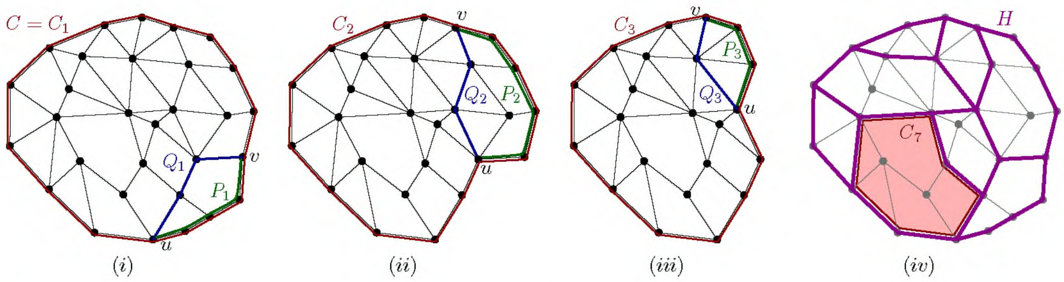

Figure 3 Greedy filling for the face with boundary cycle C. (i)-(iii) depicts the first few steps of the filling. (iv) shows the final filling H. The greedy-filling procedure terminated when it found C_7, which is a geodesic cycle.

of length $\mathcal{O}(\delta)$. See Figure 3 for a pictorial description of the procedure. Since computing the greedy filling with the minimum number of faces may be difficult, we show a generalization of the isoperimetric inequality for a fixed greedy filling that we can compute (see the full version of this paper [54] for precise definitions). In a fixed planar embedding of G we say that a cycle γ *interacts* with a face F if γ intersects F or if the bounded region defined by γ completely contains F . We can also define interaction for a sphere embedding of G when a face C plays the role of an outer face.

▶ **Lemma 11** (Isoperimetric Inequality for Greedy Fillings). *Let G be 2-connected graph*[2] *embedded on $\mathbb{S}^2$ and let H be some* greedy k-filling *of C inside some subgraph*[3] *G' of G where C is contained in G'. Then any simple cycle in G of length ℓ interacts with at most $(k+1)\cdot\ell$ faces of H. Moreover, if the cycle is in H, then it interacts with at most ℓ faces.*

We apply this filling procedure in an iterative manner with the goal of arriving at an $\mathcal{O}(\delta)$-filling where each face of the filling contains "few" vertices of G. First, we apply the filling procedure on the longest face in G itself. If one of the faces of the resulting $\mathcal{O}(\delta)$-filling covers a region F that contains "many" vertices of G, then we apply it iteratively on the longest face of this region. In this new iteration, the rest of the graph (formed by vertices in the interior of $\mathbb{S}^2 \setminus F$) is removed. This region becomes a face, which is assigned a weight equal to the number of vertices of G outside F, of which there are "few" by the fact that there are "many" vertices of G inside F. In this way, we slowly and iteratively proceed towards our stated goal. We only terminate prematurely if along the way a suitable separator is already found (see Outcome 1 below).

Let G' be the graph after the final iteration of our algorithm (this may be after the above procedure fully finishes or is terminated prematurely). We can terminate with one of three outcomes:

Outcome 1: One of the cycles of the current filling of G' already has a good enough balance.

Outcome 2: The maximum face length of G' is $\mathcal{O}(\delta)$, i.e., a new greedy filling procedure would terminate with the trivial filling consisting only of the initial face cycle.

Outcome 3: All faces of the current filling have few vertices of G inside.

[2] We note that the lemma does not use the hyperbolicity of G directly, only indirectly, namely in the fact that the greedy filling H turned out to be a k-filling. Generally, the filling face that is created last may have arbitrary length.

[3] We need to consider a subgraph G' for technical reasons; in a typical application one should think of $G' = G$.

Note that when looking at the number of vertices of G inside a face of the current filling, or in other words at the balance of this face, we also account for the newly assigned weights to (some of) the faces. We describe how we deal with each of these outcomes in turn.

In **Outcome 1**, there is a cycle in the filling with a good enough balance. In general, we can prove that if we encounter a cycle separator of length ℓ with balance α, then we can compute a geodesic cycle separator of length $\mathcal{O}(\delta)$ with balance $\alpha/2^{\mathcal{O}(\ell)}$. This can be obtained by iteratively carving away a constant fraction of the vertices inside (or outside) the cycle while reducing the length of the separating cycle by at least 1. Applying this shortcutting procedure to the assumed cycle, we obtain a geodesic cycle separator.

In **Outcome 2**, all faces of G' have length $\mathcal{O}(\delta)$. This is the case, for example, for δ-cylinders. In a δ-cylinder, we can directly find a geodesic cycle separator roughly in the middle of the cylinder. However, this intuition does not immediately carry over to general planar δ-hyperbolic graphs with short faces.

We first find a separator S of size $\mathcal{O}(\delta \log n)$ by Corollary 4. However, this separator is possibly not geodesic nor a path or cycle. Next, our goal is to transform the separator S into a separator $\bar{S}$ that has a good *split balance*, meaning that each face of the graph induced by $\bar{S}$ contains at most a constant proportion of the vertices of G'. This transformation is non-trivial and is done with the help of an auxiliary graph. Once the separator $\bar{S}$ with a good split balance is found, we can find some collection of faces in $G'[\bar{S}]$ whose union U has a boundary ∂U that gives a constant-balanced separator. However, the boundary ∂U may consist of several cycles. We then find a single component cycle of ∂U with a good split balance. Here, we need to offset the split balance of the cycle against its length. Hence, we select the component cycle γ_i of ∂U with the best ratio of balance to length. We then use a more involved shortening procedure on this cycle to find a geodesic cycle with the desired balance. For details, see the full version of this paper [54].

Finally, we can end up in **Outcome 3**. In this case we think of the graph as embedded on the plane with the outer face being the starting cycle of the final filling. Recall that in this case, each filling face (except the outer face) has only a few vertices of G inside. This case would be the outcome if the initial graph is a patch of the pentagonal grid, and the filling is based on the cycle around the perimeter of the patch, i.e., the boundary of the outer face. We now claim that we can find two "antipodal" vertices on the outer face such that some shortest path between them is a balanced separator.

It is far from clear in general why some shortest path connecting two "antipodal" vertices of the outer face has constant balance. We begin this proof by defining *layers* on the filling faces: a face is in layer i if its distance to the outer face is i. Roughly, we aim to show that there are only $\mathcal{O}(\log n)$ layers.

To bound the number of layers, we prove a variant of the isoperimetric inequality for our purposes, which may be of independent interest. In general, consider a planar δ-hyperbolic graph with a $\mathcal{O}(\delta)$-filling H of a cycle C of G, where C is the cycle along a face in some fixed embedding of G. We then consider an arbitrary cycle γ in G. Recall that our *isoperimetric inequality for greedy fillings* shows that γ interacts with $\mathcal{O}(\delta|\gamma|)$ faces of the filling H. To bound the number of layers, one can show that the outer face cycle touches a constant proportion of all faces, i.e., the outermost layer has a constant proportion of all the faces of the filling. Iterating this argument shows that the number of layers is $\mathcal{O}(\log n)$.

Then, in the last layer, we find two vertices that are as far from the outer face as possible. Using an auxiliary tree in the planar dual graph, we can select a good balanced cut edge. The endpoints of the corresponding primal edge are connected to their respective nearest vertices a and b on the outer face. The path we obtain this way from a to b is a balanced separator, but unfortunately, it is not a shortest path. We need to argue about the balance of a shortest path from a to b in G instead. To prove that some shortest ab path is also a balanced separator, we crucially rely on the isoperimetric inequality on greedy fillings again. The relatively short closed walk given by the initial path and the shortest path can only interact with a small number of filling faces due to the isoperimetric inequality. Since we are in the case where each filling face contains only a few vertices inside, we can upper bound the balance shift between the initial path and the shortest path. This then gives the geodesic path separator with the desired balance. For details, see the full version of this paper [54].

With the above outcomes handled, the only missing piece of the proof of Theorem 1 is the case of graphs that are not 2-connected. Again, since hyperbolicity is very sensitive to changes in the graph, we have a slightly more technical reduction from the general case to the 2-connected case. Intuitively, it is enough to find a separator of a "central" 2-connected component, but this would not immediately account for the number of vertices in other components and thus potentially lead to an imbalanced separator. We represent all the non-central 2-connected components of G by attaching wheel graphs to the central 2-connected component, which (i) ensures that a balanced separator of the reduced 2-connected instance corresponds to a balanced separator in the original graph and (ii) does not increase the hyperbolicity significantly.

2.2 Main Ideas and Techniques for the Approximation Algorithms

We rely on the notion of a (weak) r-division of a graph G. This is a family of $\Theta(n/r)$ subsets of $V(G)$ (called groups) that each have size at most r, jointly cover $V(G)$, and the total number of edges between the groups is $O(n/\sqrt{r})$. Using Corollary 2, we can show that a weak r-division of a planar δ-hyperbolic graph can be computed in $2^{O(\delta)} \cdot n \log^6 n$ time such that each group induces again a planar δ-hyperbolic graph. To obtain the latter property, it is crucial that our separator is in-class.

We can derive the algorithm for Maximum Independent Set by adapting a proof of Chiba et al. [23] and Lipton and Tarjan [63]. We obtain a weak r-division for $r = 1/\varepsilon^2$. Since each group of the weak r-division is planar δ-hyperbolic, we can use Proposition 3 to bound the treewidth of each group and use the known algorithm for Maximum Independent Set on graphs of bounded treewidth. Joining the solutions appropriately then yields Theorem 5.

For the Traveling Salesperson problem, extra care is needed to patch together the solutions for the different groups and obtain Theorem 6.

2.3 Main Ideas and Techniques for the Lower Bound

For the sake of convenience, we restate the lower bound:

▶ **Theorem 9.** *There is no $n^{o(\delta)}$-time algorithm for Maximum Independent Set in planar δ-hyperbolic graphs, unless ETH fails.*

The proof is based on embedding a subdivision of a Euclidean grid (with some diagonals) into a planar δ-hyperbolic graph. It is known that Maximum Independent Set in subgraphs of the $n \times n$ grid (with some diagonals) has a $2^{o(n)}$-time lower bound under ETH [30], which we use as our starting point.

Let G be a given subgraph of an $n \times n$ grid with diagonals, which we denote by Grid_n. Observe that subdividing each edge of G an even number of times gives an instance that is equivalent to G [67]. We are then left with two tasks: (a) create a planar δ-hyperbolic host graph of size $2^{O(n/\delta)}$ that "surrounds" an even subdivision of G, and (b) make sure that the surrounding parts of the host graph created in (a) do not impact the hardness proof.

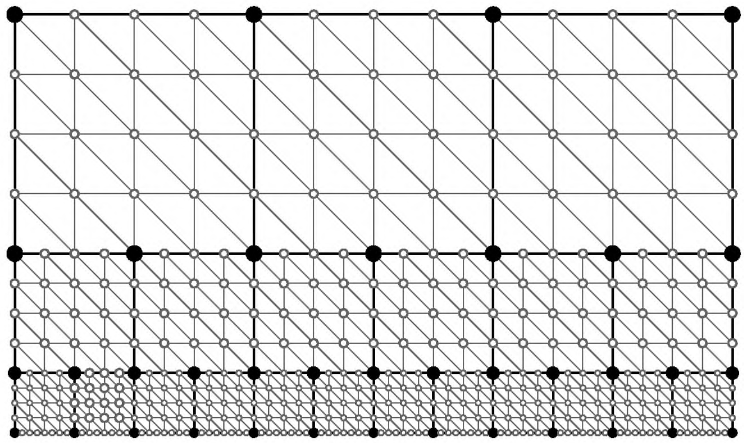

Figure 4 A part of a binary tiling (thick edges) with small $\delta \times \delta$ grids (with diagonals) embedded in each face except the outer face.

Task (a) requires a thorough, technical approach. It is in fact easier to think of embedding Grid_n itself instead of focusing on some custom graph G. Our construction is based on the so-called *binary tiling* of Böröczky [17], which is a tiling of the hyperbolic plane; see Figure 4. The underlying infinite planar graph of this tiling is known to be constant hyperbolic. We carefully choose a portion B_1 of this graph that has size $2^{\mathcal{O}(n/\delta)}$ and is still constant-hyperbolic. By inserting $\delta \times \delta$ grids into the faces of B_1 (except its outer face), we can show that we get an $\mathcal{O}(\delta)$-hyperbolic graph B_{Grid}, which contains some subdivision of Grid_n. A further modification of B_{Grid} leads to a graph B that is $\mathcal{O}(\delta)$-hyperbolic, is of size $2^{\mathcal{O}(n/\delta)}$, and contains a subgraph that is an even subdivision of G. It follows that B contains a hard instance of Maximum Independent Set as a subgraph and we have achieved (a).

To achieve (b), we cannot just remove unwanted parts of the $2^{\mathcal{O}(n/\delta)}$-hyperbolic host graph we just constructed, as that would change the hyperbolicity. However, in the case of Maximum Independent Set, we show how to achieve (b) using a simple local modification, attaching a small gadget to vertices of B that are not in the hard instance, that does not impact hyperbolicity. This gives the desired graph whose maximum independent sets can be related to the maximum independent sets of G and completes the reduction.

3 Discussion and Open Problems

We conclude this paper with a discussion of future directions that have sprung up from our work. A first natural question is whether there is a $n^{o(\delta)}$ time lower bound to solve the Traveling Salesperson problem on (planar) δ-hyperbolic graphs under the Exponential Time Hypothesis (ETH). This would match the running time of the exact algorithm of Corollary 8 and also imply that the running time of the approximation scheme of Theorem 6 is essentially best possible. One might expect that this could be possible by suitably adjusting the construction of Theorem 9. However, we recall that a crucial aspect of this construction was to plant a subdivision of a grid inside a large δ-hyperbolic graph, without this affecting the hardness proof. While for Maximum Independent Set this was possible by some local modifications, it seems much harder to achieve this for the Traveling Salesperson problem.

A next algorithmic question is whether there exist fast approximation schemes (i.e., a near-linear time FPTAS) for other problems on planar δ-hyperbolic graphs. This paper develops such a scheme for Maximum Independent Set by employing ideas of the recursive separator approach of Lipton and Tarjan [63]. It is known, however, that this approach can also lead to approximation schemes for e.g. Minimum Vertex Cover [23] and Minimum Feedback Vertex Set [56]. Both these algorithms rely on the contraction of edges to ensure the solution size is at least linear in the number of vertices. However, planar δ-hyperbolic graphs are not closed under edge contraction and thus this approach cannot succeed. The non-hereditary nature of (planar) δ-hyperbolic graphs stands in the way of other approaches for these problems as well (e.g., [4]; [5, 47]; or [18]).

In the same vein, we wonder about approximation schemes for other network design problems. This would extend our scheme for the Traveling Salesperson problem. Therefore, we ask about a near-linear time FPTAS for problems such as Subset TSP and Steiner Tree. We deem the existence of such schemes to be highly plausible.

Turning to our separator theorem, we notice that the geodesic cycle separator that is returned by Theorem 1 has size $\mathcal{O}(\delta)$ but balance $2^{-\mathcal{O}(\delta)}/\log n$. While we know that the size bound cannot be improved by the example of a δ-cylinder, we do not know of any example that shows that the balance factor should be $2^{-\mathcal{O}(\delta)}/\log n$. Therefore, we ask whether we can find, in near-linear time, a geodesic cycle separator of balance equal to some constant.

Finally, we consider how to determine the hyperbolicity of a graph. This can be naively done in $\mathcal{O}(n^4)$ time by following the definition of hyperbolicity. Fournier et al. [41] improved this to $\mathcal{O}(n^{3.69})$ and even gave a 2-approximation that runs in $\mathcal{O}(n^{2.69})$ time. Considering the context of planarity, Borassi et al. [16] proved that the hyperbolicity of a general sparse graph cannot be computed in subquadratic time under the Strong Exponential Time Hypothesis (SETH). However, this general result does not exclude the existence of a subquadratic algorithm on planar graphs. A potentially encouraging sign in this direction is the known linear-time algorithm for outerplanar graphs [24]. Hence, we repeat the question of Cohen et al. [24] and ask for a linear-time algorithm that computes the hyperbolicity of planar graphs.

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